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SURVEY

Smooth unimodal maps in the 1990s

JACEK GRACZYK†§ and GRZEGORZ ŚWIĄTEK‡¶

† *Department of Mathematics, Royal Institute of Technology, 100 44 Stockholm, Sweden*
‡ *Department of Mathematics, Pennsylvania State University, University Park, PA 16802, USA*

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Abstract. The purpose of this paper is to survey the key ideas which have played a role in the development of the theory of smooth unimodal maps in the 1990s so as to provide a starting point for more detailed study. Most papers underlying this survey are distinguished by great technical complexity; we have therefore attempted to extract their essence and present it in a way accessible to a non-specialist. For further study we compiled a long list of original references. Another goal of this paper is to present the multi-directional process in which ideas flowed and developed, placed in a historical order.

1. *Overview and sources of progress*

Among non-linear smooth dynamical systems the theory of one-dimensional maps occupies a special place. This is a class of relatively simple transformations with rich dynamics. The simplicity comes from topological triviality of the real line. The hard part lies in delicate estimates of real and complex nature, often involving combinatorial analysis of sequences of dynamically defined points.

Many phenomena present in higher dimensions are impossible on the real line, but computational accessibility makes one-dimensional maps uniquely useful to model and study rigorously properties of non-hyperbolic systems which may be too hard to approach in higher dimensions. Such phenomena include transition to chaos, smoothness of conjugation, problems of stability and abundance of prototype systems. Smooth unimodal mappings of an interval are a particularly simple class from the point of view of topological dynamics, yet rich enough to show a wide range of behaviour, from deterministic to chaotic. In addition, the quintessential examples of such maps, quadratic polynomials, are also the simplest analytic dynamical systems on the Riemann sphere.

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In the 1980s an impressive theory of one-dimensional maps with negative Schwarzian was developed, see [**MS1**] for an overview and details, which made it possible to study the critical phenomena. Also beginning from the mid-1980s, the idea of treating jointly real and complex systems became increasingly popular. Since then, complex dynamics have contributed to the development of unimodal theory.

The purpose of this paper is to survey the development in the field of smooth unimodal dynamics in the 1990s. We have made an effort to cover the developments through to the beginning of 1998. Recent progress in understanding topological aspects of one dimensional dynamics (real and complex) and in inventing appropriate analytic tools resulted in complete or partial solutions of a few conjectures which have challenged the researchers for some time.

Smooth unimodal maps. A differentiable map f of the interval I is called unimodal if it has exactly one critical point, $f'(c) = 0$, and maps the boundary of I into itself. All unimodal maps considered in this paper are symmetric with respect to c and have the singularity of a polynomial type, i.e. $\lim_{x\to c} |f'(x)|/|x - c|^{\ell-1}$ exists and is different from 0 for a positive integer ℓ. The number ℓ is called the *order* of the critical point c. S-unimodal maps are C^3 and have negative Schwarzian derivative $Sf = f'''(x)/f'(x) - (3/2)(f''(x)/f'(x))^2 < 0$ for $x \in I \setminus \{c\}$. Diffeomorphisms with negative Schwarzian behave similarly to univalent maps in the complex plane. In particular, they expand the Poincaré distance. There is also an abundance of real distortion estimates for S-unimodal maps. They are usually named after Köbe, the most likely to emphasize similarities to the classical Köbe distortion theorem. Our estimates are based on the following distortion lemma.

LEMMA 1.1. *Suppose that f is a diffeomorphism onto its image, either defined on the interval $(0, 1)$ with non-negative Schwarzian derivative, or univalent on the unit disk. Suppose that x, y are points in the domain of f and are at a distance at least δ from the boundary. Then*

$$\frac{|f'(x)|}{|f'(y)|} < K\delta^{-4}$$

where K is a constant.

Proof. In the holomorphic case, this is a direct corollary to the Köbe distortion lemma. In the negative Schwarzian case, it is a corollary to the so-called real Köbe lemma, see Lemma 1 in [**Gu2**]. □

Main problems. The first of these was the problem of measure theoretical attractors, see [**Mi1**]. A forward invariant set A will be called a *metric attractor* for some dynamics, provided that $\omega(x) \subset A$ for a set of x of positive Lebesgue measure, and A has no proper subset with the same property. By changing the words 'metric' to 'topological' and 'positive Lebesgue measure' to 'residual', we get the definition of a *topological attractor*. The question is whether these two must be equal. At the beginning of the 1990s, the question was open for the class of unimodal mappings with reasonable smoothness.

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The second problem was the real Fatou's conjecture, see [**MiT**]. The conjecture held that the set of parameter values in the logistic family for which there is an attracting periodic orbit is dense. Another famous problem which was solved recently, concerns the topological invariance of the so-called Collet–Eckmann condition for S-unimodal maps. This problem was posed in [**S**] as well as by Guckenheimer and Misiurewicz in the early 1980s. Yet another problem concerned the geometry of infinitely renormalizable real-analytic unimodal maps, the so-called complex bounds. This is an important technical property with numerous consequences.

1.1. *Main concepts.* These solutions involved the work of many mathematicians working in the area. These people came with varying backgrounds and motivations for working in one dimension. Not surprisingly, it is hard to pinpoint a single concept joining all these results. We will concentrate on four sources of ideas. Interaction with complex dynamics is one. Another idea of crucial importance was the quest for induced hyperbolicity. Renormalization was another useful concept, as were ideas coming from ergodic theory. We will proceed to briefly discuss these approaches. Among new techniques, cross-ratio distortion deserves to be emphasized. This tool appeared in the 1980s in the works of several authors; see [**Yo1, Sw, MS2**].

Interaction with complex dynamics. A quintessential unimodal map is the quadratic polynomial $x \to ax(1-x)$. It is a good idea to view it as an analytic dynamical system in the Riemann sphere. This will enhance the set of tools which can be used. Also, the great simplicity of the system can be more readily used in the complex setting. Of particular importance here is the method of conformal deformations introduced to dynamics by Sullivan, see [**Su1**]. This approach gives checkable conditions for analytic structural instability in a narrow set of mappings, for example, in the set of quadratic polynomials. Nothing like this is available if unimodal maps are viewed purely from the real point of view, because sufficiently smooth perturbations of non-hyperbolic systems have global and generally unpredictable consequences. What is also interesting is that many results, once they have been obtained for holomorphic maps, or just polynomials, can later be extended to non-analytic cases. This is exemplified by recent progress in the study of S-unimodal Collet–Eckmann maps. First, Przytycki advanced a technique of estimating distortion for analytic CE maps, introducing a concept of 'shrinking neighborhoods' [**P**]. In particular, this and certain inductive methods of decomposing orbits led to the global estimates of expansion for complex unimodal polynomials [**GSm**]. In consequence, the Collet–Eckmann condition turned out to be equivalent to some other analytical conditions [**GSm**] and [**NSa**]. Of special importance was the equivalence with a uniform expansion on periodic orbits proved in [**NSa**]. The final conceptual step was achieved in [**PR**] where the so-called topological Collet–Eckmann condition was formulated (without naming it so) for rational maps. The proof that this topological condition is implied by the Collet–Eckmann condition was then translated in [**NP2**] to the real S-unimodal setting where the results of [**NSa**] were valid and the topological invariance of the Collet–Eckmann condition was established.

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Induced hyperbolicity. An approach that we personally favour is looking for induced hyperbolicity. Given a non-hyperbolic system one tries to make it hyperbolic by taking only pieces of the phase space and a high iterate of the map on each piece. Jakobson used this approach to prove his theorem on the abundance of absolutely continuous invariant measures [**J1**]. The construction of Yoccoz for quadratic polynomials is concentrated on partitions which are not quite Markov and the thrust is in the direction of making the non-Markov part small. On the real line and for real polynomials, the Yoccoz and Jakobson constructions give equivalent partitions. The approach of Benedicks and Carleson is loosely similar in the sense that they also look for pieces of the phase space with nice behaviour of high iterations, see [**BeC1**]. Their work on the Hénon map also underscores the more general nature of the idea and its usefulness in higher dimensions [**BeC2**]. In most examples, induced hyperbolicity can only be obtained if many parameter values are excluded. A particular feature of the unimodal case is that it makes sense to look for induced hyperbolicity for all parameters. Even in cases which are not expansive, such as infinitely renormalizable maps, the constructions involved in the quest for induced hyperbolicity remain useful.

Canonical Markov extensions. We already mentioned how it is useful to look for hyperbolicity even when it is absent in the original system. There is another way of doing this through so-called *Markov extensions*. A Markov extension is a Markov mapping F, usually defined on a countable disjoint union of intervals, with a projection π onto the interval which satisfies

$$f \circ \pi = \pi \circ F$$

where f is the original map. In 1979 Hofbauer introduced the idea of using certain countable-state Markov chains, called Markov diagrams, which were induced by the original system and he showed how their properties were connected with the asymptotic topological dynamics of the original system [**H**]. About a decade later, this idea was revisited by Keller [**K1**], with more emphasis on asymptotic metric properties. In [**K2**], canonical Markov extensions were studied for polymodal maps of the interval with negative Schwarzian derivative, with strong results about the ergodic properties of such maps and the attractor problem. In the 1990s, canonical Markov extensions became quite well-known and were often referred to as *Hofbauer towers*. The idea of cutting times which also comes from the Hofbauer tower construction was applied by Nowicki and Sands in their paper about Collet–Eckmann maps [**NSa**]. In fact, that paper coupled the formalism of cutting times with cross-ratio estimates to develop a technique for getting strong estimates on the geometry of unimodal maps. This technique works for a large scope of cases including infinitely renormalizable maps.

2. *Method of inducing*

As we have already mentioned, there were two types of inducing constructions which were successfully used in the 1990s. One was derived from the Markov extension. The word ‘derived’ means that iterates of the Markov extension can be considered and projected to the base space. The other one was based on Jakobson’s work [**J1**]. The basic difference

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between these two approaches is that the first one determines the domains of the induced map by tracking only the orbit of the critical point. The second one looks instead at the backward orbit of the fixed point with negative derivative and places the critical orbit with respect to this partition. Obviously, the first one is formally simpler and so might allow easier proofs or sharper estimates. The second approach presumably is more complete as it keeps track of more data. Its main apparent advantage, however, is in its connection with complex dynamics, through the construction of Yoccoz partitions. We will now describe the Jakobson–Yoccoz approach in some detail. An induced map derived from a Markov extension will be discussed with the solution of the attractor problem.

