construction of the first branch of the hyperbola. We deduce again that $|PF| = e|PQ|$ with same directrix $S \mid F$. The existence of a second directrix may be established by the same means as for the ellipse.

The parabola may be obtained as the limiting case of either the ellipse or the hyperbola in which the radius of the circle increases without bound while keeping the circle at a finite distance from F. The definition of parabola as the locus of points equidistant to a fixed point and a straight line, and the corresponding geometric construction, follows from this.

The equation of a conic

So far we have employed exclusively the methods of classical Euclidean geometry. No use has been made of coordinate systems or equations other than the equality of certain lengths or ratios of lengths. This was a deliberate choice, as the subject matter predates the application of Algebra to Geometry. That notwithstanding, and in acknowledgement of a suggestion by a referee, I shall indicate how the construction presented above leads directly to the equation of the ellipse. Identical arguments apply to the other conics.

Write $|AC| = 2a$ and $|FP| = r$. Let $\theta = \angle PFC$. Because DFQ is similar to FGH and $|FH| = e|QF|$ we have $|FG| = e|DF|$. Because $|SC| = 2a/e$ and $|DF| = 2ae$ we have $|DF| = a(1/e - e)$. Therefore $|FG| = a(1 - e^2)$. From $r = |FG| + |HP| \cos\theta$ and $|HP| = e|AP| = er$ we find $r(1 - e\cos\theta) = |FG|$. In conclusion, we have deduced that $r(1 - e\cos\theta) = a(1 - e^2)$.

10.1017/mag.2017.143

MARCOS ALVAREZ
Department of Mathematics, City, University of London, Northampton Square, London EC1V 0HB
e-mail: *m.alvarez@city.ac.uk*

101.42 The Steiner point revisited

The circumcircle and the outer Steiner ellipse of a triangle meet in four points, namely, the vertices A, B, C and the Steiner point S.

In this Note we will use a linear transformation T on (the areal coordinates of) four concyclic points of the triangle to locate another circle passing through S.

The coordinates

Using $\Delta ABC = \Delta$ to denote the area of the triangle of reference, a point $P[x, y, z]$ has actual areal coordinates $x = \dfrac{\Delta PBC}{\Delta}$, $y = \dfrac{\Delta PCA}{\Delta}$, $z = \dfrac{\Delta PAB}{\Delta}$, where $x + y + z = 1$. We also use the cotangent notation $\alpha = \cot A, \beta = \cot B, \gamma = \cot C$, where $\beta\gamma + \gamma\alpha + \alpha\beta = 1$.

https://doi.org/10.1017/mag.2017.144 Published online by Cambridge University Press

The linear transformation

The matrix $T = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$ represents the linear transformation T which maps the medial triangle $A'B'C'$ onto the triangle ABC, that is an enlargement with centre G and scale factor -2. Some important results are now listed below where round brackets denote relative areal coordinates. Thus

$$\begin{aligned} A'\left[0, \frac{1}{2}, \frac{1}{2}\right] &\to A\,[1,\ 0,\ 0] \\ B'\left[\frac{1}{2},\ 0, \frac{1}{2}\right] &\to B\,[0,\ 1,\ 0] \\ C'\left[\frac{1}{2}, \frac{1}{2},\ 0\right] &\to C\,[0,\ 0,\ 1] \\ O\big(1-\beta\gamma,\ 1-\gamma\alpha,\ 1-\alpha\beta\big) &\to H\left[\beta\gamma,\ \gamma\alpha,\ \alpha\beta\right] \\ N\big(1+\beta\gamma,\ 1+\gamma\alpha,\ 1+\alpha\beta\big) &\to O\,\big(1\ -\ \beta\gamma,\ 1\ -\ \gamma\alpha,\ 1\ -\ \alpha\beta\big) \\ L\big(\beta\ +\ \gamma,\ \gamma\ +\ \alpha,\ \alpha\ +\ \beta\big) &\to H'\,(\alpha,\ \beta,\ \gamma) \\ K\big((\beta-\gamma)^2, (\gamma-\alpha)^2, (\alpha-\beta^2\big) &\to S\big((\gamma-\alpha)(\alpha-\beta), (\alpha-\beta)(\beta-\gamma), (\beta-\gamma)(\gamma-\alpha\big) \end{aligned}$$

where (in the usual notation) we have the circumcentre O, orthocentre H, nine-points centre N, Lemoine (symmedian) point L, Kiepert centre K and Steiner point S. Less well-known perhaps is H', the isotomic conjugate of H.

The circle

Now it may be shown, by means of power lemma, that the points O, N, L, K lie on a circle, which is somewhat reminiscent of the more widely known Lester circle ([1]). Indeed, both circles may be established this way using the point $M\,(1\ +\ 3\beta\gamma,\ 1\ +\ 3\gamma\alpha,\ 1\ +\ 3\alpha\beta)$, the meet of the Euler and Fermat lines (although the Cartesian coordinates used in reference [2] may seem preferable). Thus, given $ML \times MK = MN \times MO$, the linear transformation T maps the circle $ONLK$ onto the circle $HOH'S$.

Note finally that the image $T\,(M)$ is the point $P\,(2\ -\ 3\beta\gamma,\ 2\ -\ 3\gamma\alpha,\ 2\ -\ 3\alpha\beta)$. Hence, having first established the collinearities of P, H, O and P, H', S by determinant, the cotangent metric might be used to establish the power equality $PO \times PH = PH' \times PS$, thus providing a direct proof for the circle $HOH'S$.

References

1. J. A. Scott, Two more proofs of Lester's theorem, *Math. Gaz.* **87** (November 2003) pp. 553-556.
2. J. A. Scott, A nine-point hyperbola, *Math. Gaz.* **89** (March 2005) pp. 93-96.

10.1017/mag.2017.144

J. A. SCOTT
1 Shiptons Lane, Great Somerford, Chippenham SN15 5EJ

https://doi.org/10.1017/mag.2017.144 Published online by Cambridge University Press