THE MARKOV OSCILLATION PROBLEM IN DISCRETE TIME

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ABSTRACT

Consider a discrete-time regenerative phenomenon with associated renewal sequence u_n. General results for the supremum of $u_{m+1}, \ldots, u_{m+n}$ are developed for those renewal sequences $\{u_n\}$ for which the first $m+1$ elements match those of a fixed renewal sequence $\{v_n\}$, that is, $u_0 = v_0, \ldots, u_m = v_m$. A series of associated lemmas are developed in the process.

1. *Introduction and statement of key theorems*

Classical probability theory is concerned with transition probabilities $p_{ij}(t)$, where the time $t = 0, 1, 2, \ldots$ or $0 \leqslant t < \infty$ and i and j run over the set S of possible states, and it consists of consequences of the conditions

$$p_{ij}(t) \geqslant 0, \quad \sum_{j \in S} p_{ij}(t) = 1 \quad \text{and} \quad p_{ij}(s+t) = \sum_{k \in S} p_{ik}(s)\, p_{kj}(t) \tag{1.1}$$

[**2**]. If, however, interest centres on the specific function $p_{ii}(t)$ for some particular state i, then surprisingly little information is lost if the other states $j \neq i$ are lumped together in a composite state 'not i'. The result is a two-state process which is not usually Markovian, but has nevertheless a simple stochastic structure. It is called a *regenerative phenomenon* [**11**].

Specific practical examples include the probability $p_{ii}(t)$ of returning to (a) an empty state ($i = 0$) in a recurrent epidemic or queueing scenario, (b) the deterministic carrying capacity ($i = K > 0$) in a stochastic logistic process, and similarly the deterministic threshold population level in a general stochastic epidemic, and (c) the initial financial stake ($i = i_0$) in a stockmarket investment. The development of the associated theory is clearly extremely important, but progress has been limited by the stubborn resistance of mathematical proposals to theoretical attack.

Davidson [**6**, **7**] posed one particularly tantalising gem (see also [**4**]). For a given state of interest i write $p(t) = p_{ii}(t)$, and then for any $t > 0$ put $p(t) = M$. How small is $m = \inf\{p(s) : s < t\}$? In other words, what pairs (m, M) can occur? Consider the following special case of this problem. Let $\mathscr{P}$ denote the set of standard p-functions. For $p \in \mathscr{P}$ and $0 < M < 1$, define $m(p) = \inf\{p(s) : s < 1\}$ with $I(M) = \inf\{m(p) : p \in \mathscr{P}, p(1) = M\}$ and $v_0 = \inf\{M : I(M) > 0\}$. Davidson first proved that $v_0 \geqslant e^{-1}$, and then conjectured that $v_0 = e^{-1}$. Although this conjecture remained unproved for over two decades, successive reductions in an upper bound for v_0 have been obtained:

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(1) 1968: $v_0 \leqslant 3/4$ (Davidson [**6**, **7**] and Blackwell and Freedman [**1**]);
(2) 1973: $v_0 \leqslant 2/3$ (Griffeath [**8**]);
(3) 1973: $v_0 \leqslant 1-e^{-1}$ (Cornish [**3**, **4**]);
(4) 1975: $v_0 \leqslant 0.59$ (Joshi [**9**]);
(5) 1977: $v_0 \leqslant 0.56$ (Joshi [**10**]);
(6) 1984: $v_0 \leqslant 0.515$ (Yu [**12**]);
(7) 1986: $v_0 \leqslant 1/2$ (Zou [**13**]).

The required upper bound was eventually achieved in 1994 when Dai proved the following result [**5**].

PROPOSITION 1. *For any* $p \in \mathscr{P}$,

$$p(t) \leqslant p(s) + e^{-1} \quad (0 \leqslant s \leqslant t \leqslant \infty). \tag{1.2}$$

Davidson's conjecture that $v_0 = e^{-1}$ follows as a direct consequence of this theorem.

However, the proof of Proposition 1 is extremely long and complicated, involving 44 lemmas over 41 pages. Although a simple direct proof that $v_0 = e^{-1}$ still eludes us, substantial simplification of Dai's proof can, nevertheless, be made. One of the key steps in the proof of Proposition 1 is the following Proposition 2. Let p_0 be a fixed standard p-function, and for any $T \geqslant u > 0$ define

$$L(p_0, u, T) = \sup\{p(t) : p \in \mathscr{P},\ p(s) = p_0(s),\ 0 \leqslant s \leqslant u \leqslant t \leqslant T\}. \tag{1.3}$$

PROPOSITION 2. *We have*

$$L(p_0, u, T) = L(p_0, u, 3u) \quad (T \geqslant 3u). \tag{1.4}$$

In particular,

$$\lim_{T \to \infty} L(p_0, u, T) = L(p_0, u, 3u). \tag{1.5}$$

In this paper we shall present a greatly simplified proof of Proposition 2, which in turn greatly simplifies the proof of Proposition 1, and thereby that of Davidson's conjecture. We accept that 3 is perhaps not the best possible constant one might use; for example, we conjecture that $L(p_0, u, T) = L(p_0, u, 2u)$ for $T \geqslant 2u$. However, the proof is likely to be difficult, and so we shall retain the constant 3 for our needs here.

