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Coherent sheaves, sheared D-modules, and Hochschild cochains

Dario Beraldo, Kevin Lin and Wyatt Reeves

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Abstract

We show that the category of ind-coherent sheaves on a quasi-smooth scheme is naturally tensored over the category of sheared D-modules on its shifted cotangent bundle, commuting with its natural action of categorified Hoschschild cochains. We prove that it defines a Morita equivalence as such. We then extend these results to quasi-smooth Artin stacks. As a consequence of our formalism, we are able to articulate a precise sense in which the space of unramified automorphic functions over a function field localizes over the stack of arithmetic Arthur parameters.

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> The anatomist presents to the eye the most hideous and disagreeable objects; but his science is useful to the painter in delineating even a Venus or an Helen.
>
> (David Hume, *An Enquiry Concerning Human Understanding*)

1. Introduction

1.1 Motivation

1.1.1. The question of the existence of finite projective resolutions for modules was first raised by Hilbert, who showed in his famous 'syzygy theorem' that the polynomial ring $k[x_1, \ldots, x_n]$ has global dimension n. Another major result in the subject, due to Auslander, Buchsbaum, and Serre, says that a Noetherian local ring has bounded global dimension if and only if it is regular. This indicates that the failure of a coherent sheaf to have a finite projective resolution is entirely determined by its behavior around the singular points of the underlying variety.

1.1.2. If Y is a local complete intersection scheme, one can even produce a quantitative measure of how badly a coherent sheaf on Y fails to have a finite projective resolution. Namely, there is a scheme Sing(Y) lying over Y, which is isomorphic to Y over the smooth locus, but which has positive fiber dimension over the singular locus. (The fiber dimension at a singular point measures 'how pathological' the singularity is.) Sing(Y) carries a contracting action of $\mathbf{G}_m$ and the fixed locus for this action is isomorphic to Y.

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For any coherent sheaf $\mathcal{F} \in \mathrm{Coh}(Y)$, one can produce a closed conical subset $\mathrm{supp}(\mathcal{F}) \subseteq \mathrm{Sing}(Y)$, which is contained in the zero section if and only if $\mathcal{F}$ admits a finite projective resolution. The assignment $\mathcal{F} \mapsto \mathrm{supp}(\mathcal{F})$ gives a notion of 'microlocal support' for coherent sheaves.

1.1.3. Recently, the microlocal geometry of coherent sheaves has played a key role in the resolution of the geometric Langlands conjecture:

$$\mathrm{D}(\mathrm{Bun}_G) \simeq \mathrm{IndCoh}_{\mathcal{N}}(\mathrm{LS}_{\check{G}}).$$

Evidently, the theory of microsupport of coherent sheaves is required to even state the result. One should think that the theorem gives a 'spectral decomposition' of the category $\mathrm{D}(\mathrm{Bun}_G)$. To first order, $\mathrm{D}(\mathrm{Bun}_G)$ is tensored over $\mathrm{QCoh}(\mathrm{LS}_{\check{G}})$, which gives the decomposition of $\mathrm{D}(\mathrm{Bun}_G)$ by Langlands parameters. However, the full statement of the equivalence shows that D-modules on Bun_G have a more refined decomposition in terms of coherent microsupport on $\mathrm{LS}_{\check{G}}$.

1.1.4. The microsupport filtration on $\mathrm{IndCoh}_{\mathcal{N}}(\mathrm{LS}_{\check{G}})$ is expected to have the following 'experimental signature': if X is a curve over $\mathbf{F}_q$, then the stack $\mathrm{LS}^{\mathrm{res}}(X)$ of ℓ-adic local systems with restricted variation was introduced in [AGK+20]. The authors conjecture a restricted version of the geometric Langlands equivalence

$$\mathrm{Sh}_{\mathcal{N}}(\mathrm{Bun}_G) \overset{?}{\simeq} \mathrm{IndCoh}_{\mathcal{N}}(\mathrm{LS}^{\mathrm{res}}_{\check{G}})$$

which is expected to be closely related to the de Rham form of the geometric Langlands correspondence. This equivalence is expected to be equivariant for the natural Frobenius endomorphism on each side, and one should be able to recover the unramified part of the Langlands correspondence for function fields

$$\mathrm{Fun}_c(\mathrm{Bun}_G(\mathbf{F}_q); \overline{\mathbf{Q}}_\ell) \simeq \Gamma(\mathrm{LS}^{\mathrm{arith}}_{\check{G}}; \omega)$$

by taking the categorical trace of both sides. Now, the filtration of $\mathrm{IndCoh}_{\mathcal{N}}(\mathrm{LS}^{\mathrm{res}}_{\check{G}})$ by microsupport decategorifies to a filtration on unramified automorphic functions, which is expected to encapsulate the theory of *Arthur parameters*.

1.1.5. Motivated by these considerations, in this paper we undertake a study of the microlocal geometry of coherent sheaves from a categorical perspective; and especially with the goal of computing categorical traces.

1.2 Statement of results

1.2.1. For the rest of the introduction, Y will denote a quasi-smooth (derived) scheme unless otherwise stated.

1.2.2 *Microlocalization.* Our first goal is to make precise the sense in which coherent sheaves lie over $\mathrm{Sing}(Y)$. Recall the so-called category of singularities

$$\mathrm{Ind}\mathring{\mathrm{Coh}}(Y) = \mathrm{IndCoh}(Y)/\mathrm{QCoh}(Y).$$

One of the main accomplishments of [AG18] was using the theory of singular support from [AG15] to equip $\mathrm{IndCoh}(Y)$ with the following additional structure.

Construction 1.1 [AG18, Theorem 0.4.2]. The category $\mathrm{Ind}\mathring{\mathrm{Coh}}(Y)$ is naturally a crystal of categories over $\mathbf{P}\mathrm{Sing}(Y)$.

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1.2.3. It is natural to wonder whether the theory of singular support actually encodes some additional structure on IndCoh(Y) itself. In fact, this question was already raised in [AG18, Remark 1.4.3]. We offer the following answer.

Construction 1.2 (Construction 2.1). The category IndCoh(Y) is naturally tensored over $\mathrm{D}(\mathrm{Sing}(\mathrm{Y}))^{\Rightarrow}$.