2.1. *The construction.* Consider a smooth unimodal map f of the interval $[-1, 1]$ into itself, normalized so that 0 is the critical point and -1 is fixed. Assume that there is another repelling fixed point $0 < q < 1$. This setting will be called the 'real case'. A separate 'complex case' occurs when f satisfies the conditions of the real case on the interval and is a complex quadratic polynomial. Then we view f as defined on the Riemann sphere and not just the interval.

To begin the construction, look for an open topological disk U contained in the domain of the map, which satisfies the following:

- $0 \in U$;
- for every $n > 0$, $f^n(\partial U) \cap U = \emptyset$;
- in the complex case, U is a Jordan domain.

If U satisfies all these conditions, we will call it an *inducing domain*.

We understand that in the real case, an open topological disk is simply an open interval. The second condition was suggested in [**Ma2**] and allows a more concise presentation of the construction. The last condition is for simplicity and possibly superfluous. There is a canonical way of finding U. Note first that the fixed point q has another preimage $q' < 0$. So, the interval (q', q) can be taken as U in the real case. In the complex case, q is the limiting point of two rays. These rays are open disjoint Jordan arcs in $\hat{\mathbb{C}}$ with endpoints at ∞ and q and are interchanged by the dynamics of f. Let $\mathcal{R}_q$ denote the Jordan arc in the plane which is the union of the rays and q. The preimage $f^{-1}(\mathcal{R}_q)$ consists of $\mathcal{R}_q$ and another Jordan arc disjoint with $\mathcal{R}_q$ which passes through q'. Call it $\mathcal{R}_{q'}$. Finally, find a Jordan curve w which separates 0 from ∞ and is mapped by f in the unbounded connected component of its complement. A circle with large radius would work as w. Then $\mathcal{R}_q$, $\mathcal{R}_{q'}$ and w cut off a Jordan domain which can be used as U.

The inducing lemma. If X is any set, $U \subset X$ and $g : X \to X$, then the *first entry time* $e(x)$ of $x \in X$ into U is

$$e(x) := \inf\{k = 0, 1, \ldots, : g^k \in U\}.$$

Note that $g^k(x)$ may not be defined for all k and the set of integers $\{k : g^k(x) \in U\}$ might be empty. In this case, following the usual convention we set $e(x) := \infty$. The mapping $\varphi : x \to g^{e(x)}(x)$ is called the *first entry map* and is defined wherever the first entry time is finite.

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Suppose that U is an inducing domain and $f^n(0) \in U$ for some $n > 0$. Let $\varphi : W \to \mathbb{C}$ denote the first entry map into U and let $W_0, \ldots,$ be the list of connected components of W, which may be finite or infinite, enumerated so that $0 \in W_0$. Then:

- *in the complex case, all W_i are Jordan domains;*
- *for every i, $\varphi_{|W_i}$ is a fixed iterate of f;*
- *for every $i \neq 0$, $\varphi_{|W_i}$ is one-to-one and onto U;*
- *$\varphi_{|W_0}$ has a single critical point at 0, of the same order as f, and is proper into U, i.e. for every compact subset of U its preimage is compact;*
- *W_0 is an inducing domain.*

For the proof, see [**GSw5**, Chapter 1]. The first four properties show in what sense φ is a simple mapping, and the last one allows one to continue inductively. That is, suppose that an inducing domain B_n is given and the critical orbit returns to B_n. Then we can apply the inducing lemma with $U := B_n$ to construct $B_{n+1} := W_0$ and the induced map $\phi_{n+1} := \varphi$. This construction can be begun with any inducing domain B_0 and if the critical orbit of f is recurrent, it can be continued indefinitely. Any sequence of mappings $\phi_1, \ldots,$ obtained in this way from some inducing domain B_0 will be called an *induced sequence* of f. In particular, by choosing B_0 in the way described in the preceding paragraph, we get the *canonical induced sequence*.

Generating induced sequences. In some cases, not only is the induced sequence infinite, but also

$$\bigcap_{n=0}^{\infty} B_n = \{0\}.$$

We will then say that the induced sequence is generating. For maps with negative Schwarzian derivative and with recurrent critical orbits, the picture is as follows. If f is topologically transitive, then every induced sequence is generating. A unimodal map is called *weakly expansive* if the image of any interval under forward iteration eventually grows to a length larger than a fixed positive constant (a stronger notion of an expansive map demands that for every two points $x \neq y$ there exists $n > 0$ such that the distance between their nth images exceeds $\epsilon > 0$). For maps which are weakly expansive but not topologically transitive some inducing sequences are generating and some are not, in particular the canonical one is not generating. For maps which are not weakly expansive, there are no generating induced sequences.

Close returns. Consider an induced sequence ϕ_n arranged so that ϕ_n is the first entry map into an inducing domain B_{n-1}. We say that ϕ_n makes a *close return* if and only if $\phi_n(0) \in B_n$. Otherwise, it is a *non-close* return.

If f has negative Schwarzian derivative, it is known that an induced sequence ϕ_n is generating iff ϕ_n makes a non-close return for infinitely many n. In any case, let n_j be the subsequence of all n for which ϕ_n shows a non-close return.

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Decay of geometry. Suppose that the critical orbit of f is recurrent. Consider the subsequence n_j of all non-close returns introduced in the preceding paragraph. Set

$$\alpha_j := \log \frac{|B_{n_j}|}{|B_{n_j+1}|}$$

in the real case and

$$\alpha_j := \mathrm{mod}(B_{n_j} \setminus \overline{B}_{n_j+1})$$

in the complex case.

Definition 2.1. An induced sequence shows the *decay of geometry* provided that $\alpha_j \geq Cj$ for some $C > 0$ and all j such that n_j is well-defined.

The constant C may depend on f or B_0, but this dependence is rather mild. When the critical point is of order two, C typically only depends on α_1, in which case we talk of *uniform decay of geometry*. The way we stated Definition 2.1, when the sequence (n_j) is finite, the decay of geometry holds trivially. However, the uniform decay of geometry is meaningful in that case. The decay of geometry occurs when the order of the critical point is two and is believed to occur also when it is between one and two. When the order of the critical point is larger than two, there exist examples of generating induced sequences for which α_j remain bounded for all j; this situation is often referred to as *bounded geometry*.

Connection with induced expansion. In the real case, we will say that f *induces expansion* provided that there is an open subset Γ of $[-1, 1]$ with full measure and a fixed neighborhood T of zero so that for each connected component γ of Γ there is a positive integer $n(\gamma)$ and $f^{n(\gamma)}$ is one-to-one on γ, expanding with derivative bounded away from one independently of γ, and onto T. Induced expansion clearly implies that Lebesgue-a.e. orbit is dense. There is a connection with decay of geometry, namely, that the existence of a generating induced sequence with the decay of geometry implies induced expansion.

2.2. *Inducing: a historical perspective.* In 1981 Jakobson proved his theorem about absolutely continuous invariant measures [**J1**]. The proof relies on showing that many unimodal maps induce expansion. However, Jakobson did not prove induced expansion for all unimodal maps. His procedure was based on parameter exclusion, which however, left a set of positive measure. A discovery of the 1990s was that induced expansion was actually always valid for weakly expansive real quadratic polynomials (and some other unimodal maps). Next, it was observed that the same mechanism which causes induced expansion in non-renormalizable maps applies to infinitely-renormalizable ones. In this way the uniform decay of geometry was discovered [**GSw3**]. For maps which are topologically mixing on some intervals the problem of measure theoretical attractors is equivalent to asking whether induced expansion holds [**Ma2**].

The first thing needed for the proof of induced expansion was a construction of the open set of full measure on which induced expansion occurs. Such a construction was suggested in [**GuJ**], however that work did not prove induced expansion for non-renormalizable maps except in special cases and under special assumptions. The next step was [**JSw1**].

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Here, a slightly different construction was given and induced expansion was expressly conjectured. By an inductive argument the decay of geometry, and hence, induced expansion, was proved. However, the first step of the induction was based on an estimate (starting condition) which was not verified except in special cases. Interestingly, a similar construction was discovered in [**Ma2**]. In a rather unrelated problem about circle mappings a similar inductive argument was developed before in [**G**] and [**TV**]. Significantly, on the circle the decay of geometry was only observed if the second derivative at a critical point did not vanish. In connection with the problem of measure theoretical attractors the so-called *Fibonacci map* was suggested as a candidate for a counterexample [**HK2**]. The dynamics of this map are reminiscent of the golden mean circle rotation. The Fibonacci map was one of those not handled in [**JSw1**]. In 1991, after the work of Yoccoz [**Yo2**], which somehow dealt with all non-renormalizable cases, it was natural to look for a solution in complex dynamics. A proof that the Fibonacci map has induced expansion was given in [**LyMi**]. The proof was based on a complex technique of linear models, ultimately reducing the problem to a computation done in [**BrH**] for cubic polynomials. A different proof was given in [**KN2**]. This paper used a real technique and so handled Fibonacci maps with local behaviour near the critical point other than quadratic. The same phenomenon was observed for circle maps, namely that the decay of geometry occurred exactly for the singularity type $|x|^\ell$ with $\ell \leq 2$. The next step was [**Ly3**]. This paper proved induced expansion for all non-renormalizable quadratic polynomials using an extension of the technique of linear models. An attempt to extend the result to S-unimodal maps with critical point of order two, by taking limits in a certain way and proving that they are analytic, is less convincing. This part of the proof, particularly Lemma 3.8, is very sketchy and likely to be incomplete, however, the theorem is widely believed to be true.

At this point the decay of geometry became interesting for infinitely renormalizable quadratic polynomials because it was needed in the proof of the real Fatou conjecture [**GSw2**]. More will be said about this in the next section. After an early argument which used the method of linear models, a different method based on an inductive argument using complex moduli was devised. The idea of counting moduli had previously been used in complex dynamics [**BrH, Yo2**]. A proof based on this idea was given in [**GSw3**]. This provided an independent verification of induced expansion for quadratic polynomials. In order to study the decay of geometry with reference to the combinatorics of polynomials, a new class of systems called *box mappings* was suggested in [**GSw3**].