A stochastic sequence $Z = \{Z_n : n \geqslant 0\}$ taking the values 0 and 1 is called a discrete-time regenerative phenomenon if $Z_0 = 1$ and there exists a non-negative sequence $\{u_n : n \geqslant 0\}$ with $u_0 = 1$ such that

$$\Pr(Z_{t_1} = \ldots = Z_{t_n} = 1) = \prod_{r=1}^{n} u_{t_r - t_{r-1}} \tag{1.6}$$

for all integers $t_0, t_1, \ldots, t_n$ satisfying $0 = t_0 < t_1 < \ldots < t_n$. Define $\lambda_1 = \Pr(Z_1 = 1)$ and $\lambda_n = \Pr(Z_1 = 0 = \ldots = Z_{n-1} = 0,\ Z_n = 1)$ $(n \geqslant 2)$. Then $\lambda_n \geqslant 0$ and $\sum_{n=1}^{\infty} \lambda_n \leqslant 1$, and it is easy to see that

$$u_0 = 1 \quad \text{and} \quad u_n = \sum_{k=1}^{n} \lambda_k u_{n-k} \quad (n \geqslant 1). \tag{1.7}$$

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The non-negative sequence $u = \{u_n : n \geqslant 0\}$ satisfying (1.7) is called a *renewal sequence*, and $\lambda = \{\lambda_n : n \geqslant 1\}$ is called the associated *F-sequence*. Equations (1.7) establish a one-to-one correspondence between the set of all renewal sequences and regenerative phenomena.

Denote

$$U(z) \equiv \sum_{n=0}^{\infty} u_n z^n \quad \text{and} \quad \Lambda(z) \equiv \sum_{n=1}^{\infty} \lambda_n z^n \quad (0 \leqslant z < 1). \tag{1.8}$$

Then it is easy to prove the following proposition.

PROPOSITION 3. *The following statements are equivalent:*
(1) $u = \{u_n : n \geqslant 0\}$ *is a renewal sequence with F-sequence* $\lambda = \{\lambda_n : n \geqslant 1\}$.
(2) $U(z) = 1/(1-\Lambda(z)) \,\forall\, 0 \leqslant z < 1$.
(3) *If* $\lambda^{(k)}$ *denotes the k th convolution of* λ *with itself, then*

$$u_0 = 1 \text{ and } u_n = \sum_{k=1}^{n} \lambda_n^{(k)} \qquad (n \geqslant 1). \tag{1.9}$$

Note that since

$$\lambda_n^{(k)} = \sum_{i_1+\ldots+i_k=n} \lambda_{i_1} \ldots \lambda_{i_k}, \tag{1.10}$$

on using (1.9) we have

$$u_0 = 1 \quad \text{and} \quad u_n = \sum_{k=1}^{n} \sum_{i_1+\ldots+i_k=n} \lambda_{i_1} \ldots \lambda_{i_k} \quad (n \geqslant 1). \tag{1.11}$$

The study of renewal sequences has a long history, and although many problems have been solved completely, the objective of this paper is to raise and discuss some new issues concerning the supremum of $\{u_n\}$. Specifically, we shall examine the supremum of renewal sequences $\{u_n\}$ whose first m elements coincide with those of a fixed sequence $\{v_n\}$. Thus, given a renewal sequence with known initial structure, we are interested in the supremum of subsequent elements.

Consider the fixed renewal sequence $v = \{v_n : n \geqslant 0\}$, with F-sequence $g = \{g_n : n \geqslant 1\}$, satisfying the condition that there is a positive integer m such that $g_n = 0$ for all $n \geqslant m+1$. That is, $g = (g_1, \ldots, g_m, 0, \ldots)$. Let $a = \sum_{i=1}^{m} g_i$ with $g_\infty = 1-a$. Denote $\mathscr{R}$ to be the totality of all renewal sequences, and $\mathscr{R}(v)$ to be the subset of renewal sequences such that the first $m+1$ elements coincide with v. In other words,

$$\mathscr{R}(v) = \{u : u \in \mathscr{R}, u_i = v_i, 0 \leqslant i \leqslant m\}. \tag{1.12}$$

Clearly, $v \in \mathscr{R}(v)$. It is easy to see that if $u \in \mathscr{R}(v)$ then its F-sequence $\lambda = \{\lambda_n : n \geqslant 1\}$ satisfies

$$\lambda_i = g_i \quad (0 \leqslant i \leqslant m). \tag{1.13}$$

Conversely, if the F-sequence λ satisfies (1.13), then $u \in \mathscr{R}(v)$. Moreover, it follows that if $u \in \mathscr{R}(v)$ then $u_i \geqslant v_i$, $\forall\, i \geqslant m+1$. Thus v is the minimal element in $\mathscr{R}(v)$.

Set

$$L(v, m, m+n) = \sup\{u_k : u \in \mathscr{R}(v),\ m+1 \leqslant k \leqslant m+n\}. \tag{1.14}$$

Thus $L(v, m, m+n)$ is the supremum of values which are taken by all elements of $\mathscr{R}(v)$ in $[m+1, m+n]$. Later we will prove that there is a $u \in \mathscr{R}(v)$ which attains this supremum. Clearly, $L(v, m, m+n)$ is a nondecreasing function of n. Therefore,

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$L(v, m, \infty) = \lim_{n\to\infty} L(v, m, m+n)$ exists. Although we cannot obtain a precise value of $L(v, m, \infty)$ for any fixed v, we can derive an explicit expression for $L(v, m, m+n)$ which depends only on v. We first state our two main results; proofs are provided at the end of the paper following the development of a series of intermediate lemmas.

THEOREM 1. *The supremum*

$$L(v, m, m+n) = L(v, m, 3m) \quad \forall\ n \geqslant 2m. \tag{1.15}$$

Proposition 2 can now be deduced as a direct corollary of this theorem by using the discrete skeleton method.

To obtain an explicit expression for $L(v, m, m+n)$, define

$$h(s) = \begin{cases} 1 & s = 0 \\ \sum_{j=1}^{s} (j+1)\, g_s^{(j)} & s \geqslant 1. \end{cases} \tag{1.16}$$

Then we may prove the following result.