1.2.4. Here, the notation $(-)^{\Rightarrow}$ refers to the *shearing* functor, which is an automorphism of the 2-category of categories acted on (weakly) by $\mathbf{G}_m$ (see [AG15, § A.2] or [Ber20]). The key example to bear in mind is that if A is a graded algebra, then the category A-mod is acted on $\mathbf{G}_m$, and

$$(\mathrm{A}\text{-mod})^{\Rightarrow} \simeq (\mathrm{A}^{\Rightarrow})\text{-mod},$$

where $\mathrm{A}^{\Rightarrow}$ is the graded algebra whose ith graded piece is $\mathrm{A}_i[-2i]$. Because we do not seriously consider questions of t-exactness in this paper, for us, shearing plays a technically important but conceptually minor role. Let us note for future reference another important property of shearing, which is that it induces an equivalence on categories of equivariant objects:

$$\mathcal{C}^{\mathbf{G}_m} \simeq (\mathcal{C}^{\Rightarrow})^{\mathbf{G}_m}. \tag{1.1}$$

1.2.5. When Y is a global complete intersection scheme, an action of $\mathrm{D}(\mathrm{Sing}(\mathrm{Y}))^{\Rightarrow}$ on IndCoh(Y) was previously constructed in [Ber20, § 2.2.5]. One should think of this as giving a construction of the action in local coordinates. We should emphasize that in the setting of derived algebraic geometry, it is highly non-trivial to glue constructions along coordinate patches, due to the presence of infinite towers of higher homotopy coherences. In fact, our Construction 1.2 proceeds by a method quite different from that in [Ber20, § 2.2.5].

1.2.6 *Morita equivalence.* To state our next result, we should first recall the category $\mathbf{H}(\mathrm{Y})$, introduced in [Ber21]. For any eventually coconnective scheme with perfect cotangent complex, the HKR theorem (see [AG15, Proposition F.1.5] and [AG15, Corollary G.2.7(b)] for a more precise reference) gives an isomorphism of $\mathbf{E}_1$-algebras between the algebra HC(Y) of Hochschild cochains and $\mathcal{U}(\mathbf{T}_{\mathrm{Y}}[-1])$, the universal enveloping algebra of the shifted tangent sheaf of Y.[1] As a result, the category $\mathbf{H}(\mathrm{Y})$ of left modules for HC(Y) can be described explicitly as

$$\mathrm{IndCoh}_0((\mathrm{Y}\times\mathrm{Y})^{\wedge}_{\mathrm{Y}}) = \mathrm{IndCoh}((\mathrm{Y}\times\mathrm{Y})^{\wedge}_{\mathrm{Y}}) \underset{\mathrm{IndCoh}(\mathrm{Y})}{\times} \mathrm{QCoh}(\mathrm{Y}). \tag{1.2}$$

The category $\mathbf{H}(\mathrm{Y})$ may be thought of as a categorification of either Hochschild cochains or the ring of differential operators. From the explicit description (1.2), we see that $\mathbf{H}(\mathrm{Y})$ itself has a monoidal structure, coming from convolution. In fact, $\mathbf{H}(\mathrm{Y})$ acts on IndCoh(Y) via convolution, and this action is known to encode important facts about singular support for IndCoh.

1.2.7. In § 4.1, we show that the action of $\mathrm{D}(\mathrm{Sing}(\mathrm{Y}))^{\Rightarrow}$ on IndCoh(Y) from Construction 1.2 commutes with the well-known convolution action of $\mathbf{H}(\mathrm{Y})$ on IndCoh(Y). The main theorem of this paper is that IndCoh(Y), as a $(\mathrm{D}(\mathrm{Sing}(\mathrm{Y}))^{\Rightarrow}, \mathbf{H}(\mathrm{Y}))$-bimodule, defines a Morita equivalence.

Theorem 1.1 (Theorem 4.1).

$$\mathrm{D}(\mathrm{Sing}(\mathrm{Y}))^{\Rightarrow}\textbf{-mod} \simeq \mathbf{H}(\mathrm{Y})\textbf{-mod}.$$

[1] Here we are regarding $\mathbf{T}_{\mathrm{Y}}$ as an object of QCoh(Y) and forming the universal enveloping algebra inside of QCoh(Y). Likewise, HC(Y) refers to the quasi-coherent sheaf of Hochschild cochains.

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1.2.8. Here is a simple result one category level down which captures the spirit of our Theorem 1.1: if Y is a smooth affine variety over a field of characteristic zero, then $\mathcal{O}_Y$ is a bimodule for the action of locally constant functions $\Gamma(\underline{k}_Y)$ and the ring of differential operators $\mathcal{D}_Y$. Famously, locally constant functions are precisely the ones that commute with the action of every differential operator:

$$\mathrm{End}_{\mathcal{D}_Y}(\mathcal{O}_Y) \simeq \Gamma(\underline{k}_Y),$$

so the bimodule $\mathcal{O}_Y$ defines a fully faithful functor

$$\Gamma(\underline{k}_Y)\text{-mod} \hookrightarrow \mathcal{D}_Y\text{-mod}. \tag{1.3}$$

One should think that there are analogies.

Theorem 1.1	§ 1.2.8
$\mathrm{D}(\mathrm{Sing}(Y))^{\Rightarrow}$	$\Gamma(\underline{k}_Y)$
$\mathbf{H}(Y)$	$\mathcal{D}_Y$
$\mathrm{IndCoh}(Y)$	$\mathcal{O}_Y$

(See § 1.2.10 for more discussion of this point.)

1.2.9. It is notably almost never the case that $\mathcal{D}_X$ can be identified with endomorphisms of $\mathcal{O}_X$ as a module for locally constant functions, and for this reason the reader might be surprised by the comparison between our Theorem 1.1, where IndCoh(Y) defines a Morita equivalence, and the toy model of § 1.2.8, where $\mathcal{O}_Y$ merely defines an embedding in one direction.

The key mechanism that improves the situation one category level up is Gaitsgory's *1-affineness theorem* for de Rham prestacks [Gai15, Theorem 2.6.3]: if we identify $\underline{k}_Y$ with $\mathcal{O}_{Y_{\mathrm{dR}}}$, then failure of the map (1.3) to be an equivalence amounts to a failure of

$$\Gamma(\mathcal{O}_{Y_{\mathrm{dR}}})\text{-mod} \hookrightarrow \mathrm{QCoh}(Y_{\mathrm{dR}})$$

to be an equivalence, which says that Y_{dR} is not 0-affine. What the 1-affineness result of [Gai15, Theorem 2.6.3] shows is that this obstruction goes away in our categorified setting:

$$\mathrm{D}(Y)\textbf{-mod} \simeq \mathrm{ShCat}(Y_{\mathrm{dR}}).$$

1.2.10. Let us try to briefly justify the table of analogies in § 1.2.8. Recall that for a smooth scheme Y, there are canonical equivalences

$$\begin{aligned} \mathrm{D}(\mathrm{Sing}(Y))^{\Rightarrow} &\simeq \mathrm{D}(Y), \\ \mathbf{H}(Y) &\simeq \mathrm{QCoh}((Y \times Y)^{\wedge}_{Y}), \\ \mathrm{IndCoh}(Y) &\simeq \mathrm{QCoh}(Y). \end{aligned}$$

The analogy between QCoh(Y) and $\mathcal{O}_Y$ is well-known (see e.g. [Gai15]). The assertion that D(Y) categorifies $\Gamma(\underline{k}_X)$ says that there is an analogy between a function being locally constant and a sheaf admitting a flat connection. The assertion that $\mathrm{QCoh}((Y \times Y)^{\wedge}_{Y})$ categorifies $\mathcal{D}_X$ has the following interpretation: we should perceive the operation of acting by a differential operator as being like convolving with an integral kernel 'supported on the formal neighborhood of the diagonal'.