The decay of geometry always holds [**GSw3**] for real box mappings with quadratic singularity. For weakly expansive polynomials, this implies induced expansion. For renormalizable ones, the decay of geometry implies that the renormalized map becomes quadratic when the return time to the restrictive domain goes to infinity. Hence, surprisingly in many cases, the theory of renormalization for quadratic polynomials is much simpler in the presence of unbounded combinatorics.

Afterwards a related result including purely complex cases and likewise based on the induction on complex moduli was given in [**Ly1**]. Later, the results of [**GSw3**] were generalized in [**GSw5**] to complex box mappings. However, the situation in the complex domain is more delicate and the decay of geometry does not yield the density of the hyperbolic maps in the complex domain.

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In [**GSw3**] the proof proceeds most of the time in the complex plane, except in one particularly hard case called 'rotation-like' when both complex and real estimates are used. Once we know that the moduli grow linearly for real box mappings, the combinatorics of rotation-like box mappings allows us to build a quasiconformal equivalence between real and complex box rotation-like mappings. Since the property of a linear growth of moduli is a quasiconformal invariant, estimates for complex box mappings can be derived from their real counterparts.

It should be emphasized that the decay of geometry depends on the critical point being non-degenerate. Counterexamples are known if this hypothesis is dropped, [**KN2**].

2.3. *The attractor problem.* The problem of measure theoretical attractors remained unsettled except in the special case when the order of the critical point was two. In general, decay of geometry is absent and the inducing construction does not give a definite answer. A solution was obtained by completely different methods. Let us begin by briefly explaining the setting which was derived from canonical Markov extensions.

Consider a unimodal mapping f normalized as at the beginning of §2.1. In addition, suppose that f is topologically transitive and, to avoid technical complications, that it's Schwarzian derivative is negative.

Cutting times. For a very readable description of cutting times, the function Q and their connection with Hofbauer towers see [**B2**]. Suppose that for some $n > 0$ there is a point α, $0 \le \alpha < f(0)$ such that f^n is monotone on $[\alpha, f(0)]$, but its derivative at α vanishes. Then we will say that n is a *cutting time*. Our assumptions imply that f has no homterval, that is iterates of f on every interval are eventually not monotonic. Hence, there is a infinite sequence of cutting times. Let us denote cutting times by S_k, starting from $S_0 = 1$. Let α_k denote the point nearest to $f(0)$ and to the left of $f(0)$ where the derivative of f^{S_k} vanishes. Then α_k form an increasing sequence with $\alpha_0 = 0$ and the limit equal to $f(0)$. For $x \in (-1, f(0))$ let us define $D(x)$ as the smallest integer j for which $x \le \alpha_j$.

The key observation is that for $k > 0$, $S_k - S_{k-1}$ is again a cutting time which is denoted with $S_{Q(k)}$. The function Q defined by this relation maps positive integers into non-negative integers and satisfies $Q(k) < k$. The interval between 0 and $f^{S_k}(0)$ is mapped diffeomorphically by the following $Q(k+1)$ iterates. Hence, $f^{S_k+1}(0) > 0$ provided that $Q(k+1) > 0$.

Let us now define an induced map ψ on $(-1, f(0))$. For x in this interval, look for the smallest n for which the derivative of f^n vanishes on $[x, f(0))$. Then $\psi(x) := f^n(x)$. To understand ψ, recall $j := D(x)$. Then $n = S_j$. If $j > 0$, then the largest interval containing x on which ψ is continuous and monotone is $(\alpha_{j-1}, \alpha_j]$ and its image is $(f^{S_{Q(j)}+1}(0), f(0)]$. Observe that $f^{S_{Q(j)}+1}(0) > 0$ provided that $Q(Q(j)+1) > 0$. Note that ψ can be iterated infinitely many times except at preimages of 0.

LEMMA 2.1. *Suppose that $x \in (0, f(0))$ is not a preimage of 0 and $\psi^n(x)$ tends to $f(0)$. Also let $\lim_{k\to\infty} Q(k) = \infty$. Then x does not have a base of connected open neighborhoods each of which is mapped onto $(f^2(0), f(0))$ diffeomorphically by an iterate of f.*

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Proof. Let J be an open interval containing x. The only iterates of f which may map it diffeomorphically onto $(f^2(0), f(0))$ coincide with either ψ^q or $f \circ \psi^q$ on J, for some $q \geq 0$. Without loss of generality, $\overline{J} \subset (0, f(0))$ and hence, $q > 0$. Denoting $j := D(\psi^{q-1})(x)$, we see that the largest interval on which J can be mapped diffeomorphically by ψ^q is $(f^{S_{Q(j)}+1}(0), f(0))$. If this or the next iterate of f sends J onto $(f^2(0), f(0))$, we must have $f^{S_{Q(j)}+1}(0) \leq 0$, and this implies $Q(Q(j)+1) > 0$. Since Q tends to ∞, this is only possible for finitely many j. Then $D(\psi^{q-1}(x))$ tends to ∞ with q because $\psi^{q-1}(x)$ tends to $f(0)$. Thus, there are only finitely many iterates of f which could map J onto $(f^2(0), f(0))$, and so there is no base of such J. □

Strategy of the counterexample. If the measure-theoretical attractor for f is the entire interval $(f^2(0), f(0))$, then Lebesgue-a.e. point of the interval has a base of connected neighborhoods which are mapped diffeomorphically onto $(f^2(0), f(0))$ by iterates of f. So to find an example when the measure-theoretical attractor is something else, it is enough to show a situation when orbits by ψ tend to $f(0)$ for a set of points of positive measure. This strategy is natural in the context of Markov extensions, in particular, in view of the fact that such an extension must be conservative if f is topologically transitive and its topological attractor is the same as the measure-theoretical one. In [**HK2**], a function Q was suggested which would make a counterexample most plausible. Even from our discussion, it is clear that since ψ 'sets x back' at worst to $S^{Q(D(x))+1}$, the best candidate would be a function for which $k - Q(k)$ is as small as possible. While $k - Q(k) = 1$ leads to the infinitely renormalizable Feigenbaum polynomial, $k - Q(k) = 2$ indeed defines a topologically transitive 'Fibonacci' map. The solution was finally achieved by [**BKNS**]. It should be noted that they do not work with the map ψ defined here, but with a related induced map. This point is important for distortion estimates, but on the level of precision of our discussion, it does not matter, so we will continue writing ψ.

The random walk argument. We think of the interval, with the Lebesgue measure, as a probabilistic space, and the iterates ψ^n as defining a random process. An integer-valued process is obtained by taking $D(\psi^n(x))$ and our task is to show that it tends to ∞ with positive probability. It turns out that for a sufficiently high order of the singularity this integer-valued process is a submartingale with bounded variance. The key submartingale property can be expressed as follows. If $H := \{x : D(x) = j\}$ for some j, and $n > 0$, then

$$\int_H [D(\psi^n(x)) - D(x)]\, d\lambda \geq 0.$$

If ψ^n is viewed as a composition of f on various intervals, then the last map in this composition introduces very strong distortion and tends to push the measure towards $f(0)$. This effect is conducive to the submartingale condition being satisfied, and is getting stronger as the order of the critical point increases. What remains is proving that increasing the order of the critical point leaves other aspects of the map, most importantly the rate at which α_k tends to $f(0)$ and the distortion of the iterates of f that come in ϕ^n before the last one, suitably bounded.

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With the submartingale and bounded variance conditions secured, a martingale theorem proved in [**BKNS**] implies that with positive probability, trajectories under ψ converge to $f(0)$ which ends the demonstration. By [**K2**] and also by the paper [**BLy1**], the measure-theoretical attractor turns out to be equal to $\omega(c)$.

The main contribution of [**BKNS**] lies in a combination of geometric estimates and a probabilistic argument. Let us remark here that a probabilistic argument had been used in the theory of circle mappings to establish universal estimates of the Hausdorff dimension of invariant singular measures, see [**GSw1**]. It would be interesting to try to combine these two approaches to explore questions about the Hausdorff dimensions of invariant sets of unimodal maps or complex polynomials.

2.4. *The real Fatou conjecture.* Consider the *logistic family*

$$f_a : x \mapsto ax(1-x),$$

where $x \in [0,1]$ and a is a parameter in $(0,4]$. The real Fatou conjecture was that for a dense set of parameter values a the corresponding mapping has an attracting periodic orbit. This problem was solved in the 1990s, using a combination of various techniques with heavy reliance on methods of complex dynamics. We will now discuss essential steps of the proof. Details can be found in [**GSw5**].

Preliminary reductions. It is rather easy to show that neutral cycles exist only for countably many parameters. Hence, it remains to prove that the set of those parameters for which all orbits are repelling is nowhere dense. Easy considerations based on the kneading sequence [**MiT, Gu1**] show that if f_{a_0} has only repelling cycles, then the set of all a for which f_a is topologically conjugate to f_{a_0} (later referred to as the topological class of a_0) is closed. Furthermore, if f_{a_1} and f_{a_2} are in different topological classes, then there is a parameter between them for which an attracting cycle exists. Thus, the real Fatou conjecture follows once it is demonstrated that the topological class of every polynomial with only repelling periodic orbits is just one point.

Quasiconformal deformations. The strategy of an attack on this problem was suggested by Sullivan in the 1980s. Based on the argument of quasiconformal deformations [**Su1**] one can show that the set of parameters a for which the mappings f_a are quasisymmetrically conjugate to a given f_{a_0} (the quasisymmetric class of a_0), is either a point or open. In view of what was said in the preceding paragraph, the proof is over once the following theorem is shown.