THEOREM 2. *For any $1 \leqslant n \leqslant m$, there exists n^*, where $1 \leqslant n^* \leqslant n$, such that*

$$L(v, m, m+n) = v(m+n^*) + (1-a) \max_{0 \leqslant s \leqslant n^*-1} h(s); \tag{1.17}$$

whilst for any $m+1 \leqslant n \leqslant 2m$, there exists n^*, where $m+1 \leqslant n^* \leqslant n$, such that

$$L(v, m, m+n) = v(m+n^*) + (1-a) \max_{0 \leqslant s \leqslant m} h(s). \tag{1.18}$$

2. *Convolution*

Let $\lambda = \{\lambda_n : n \geqslant 1\}$ and $\eta = \{\eta_n : n \geqslant 1\}$ be two probability distributions on $[1, 2, \ldots, \infty]$ (or F-sequences of two renewal sequences). Then their *convolution* $\lambda * \eta$ is defined as

$$(\lambda * \eta)_n = \sum_{s=0}^{n} \lambda_s \eta_{n-s};$$

we denote $\lambda_0 = 0$ if λ is an F-sequence.

Let $u \in \mathscr{R}(v)$ with F-sequence $\lambda = \{\lambda_n : n \geqslant 1\}$. Then we can write $\lambda = \{g_1, \ldots, g_m, \lambda_{m+1}, \ldots\}$. Set $\eta = \lambda - g = (0, \ldots, 0, \lambda_{m+1}, \ldots)$, and note that η is independent of $g = \{g_n : n \geqslant 1\}$. From (1.9) we have

$$u_n - v_n = \sum_{k=1}^{n} (\lambda_n^{(k)} - g_n^{(k)}). \tag{2.1}$$

We shall use the usual convolution formula, namely

$$\lambda^{(k)} = (g+\eta)^{(k)} = \sum_{j=0}^{k} \binom{k}{j} (g^{(j)} * \eta^{(k-j)}), \tag{2.2}$$

where by convention $g^0 = \eta^0 = \delta$ denotes the Dirac measure on $\{0\}$ (that is, $\delta_0 = 1$, $\delta_n = 0$, $\forall\ n \geqslant 1$). Clearly, $\lambda * \delta = \lambda$ for any F-sequence λ. Thus from (2.1)

$$u_n - v_n = \sum_{k=1}^{n} \sum_{j=0}^{k-1} \binom{k}{j} (g^{(j)} * \eta^{(k-j)})_n. \tag{2.3}$$

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We shall now consider some particular cases.

Case 1. $m+1 \leqslant n \leqslant 2m+1$. Since $\eta_n^{(j)} = 0$ $(j \geqslant 2)$, (2.3) becomes

$$u_n - v_n = \sum_{k=1}^{n} k(g^{(k-1)} * \eta)_n \quad (m+1 \leqslant n \leqslant 2m+1). \tag{2.4}$$

Moreover, when $k = 1$,

$$(g^{(k-1)} * \eta)_n = (\delta * \eta)_n = \lambda_n \quad (m+1 \leqslant n \leqslant 2m+1); \tag{2.5}$$

and when $k > 1$,

$$(g^{(k-1)} * \eta)_n = \sum_{s=1}^{n-1} g_{n-s}^{(k-1)} \eta_s = \sum_{s=m+1}^{n-1} \lambda_s g_{n-s}^{(k-1)} \quad (m+1 \leqslant n \leqslant 2m+1). \tag{2.6}$$

Substituting (2.5) and (2.6) back into (2.4) then gives

$$\begin{aligned} u_n - v_n &= \lambda_n + \sum_{k=2}^{n} k \sum_{s=m+1}^{n-1} \lambda_s g_{n-s}^{(k-1)} = \lambda_n + \sum_{s=m+1}^{n-1} \lambda_s \sum_{k=2}^{n-s+1} k g_{n-s}^{(k-1)} \\ &= \sum_{s=m+1}^{n-1} \lambda_s h(n-s) + \lambda_n = \sum_{s=m+1}^{n} \lambda_s h(n-s), \end{aligned} \tag{2.7}$$

where $h(n-s)$ is defined by (1.16).

Case 2. $2m+2 \leqslant n \leqslant 3m+2$. In this case $\eta_n^{(j)} = 0$ $(j \geqslant 3)$, so

$$\begin{aligned} u_n - v_n &= \sum_{k=1}^{n} k(g^{(k-1)} * \eta)_n + \sum_{k=2}^{n} \frac{k(k-1)}{2} (g^{(k-2)} * \eta^{(2)})_n \\ &= \sum_{s=m+1}^{n} \lambda_s h(n-s) + \sum_{k=2}^{n} \frac{k(k-1)}{2} (g^{(k-2)} * \eta^{(2)})_n. \end{aligned} \tag{2.8}$$

Here we used $\eta_n^{(2)} = 0$ for $1 \leqslant n \leqslant 2m+1$. When $k = 2$

$$(g^{(k-2)} * \eta^{(2)})_n = \lambda_n^{(2)}, \tag{2.9}$$

and when $k > 2$

$$(g^{(k-2)} * \eta^{(2)})_n = \sum_{s=2m+2}^{n-1} \lambda_s^{(2)} g_{n-s}^{(k-2)}, \tag{2.10}$$

whence inserting (2.9) and (2.10) back into (2.8) yields

$$u_n - v_n = \sum_{s=m+1}^{n} \lambda_s h(n-s) + \sum_{s=2m+2}^{n} \lambda_s^{(2)} d(n-s), \tag{2.11}$$

where

$$d(s) = \sum_{k=0}^{s} \frac{(k+1)(k+2)}{2} g_s^{(k)}. \tag{2.12}$$

Case 3. $3m+3 \leqslant n \leqslant 4m+3$. Applying the same method produces

$$u_n - v_n = \sum_{s=m+1}^{n} \lambda_s h(n-s) + \sum_{s=2m+2}^{n} \lambda_s^{(2)} d(n-s) + \sum_{s=3m+3}^{n} \lambda_s^{(3)} e(n-s), \tag{2.13}$$

where

$$e(s) = \sum_{k=0}^{s} \frac{(k+1)(k+2)(k+3)}{6} g_s^{(k)}. \tag{2.14}$$

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Results (2.7), (2.11) and (2.13) are sufficient for our future needs.