1.2.11. For a smooth scheme, it is well-known that QCoh(Y) defines a Morita equivalence between D(Y)**-mod** and $\mathbf{H}(Y)$**-mod**. In fact, for any scheme, it was shown in [Ber19,

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Lemma 4.2.7] that IndCoh(Y) defines a Morita equivalence

$$\mathrm{D}(\mathrm{Y})\textbf{-mod} \simeq \mathrm{IndCoh}((\mathrm{Y}\times\mathrm{Y})^{\wedge}_{\mathrm{Y}})\textbf{-mod}.$$

Our Theorem 1.1 amounts to saying that for a quasi-smooth scheme Y, if you 'increase the size' of D(Y) to $\mathrm{D}(\mathrm{Sing}(\mathrm{Y}))^{\Rightarrow}$, then on the dual side $\mathrm{IndCoh}((\mathrm{Y}\times\mathrm{Y})^{\wedge}_{\mathrm{Y}})$ 'decreases in size' to precisely become $\mathbf{H}(\mathrm{Y})$.

1.2.12 *Extension to stacks.* Let us emphasize now that so far in this introduction, Y has always been a *scheme*. In fact, our Theorem 1.1, as stated, fails for even the simplest stack BG (see § 1.3.7). However, the equivalence of Theorem 1.1 is natural enough that we are able to produce an equivalence.

Construction 1.3 (Construction 5.1).

$$\mathrm{ShCat}_{\mathbf{S}^{\Rightarrow}}(\mathcal{Y}) \simeq \mathrm{ShCat}_{\mathbf{H}}(\mathcal{Y})$$

for any quasi-smooth Artin stack $\mathcal{Y}$.[2]

1.2.13. To make sense of the statement of Construction 1.3, we need to recall the notion of a sheaf of categories with coefficients, introduced in [Ber19] (although let us say right away that our notion of a sheaf of categories with coefficients will be slightly different from that of [Ber19]).

First, in [Ber19], a coefficient system is defined to be a functor

$$\mathbf{A} : \mathrm{AffSch} \to \mathrm{Mo}(\mathrm{dgCat})$$

where Mo(dgCat) is the Morita bimodule category whose objects are monoidal dg categories, morphisms are bimodule categories, etc. The composite functor

$$\mathrm{AffSch} \xrightarrow{\mathbf{A}} \mathrm{Mo}(\mathrm{dgCat}) \xrightarrow{(-)\textbf{-mod}} \textbf{2-dgCat}$$

can then be right Kan extended from AffSch to PreStk, producing a functor called $\mathrm{ShCat}_{\mathbf{A}}$.[3] More explicitly, if $\mathcal{Y}$ is any prestack,

$$\mathrm{ShCat}_{\mathbf{A}}(\mathcal{Y}) = \lim_{\mathrm{S}\to\mathcal{Y}} \mathbf{A}(\mathrm{S})\textbf{-mod},$$

where the limit is formed over affine schemes S mapping to $\mathcal{Y}$.

1.2.14. The definition of the 2-category $\mathrm{ShCat}_{\mathbf{H}}(\mathcal{Y})$ documented in [Ber19] follows the construction above. Our construction of $\mathrm{ShCat}_{\mathbf{S}^{\Rightarrow}}(\mathcal{Y})$ doesn't quite fit into the pattern outlined above; instead, for a quasi-smooth Artin stack $\mathcal{Y}$, we start with the functor

$$\mathbf{S}^{\Rightarrow}\textbf{-mod} : (\mathrm{Sm}_{/\mathcal{Y}})^{\mathrm{op}} \xrightarrow{\mathrm{D}(\mathrm{Sing}(-))^{\Rightarrow}\textbf{-mod}} \textbf{2-dgCat}$$

and then set

$$\mathrm{ShCat}_{\mathbf{S}^{\Rightarrow}}(\mathcal{Y}) = \lim_{\mathrm{S}\to\mathcal{Y}} \mathrm{D}(\mathrm{Sing}(\mathrm{S}))^{\Rightarrow}\textbf{-mod},$$

where the limit is formed over the smooth site of $\mathcal{Y}$.

1.2.15. Although this construction of $\mathrm{ShCat}_{\mathbf{S}^{\Rightarrow}}(\mathcal{Y})$ is not obviously functorial in $\mathcal{Y}$, our Construction 1.3 makes $\mathrm{ShCat}_{\mathbf{S}^{\Rightarrow}}(\mathcal{Y})$ into a functor out of quasi-smooth Artin stacks by transport of structure from $\mathrm{ShCat}_{\mathbf{H}}$. We characterize this functoriality by showing that our Morita

[2] For our precise assumptions on $\mathcal{Y}$, see § 1.5.6.

[3] In [Ber19], this object is actually called $\mathrm{ShCat}^{\mathbf{A}}$. We apologize for the minor inconsistency in notation.

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equivalence matches the $(\mathbf{H}(\mathcal{Z}), \mathbf{H}(\mathcal{Y}))$-bimodule $\mathbf{H}_{\mathcal{Z}\to\mathcal{Y}}$ with the sheared crystal $\underline{\mathrm{D}}(\mathrm{Sing}(\mathcal{Y})_{\mathcal{Z}})^{\Rightarrow}$ over $\mathrm{Sing}(\mathcal{Z}) \times \mathrm{Sing}(\mathcal{Y})$.

1.2.16 *Application to traces.* Let us now recall the formalism of categorical traces, which has been considered in [BN21, KP20, GKR$^+$22]. If V is a finite-dimensional vector space over k, then the trace map is

$$\mathrm{tr} : \mathrm{End}(\mathrm{V}) \simeq \mathrm{V} \otimes \mathrm{V}^\vee \xrightarrow{\mathrm{ev}} k.$$

If ϕ is a particular endomorphism of V, then we can further rewrite the formula for the trace of ϕ as

$$\mathrm{tr}(\phi; \mathrm{V}) = \{k \xrightarrow{\mathrm{unit}} \mathrm{V} \otimes \mathrm{V}^\vee \xrightarrow{\phi\otimes\mathbf{1}} \mathrm{V} \otimes \mathrm{V}^\vee \xrightarrow{\mathrm{ev}} k\},$$

where we have implicitly used the canonical identification $k \simeq \mathrm{End}(k)$. This formula for the trace admits a substantial generalization: if $\mathcal{C}$ is any symmetric monoidal category, and $c \in \mathcal{C}$ is a *dualizable object* of $\mathcal{C}$, then for any endomorphism $\phi \in \mathrm{End}(c)$, we can form the trace $\mathrm{tr}(\phi; c)$, which is an element of $\mathrm{End}(1_{\mathcal{C}})$:

$$\mathrm{tr}(\phi; c) = \{1_{\mathcal{C}} \xrightarrow{\mathrm{unit}_c} c \otimes c^\vee \xrightarrow{\phi\otimes\mathbf{1}} c \otimes c^\vee \xrightarrow{\mathrm{ev}_c} 1_{\mathcal{C}}\}.$$