THEOREM 2.1. *Suppose that f_{a_0} has only repelling orbits and f_{a_1} is topologically conjugate to it. Then the conjugacy is quasisymmetric.*

A historical perspective. Beginning in the late 1980s, the efforts of several researchers were devoted to the proof of Theorem 2.1. First, it was proved for infinitely renormalizable polynomials with bounded combinatorics [**Su2**]. The next proof was for those maps for which the critical orbit is non-recurrent [**J2**]. The result of Yoccoz about local

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connectivity of the Mandelbrot set solved the problem for all non-renormalizable maps. Then [**JSw2**] proved Theorem 2.1 for a class of infinitely renormalizable mappings with unbounded combinatorics. The technique used in [**JSw2**] relied on complex pull-back to build quasisymmetric conjugacy. First, the paper follows Sullivan's idea of tangent extensions to build approximately holomorphic extensions of S-unimodal maps to the complex plane, it then develops the method of combined application of pullback and partial conjugacies (called branchwise equivalences) to construct in a sequence of inducing steps the quasisymmetric conjugacies for unbounded combinatorics. For the first time the renormalizable and non-renormalizable maps were treated alike only if the return time to the restrictive interval was long enough. The same paper also showed how the construction of the quasisymmetric conjugacy between infinitely renormalizable maps could be reduced to partial conjugacies on each stage of renormalization, which were later glued. The gluing was carried out by real variable methods.

Finally, [**GSw2**] solved the problem in general by combining the pull-back constructions of [**JSw2**] with the result about the decay of geometry [**GSw3**]. In [**Ly1**] a more general theorem was presented, including a generalization to an uncountable class of complex quadratic polynomials.

Both approaches rely on the decay of geometry (recall Definition 2.1) which only holds if the critical point is non-degenerate. It is widely expected that the real Fatou conjecture should be true for families of polynomials of higher degrees. However, the proof will have to be different.

Inducing and branchwise equivalences. In the setting of Theorem 2.1 we can assume without loss of generality, that the critical orbits are recurrent, otherwise the result follows from the already mentioned work of [**J2**]. Let us build the sequences of induced maps, ϕ_n derived from f_{a_1} and $\hat{\phi}_n$ corresponding to f_{a_2}. A branchwise equivalence between ϕ_n and $\hat{\phi}_n$ is any homeomorphism Υ_n of the plane onto itself which satisfies the conjugacy condition:

$$\Upsilon_n(f_{a_1}(z)) = f_{a_2}(\Upsilon_n(z))$$

for all z not in the domain of ϕ_n. If ϕ_n and $\hat{\phi}_n$ are generating induced sequences, then maps Υ_n tend to the conjugacy uniformly. The idea of the proof is to construct a sequence of branchwise equivalences Υ_n which are all quasiconformal with the same bound. This poses four problems.

(1) Construct Υ_0 which is quasiconformal.
(2) Show that, given Υ_n which is K-quasiconformal, one can build Υ_{n+1} which is K' quasiconformal, with K' depending only on K.
(3) Deal with the divergence of the quasiconformal bound as n goes to ∞.
(4) Deal with the possibility that the inducing sequences may not be generating; for real polynomials this is equivalent to the existence of a restrictive interval.

Among these, the second problem was solved in [**JSw2**]. More can be proved, however. Namely, if n_j denotes the smallest non-close return greater or equal to n, and if a K-quasiconformal branchwise equivalence Υ_n is given, then Υ_{n_j+1} can be constructed which is K'-quasiconformal where K' only depends on K.

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The fourth problem is handled as follows. Suppose that ϕ_{n_j} is the last non-close return. Then Υ_{n_j+1} is given, quasiconformal, although from what we have seen so far its bound may depend on j. Our goal is to show that in fact this quasiconformal bound depends neither on j nor even on the choice of polynomials f_{a_1} and f_{a_2}. This is something that will follow from steps one and three. We postpone justification of these two steps until the λ-lemma is formulated. Now consider the first return maps onto the restrictive intervals for f_{a_1} and f_{a_2}. By the property of complex bounds, discussed later in this paper, these first return maps are quasisymmetrically conjugated to a pair of polynomials, $f_{a_1^1}$ and $f_{a_2^1}$, and the quasisymmetric bound of this conjugacy is again universal. Now, even if restrictive intervals exist again for $f_{a_1^1}$ and $f_{a_2^1}$, and so infinitely many times, by repeating this argument we get a sequence of branchwise equivalences $\Upsilon_{n_{j_k}}$, all quasisymmetric with the same bound. These can be glued together to the conjugacy using the technique introduced in [**JSw2**]. A different technique for gluing was used in [**Ly1**].

The λ-lemma. Steps one and three follow from the λ-lemma. Let us state it here.

THEOREM 2.2. *Suppose that a function of two complex variables is given, $\Psi(z,t)$, with t in the open unit disk and z from an arbitrary subset S of the Riemann sphere. Let Ψ be holomorphic with respect to t for any fixed $z \in S$, suppose that $\Psi(z_1,t) = \Psi(z_2,t)$ implies $z_1 = z_2$ for every t in the unit disk and $z_1, z_2 \in S$ and that $\Psi(z,0) = z$ for every $z \in S$.*

Then Ψ extends to a map $\tilde{\Psi}$ with the same properties, but with S replaced by the whole Riemann sphere. It follows that the map $\psi_t(z) := \tilde{\Psi}(z,t)$ is $((1+|t|)/(1-|t|))$-quasiconformal for every t in the unit disk.

This result, purely analytic in nature, was inspired by problems in holomorphic dynamics. The estimate $(1+|t|)/(1-|t|)$ on the quasiconformal dilatation of $\psi_t(z)$ in the statement of Theorem 2.2 was proved in [**MñSS**]. The rest, which is proving that Ψ extends to $\tilde{\Psi}$, took more time, with the final solution provided in [**Sl**].

Now step one follows, because we can cover the parametric interval, viewed as embedded in the complex parametric plane, by finitely many disks on which the hypotheses of the λ-lemma are satisfied for S equal to the complement of the domain of ϕ_0. Recall that the boundary of this set is very simple, consisting of just two rays and an equipotential, so that the hypothesis is easy to verify. This will give us Υ_0 which is quasiconformal with a uniform bound.

Step three, in addition to the λ-lemma, requires the decay of geometry and the fact that large annuli from the phase space can be lifted to large annuli in the parameter space. What follows is that for j large enough, the λ-lemma can be used on a disk which contains both a_1 and a_2 and for which S can be taken as the complement of the domain of ϕ_{n_j+1}. By now the difficulty of step three is resolved, since for j large enough we no longer need to rely on the inductive use of step two, but rather can build the branchwise equivalence directly from the λ-lemma.

Other related problems. A related problem was the monotonicity conjecture for the bimodal cubic family [**DGMT**]. The conjecture says that if the bimodal family is naturally

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parametrized by its critical values, then the sets of constant entropy are Jordan arcs. The missing link was ruling out closed loops. This was done in [**He**] by methods of [**GSw2**]. Thus, the monotonicity conjecture became a theorem.

Recently, a study was undertaken about what behaviour in the logistic family is typical in the sense of measure. It was shown that almost all weakly expansive maps have an absolutely continuous invariant measure. Recall that by Jakobson's theorem [**J1**] the set of parameters which correspond to weakly expansive maps has positive measure. This should be contrasted with examples of weakly expansive maps without absolutely continuous invariant measures [**Jo**]. In fact, statistical behaviour of orbits for weakly expansive maps might be quite surprising. There are transitive maps in the logistic family so that probabilistic measures μ_n equally distributed along the orbits $x, f_a(x), \ldots, f_a^n(x)$ converge in a weak topology to a point mass in a repelling fixed point for almost every x with respect to the Lebesgue measure, [**HK1**]. In the work by [**MaN**], a topological condition is given, based on the property of decaying geometry, which implies the existence of an absolutely continuous invariant measure. In [**Ly4**] this result and more from the decay of geometry are coupled with a 'parapuzzle construction' to prove that almost all weakly expansive maps have an absolutely continuous invariant measure. This also amounts to the statement that in the logistic family, f_a has an SRB measure for almost all parameters a. The parapuzzle construction comes from complex dynamics [**BrH, Yo2**]. An invariant measure is called a *Sinai–Ruelle–Bowen*, or SRB for short, measure if it is a weak accumulation point of μ_n for x in a positive Lebesgue measure set.

3. *Collet–Eckmann maps*

The Collet–Eckmann condition in unimodal dynamics was introduced in [**CE**] and demands that the derivatives grow exponentially fast along the orbit of the critical value. There were many papers about Collet–Eckmann maps in the 1980s. It was proved that CE maps share some properties with Misiurewicz maps defined by the condition that the critical point is not recurrent and all periodic orbits are repelling. On the other hand it is known [**BeC1**] that the maps in the quadratic family $f_a = ax(1-x)$, $0 < a < 4$, which satisfy the Collet–Eckmann condition have positive Lebesgue measure in the parameter space. The result also applies to more general families of unimodal maps [**TTY**]. In the families considered in [**TTY**] every a^* which corresponds to a Misiurewicz map is a Lebesgue density point of CE parameters provided the family satisfies a certain transversally condition at a^*. Furthermore, by [**T1**], the same holds in generic monotone one-parameter families of polymodal maps. Being 'almost hyperbolic' and abundant, Collet–Eckmann maps drew the interest of many researchers.

The problem of topological invariance of the Collet–Eckmann condition appeared naturally, but was not easy to solve. The research focused on S-unimodal maps with a critical point of polynomial type. Initial progress was reported in [**JSw2**]. The Collet–Eckmann condition was proved to be topologically invariant for a subset of S-unimodal maps represented by a positive measure set of parameters in the logistic family.

The problem of topological invariance recently underwent a rapid development mainly due to unexpected interaction with regularity problems in holomorphic dynamics. The study of regularity of Fatou components for rational functions was initiated in [**CJY**]

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and continued by others. In [**P**] rational Collet–Eckmann maps were studied for the first time, mainly from the point of view of ergodic and measure theoretical properties. A rational map satisfies the CE condition if for every critical point c of f which belongs or accumulates to the Julia set, there exists n_c so that the Lyapunov exponent at $f^{n_c}(c)$ is positive. It is consistent with the definition of unimodal CE maps with one exception—we allow critical points to be attracted to (super) attractive cycles. For example, the real map $x \to x^2$ is not CE in the real sense but its complex counterpart $z \to z^2$ is a rational CE map. A new and interesting technique of estimating distortion was introduced in [**P**]. This method, called 'shrinking neighborhoods' was applied in [**GSm**] to prove Hölder regularity of the Julia sets and estimate the Hausdorff dimension.