3. *Intermediate lemmas*

In order to prove Theorems 1 and 2 we need to prove some preliminary results. Let us first consider two extreme cases.

LEMMA 1. *If $a = 1$, that is, $g_1 + \ldots + g_m = 1$, then*

$$L(v, m, n) = L(v, m, 2m) \quad (n \geqslant 2m). \tag{3.1}$$

Proof. In this case $\mathscr{R}(v)$ contains only a single element, that is, $\mathscr{R}(v) = \{v\}$. Now when $k \geqslant m+1$,

$$v_k = \sum_{j=1}^{m} g_j v_{k-j}. \tag{3.2}$$

Set $\xi = \max_{1 \leqslant j \leqslant m} \{v_{m+j}\}$, whence $v_{2m+1} = \sum_{j=1}^{m} g_j v_{2m+1-j} \leqslant \xi$. By using the recursion formula (3.2) we can then easily show that $v_{2m+j} \leqslant \xi$ for any $j \geqslant 1$, whence (3.1) follows. □

For the second extreme case, that is, $a = 0$, we need only consider the renewal sequence $u = \{u_n : n \geqslant 0\}$ with F-sequence $\lambda = \{\lambda_n : n \geqslant 1\}$ satisfying $\lambda_{m+1} = 1$ and $\lambda_k = 0$ if $k \neq m+1$. It is clear that (3.1) holds.

In the following discussion we shall therefore always suppose that $0 < a < 1$. We now prove a series of general lemmas.

LEMMA 2. *Suppose that $u \in \mathscr{R}(v)$ with F-sequence λ satisfying $\lambda_\infty = 1 - \sum_{i=1}^{\infty} \lambda_i > 0$. Let $n > m$, and define*

$$\mu_i = \begin{cases} \lambda_i & i \neq n, \\ \lambda_n + r & i = n, \end{cases} \tag{3.3}$$

where $r \leqslant \lambda_\infty$. Then if w is the renewal sequence with F-sequence μ,

$$w_{n+j} = u_{n+j} + rh(j) \quad (\forall\, 0 \leqslant j \leqslant m). \tag{3.4}$$

Proof. This parallels the development of (2.7) □

LEMMA 3. *For any $n \geqslant 1$, there exists a renewal sequence $u = \{u_n : n \geqslant 0\} \in \mathscr{R}(v)$ and an integer n^*, where $1 \leqslant n^* \leqslant n$, such that*

$$u_{m+n^*} = L(v, m, m+n). \tag{3.5}$$

Proof. For any $j \geqslant 1$, take $u(j) \in \mathscr{R}(v)$ such that

$$L(v, m, m+n) - \max_{1 \leqslant s \leqslant n} u(j)_{m+s} < 1/j. \tag{3.6}$$

Then by the compactness of $\mathscr{R}$ (see [**11**, Theorem 1.3]), there exists $\bar{u} \in \mathscr{R}$ and a subsequence $u(j_k)$ of $u(j)$ such that

$$\lim_{k \to \infty} u(j_k)_n = \bar{u}_n.$$

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Since $u(j)\in\mathscr{R}(v)$, we have $\bar{u}_s = u(j)_s = v_s$ for $0\leqslant s\leqslant m$, and so $\bar{u}\in\mathscr{R}(v)$, whence, by (3.6), $L(v,m,m+n) = \max_{1\leqslant s\leqslant n}\bar{u}_{m+s}$. On choosing n^* $(1\leqslant n^*\leqslant n)$ such that $\bar{u}_{m+n^*} = \max_{1\leqslant s\leqslant n}\bar{u}_{m+s}$, we then obtain (3.5). □

LEMMA 4. *Let $u\in\mathscr{R}(v)$ with F-sequence $\lambda = \{\lambda_n : n\geqslant 1\}$, where $1\leqslant n^*\leqslant n$, such that*

$$u_{m+n^*} = L(v,m,m+n). \tag{3.7}$$

Then $\lambda_{m+j} = 0$ for any $j\geqslant n^*+1$.

Proof. We must prove that $c = a+\sum_{s=1}^{n^*}\lambda_{m+s} = 1$. Assume that $c<1$, and set

$$\bar{\lambda}_j = \begin{cases} \lambda_j & 1\leqslant j\leqslant m+n^*-1 \\ \lambda_{m+n^*}+1-c & j = m+n^* \\ 0 & j\geqslant m+n^*+1. \end{cases} \tag{3.8}$$

If $\bar{u}\in\mathscr{R}(v)$ with F-sequence $\bar{\lambda}$, then $\lambda_i = \bar{\lambda}_i$ and $u_i = \bar{u}_i$ $(0\leqslant i\leqslant m+n^*-1)$. Hence

$$\begin{aligned}\bar{u}_{m+n^*} &= \sum_{s=1}^{m+n^*}\bar{\lambda}_s\bar{u}_{m+n^*-s} = \sum_{s=1}^{m+n^*-1}\lambda_s u_{m+n^*-s}+\bar{\lambda}_{m+n^*} \\ &= u_{m+n^*}+1-c > u_{m+n^*} = L(v,m,m+n).\end{aligned}$$

This is contrary to the definition of $L(v,m,m+n)$, and so $c=1$. □

LEMMA 5. *If $1\leqslant n\leqslant m+1$, then there is an n^* satisfying $1\leqslant n^*\leqslant n$ such that*

$$L(v,m,m+n) = v_{m+n^*}+(1-a)\max_{0\leqslant s\leqslant n^*-1}h(s). \tag{3.9}$$