1.2.17. If $\mathcal{C}$ is itself a 2-category, then $\mathrm{End}(1_{\mathcal{C}})$ is a 1-category, and the categorical trace enjoys the following functoriality: if c and c' are dualizable objects of $\mathcal{C}$, equipped with endomorphisms ϕ and ψ, respectively, then for every *right adjointable* morphism f, and every natural transformation α as in the following diagram, we obtain a *Chern class map*

$$\mathrm{cl}(f, \alpha) : \mathrm{tr}(\phi; c) \to \mathrm{tr}(\psi; c').$$

$$\begin{array}{ccc} c & \xrightarrow{f} & c' \\ \downarrow{\scriptstyle \phi} & \overset{\alpha}{\Rightarrow} & \downarrow{\scriptstyle \psi} \\ c & \xrightarrow[f]{} & c' \end{array}$$

1.2.18. The starting point for the theory of enhanced traces is the following tautological observation: the category Vect is dualizable as an object of dgCat, and

$$\mathrm{tr}(\mathbf{1}_{\mathrm{Vect}}; \mathrm{Vect}) \simeq k.$$

Moreover, for any finite-dimensional vector space V with an endomorphism ϕ, we obtain a lax-commutative diagram in dgCat.

$$\begin{array}{ccc} \mathrm{Vect} & \xrightarrow{\mathrm{V}\otimes(-)} & \mathrm{Vect} \\ {\scriptstyle \mathbf{1}_{\mathrm{Vect}}}\downarrow & \overset{\phi}{\Rightarrow} & \downarrow{\scriptstyle \mathbf{1}_{\mathrm{Vect}}} \\ \mathrm{Vect} & \xrightarrow[\mathrm{V}\otimes(-)]{} & \mathrm{Vect} \end{array}$$

Also,

$$\mathrm{cl}(\mathrm{V}, \phi) = \mathrm{tr}(\phi; \mathrm{V})$$

in $\mathrm{End}(k)$.

1.2.19. Now let $\mathcal{F}$ be a $\overline{\mathbf{Q}}_\ell$ Weil sheaf on a scheme $\mathcal{Y}/\overline{\mathbf{F}}_q$. By § 1.2.18, the trace of Frobenius on the compactly supported cohomology of $\mathcal{F}$ may be written as $\mathrm{cl}(\mathrm{H}^*_c(\mathcal{Y}; \mathcal{F}), \mathrm{Fr}_q)$. We can factor

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$\mathrm{H}^*_c(\mathcal{Y};\mathcal{F}) : \mathrm{Vect} \to \mathrm{Vect}$ as the composite

$$\mathrm{Vect} \xrightarrow{\mathcal{F}} \mathrm{Sh}(\mathcal{Y}) \xrightarrow{\Gamma_c} \mathrm{Vect}$$

and using the Weil structure on $\mathcal{F}$, we can form the following lax-commutative diagram.

$$\begin{array}{ccccc} \mathrm{Vect} & \xrightarrow{\mathcal{F}} & \mathrm{Sh}(\mathcal{Y}) & \xrightarrow{\Gamma_c} & \mathrm{Vect} \\ {\scriptstyle \mathbf{1}_{\mathrm{Vect}}}\downarrow & \overset{\alpha}{\nearrow} & \downarrow{\scriptstyle \mathrm{Fr}_q^*} & \nearrow & \downarrow{\scriptstyle \mathbf{1}_{\mathrm{Vect}}} \\ \mathrm{Vect} & \xrightarrow{\mathcal{F}} & \mathrm{Sh}(\mathcal{Y}) & \xrightarrow{\Gamma_c} & \mathrm{Vect} \end{array}$$

The class $\mathrm{cl}(\mathcal{F},\alpha)$ in $\mathrm{tr}(\mathrm{Fr}_q;\mathrm{Sh}(\mathcal{Y})) = \mathrm{Fun}(\mathcal{Y}(\mathbf{F}_q);\overline{\mathbf{Q}}_\ell)$ is the usual function attached to $\mathcal{F}$ under the sheaf-function correspondence, and the class map associated to the second square in the diagram is given by summing over the $\mathbf{F}_q$ points of $\mathcal{Y}$.

1.2.20. One category level up, if a dg-category $\mathcal{C}$ arises as the global sections of a 'Weil' sheaf of categories $\underline{\mathcal{C}}$ over a space $\mathcal{Y}$ equipped with an endomorphism ϕ, then we can form the object

$$\mathrm{tr}^{\mathrm{enh}}(\phi;\underline{\mathcal{C}}) \in \mathrm{tr}(\phi;\mathrm{ShCat}(\mathcal{Y})) = \mathrm{QCoh}(\mathcal{Y}^\phi),$$

which is by definition $\mathrm{cl}(\mathcal{C},\alpha)$, where α is the Weil structure for $\mathcal{C}$. This enhanced trace has the property that

$$\Gamma(\mathcal{Y}^\phi;\mathrm{tr}^{\mathrm{enh}}(\phi;\underline{\mathcal{C}})) \simeq \mathrm{tr}(\phi;\mathcal{C}).$$

1.2.21. Morally, the previous paragraphs 1.2.19 and 1.2.20 show that the enhanced trace of a Weil object is like a version of the trace that 'lives over $\mathcal{Y}$'. Following this pattern, one should expect that the enhanced trace of a category $\mathcal{C}$ acted on by $\mathbf{H}(\mathcal{Y})$ should be an upgrade of the usual trace that also remembers how $\mathcal{C}$ lives over the stack of singularities. The final results of our paper confirm this expectation.

Computation 1.1 (Computation 6.2). Let ϕ be a schematic endomorphism of a quasi-smooth stack. Then

$$\mathrm{tr}(\mathbf{H}_\phi(\mathcal{Y}),\mathbf{H}(\mathcal{Y})\textbf{-mod}) \simeq \mathrm{D}(\mathrm{Sing}_2(\mathcal{Y}^\phi))^{\Rightarrow}.$$

Computation 1.2 (Computation 6.4). If $\mathcal{N}$ is a subset of $\mathrm{Sing}(\mathcal{Y})$ stable under ϕ and α is the natural equivariance data for $\mathrm{IndCoh}_{\mathcal{N}}(\mathcal{Y})$ as an object of $\mathbf{H}(\mathcal{Y})$**-mod**, then

$$\mathrm{cl}(\mathrm{IndCoh}_{\mathcal{N}}(\mathcal{Y}),\alpha) \simeq (\omega_{\mathcal{N}^\phi})^{\Rightarrow}$$

as an object of $\mathrm{D}(\mathrm{Sing}_2(\mathcal{Y}^\phi))^{\Rightarrow}$.