3.1. *Outline of methods.*

Shrinking neighborhoods. We will explain this technique both for real and complex maps. Slightly paradoxically, it enables one to accumulate and control expansion along pieces of orbits which pass very close to the critical points. This leads to global expansion properties of CE maps. For example, suppose that we have a cover $\mathcal{B}$ of the interval I by balls of radius r. Denote by d_n the supremum of the diameters of components of $f^{-n}(\mathcal{B})$. Similar covers can be considered for Julia sets. A global expansion would manifest itself if d_n converged (exponentially) to zero. Already, a classical result in this direction has been obtained by Mañé for rational maps [**Mñ**]. In the S-unimodal case, a version of Mañé's lemma says that if f has no neutral points, then the diameters of components of $f^{-n}(\mathcal{B})$ go to zero with $n \to \infty$, at a uniform rate for each N, provided that f^n has at most N critical points in these components. The technique of shrinking neighborhoods yields much stronger results for CE maps, in particular an affirmative answer to the question about an exponential decay of d_n.

We fix a decreasing sequence of positive numbers $\{\delta_n\}$ which satisfy $\prod_n(1-\delta_n) > \frac{1}{2}$. Set $\Delta_n := \prod_{k\le n}(1-\delta_k)$. Let B_r be a round ball at a point z (an interval centered at z on the real line) and $\{f^{-n}(z)\}$ be a *sequence of preimages* of z, chosen so that

$$f(f^{-n-1}(z)) = f^{-n}(z)$$

for all $n \ge 0$. We define U_n and U_n' as the connected components of $f^{-n}B_{r\Delta_n}$ and $f^{-n}B_{r\Delta_{n+1}}$ respectively which contain $f^{-n}z$. Clearly,

$$fU_{n+1} \subset U_n' \subset U_n.$$

If U_k, for $1 \le k \le n$, do not contain critical points then distortion of $f^n : U_n' \to B_{r\Delta_{n+1}}$ is bounded by a power of $K(1/\delta_{n+1}^4)$ according to Lemma 1.1.

Exponential decay of preimages. The next estimate is well known for S-unimodal maps and can be traced back to [**NP1**]. It is also known for rational maps [**GSm, PR**]. We will sketch a modern proof to illustrate the usefulness of the technique of shrinking neighborhoods.

Fact 3.1. If f is an S-unimodal map which satisfies the CE condition, then there are $\delta > 0$ and $\xi < 1$ so that the diameter of every component of $f^{-i}(B_\delta)$ is less than ξ^i, where B_δ is any ball of radius δ.

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Proof. For simplicity, we will confine ourselves to S-unimodal real polynomials. The proof does not use any combinatorial analysis and can be easily adapted for rational maps [**GSm, PR**]. By an argument which is omitted here, Fact 3.1 is reduced to the following claim: there exists $R > 0$ so that for every $r < R$ we have

$$\operatorname{diam} \operatorname{Comp}_c f^{-i}(B_r(c)) < \xi^i r.$$

Suppose that the order of the critical point c is ℓ. Consider a sequence of preimages of the critical point c, $f^{-1}(c), \dots, f^{-n}(c)$ such that shrinking neighborhoods U_k for $B_r(c)$, $1 \le k \le n-1$ avoid c, $r < R$, and $c \in U_n$.

From the CE condition, we have that $(f^n)'(f(c)) \ge \lambda^n$ for large enough n. We set δ_n in the definition of shrinking neighborhoods to decrease subexponentially fast, in particular, for large enough n, $\delta_n^{-1} \ge \lambda^{\epsilon n}$ for any positive ϵ. By the Köbe distortion lemma applied to $f^{-(n-1)} : B_{r\Delta_{n-1}}(c) \to U_{n-1}$ we obtain that up to a positive constant

$$\begin{aligned} \operatorname{diam}(U_n)^\ell &\le \operatorname{diam}(U'_{n-1}) \le \delta_n^{-4} r |(f^{n-1})'(f(c))|^{-1} \\ &\le \lambda^{4n\epsilon} r |f^n(c)|^{\ell-1} |(f^n)'(f(c))|^{-1} \le r^\ell \lambda^{-n(1-4\epsilon)}. \end{aligned}$$

□

An interesting consequence of Fact 3.1 is that an S-unimodal map which satisfies the CE condition cannot have restrictive intervals shorter than δ and hence, can only be finitely many times renormalizable. The same corollary is valid for complex quadratic polynomials. Indeed, suppose that f is infinitely renormalizable. Then there is an arbitrary large n and two topological disks $\overline{U} \subset V$ so that $f^n : U \to V$ is proper of degree ℓ and $c \in \cap_{i=0}^{\infty} f^{-ni}(U)$ is a non-trivial continuum. We cover V by M/δ^2 balls of the radius δ supplied by the complex counterpart of Fact 3.1. Then,

$$\operatorname{diam} f^{-n}(V) \le \ell M \delta^{-2} \xi^n \le \delta$$

and $\lim_{n\to\infty} \operatorname{diam} f^{-n}(V) = 0$, contradiction.

Various conditions equivalent to CE. On a technical level the proof in [**GSm**] relied on the equivalence between the Collet–Eckmann condition CE and the so-called second Collet–Eckmann condition CE_2.

Definition 3.1. We say that a map f satisfies the CE_2 condition if there are positive constants C and $\lambda > 1$ so that for every preimage $y \in f^{-n}(c)$ we have $|f^{n\prime}(y)| > C\lambda^n$.

The equivalence of CE and CE_2 conditions for real S-unimodal maps had been open since the early 1980s. The proof was only given in [**NSa**]. The implication CE $\Rightarrow$ CE_2 was already known in the 1980s, [**N**]. The second Collet–Eckmann condition for complex maps seems stronger than the second Collet–Eckmann condition in the real case since some preimages are missing on the real line. In [**NSa**] another technically hard fact was established, namely that the Collet–Eckmann condition is equivalent with the uniform hyperbolicity on cycles.

Definition 3.2. We say that a map f is uniformly hyperbolic on cycles if there is a constant $Q > 1$, such that for every periodic point x, $f^m(x) = x$, the inequality $|(f^m)'(x)| > Q^m$ holds.

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This property means that the Lyapunov exponent is bounded from below by a positive constant for all periodic orbits. The validity of this equivalence is not known for complex quadratic polynomials, which is somewhat paradoxical since again, there are more periodic orbits in the complex plane.

The proof that uniform hyperbolicity on cycles, or UHC for short, implies CE conditions for S-unimodal maps is too complicated to be stated here. It also uses in a substantial way, real constructions. The opposite implication is also true for rational maps. It follows immediately from CE_2. Indeed, if x is a repelling periodic point, $f^m(x) = x$, then there exists a neighborhood U of x so that the inverse branch of f^m fixing x is diffeomorphically equivalent to the multiplication by $1/(f^m)'(x)$. The backward orbit of the critical point c is dense in I (or in the Julia set if f is a rational map) and hence, there exists $y = f^{-k}(c)$ which belongs to U. By bounded distortion on U, there exists a constant $Q > 1$ so that

$$|(f^m)'(x)| \geq \liminf_{s\to\infty} |(f^{sm+k})'(f^{-k-sm}(c))|^{1/s} \geq \liminf_{s\to\infty} Q^{k+sm/s} = Q^m.$$

3.2. *The topological CE condition.* The definition of TCE (first formulated in [**PR**] for rational maps, see also Condition 4 in [**NP2**]) can be stated as follows. For a given positive δ and L define $G(x, \delta, L)$ to be the set of positive integers n such that

$$\#\{i : 0 \leq i \leq n, c \in \mathrm{Comp}_{f^i(x)} f^{-n+i}(B(f^n(x), \delta))\} \leq L,$$

where $\mathrm{Comp}_y(A)$ stands for the component of A which contains y.

Definition 3.3. A unimodal map (rational) f satisfies TCE if there are positive δ, $L < \infty$ and η so that for every point $x \in I$ (x belongs to the Julia set) we have

$$\inf_n \frac{\#G(x, \delta, L) \cap [1, n]}{n} \geq \eta.$$

On the real line the definition of $G(x, \delta, L)$ is in fact stronger than the statement that $n \in G(x, \delta, L)$ if f^n on $\mathrm{Comp}_x f^{-n}(B_\delta(f^n(x))$ has at most L critical points. This is so because near the critical value a part of an interval might have a preimage in the complex plane. In the complex setting these two conditions are equivalent.

TCE is obviously a topological invariant. The strategy of the proof that the CE condition is a topological invariant relies now on proving two implications: CE $\Rightarrow$ TCE $\Rightarrow$ UHC.

It is easy to see that TCE implies uniform hyperbolicity on cycles. We sketch a proof for S-unimodal maps to illustrate the so-called 'telescope' construction (see [**PR**], Proposition 3.1 and [**NP2**]).