Proof. By Lemma 3, there exist $u\in\mathscr{R}(v)$ and n^* satisfying $1\leqslant n^*\leqslant n$, such that

$$u_{m+n^*} = L(v,m,m+n).$$

If λ is the F-sequence of u, then, by Lemma 4, $a+\sum_{s=1}^{n^*}\lambda_{m+j} = 1$. From (2.7)

$$u_{m+n^*}-v_{m+n^*} = \sum_{j=1}^{n^*}\lambda_{m+j}h(n^*-j);$$

hence

$$L(v,m,m+n)\leqslant v_{m+n^*}+(1-a)\max_{1\leqslant j\leqslant n^*}h(n^*-j) = v_{m+n^*}+(1-a)\max_{0\leqslant s\leqslant n^*-1}h(s).$$

Now choose i_0 $(1\leqslant i_0\leqslant n^*)$ satisfying $h(n^*-i_0) = \max_{1\leqslant j\leqslant n^*}h(n^*-j)$, and set

$$\bar{\lambda}_i = \begin{cases} g_i & 1\leqslant i\leqslant m \\ 1-a & i = m+i_0 \\ 0 & i\geqslant m+1,\ i\neq m+i_0. \end{cases}$$

Let $\bar{u}\in\mathscr{R}(v)$ with F-sequence $\bar{\lambda}$. Then $\bar{u}\in\mathscr{R}(v)$ and

$$\bar{u}_{m+n^*} = v_{m+n^*}+\bar{\lambda}_{m+i_0}h(n^*-i_0).$$

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Hence by definition we have $\bar{u}_{m+n^*} = L(v,m,m+n) = u_{m+n^*}$. □

If we let

$$I(v,m) = \{n : n \geqslant 2, L(v,m,m+n-1) < L(v,m,m+n)\}, \tag{3.10}$$

then it is easy to see that, if $n \in I(v,m)$, then the result in Lemma 5 becomes $n^* = n$. Thus there exists $u \in \mathscr{R}(v,m)$ such that $u_{m+n} = L(v,m,m+n)$.

LEMMA 6. *Let $n \geqslant m+1$ and $n \in I(v,m)$. If $u \in \mathscr{R}(v)$ with F-sequence λ, then*

$$\max\{\lambda_n, \ldots, \lambda_{m+n}\} > 0. \tag{3.11}$$

Proof. Assume that $\lambda_{m+j} = 0$ for all $j \in [n-m, n]$. Then

$$u_{m+n} = \sum_{j=1}^{m+n} \lambda_j u_{m+n-j} = \sum_{j=1}^{n-1} \lambda_j u_{m+n-j} \tag{3.12}$$

$$\leqslant \max_{1 \leqslant j \leqslant n-1} \{u_{m+n-j}\} = \max\{u_{m+1}, \ldots, u_{m+n-1}\} \tag{3.13}$$

$$\leqslant L(v,m,m+n-1). \tag{3.14}$$

This is contrary to $n \in I(v,m)$; hence (3.11) holds. □

In the following lemma we let $m_0 \in [0,m]$ be such that

$$h(m_0) = \max\{h(0), h(1), \ldots, h(m)\}. \tag{3.15}$$

LEMMA 7. *Let $n \geqslant m+1$ and $n \in I(v,m)$. Then we can choose $\bar{u} \in \mathscr{R}(v)$ such that $\bar{u}_{m+n} = L(v,m,m+n)$, and its F-sequence $\bar{\lambda}$ satisfies*

$$\bar{\lambda}_i = \begin{cases} g_i & 1 \leqslant i \leqslant m \\ \bar{\lambda}_i & m+1 \leqslant i \leqslant n-1 \\ 1-a-\sum_{s=m+1}^{n-1} \bar{\lambda}_s & i = m+n-m_0 \\ 0 & i \geqslant n, i \neq m+n-m_0, \end{cases} \tag{3.16}$$

where $\sum_{s=m+1}^{n-1} \bar{\lambda}_s < 1-a$, that is, $\bar{\lambda}_{m+n-m_0} > 0$.

Proof. By Lemma 6, if $u \in \mathscr{R}(v)$ with F-sequence λ such that $u_{m+n} = L(v,m,m+n)$, then $\max\{\lambda_n, \ldots, \lambda_{m+n}\} > 0$. Now define $\bar{\lambda}$ as in (3.16), where $\bar{\lambda}_i = \lambda_i$ if $m+1 \leqslant i \leqslant n-1$, and let $\bar{u} \in \mathscr{R}(v)$ with F-sequence $\bar{\lambda}$. Then it follows from (3.15) that $\bar{u}$ and $\bar{\lambda}$ satisfy the conditions of Lemma 7. □

LEMMA 8. *Let $n \geqslant m+1$ and $n \in I(v,m)$, with $\bar{u}$ and $\bar{\lambda}$ as defined in Lemma 7. Then*

$$\bar{\lambda}_r = 0 \quad \text{for } \left[\frac{m+n}{2}\right] + 1 \leqslant r \leqslant m+n. \tag{3.17}$$

Proof. Denote

$$\tilde{\lambda}_i = \begin{cases} g_i & 1 \leqslant i \leqslant m \\ \bar{\lambda}_i & m+1 \leqslant i \leqslant [(m+n)/2] \\ 0 & i \geqslant [(m+n)/2]+1, \end{cases} \tag{3.18}$$

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and let $\tilde{u} \in \mathscr{R}(v)$ with F-sequence $\tilde{\lambda}$. Now replace v and m in the definition of $L(v, m, m+n)$ by $\tilde{u}$ and $[(m+n)/2]$, respectively. Then, on considering $L(\tilde{u}, [(m+n)/2], m+n)$, and applying (2.7), we obtain

$$\bar{u}_{m+n} = \tilde{u}_{m+n} + \sum_{s=[(m+n)/2]+1}^{n-1} \bar{\lambda}_s \tilde{h}(m+n-s) + \bar{\lambda}_{m+n-m_0} h(m_0). \tag{3.19}$$