In the statements of Computations 1.1 and 1.2, the notation $\mathrm{Sing}_2(\mathcal{X})$ refers to the 'total space' of the 2-shifted cotangent complex $\mathbf{L}_{\mathcal{X}}[-2]$. (See § 6.2.2 for more discussion.)

In the statement of Computation 1.2, the notation $(\omega_{\mathcal{N}^\phi})^{\Rightarrow}$ denotes the object which corresponds to $\omega_{\mathcal{N}^\phi}$ in $\mathrm{D}(\mathrm{Sing}_2(\mathcal{Y}^\phi))$ under the equivalence (1.1). Here, $\omega_{\mathcal{N}^\phi}$ is the (naturally $\mathbf{G}_m$-equivariant) object of $\mathrm{D}(\mathrm{Sing}_2(\mathcal{Y}^\phi))$ which is the pushforward of the dualizing sheaf on the fixed-point stack

$$\mathcal{N}^\phi = \mathcal{N} \times_{\mathrm{Sing}(\mathcal{Y}\times\mathcal{Y})} \mathrm{Sing}_\phi(\mathcal{Y}).$$

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1.2.22. Now, let us fix a smooth proper curve X over $\overline{\mathbf{F}}_q$ and set $\mathcal{Y} = \mathrm{LS}^{\mathrm{res}}_{\check{\mathrm{G}}}(\mathrm{X})$. Let ϕ be the endomorphism of $\mathrm{LS}^{\mathrm{res}}_{\check{\mathrm{G}}}$ coming by transport of structure from the geometric Frobenius endomorphism of X. Then $\mathrm{Sing}_2(\mathcal{Y}^{\phi})$ is the stack of arithmetic Arthur parameters (see [AGK+20, § 24.5.5]). Strictly speaking, our Computations 1.1 and 1.2 do not apply as stated in this situation because $\mathcal{Y}$ is a formal stack rather than an honest Artin stack. However, the singular support theory of formal stacks is expected to be very similar to the theory of Artin stacks (see e.g. [AGK+20, § 24.7.5]). We expect that our Computations 1.1 and 1.2 extend 'as-is' to formal stacks; this gives a precise expectation for how the space of automorphic functions localizes over the stack of Arthur parameters.

1.3 Examples

1.3.1. In this section we illustrate what our main results amount to in some simple examples.

1.3.2 *The case of a shifted vector space.* Let us first consider our Morita equivalence Theorem 1.1 for the simplest quasi-smooth scheme, $\Omega\mathrm{V}$. Let V be a vector space, viewed as an affine scheme. Then

$$\Omega\mathrm{V} = \mathrm{pt} \underset{\mathrm{V}}{\times} \mathrm{pt}$$

is a quasi-smooth affine scheme with

$$\mathrm{Sing}(\Omega\mathrm{V})_{\mathrm{dR}} = (\mathrm{V}^{\vee})_{\mathrm{dR}}.$$

The main results of this paper are particularly transparent when one takes advantage of the Koszul duality equivalence

$$\mathrm{KD} : \mathrm{IndCoh}(\Omega\mathrm{V}) \simeq \mathrm{Sym}(\mathrm{V}[-2])\text{-mod} \simeq \mathrm{QCoh}(\mathrm{V}^{\vee})^{\Rightarrow}. \tag{1.4}$$

1.3.3. Let us recall some basic features of the equivalence (1.4) for future reference.

(i) Koszul duality intertwines the action of $\mathbf{G}_m$ on $\Omega\mathrm{V}$ by dilation with the action of $\mathbf{G}_m$ on $\mathrm{V}^{\vee}$ by *inverse* dilation.

(ii) Koszul duality intertwines the Serre self-duality of $\mathrm{IndCoh}(\Omega\mathrm{V})$ with the self-duality of $\mathrm{QCoh}(\mathrm{V}^{\vee})^{\Rightarrow}$, *up to the negation involution of* V

$$\tau : \mathrm{V} \to \mathrm{V}, \quad v \mapsto -v.$$

Observe that V, and hence $\Omega\mathrm{V}$, is a commutative group scheme. Therefore, $\mathrm{IndCoh}(\Omega\mathrm{V})$ acquires the structure of a symmetric monoidal category under convolution.

(iii) Koszul duality intertwines the convolution symmetric monoidal structure on $\mathrm{IndCoh}(\Omega\mathrm{V})$ with the pointwise tensor symmetric monoidal structure on $\mathrm{QCoh}(\mathrm{V}^{\vee})$.

(iv) Koszul duality intertwines the pointwise tensor symmetric monoidal structure on $\mathrm{IndCoh}(\Omega\mathrm{V})$ with the convolution symmetric monoidal structure on $\mathrm{QCoh}(\mathrm{V}^{\vee})$.

1.3.4. In terms of Koszul duality, the action of $\mathrm{D}(\mathrm{Sing}(\Omega\mathrm{V}))^{\Rightarrow}$ on $\mathrm{IndCoh}(\Omega\mathrm{V})$ is the action of $\mathrm{D}(\mathrm{V}^{\vee})^{\Rightarrow}$ on $\mathrm{QCoh}(\mathrm{V}^{\vee})^{\Rightarrow}$ induced by the symmetric monoidal functor

$$\mathrm{oblv}^{\Rightarrow} : \mathrm{D}(\mathrm{V}^{\vee})^{\Rightarrow} \to \mathrm{QCoh}(\mathrm{V}^{\vee})^{\Rightarrow}.$$

1.3.5. By § 1.3.3(iii) and (iv), the equivalence

$$\mathrm{IndCoh}(\Omega\mathrm{V} \times \Omega\mathrm{V}) \overset{\mathrm{KD}\otimes\mathrm{KD}}{\longrightarrow} \mathrm{QCoh}(\mathrm{V}^{\vee} \times \mathrm{V}^{\vee})^{\Rightarrow} \overset{\mathbf{1}\otimes\tau}{\longrightarrow} \mathrm{QCoh}(\mathrm{V}^{\vee} \times \mathrm{V}^{\vee})^{\Rightarrow} \tag{1.5}$$

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intertwines the convolution monoidal structures on its source and target, compatibly with their actions on the two sides of (1.4).