Telescope construction. By a compactness argument, we have that for every $\epsilon > 0$ there exists $k(\epsilon)$ so that for every $n \geq k(\epsilon)$ and $n \in G(x, \delta, L)$, the diameter of $\mathrm{Comp}_x f^{-n}(B_\delta(f^n(x)))$ is less than ϵ. Otherwise, there would exist an interval J of length ϵ/L such that $c \notin \cup_{i=1}^{\infty} f^i(J)$ (J with this property is called a wandering interval). This is not possible for S-unimodal maps [**Gu1**]. Let $\epsilon := \delta$ and $k := k(\delta)$. We fix x in I and take $n \in G(x, \delta, L)$. There exists a decreasing sequence $n = n_0 > n_1 > \cdots > n_m$ of numbers from $G(x, \delta, L)$ where m is comparable with $n\eta/k$ and any two consecutive numbers from

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the sequence are at distance larger than k. Generally, the sequence n_i need not be 'evenly' distributed in $[0, n]$ since $|n_{i+1} - n_i|$ might range between k and $n\eta$. By the choice of k we have that

$$\mathrm{Comp}_{f^{n_{i+1}}(x)} f^{-n_i+n_{i+1}}(B_\delta(f^{n_i}(x)) \subset B_{\delta/2}(f^{n_{i+1}}(x)).$$

In this way we have built a 'telescope' which has $m+1$ tubes and each tube has length at least k. This construction and standard estimates based on the concept of Poincaré length (in the analytic setting one can use moduli of annuli) allow us to estimate the rate of decay of the intervals

$$\mathrm{Comp}_x f^{-n}(B_\delta(f^n(x))) \subset \mathrm{Comp}_x f^{-n_1}(B_\delta(f^{n_1}(x))) \subset \cdots \subset I.$$

The Poincaré length of J_1 inside J_2 is an affine invariant and after normalization of J_2 to $[-1, 1]$ is equal to $\int_{J_1}(dx/1 - |x|^2)$. The crucial property is that diffeomorphisms with non-negative Schwarzian contract the Poincaré length. Also, a pull-back by a folding branch with the critical point of polynomial type can increase the Poincaré length by at most a bounded multiple. Hence, by the choice of n_i, the Poincaré length of $\mathrm{Comp}_x f^{-n_i}(B_{\delta/2}(f^{n_i}(x)))$ inside $\mathrm{Comp}_x f^{-n_i}(B_\delta(f^{n_i}(x)))$ is bounded by a constant which depends only on L. Now we see that the Poincaré diameter of $\mathrm{Comp}_x f^{-n}(B_\delta(f^n(x)))$ inside I is bounded by a constant multiple of $\eta n/k$. In conclusion, we can find a number $\xi < 1$ which depends on δ, η, and L but not on x so that

$$\mathrm{diam}[\mathrm{Comp}_x f^{-n}(B_\delta(f^n(x))] \le \xi^n$$

for any $n \in G(x, \delta, L)$. In fact, the argument using the Poincaré length yields that for any $r \le \delta$

$$\mathrm{diam}[\mathrm{Comp}_x f^{-n}(B_r(f^n(x))] \le \mathrm{const.}\, r^\kappa \xi^n,$$

where κ depends only on L. Now every periodic point x of f must be repelling and its Lyapunov exponent is positive and depends only on η, δ and L.

A global estimate. The next fact (Lemma 1.1 in [**NP2**]) deals with differentiable maps f of the unit interval I which have Hölder continuous derivative at critical points $c \in \mathrm{Crit}$, i.e. there exists $\gamma > 0$ so that $|f'(x) - f'(c)| < C|x - c|^\gamma$ for every $x \in I$ and $c \in \mathrm{Crit}$. Fact 3.2 was first proved in the purely complex setting in [**DPU**]. If $c \in \mathrm{Crit}$, we set

$$\phi(x, n) = -\log \mathrm{dist}(f^n(x), \mathrm{Crit}),$$

for every x and every positive integer n. Due to our assumption that $|I| = 1$ we have that $\phi(x, n) \ge 0$.

Fact 3.2. Let f be a differentiable map of the unit interval so that $\#\mathrm{Crit} < \infty$ and the derivative is Hölder continuous at critical points $c \in \mathrm{Crit}$. If f does not have attracting periodic points, then there exists a constant C such that for every n and $x \in I$,

$$\sum_{j=0}^{n}{}' \phi(x, j) \le nC,$$

where $\sum'$ denotes summation over all but at most $\#\mathrm{Crit}$ indices.

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The removed indices correspond to the closest approaches of the orbit of x to the critical points. For example, if x is very close to Crit then the first index in the sum $\sum'$ will be removed. The general idea of the proof is to study $\phi(x,n)$ on pieces of the orbit of x between closest approaches to a given critical point c. In this situation, some definite estimate on $\sum \phi(x,n)$ is expected, otherwise the derivative of the corresponding iterate of f would be small on some interval around c. The interval would be mapped into itself procuring an attracting periodic orbit. The proof of Fact 3.2 uses an inductive construction and can be found in [**NP2**].

To prove that CE $\Rightarrow$ TCE for S-unimodal maps, we follow closely the reasoning of [**PR**] and [**NP2**]. First, observe that if $\mathrm{Comp}_{f^i(x)} f^{-n+i}(B_\delta(f^n(x)))$ contains c then, by Fact 3.1, $|f^i(x) - c| \le \xi^{n-i}$ and $n - i \le \phi(x,i)/\log(1/\xi)$. In other words, $n \in S_i = (i, i + \phi(x,i)/K]$ where $K = 1/\log(1/\xi)$. Now for every n, define

$$A_n = \{k \le n : k \text{ belongs to at most } N \text{ sets } S_i\}.$$

From our previous remark, we see that $G(x,\delta,N) \supset A_n$. Hence, TCE will follow if we show that $\#A_n/n$ is greater than a positive constant for all n and a large enough N. Since $|S_j| = K\phi(x,j)$ the claim of Fact 3.2 states that

$$n \cdot CK \ge \sum_{j=0}^{n-1}{}' |S_j \cap [0,n]| \ge -n + \sum_{j=0}^{n-1} |S_j \cap [0,n]|.$$

By the definition of A_n, each integer from the set $[0,n] \setminus A_n$ is covered by at least N different sets S_j. Thus, $\sum_{j=0}^{n-1} |S_j \cap [0,n]|$ must be larger than $\#([1,n] \setminus A_n)N$ and one gets

$$n \cdot CK \ge (n - \#A_n)N - n$$

or

$$N \frac{\#A_n}{n} \ge N - 1 - CK$$

which gives the desired estimate for large N.

4. *Other results*

4.1. *Results related to invariant measures.* A related question is whether the existence of a probabilistic, absolutely continuous, invariant measure is a topological invariant for S-unimodal maps. A negative answer was given in [**B1**]. The counterexample relies on a judicious choice of two topologically conjugate long branched S-unimodal maps exhibiting on the one hand almost parabolic dynamics but on the other, showing different expansion along the critical orbits. This is achieved by forcing the critical orbit to frequently visit a proximity of a repelling fixed point. A counterexample is obtained by taking a pair of maps with different multipliers at the fixed point by which one map is caused to satisfy a condition which is equivalent to the existence of a probabilistic invariant measure absolutely continuous with respect to the Lebesgue measure, and the other to fail it. The condition is that for the induced expanding map χ, the return time function r is integrable. An integer valued function r is defined on the domain of χ by the condition that

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$\chi(x) = f^{r(x)}(x)$, where f is the original map which induces χ. These delicate estimates were possible mainly due to a proper setting given by the formalism of cutting times based on the Hofbauer tower construction.

There were also interesting results relating to invariant measures in families of unimodal maps. In [**T2**] the question of the weak continuity of SRB measures with respect to the parameter was addressed. Although the mapping is not generally continuous, it is continuous for a large set of parameter values. Rychlik's proof of Jakobson's theorem [**Ry**], determines a positive measure set of parameters in the logistic family the map has an absolutely continuous invariant measure with a density $\rho_a \in L^p$, $p \in [1, 2)$. On this set of parameters a function $a \mapsto \rho_a$ is continuous in L^p, $p \in [1, 2)$, at Misiurewicz points [**RySo**]. A new estimate of density of the set of parameters corresponding to the maps with absolutely continuous invariant measures was given in [**T1**]. Namely, in the family $g_a^\gamma : x \mapsto 1 - a|x|^\gamma$, γ is a parameter bigger than 1, we have that

$$\lim_{\epsilon \to +0} \frac{\log |[2-\epsilon, 2] \cap W|}{\log \epsilon} = \lim_{\epsilon \to +0} \frac{\log |[2-\epsilon, 2] \setminus E|}{\log \epsilon} = \frac{\gamma}{\gamma - 1},$$

where W is the set of parameters a for which g_a^γ has an attractive cycle, $E = \{a \in [0, 2] : g_a^\gamma$ is a CE map and $\liminf_{n\to\infty} \operatorname{dist}(0, (g_a^\gamma)^n(0)) = 0\}$, and $|\cdot|$ stands for the Lebesgue measure. A generalization of the above estimate for generic families of C^2 polymodal maps is also provided.

The work [**BeY**] shows stochastic stability of absolutely continuous invariant measures for a positive measure set A of parameters in the quadratic family $f_a(x) = 1 - ax^2$, $x \in [-1, 1]$. The orbits of $f := f_a$, $a \in A$, can be interpreted as sample paths of a stationary stochastic process with the probability space corresponding to an absolutely continuous invariant measure μ. Under additional assumptions, this process has genuinely random stochastic features, the correlation functions decay exponentially fast and the central limit theorem holds, [**Y, KN1**]. The random perturbations considered in [**BeY**] amount to taking iterates of $f + \beta_i$ instead of f, where f is viewed as a map of $\mathbb{R}$ into itself and β_i are independently chosen numbers from the range $(-\epsilon, \epsilon)$. Let δ_x denote the Dirac measure at x. Then, by the theory of Markov chains, there is an asymptotic distribution λ^ϵ to which $n^{-1} \sum_{i=1}^{n-1} \delta_{x_i}$ converges for almost every sequence $x_i = (f + \beta_i) \circ \cdots \circ (f + \beta_1)(x_0)$ starting at any $x_0 \in [-1, 1]$. The assertion is that the distributions λ^ϵ converge weakly to μ if we decrease ϵ to zero. Later, it was proved in [**BaV**] that the distributions λ^ϵ converge to μ in L^1 norm.

Abundance of classes of S-unimodal CE maps parametrized by the slopes of continuous piecewise-linear transformations of the interval (the so-called 'tent maps') was discussed in [**Sa1**]. By the work of Milnor and Thurston, every unimodal map which has at least one odd periodic orbit is semi-conjugate to a continuous, piecewise linear map of the unit interval with a slope s. One of the main results of [**Sa1**] asserts that for almost every value of s corresponding S-unimodal maps satisfy the CE condition. This result also admits a topological interpretation since s is known to be a logarithm of the topological entropy and consequently a topological invariant.