Here $\tilde{h}(j) = h(j)$ if $0 \leqslant j \leqslant m-1$. Clearly,

$$\bar{u}_{m+n} = L(v, m, m+n) = L\left(\tilde{u}, \left[\frac{m+n}{2}\right], m+n\right).$$

Moreover, using Lemma 6 we can show that

$$\tilde{h}(m+n-s) < h(m_0) \quad \forall\, s \in \left[\left[\frac{m+n}{2}\right], n-1\right]. \tag{3.20}$$

Indeed, if there is some $s_0 \in [[(m+n)/2], n-1]$ such that $\tilde{h}(m+n-s_0) = h(m_0)$, then on taking

$$\hat{\lambda}_{s_0} = 1 - a - \sum_{s=m+1}^{[(m+n)/2]} \tilde{\lambda}_s$$

we obtain

$$u_{m+n} = \tilde{u}_{m+n} + \hat{\lambda}_{s_0} \tilde{h}_{m+n-s_0} = L(v, m, m+n)$$

which is contrary to Lemma 6. We therefore have

$$u_{m+n} = \tilde{u}_{m+n} + \bar{\lambda}_{m+n-m_0} h(m_0) \tag{3.21}$$

which completes the proof. □

The following lemma is a key result of this paper.

LEMMA 9. *Let m be a positive integer and let $c > 0$. Denote by $\mathscr{M}$ all the nonnegative measures on $\{m+1, \ldots, 3m+2\}$ satisfying the condition that $\lambda \in \mathscr{M}$ implies that $\sum_{s=m+1}^{3m+2} \lambda_s = c$. Let h and d be nonnegative functions on $\{m+1, \ldots, 3m+2\}$. Then for any $\lambda \in \mathscr{M}$ and $2m+2 \leqslant T \leqslant 3m+2$, either*

$$\sum_{s=m+1}^{T} h(s)\lambda_s + \sum_{s=2m+2}^{T} d(s)\lambda_s^{(2)} \leqslant c \max_{T-m+1 \leqslant s \leqslant T} h(s), \tag{3.22}$$

or else there exists $\mu \in \mathscr{M}$ such that

$$\mu_s = 0 \quad (\forall (T-m+1) \leqslant s \leqslant T) \tag{3.23}$$

and

$$\sum_{s=m+1}^{T} h(s)\mu_s + \sum_{s=2m+2}^{T} d(s)\mu_s^{(2)} \geqslant \sum_{s=m+1}^{T} h(s)\lambda_s + \sum_{s=2m+2}^{T} d(s)\lambda_s^{(2)}. \tag{3.24}$$

Here $\lambda^{(2)}$ ($\mu^{(2)}$) denotes the convolution of λ (μ) with itself.

Proof. If (3.22) is not true, then

$$\sum_{s=m+1}^{T} h(s)\lambda_s + \sum_{s=2m+2}^{T} d(s)\lambda_s^{(2)} > c \max_{T-m+1 \leqslant s \leqslant T} h(s). \tag{3.25}$$

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Now we may prove that $\sum_{s=m+1}^{T-m}\lambda(s)>0$ by using the definition of convolution and noting that $2(T-m)>T$ and $\lambda_s^{(2)}=0$ for all $s\in[T-m+1,T]$. Thus

$$\sum_{s=m+1}^{T}h(s)\,\lambda_s=\sum_{s=T-m+1}^{T}h(s)\,\lambda_s\leqslant c\max_{T-m+1\leqslant s\leqslant T}h(s)$$

which contradicts (3.25). Set

$$\mu_s=\begin{cases}c\lambda_s/R & m+1\leqslant s\leqslant T-m\\ 0 & T-m+1\leqslant s\leqslant 3m+2,\end{cases}\tag{3.26}$$

where $R=\sum_{s=m+1}^{T-m}\lambda_s$. Then μ is a measure on $\{m+1,\ldots,3m+2\}$ and $\sum_{s=m+1}^{3m+2}\mu_s=c$, so $\mu\in\mathscr{M}$. Using (3.26) it is easy to see that (3.23) holds for μ. We shall now prove that (3.24) also holds. We have

$$\begin{aligned}\sum_{s=m+1}^{T-m}h(s)\,\mu_s+\sum_{s=2m+2}^{T}d(s)\,\mu_s^{(2)}&=\frac{c}{R}\sum_{s=m+1}^{T-m}h(s)\,\lambda_s+\left(\frac{c}{R}\right)^2\sum_{s=2m+2}^{T}d(s)\,\lambda_s^{(2)}\\&\geqslant\frac{c}{R}\left[\sum_{s=m+1}^{T}h(s)\,\lambda_s+\sum_{s=2m+2}^{T}d(s)\,\lambda_s^{(2)}-\sum_{s=T-m+1}^{T}h(s)\,\lambda_s\right],\end{aligned}$$

which by (3.25) is

$$\begin{aligned}&\geqslant\frac{c}{R}\left[1-(1/c)\sum_{s=T-m+1}^{3m+2}\lambda_s\right]\left[\sum_{s=m+1}^{T}h(s)\,\lambda_s+\sum_{s=2m+2}^{T}d(s)\,\lambda_s^{(2)}\right]\\&=\sum_{s=m+1}^{T}h(s)\,\lambda_s+\sum_{s=2m+2}^{T}d(s)\,\lambda_s^{(2)}.\end{aligned}$$

Hence (3.24) holds for μ. □

Using the same method we can also prove the following result.