It follows that (1.5) restricts to an equivalence

$$\mathbf{H}(\Omega \mathrm{V}) \simeq \mathrm{QCoh}((\mathrm{V}^\vee \times \mathrm{V}^\vee)^\wedge_{\mathrm{V}^\vee})^{\Rightarrow}$$

intertwining the action of $\mathbf{H}(\mathcal{Y})$ on $\mathrm{IndCoh}(\mathcal{Y})$ with the action of

$$\mathrm{QCoh}((\mathrm{V}^\vee \times \mathrm{V}^\vee)^\wedge_{\mathrm{V}^\vee})^{\Rightarrow}$$

on $\mathrm{QCoh}(\mathrm{V}^\vee)^{\Rightarrow}$ obtained by recognizing

$$(\mathrm{V}^\vee \times \mathrm{V}^\vee)^\wedge_{\mathrm{V}^\vee} \simeq \mathrm{V}^\vee \underset{(\mathrm{V}^\vee)_{\mathrm{dR}}}{\times} \mathrm{V}^\vee$$

as the infinitesimal groupoid of $\mathrm{V}^\vee$.

So, the Morita equivalence of Theorem 1.1 in this case is a manifestation of the well-known fact that, for any smooth scheme S, the category $\mathrm{QCoh}(\mathrm{S})$ is a Morita bimodule for $\mathrm{D}(\mathrm{S})$ and $\mathrm{QCoh}((\mathrm{S} \times \mathrm{S})^\wedge_{\mathrm{S}})$.

1.3.6 *The case of a classifying stack.* Let us now examine the equivalence

$$\mathrm{ShCat}_{\mathbf{H}} \simeq \mathrm{ShCat}_{\mathbf{S}^{\Rightarrow}}$$

for the smooth stack $\mathcal{Y} = \mathrm{BG}$. In this subsection we will only consider *strong* actions of groups on categories.

1.3.7. In this case, both sides of the equivalence are familiar to geometric representation theory.

– On the one hand,

$$\mathrm{ShCat}_{\mathbf{S}^{\Rightarrow}}(\mathrm{BG}) \simeq \mathrm{G}\textbf{-mod}$$

identifies with the $(\infty, 2)$-category of dg categories with a strong action of G.

– On the other hand,

$$\mathbf{H}(\mathrm{BG}) \simeq \mathrm{IndCoh}((\mathrm{BG} \times \mathrm{BG})^\wedge_{\mathrm{BG}}) \simeq (\mathfrak{g}\text{-mod} \otimes \mathfrak{g}\text{-mod})^{\mathrm{G}}$$

is the category of Harish-Chandra bimodules for G.

Therefore, the equivalence (5.6) in this case recovers the well-known equivalence (see [Ber19, Equation (2.6)]) between categories acted on strongly by G and modules for the category of Harish-Chandra bimodules for G.

1.3.8. In the case at hand, the adjunction

$$\mathbf{Loc}_{\mathbf{S}^{\Rightarrow}} : \mathrm{D}(\mathrm{Sing}(\mathcal{Y}))^{\Rightarrow}\textbf{-mod} \rightleftharpoons \mathrm{ShCat}_{\mathbf{S}^{\Rightarrow}}(\mathcal{Y}) : \mathbf{\Gamma}_{\mathbf{S}^{\Rightarrow}}$$

identifies with the adjunction

$$\mathrm{D}(\mathrm{BG})\textbf{-mod} \rightleftharpoons \mathrm{G}\textbf{-mod}$$

defined by the trivial bimodule

$$\mathrm{Vect}.$$

This adjunction is a colocalization because

$$\mathrm{End}_{\mathrm{G}}(\mathrm{Vect}) \simeq \mathrm{Vect}^{\mathrm{G}} \simeq \mathrm{D}(\mathrm{BG}).$$

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On the other hand,

$$\mathrm{End}_{\mathrm{D(BG)}}(\mathrm{Vect}) \simeq \mathrm{Vect} \underset{\mathrm{D(BG)}}{\otimes} \mathrm{Vect} \simeq \mathrm{D}(\mathrm{G})_0$$

is the full subcategory of D(G) generated by the constant sheaf.

1.3.9. To summarize, the failure of the $(\mathbf{Loc}_{\mathbf{S}^{\Rightarrow}}, \mathbf{\Gamma}_{\mathbf{S}^{\Rightarrow}})$ adjunction to be an equivalence is tied to the failure of D-affineness for stacks, which in turn manifests as the failure of the relative categorical Kunneth formula for the fiber product.

$$\begin{array}{ccc} \mathrm{G} & \longrightarrow & \mathrm{pt} \\ \downarrow & & \downarrow \\ \mathrm{pt} & \longrightarrow & \mathrm{BG} \end{array}$$

1.4 Miscellania

1.4.1. In this section we review some of the important technical ideas in this paper that may be of interest to experts.

1.4.2. At various points in this paper, homotopy coherence issues arise in the context of crystals of categories. Our Proposition 2.3 provides an approach for dealing with these issues. A simplified version of the statement is that if $\mathcal{Y}$ is a scheme and $\mathcal{C}_1$ and $\mathcal{C}_2$ are two module categories for $\mathrm{D}(\mathcal{Y})$, then it is a *property* whether a functor $\mathrm{F} : \mathcal{C}_1 \to \mathcal{C}_2$ is $\mathrm{D}(\mathcal{Y})$ linear. We use this proposition in a crucial way both when we construct an action of $\mathrm{D}(\mathrm{Sing}(\mathcal{Y}))^{\Rightarrow}$ on $\mathrm{IndCoh}(\mathcal{Y})$ and when we extend our theory from schemes to stacks.

We should note that this idea is implicit in the main construction of [AG18], which gives the category of singularities $\mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(\mathrm{Z})$ the structure of a crystal of categories over $\mathbf{P}(\mathrm{Sing}(\mathrm{Z}))$. In this construction, the crystal structure is produced by specifying, for each $\mathrm{S} \to \mathbf{P}(\mathrm{Sing}(\mathrm{Z}))$, a *full subcategory* of $\mathrm{QCoh}(\mathrm{S}) \otimes \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(\mathrm{Z})$. An even earlier antecedent for this sort of idea is [Gai15, § 4.2.1], which observes that for a closed embedding of schemes, the functor of coinduction of crystals of categories is actually fully faithful.

1.4.3. The main idea of our construction of the action of $\mathrm{D}(\mathrm{Sing}(\mathcal{Y}))^{\Rightarrow}$ on $\mathrm{IndCoh}(\mathcal{Y})$ is that there is a close relationship between $\mathrm{IndCoh}(\mathcal{Y})$ and the category of singularities $\mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(\mathcal{Y} \times \Omega\mathbf{A}^1)$, which allows us to bootstrap from the results of [AG18]. The idea of relating IndCoh on one variety to the category of singularities on another has appeared already in the work of Isik [Isi13].

1.4.4. In the course of constructing the Morita equivalence between $\mathrm{D}(\mathrm{Sing}(\mathcal{Y}))^{\Rightarrow}$**-mod** and $\mathbf{H}(\mathcal{Y})$**-mod**, we give a new proof of an important result of Ben-Zvi, Gunningham, and Orem [BGO20, Theorem 1.3] concerning when a bimodule defines a Morita equivalence (see § 4.2).