Another direction of studies is focused on finding analytical conditions more general than the CE condition which would guarantee the existence of absolutely continuous

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measures. In [**NS**] the so-called summability condition was formulated. We say that a unimodal map f satisfies a summability condition with exponent $\beta > 0$, $f \in \mathrm{SC}(\beta)$, if

$$\sum_{i=1}^{\infty} \frac{1}{|(f^i)'(f(c))|^\beta} < \infty.$$

The main result of [**NS**] says that S-unimodal maps with the critical point c of order ℓ satisfying $\mathrm{SC}(1/\ell)$ have a probabilistic, absolutely continuous, invariant measure μ such that for every measurable set A, $\mu(A) \le K\sqrt[l]{|A|}$. It is known that this condition is not optimal [**B1**]. We say that f satisfies a strong summability condition if $f \in \mathrm{SC}(\beta)$ for any positive β. A natural question arises: is the strong summability condition a topological invariant in the class of S-unimodal maps with the same order of the critical points? The latter assumption cannot be dropped. Indeed, any two S-unimodal Fibonacci maps f_1 and f_2, with the critical point c of order two and four, respectively, are topologically conjugate. On the other hand, f_1 satisfies the strong summability condition [**LyMi**], while f_2 shows the bounded geometry, [**KN2**]. The latter implies that there is a sequence n_j and a constant C such that $|(f_2^{n_j})'(f_2(c))| \le C$, in clear violation of any summability condition.

Also, the problem of the topological invariance for S-polymodal CE maps remains open.

Another contribution to the study of CE maps was made in [**BMs**] where it is proved that smooth C^r, $r \ge 2$, Collet–Eckmann maps of the interval with finitely non-degenerate critical points are structurally unstable in C^r topology. The perturbations are constructed locally in small neighborhoods of critical points.

4.2. *Complex bounds and universality.* The property of complex bounds was introduced in [**Su2**] as a key point in universality theory. If f is an infinitely renormalizable unimodal polynomial, then f satisfies complex bounds if every renormalization can be defined on a suitable planar domain U so that it becomes a polynomial-like map in the sense of Douady and Hubbard [**DH1**], with only one critical point; moreover, so that $f(U) \setminus \overline{U}$ is an annulus with modulus bounded from below by a positive constant independent from the choice of the renormalization level.

In [**Su2**] complex bounds were proved for real infinitely renormalizable polynomials of bounded combinatorial type. This proof works for polynomials with a degenerate critical point as well and was based on a subtle estimate known as the 'sector lemma'. The result means both the compactness of the renormalization operator as well as a quasiconformal equivalence, through the work of Douady and Hubbard, between the renormalizations of combinatorially bounded type quadratic polynomials and prototype mappings $z^2 + c$.

The property of complex bounds was proved by several different methods. In [**LS**], the proof is independent of the non-degeneracy of the critical point and does not rely on the sector lemma. Instead the paper uses the technique of the 'shortest interval' and some explicit real bounds. The same paper shows how to use complex bounds to get the local connectivity of Julia sets for all real unimodal polynomials. This includes non-renormalizable ones.

The proof for quadratic polynomials based on the inducing construction was carried out in [**GSw4**]. The proof combined the sector lemma with constructions used in [**GSw3**]. These constructions in [**GSw3**] failed to produce complex bounds only in one hard 'almost

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parabolic' case. The almost parabolic mappings were singled out in [**GSw3**] as the only possible exceptions for the statement, that an increasing return time of the critical point value to the restrictive interval yields complex bounds.

Later, in [**LyYa**] the proof, also restricted to the case of a quadratic singularity, was achieved by methods based on [**Ly3**]. Finally, a short proof for unimodal polynomials of any degree was presented in [**Sa2**] (see the book [**GSw5**] for an exposition). The technique is purely real and based on cutting time formalism. Explicit real bounds which are similar to those used in [**LS**] but stronger, are obtained and give complex bounds immediately.

Universality. In the 1980s, Feigenbaum universality was proved for a certain class of analytic maps using rigorous computer-assisted estimates, see [**La**]. Still, in the 1990s a lot of effort was devoted to proofs of universality. The goal was to cover a larger range of cases, not just the Feigenbaum map, and that required proofs based on general concepts rather than case-specific estimates. The first and most influential of these papers was [**Su2**]. The strategy of the proof was to construct a functional space on which the renormalization operator has a simple nature. Elements of this space are conformal conjugacy classes of polynomial-like mappings. This was historically how complex bounds were discovered. However, to be useful this space has to be endowed with a non-standard holomorphic structure of the so-called Riemann surface lamination. This leads to technical difficulties, see [**MS1**]. A different approach was used in [**Mc**]. Here, a geometric object, called the *tower* was considered, which contained the limits of renormalization and was shown to be unique. The tower construction was further generalized in [**Hi**] to cover certain cases of infinitely renormalizable maps with unbounded combinatorics. The most ambitious is the recent approach of [**Ly5**] where the existence of a global hyperbolic attractor is announced for the renormalization operator covering all kinds of combinatorics. The proof combines elements of all approaches mentioned here, together with the decay of geometry (see Definition 2.1). The structure on the set of polynomial-like mappings was simplified by identification up to affine, rather than conformal, conjugacy. A different approach was used in [**Ma1**] where there is no passage to a conformal dynamical system, but a complicated space of infinite dimension is built to accommodate the real renormalization operator. There were also works concerned with applying the same set of ideas to establish universality for critical circle mappings [**F, FM**].

A different goal was achieved in [**LSc**]. A rigorous argument is presented based on computer-assisted estimates for the renormalization operator which shows that for two analytic unimodal maps with Feigenbaum topological type, their Cantor attractors are not generally $C^{r_0+\epsilon}$ equivalent except in a submanifold of maps of codimension 1, where they are $C^{r_1-\epsilon}$ equivalent, with $r_0 < r_1$. Constants r_0 and r_1 depend on the spectrum of the renormalization operator.

References

[BaV] V. Baladi and M. Viana. Strong stochastic stability and rate of mixing for unimodal maps. *Ann. Sci. Éc. Norm. Sup.* **4** (1996), 483–517.

[BeC1] M. Benedicks and L. Carleson. On iterations of $1 - ax^2$ on $(-1, 1)$. *Ann. Math.* **122** (1985), 1–25.

https://doi.org/10.1017/S014338579914286X Published online by Cambridge University Press

[BeC2] M. Benedicks and L. Carleson. The dynamics of the Hénon map. *Ann. Math.* **133** (1991), 73–169.

[BeY] M. Benedicks and L.-S. Young. Absolutely continuous invariant measures and random perturbations for certain one-dimensional maps. *Ergod. Th. & Dynam. Sys.* **12** (1992), 13–37.

[BLy1] A. Blokh and M. Lyubich. Attractors of maps of the interval. In *Dynamical Systems and Ergodic Theory*. Banach Center Publications, Vol. 23, 1989, pp. 427–442; see also [**BLy2**].

[BLy2] A. Blokh and M. Lyubich. Measurable dynamics of S-unimodal maps of the interval. *Ann. Sci. Éc. Norm. Sup.* **24** (1991), 545–573.

[BMs] A. Blokh and M. Misiurewicz. Collet–Eckmann maps are unstable. *Commun. Math. Phys.* **191** (1988), 61–70.

[BrH] B. Branner and J. H. Hubbard. The iteration of cubic polynomials. Part II: patterns and parapatterns. *Acta Math.* **169** (1992), 229–325.

[B1] H. Bruin. Invariant measures of interval maps. *PhD Thesis*, Technical University of Delft, 1994.

[B2] H. Bruin. Combinatorics of the kneading map. *Int. J. Bifurcation Chaos* **5** (1995), 1339–1349.

[BKNS] H. Bruin, G. Keller, T. Nowicki and S. van Strien. Wild Cantor attractors exist. *Ann. Math.* **143** (1996), 97–130; submitted in 1994.

[CJY] L. Carleson, P. Jones and J.-C. Yoccoz. Julia and John. *Bol. Soc. Bras. Mat.* **25** (1994), 1–30.

[CE] P. Collet and J.-P. Eckmann. Positive Liapunov exponents and absolute continuity for maps of the interval. *Ergod. Th. & Dynam. Sys.* **3** (1983), 13–46.

[DGMT] S. P. Dawson, R. Galeeva, J. Milnor and C. Tresser. Monotonicity conjecture for real cubic maps. In *NATO Adv. Sci. Inst. Ser. C, Math. Phys. Sci.* **464**. Kluwer, Dordrecht, 1995.

[DPU] M. Denker, F. Przytycki and M. Urbanski. On the transfer operator for rational functions on the Riemann sphere. *Ergod. Th. & Dynam. Sys.* **16** (1996), 255–266.

[DH1] A. Douady and J. H. Hubbard. On the dynamics of polynomial-like mappings. *Ann. Sci. Éc. Norm. Sup. (Paris)* **18** (1985), 287–343

[F] E. De Faria. Asymptotic rigidity of scaling ratios for critical circle mappings. *Preprint*, Stony Brook, 1996. *Ergod. Th. & Dynam. Sys.* to appear.

[FM] E. De Faria and W. De Melo. Rigidity of critical circle mappings, Parts I and II. *Preprints*, Stony Brook, 1997.

[G] J. Graczyk. Dynamics of circle maps with flat interval. Part of 1990 *PhD Thesis*. *Preprint*, Queen's University at Kingston, 1991. *Fund. Math.* to appear.

[GSm] J. Graczyk and S. Smirnov. Collet, Eckmann, & Hölder. *Invent. Math.* **133** (1988), 69–96.

[GSw1] J. Graczyk and G. Świątek. Singular measures in circle dynamics. *Commun. Math. Phys.* **157** (1993), 157–230.

[GSw2] J. Graczyk and G. Świątek. Generic hyperbolicity in the logistic family. *Ann. Math.* **146** (1997), 1–52; submitted in August 1992.

[GSw3] J. Graczyk and G. Świątek. Induced expansion for quadratic polynomials. *Ann. Sci. Éc. Norm. Sup.* **29** (1996), 399–482; first version June 1993.

[GSw4] J. Graczyk G. Świątek. Polynomial-like property for real quadratic polynomials. *Topology Proc.* **21** (1996), 33–112; first version January 1995.

[GSw5] J. Graczyk and G. Świątek. The real Fatou conjecture. *Ann. of Math. Stud.* **144**, Princeton University Press, Princeton, NJ, 1998.

[Gu1] J. Guckenheimer. Sensitive dependence on initial conditions for one dimensional maps. *Commun. Math. Phys.* **70** (1979), 133–160.

[Gu2] J. Guckenheimer. Limit sets of S-unimodal maps with zero entropy. *Commun. Math. Phys.* **110** (1987), 655–659.