LEMMA 10. *Let m be a positive integer and let $c>0$. Denote by $\mathscr{M}$ all the nonnegative measures on $\{m+1,\ldots,4m+3\}$ satisfying the condition that $\lambda\in\mathscr{M}$ implies that $\sum_{s=m+1}^{4m+3}\lambda_s=c$. Let h, d and e be nonnegative functions on $\{m+1,\ldots,4m+3\}$. Then for any $\lambda\in\mathscr{M}$ and $3m+3\leqslant T\leqslant 4m+3$, either*

$$\sum_{s=m+1}^{T}h(s)\,\lambda_s+\sum_{s=2m+2}^{T}d(s)\,\lambda_s^{(2)}+\sum_{s=3m+3}^{T}e(s)\,\lambda_s^{(3)}\leqslant c\max_{T-m+1\leqslant s\leqslant T}h(s),\tag{3.27}$$

or else there exists $\mu\in\mathscr{M}$ such that

$$\mu_s=0\quad(\forall\ (T-m+1)\leqslant s\leqslant T)\tag{3.28}$$

and

$$\begin{aligned}&\sum_{s=m+1}^{T}h(s)\,\mu_s+\sum_{s=2m+2}^{T}d(s)\,\mu_s^{(2)}+\sum_{s=3m+3}^{T}e(s)\,\mu_s^{(3)}\\&\qquad\geqslant\sum_{s=m+1}^{T}h(s)\,\lambda_s+\sum_{s=2m+2}^{T}d(s)\,\lambda_s^{(2)}+\sum_{s=3m+3}^{T}e(s)\,\lambda_s^{(3)}.\end{aligned}\tag{3.29}$$

Here $\lambda^{(3)}$ and $\mu^{(3)}$ again denote self-convolution.

LEMMA 11. *Let $m+2\leqslant n\leqslant 2m+2$. Then if $n\in I(v,m)$, there exists $u\in\mathscr{R}(v)$, with F-sequence λ, satisfying the following conditions:*

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(1) $u_{m+n} = L(v, m, m+n)$;
(2) $\lambda_{m+n-m_0} = 1-a$ *where* m_0 *is defined by* (3.15);
(3) $\lambda_j = 0$ *for any* $j \geqslant m+1$, $j \neq m+n-m_0$.

Proof. By Lemma 8, if $n \in I(v, m)$ then there exists $\bar{u} \in \mathscr{R}(v)$, with F-sequence $\bar{\lambda}$, satisfying the following conditions:
(1) $\bar{u}_{m+n} = L(v, m, m+n)$;
(2) $\bar{\lambda}_{m+n-m_0} > 0$ where m_0 is defined by (3.15);
(3) $\bar{\lambda}_j = 0$ for any $j \geqslant [(m+n)/2]+1$, $j \neq m+n-m_0$.

Therefore, by Lemma 9, either

$$\sum_{s=m+1}^{[(m+n)/2]} \bar{\lambda}_s h(m+n-s) + \bar{\lambda}_{m+n-m_0} h(m_0) + \sum_{s=2m+2}^{m+n} \bar{\lambda}_s^{(2)} d(m+n-s) \leqslant (1-a) \max_{n+1 \leqslant s \leqslant m+n} h(m+n-s) = (1-a) h(m_0), \quad (3.30)$$

or else there exists $\mu \in \mathscr{R}(v)$ such that

$$\mu_s = 0 \quad (\forall\ (n+1) \leqslant s \leqslant m+n) \quad (3.31)$$

and

$$\sum_{s=m+1}^{m+n} h(m+n-s)\mu_s + \sum_{s=2m+2}^{m+n} d(m+n-s)\mu_s^{(2)} \geqslant \sum_{s=m+1}^{m+n} h(m+n-s)\lambda_s + \sum_{s=2m+2}^{m+n-s} d(m+n-s)\lambda_s^{(2)}. \quad (3.32)$$

However, it follows from Lemma 6 that the latter cannot hold, since $\lambda_{m+n-m_0} > 0$. Therefore (3.30) must hold instead. Now put

$$\lambda_i = \begin{cases} g_i & 1 \leqslant i \leqslant m \\ 1-a & i = m+n-m_0 \\ 0 & i \geqslant m+1,\ i \neq m+n-m_0, \end{cases} \quad (3.33)$$

and let $u \in \mathscr{R}(v)$ with F-sequence λ. Then

$$u_{m+n} = v_{m+n} + (1-a) h(m_0) \geqslant \bar{u}_{m+n} = L(v, m, m+n), \quad (3.34)$$

and the proof is complete. □

Using the same method, but this time replacing Lemma 9 in the proof by Lemma 10, we can also obtain the following lemma.

LEMMA 12. *Let* $2m+3 \leqslant n \leqslant 3m+3$. *Then if* $n \in I(v, m)$ *there exists* $u \in \mathscr{R}(v)$, *with F-sequence* λ, *satisfying the following conditions:*
(1) $u_{m+n} = L(v, m, m+n)$;
(2) $\lambda_{m+n-m_0} = 1-a$ *where* m_0 *is defined by* (3.15);
(3) $\lambda_j = 0$ *for any* $j \geqslant m+1$, $j \neq m+n-m_0$.

4. *Proof of theorems*

We have now developed sufficient results to prove Theorems 1 and 2; all that is left is to show that if $n \geqslant 3m+4$ then $n \notin I(v, m)$.

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LEMMA 13. *If* $n \geqslant 3m+4$ *then* $n \notin I(v,m)$.

Proof. By Lemma 8, if $n \in I(v,m)$ then there exists $u \in \mathscr{R}(v)$ with F-sequence λ satisfying the following conditions:

(1) $u_{m+n} = L(v,m,m+n)$.

(2) $\lambda_{m+n-m_0} > 0$ where m_0 is defined by (3.15), that is, $h(m_0) = \max\{h(0), h(1), \ldots, h(m)\}$.

(3) $\lambda_j = 0$ for any $j \geqslant [(m+n)/2]+1$, $j \neq m+n-m_0$.