1.5 Conventions

1.5.1. Because this paper deals directly with issues of homotopy coherence, we are careful to distinguish between propositions and constructions. This has the following practical manifestation: citations to a 'Proposition' have a logical dependence on their statements but

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not their proofs. On the other hand, every citation to a 'Construction' refers to the particular way in which the construction is performed.

1.5.2. The notation

$$\mathsf{L} : \mathcal{C} \rightleftharpoons \mathcal{D} : \mathsf{R}$$

will always mean that (L, R) is a pair of adjoint functors. We say that an adjunction is a localization (respectively a colocalization) if the right (respectively left) adjoint is fully faithful.

1.5.3. All of the algebraic geometry in this paper is derived. In particular, all schemes and stacks are assumed to be derived. In a similar spirit, we will use the word category to refer to any $(\infty, 1)$-category, and we will use the word 2-category to refer to any $(\infty, 2)$-category. In general, we follow the notation and conventions of [GR17a] and [GR17b], with the exception that we use $\mathrm{D}(\mathrm{X})$ to refer to the category which they call $\mathrm{Dmod}(\mathrm{X})$. For a summary of the relevant notation, see [Ber21, § 0.6].

1.5.4. In the last section of this paper we will require the theory of $(\infty, 2)$-categories, as developed in [GR17a]. We note that all of the conjectures which appeared in the appendix of [GR17a] have now been resolved (see [LR25] and the references therein).

1.5.5. In this paper we will only deal with prestacks that are locally almost of finite type [GR17a, Chapter 2, § 1.7]. We remind the reader that the class of such prestacks is stable under the formation of de Rham prestacks [GR17b, Chapter 4, Proposition 1.1.4].

1.5.6. We make the running assumption (see [Ber19, § 1.3]) that our stacks are quasi-compact, perfect, bounded, and locally of finite presentation. A perfect stack has affine diagonal (see [BFN10, § 1.1]) and is therefore QCA.

1.5.7. For a quasi-smooth algebraic stack $\mathcal{Y}$, we denote

$$\mathrm{Sing}(\mathcal{Y}) = \mathrm{Spec}_{\mathcal{Y}}(\mathrm{Sym}(\mathbf{T}_{\mathcal{Y}}[1])),$$

the 'total space' of $\mathbf{L}_{\mathcal{Y}}[-1]$.[4] This notation differs slightly from the more standard notation of [AG15] and [AG18], which used this expression to denote $\mathrm{Sing}(\mathcal{Y})^{\mathrm{cl}}$.

2. Construction of the action

The goal of this section is to perform the the following.

Construction 2.1. Let Z be a quasi-smooth scheme. Then $\mathrm{IndCoh}(\mathrm{Z})$ receives a canonical action of $\mathrm{D}(\mathrm{Sing}(\mathrm{Z}))^{\Rightarrow}$.

The outline of this section is as follows.

- In § 2.1, we briefly review material about sheaves of categories from [Gai15].
- In § 2.2, we describe a general construction for equipping a category with the structure of left module over the shearing of a symmetric monoidal category acted on by $\mathbf{G}_m$. This construction is a sort of *deperiodization* result.

[4] This stack is denoted $\mathrm{T}^{\vee}[-1]\mathcal{Y}$ by some authors.

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– In § 2.3, we apply the framework of § 2.2 to carry out Construction 2.1, modulo the technical result Proposition 2.1. The main geometric input is that the vector space V is open in $\mathbf{P}(\mathrm{V} \oplus \mathbf{1})$, which allows us to reduce to the results of [AG18].
– In § 2.4, we prove a 'rigidity' theorem for relative crystals of categories, which gives us control over homotopy coherence problems.
– In § 2.5, we apply the framework of § 2.4 to prove Proposition 2.1.

2.1 Review of sheaves of categories

2.1.1. For a morphism $f : \mathrm{T} \to \mathrm{S}$ of affine schemes, we write

$$\mathbf{ind} : \mathrm{QCoh}(\mathrm{S})\textbf{-mod} \rightleftharpoons \mathrm{QCoh}(\mathrm{T})\textbf{-mod} : \mathbf{res}$$

for the adjunction defined by extension and restriction of scalars along the map of symmetric monoidal categories

$$f^* : \mathrm{QCoh}(\mathrm{S}) \to \mathrm{QCoh}(\mathrm{T}).$$

2.1.2. Let us recall some features of the theory of sheaves of categories.
To every prestack $\mathcal{Y}$, one associates the symmetric monoidal $(\infty, 2)$-category

$$\mathrm{ShCat}(\mathcal{Y}) = \lim_{\mathrm{S} \to \mathcal{Y}} \mathrm{QCoh}(\mathrm{S})\textbf{-mod}$$

of sheaves of categories on $\mathcal{Y}$. This limit is taken over test affine schemes mapping to $\mathcal{Y}$, with the transition functors given by **ind**. The value of a sheaf of categories $\mathcal{C}$ at a test affine scheme S is denoted

$$\mathbf{\Gamma}(\mathrm{S}; \mathcal{C}) \in \mathrm{QCoh}(\mathrm{S})\textbf{-mod}.$$

The monoidal unit of $\mathrm{ShCat}(\mathcal{Y})$ is denoted $\mathrm{QCoh}_{/\mathcal{Y}}$. It is given by the assignment

$$\mathbf{\Gamma}(\mathrm{S}; \mathrm{QCoh}_{/\mathcal{Y}}) = \mathrm{QCoh}(\mathrm{S}).$$

To a morphism of prestacks $f : \mathcal{Z} \to \mathcal{Y}$, one associates the adjunction

$$\mathbf{cores}_f : \mathrm{ShCat}(\mathcal{Y}) \rightleftharpoons \mathrm{ShCat}(\mathcal{Z}) : \mathbf{coind}_f. \tag{2.1}$$

For any cartesian diagram of prestacks as follows, the natural transformation

$$\mathbf{cores}_g \circ \mathbf{coind}_f \to \mathbf{coind}_{f'} \circ \mathbf{cores}_{g'}$$

obtained by adjunction from the identification

$$\mathbf{cores}_{f'} \circ \mathbf{cores}_g \xrightarrow{\sim} \mathbf{cores}_{g'} \circ \mathbf{cores}_f$$

is an isomorphism.

$$\begin{array}{ccc} \mathcal{Z}' & \xrightarrow{g'} & \mathcal{Z} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathcal{Y}' & \xrightarrow{g} & \mathcal{Y} \end{array}$$

2.1.3. Note that

$$\mathbf{coind}_f(\mathrm{QCoh}_{/\mathcal{Z}})$$

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is naturally a sheaf of symmetric monoidal categories over $\mathcal{Y}$, and the adjunction (2.1) enhances to an adjunction

$$\mathbf{Loc}_{\mathcal{Z}/\mathcal{Y}} : \mathbf{coind}_f(\mathrm{QCoh}_{/\mathcal{Z}})\text{-}\mathbf{mod}(\mathrm{ShCat}(\mathcal{Y})) \rightleftharpoons \mathrm{ShCat}(\mathcal{Z}) : \mathbf{\Gamma}_{\mathcal{Z}/\mathcal{Y}}.$$

The morphism f is called relatively 1-affine when this adjunction is an equivalence.