[GuJ] J. Guckenheimer and S. Johnson. Distortion of S-unimodal maps. *Ann. Math.* **132** (1990), 71–130.

[He] Ch. Heckman. Monotonicity and the construction of quasiconformal conjugacies in the real cubic family. *Thesis*, SUNY, Stony Brook, 1996.

[Hi] B. Hinkle. Parabolic limits of renormalization. *Preprint*, Stony Brook, 1997.

[H] F. Hofbauer. On intrinsic ergodicity of piecewise monotonic transformations with positive entropy. *Israel J. Math.* **34** (1979), 213–237.

[HK1] F. Hofbauer and G. Keller. Quadratic maps without asymptotic measure. *Commun. Math. Phys.* **127** (1990), 318–337.

[HK2] F. Hofbauer and G. Keller. Some remarks about recent results on S-unimodal maps. *Ann. Inst. de Henri Poincaré, Physique Théorique* **53** (1990), 413–425.

[J1] M. Jakobson. Absolutely continuous invariant measures for one-parameter families of one-dimensional maps. *Commun. Math. Phys.* **81** (1981), 39–88.

https://doi.org/10.1017/S014338579914286X Published online by Cambridge University Press

[J2] M. Jakobson. Quasisymmetric conjugacies for some one-dimensional maps inducing expansion. *Contemporary Mathematics* **135** (1992), 203–211; first version 1989, manuscript.

[JSw1] M. Jakobson and G. Świątek. Metric properties of non-renormalizable S-unimodal maps. *Ergod. Th. & Dynam. Sys.* **14** (1994), 721–755; first version 1991, preprint IHES.

[JSw2] M. Jakobson and G. Świątek. Metric properties of non-renormalizable S-unimodal maps: II. Quasisymmetric conjugacy classes. *Ergod. Th. & Dynam. Sys.* **15** (1995), 871–938; first version September 1991, Stony Brook preprint.

[Jo] S. Johnson. Singular measures without restrictive intervals. *Commun. Math. Phys.* **110** (1987), 185–190.

[K1] G. Keller. Markov extensions, zeta-functions and Fredholm theory for piecewise-invertible dynamical systems. *Trans. Amer. Math. Soc.* **314** (1989), 433–497.

[K2] G. Keller. Exponents, attractors and Hopf decompositions for interval maps. *Ergod. Th. & Dynam. Sys.* **10** (1990), 717–744.

[KN1] G. Keller and T. Nowicki. Spectral theory, zeta functions, and the distribution of periodic orbits for Collet–Eckmann maps. *Commun. Math. Phys.* **149** (1992), 31–69.

[KN2] G. Keller and T. Nowicki. Fibonacci maps re(al) visited. *Ergod. Th. & Dynam. Sys.* **15** (1995), 97–130.

[Ko] O. Kozlovski. Structural stability in one-dimensional dynamics. *PhD Thesis*, University of Amsterdam, 1998.

[La] O. E. Lanford. A computer-assisted proof of the Feigenbaum conjecture. *Bull. Amer. Math. Soc.* **6** (1982), 427–434.

[LS] G. Levin and S. van Strien. Local connectivity of the Julia set of real polynomials. *Ann. Math.* **147** (1998), 471–541; first version December 1994, manuscript.

[LSc] R. de la Llave and P. Schafer. Rigidity properties of one dimensional expanding maps and applications to renormalization. *Preprint*, 1996.

[Ly1] M. Lyubich. Dynamics of complex polynomials, Part I–II. *Acta Math.* **178** (1997), 185–297; first version October 1993, Stony Brook preprint.

[LyYa] M. Lyubich and M. Yampolsky. Dynamics of quadratic polynomials: complex bounds for real maps. *Ann. Inst. Fourier (Grenoble)* **47** (1997), 1219–1255; first version April 1996, preprint MSRI.

[Ly3] M. Lyubich. Combinatorics, geometry and attractors of quasi-quadratic maps. *Ann. Math.* **140** (1994), 347–404; first version November 1992, Stony Brook preprint.

[Ly4] M. Lyubich. Dynamics of quadratic maps, Part III. *Preprint*, Stony Brook, 1996.

[Ly5] M. Lyubich. Almost every real quadratic map is either regular or stochastic. *Preprint*, Stony Brook, 1997.

[LyMi] M. Lyubich and J. Milnor. The dynamics of the Fibonacci polynomial. *J. Amer. Math. Soc.* **6** (1993), 425–457; first version 1991, Stony Brook preprint

[Mñ] R. Mañé. On a theorem of Fatou. *Bol. Soc. Bras. Mat.* **24** (1993), 1–12.

[MñSS] R. Mañé, P. Sad and D. Sullivan. On the dynamics of rational maps. *Ann. Sci. Éc. Norm. Sup.* **16** (1983), 193–217.

[Ma1] M. Martens. The periodic points of renormalization. *Ann. Math.* **147** (1998), 543–584.

[Ma2] M. Martens. Distortion results and invariant Cantor sets of unimodal mappings. *Ergod. Th. & Dynam. Sys.* **14** (1994), 331–349; based on the 1990 PhD thesis, University of Delft.

[MaN] M. Martens and T. Nowicki. Invariant measures for typical quadratic maps. *Preprint*, Stony Brook, 1996.

[MS1] W. de Melo and S. van Strien. *One-Dimensional Dynamics*. Springer, New York, 1993.

[MS2] W. de Melo and S. van Strien. A structure theorem in one-dimensional dynamics. *Ann. Math.* **129** (1989), 519–546.

[Mc] C. McMullen. Renormalization and 3-manifolds which fiber over the circle. *Ann. Math. Studies* **142**. Princeton University Press, Princeton, NJ, 1996.

[Mi1] J. Milnor. On the concept of attractor. *Commun. Math. Phys.* **99** (1985), 177–195.

[Mi2] J. Milnor. The Yoccoz theorem on local connectivity of Julia sets. A proof with pictures. *Class Notes*, Stony Brook, 1991–1992.

[MiT] J. Milnor and W. Thurston. *On Iterated Maps of an Interval (Lecture Notes in Mathematics, 1342)*. Springer, Berlin, 1988, pp. 465–563.

[N] T. Nowicki. Symmetric *S*-unimodal mappings and positive Lyapunov exponents. *Ergod. Th. & Dynam. Sys.* **5** (1985), 611–616.

https://doi.org/10.1017/S014338579914286X Published online by Cambridge University Press

[NP1] T. Nowicki and F. Przytycki. The conjugacy of Collet–Eckmann's map of the interval with the tent map is Hölder continuous. *Ergod. Th. & Dynam. Sys.* **9** (1989), 379–388.

[NP2] T. Nowicki and F. Przytycki. Topological invariance of the Collet–Eckmann property for S-unimodal maps. *Fund. Math.* **155** (1998), 33–43.

[NSa] T. Nowicki and D. Sands. Non-uniform hyperbolicity and universal bounds for S-unimodal maps. *Invent. Math.* **132** (1998), 633–680.

[NS] T. Nowicki and S. van Strien. Absolutely continuous invariant measures under the summability condition. *Invent. Math.* **105** (1991), 123–136.

[P] F. Przytycki. Iterations of holomorphic Collet–Eckmann Maps: conformal and invariant measures. Appendix: on non-renormalizable quadratic polynomials. *Trans. Amer. Math. Soc.* **350** (1998), 717–742.

[PR] F. Przytycki and S. Rohde. Porosity of Collet–Eckmann Julia sets. *Fund. Math.* **155** (1998), 189–199.

[Ry] M. Rychlik. Another proof of Jakobson's theorem and related results. *Ergod. Th. & Dynam. Sys.* **8** (1988), 93–109.

[RySo] M. Rychlik and E. Sorets. Regularity and other properties of absolutely invariant measures for certain maps for the quadratic family. *Commun. Math. Phys.* **150** (1992), 217–236.

[Sa1] D. Sands. Topological conditions for positive Lyapunov exponent in Unimodal maps. *PhD Thesis*, Cambridge University, 1994, pp. 1–107.

[Sa2] D. Sands. Complex bounds. *Manuscript*, October 2, 1996.

[Sl] Z. Slodkowski. Holomorphic motions and polynomial hulls. *Proc. Amer. Math. Soc.* **111** (1991), 347–355.

[S] S. van Strien. Smooth dynamics on the interval (with an emphasis on quadratic-like maps). *New Directions in Differentiable Dynamical Systems (London Math. Soc. Lecture Note Ser., 127)*. Cambridge University Press, 1985, pp. 57–119.

[Su1] D. Sullivan. Quasiconformal homeomorphisms and dynamics I: a solution of Fatou–Julia problem on wandering domains. *Ann. Math.* **122** (1985), 401–418.

[Su2] D. Sullivan. Bounds, quadratic differentials and renormalization conjetures. In *American Mathematical Society Centennial Publications*, Vol. 2. American Mathematical Society, Providence, RI, 1991, pp. 417–466.

[Sw] G. Świątek. Rational rotation numbers for maps of the circle. *Commun. Math. Phys.* **119** (1988), 109–128.

[TV] F. Tangerman and J. Veerman. Scalings in circle maps I. *Commun. Math. Phys.* **134** (1990), 89–107.

[TTY] P. Thieullen, C. Tresser, and L.-S. Young. Exposant de Lyapunov positif dans des familles à un paramétre d'applications unimodales. *C.R. Acad. Sci. Paris Série I* **315** (1992), 69–72.

[T1] M. Tsujii. Positive Lyapunov exponents in families of one-dimensional dynamical systems. *Invent. Math.* **111** (1993), 113–137.

[T2] M. Tsujii. On continuity of Bowen–Ruelle–Sinai measures in families of one-dimensional maps. *Commun. Math. Phys.* **177** (1996), 1–11.

[Yo1] J.-C. Yoccoz. Il n'y a pas de contre-exemple de Denjoy analitique. *C.R. Acad. Sci. Paris, Série I* **298** (1984), 141–144.

[Yo2] J.-C. Yoccoz. *Personal Communication*; see also [**Mi2**].

[Y] L.-S. Young. Decay of correlations for certain quadratic maps. *Commun. Math. Phys.* **146** (1992), 123–138.

https://doi.org/10.1017/S014338579914286X Published online by Cambridge University Press