Now define

$$\mu_i = \begin{cases} g_i & 1 \leqslant i \leqslant m \\ \lambda_i & m+1 \leqslant i \leqslant [(m+n)/2] \\ 0 & i \geqslant [(m+n)/2]+1, \end{cases} \tag{4.1}$$

and let $w \in \mathscr{R}(v)$ with F-sequence μ. On using (1.7), we have

$$w_{m+n} = \sum_{s=1}^{m} g_s w_{m+n-s} + \sum_{s=m+1}^{[(m+n)/2]} \mu_s w_{m+n-s} \leqslant \max_{[(m+n)/2] \leqslant j \leqslant m+n} w_s. \tag{4.2}$$

Hence, noting that $[(m+n)/2] \geqslant 2m+2$, there is $r \in [2m+2, m+n-1]$ such that $w_r \geqslant w_{m+n}$. Set

$$\tilde{\mu}_i = \begin{cases} g_i & 1 \leqslant i \leqslant m \\ \lambda_i & m+1 \leqslant i \leqslant [(m+n)/2] \\ \lambda_{m+n-m_0} & i = r-m_0 \\ 0 & i \geqslant [(m+n)/2]+1,\ i \neq r-m_0, \end{cases} \tag{4.3}$$

and $\tilde{w} \in \mathscr{R}(v)$ with F-sequence $\tilde{\mu}$. It then follows from Lemma 2 that

$$\tilde{w}_r = w_r + \lambda_{m+n-m_0} h(m_0) \geqslant w_{m+n} + \lambda_{m+n-m_0} h_{m_0} = u_{m+n} = L(v,m,m+n), \tag{4.4}$$

which runs contrary to $n \in I(v,m)$. □

We are now in a position to prove our two main results.

Proof of Theorem 1. We first need to prove that $2m+1 \notin I(v,m)$, in other words that

$$L(v,m,3m+1) = L(v,m,3m). \tag{4.5}$$

Suppose that $2m+1 \in I(v,m)$, that is,

$$L(v,m,3m) < L(v,m,3m+1). \tag{4.6}$$

Then by Lemma 11 there exists $u \in \mathscr{R}(v)$ with F-sequence λ satisfying the following conditions:

(1) $u_{3m+1} = L(v,m,3m+1)$;

(2) $\lambda_{3m+1-m_0} = 1-a$, where m_0 is defined by (3.15);

(3) $\lambda_j = 0$ for any $j \geqslant m+1$, $j \neq 3m+1-m_0$.

Now

$$v_{3m+1} = \sum_{s=1}^{m} g_i v_{3m+1-s} \leqslant \max\{v_{2m+1}, \ldots, v_{3m}\}. \tag{4.7}$$

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Hence there exists $r \in [2m+1, 3m]$ such that $v_r \geqslant v_{3m+1}$. Set

$$\mu_i = \begin{cases} g_i & 1 \leqslant i \leqslant m \\ 1-a & i = r-m_0 \\ 0 & i \geqslant m+1,\ i \neq r-m_0. \end{cases} \tag{4.8}$$

Then on letting $w \in \mathscr{R}(v)$ with F-sequence μ, we have

$$w_r = v_r + (1-a)\,h(m_0) \geqslant u_{3m+1} = L(v, m, 3m+1), \tag{4.9}$$

which runs contrary to $2m+1 \in I(v, m)$.

Using the same method, only this time substituting $n = 2m+1$ for $n = 2m+2$, we can also prove that $2m+2 \notin I(v, m)$, whilst, using the same method, with Lemma 12 replacing Lemma 11 in the proof, we can prove that $k \notin I(v, m)$, $\forall\, 2m+3 \leqslant k \leqslant 3m+3$. This concludes the proof of Theorem 1, since the case when $n \geqslant 3m+4$ has already been considered in Lemma 13. □

Proof of Theorem 2. For $1 \leqslant n \leqslant m+1$, result (1.17) has already been proved in Lemma 6, whilst it is easy to see from the above proof of Theorem 1 that (1.18) holds for $m+1 \leqslant n \leqslant 2m$. □

5. *An example*

To conclude, we shall provide a simple example. Let $0 < a < 1$, and $v = \{v_n : n \geqslant 0\}$ be the renewal sequence

$$v = (1, a, a^2, a^3, \ldots).$$

Then the F-sequence of v is

$$g = (a, 0, 0, \ldots).$$

Let m be an arbitrary positive integer. Now, by (1.16),

$$h(s) = (s+1)\,a^s \quad (\forall\, 0 \leqslant s \leqslant m-1).$$

Therefore, by Theorem 2, for any $1 \leqslant n \leqslant m$ there exists n^*, with $1 \leqslant n^* \leqslant n$, such that

$$L(v, m, m+n) = a^{m+n^*} + (1-a) \max_{0 \leqslant s \leqslant n^*-1} (s+1)\,a^s. \tag{5.1}$$

Moreover, for any $m+1 \leqslant n \leqslant 2m$, there exists n^*, with $m+1 \leqslant n^* \leqslant n$, such that

$$L(v, m, m+n) = a^{m+n^*} + (1-a) \max_{0 \leqslant s \leqslant m-1} (s+1)\,a^s. \tag{5.2}$$

In this example, we have

$$(1-a)\,h(s) = (1-a)\,(s+1)\,a^s \leqslant \left(1 - \frac{1}{s+1}\right)^s,$$

and

$$(1-a) \max_{0 \leqslant s \leqslant m-1} h(s) \leqslant \begin{cases} 1-a & a \leqslant 1/2 \\ \left(1 - \dfrac{1}{s+1}\right)^s & \dfrac{s}{s+1} \leqslant a \leqslant \dfrac{s+1}{s+2}\ (s \geqslant 1). \end{cases}$$

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