2.1.4. A prestack $\mathcal{Y}$ is said to be 1-affine if the morphism $\mathcal{Y} \to \mathrm{pt}$ is relatively 1-affine. We remind the reader of the following.

– Every (eventually coconnective, left, with affine diagonal) algebraic stack is 1-affine.

– The de Rham prestack Z_{dR} of any finite-type scheme Z is 1-affine.

2.2 Actions of sheared categories

2.2.1. Let $\mathbf{G}_m$**-mod** denote the symmetric monoidal $(\infty, 2)$-category of dg categories with a weak action of $\mathbf{G}_m$. Recall from [AG15, § A.2] that this category is equipped with a symmetric monoidal auto-equivalence, known as shearing:

$$(-)^{\Rightarrow} : \mathbf{G}_m\text{-}\mathbf{mod} \to \mathbf{G}_m\text{-}\mathbf{mod}.$$

We write $(-)^{\Leftarrow}$ for its inverse.

2.2.2. Let $\mathcal{A}$ be a symmetric monoidal category acted on by $\mathbf{G}_m$ (i.e. $\mathcal{A}$ is a commutative algebra object of $\mathbf{G}_m$**-mod**). Since shearing is symmetric monoidal, $\mathcal{A}^{\Rightarrow}$ is another symmetric monoidal category acted on by $\mathbf{G}_m$.

The goal of this subsection is to give an answer to the question of what it takes to equip a dg category $\mathcal{M}$ with an action of $\mathcal{A}^{\Rightarrow}$ (without regard for its $\mathbf{G}_m$-action).

That is, we will give a description of

$$\mathcal{A}^{\Rightarrow}\text{-}\mathbf{mod} = \mathcal{A}^{\Rightarrow}\text{-}\mathbf{mod}(\mathrm{dgCat}).$$

2.2.3 *A toy model.* The construction we are about to give has an analogue one category level down. Namely, for a graded algebra A, and a vector space M, the following data are equivalent:

(a) a non-graded A-module structure on M;

(b) a graded $\mathrm{A}[t^{\pm 1}]$-module structure on $\mathrm{M}[t^{\pm 1}]$.

Indeed, if M is given a non-graded A-module structure, then we endow $\mathrm{M}[t^{\pm 1}]$ with the structure of a graded A-module, where a homogeneous element a of degree n acts by

$$a \cdot (mt^k) = (am)t^{n+k}.$$

This action commutes with the natural graded action of $k[t^{\pm 1}]$ on $\mathrm{M}[t^{\pm 1}]$ and hence makes $\mathrm{M}[t^{\pm 1}]$ into a graded $\mathrm{A}[t^{\pm 1}]$-module. To recover the original non-graded action of A on M, observe that A is the degree-zero part of $\mathrm{A}[t^{\pm 1}]$ and M is the degree-zero part of $\mathrm{M}[t^{\pm 1}]$.

2.2.4. Since the morphism $f : \mathrm{pt} \to \mathrm{B}\mathbf{G}_m$ is relatively 1-affine, we obtain an identification

$$\mathrm{dgCat} \simeq \mathrm{QCoh}(\mathbf{G}_m)\text{-}\mathbf{mod}(\mathbf{G}_m\text{-}\mathbf{mod}), \quad \mathcal{M} \mapsto \mathcal{M} \otimes \mathrm{QCoh}(\mathbf{G}_m),$$

where $\mathrm{QCoh}(\mathbf{G}_m)$ is viewed as a $\mathbf{G}_m$-equivariant symmetric monoidal category via its pointwise tensor product.

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Now, let $\mathcal{B}$ be any symmetric monoidal category. From the monadic adjunction

$$\mathbf{ind} : \mathrm{dgCat} \rightleftharpoons \mathcal{B}\text{-}\mathbf{mod}(\mathrm{dgCat}) : \mathbf{oblv},$$

we obtain the following commutative diagram in which both horizontal arrows are equivalences.

$$\begin{array}{ccc} \mathcal{B}\text{-}\mathbf{mod} & \xrightarrow{\sim} & (\mathcal{B} \otimes \mathrm{QCoh}(\mathbf{G}_m))\text{-}\mathbf{mod}(\mathbf{G}_m\text{-}\mathbf{mod}) \\ \downarrow{\scriptstyle \mathrm{oblv}} & & \downarrow{\scriptstyle \mathrm{oblv}} \\ \mathrm{dgCat} & \xrightarrow{\sim} & \mathrm{QCoh}(\mathbf{G}_m)\text{-}\mathbf{mod}(\mathbf{G}_m\text{-}\mathbf{mod}) \end{array} \tag{2.2}$$

2.2.5. Now, let $\mathcal{B}$ be the underlying symmetric monoidal category of $\mathcal{A}^{\Rightarrow}$. By the projection formula for sheaves of categories, we obtain the equivalences of commutative algebras

$$\begin{aligned} \mathcal{B} \otimes \mathrm{QCoh}(\mathbf{G}_m) &\simeq \mathbf{coind}_f \circ \mathbf{cores}_f(\mathcal{A}^{\Rightarrow}) \\ &\simeq \mathcal{A}^{\Rightarrow} \otimes \mathbf{coind}_f(\mathrm{Vect}) \simeq \mathcal{A}^{\Rightarrow} \otimes \mathrm{QCoh}(\mathbf{G}_m) \end{aligned}$$

in $\mathbf{G}_m$-$\mathbf{mod}$.

As a result, for our choice of $\mathcal{B}$, the diagram (2.2) rewrites as follows.

$$\begin{array}{ccc} \mathcal{A}^{\Rightarrow}\text{-}\mathbf{mod} & \xrightarrow{\sim} & (\mathcal{A}^{\Rightarrow} \otimes \mathrm{QCoh}(\mathbf{G}_m))\text{-}\mathbf{mod}(\mathbf{G}_m\text{-}\mathbf{mod}) \\ \downarrow{\scriptstyle \mathrm{oblv}} & & \downarrow{\scriptstyle \mathrm{oblv}} \\ \mathrm{dgCat} & \xrightarrow{\sim} & \mathrm{QCoh}(\mathbf{G}_m)\text{-}\mathbf{mod}(\mathbf{G}_m\text{-}\mathbf{mod}) \end{array} \tag{2.3}$$

The difference between this diagram and (2.2) is that here, the $\mathbf{G}_m$ action on $\mathcal{A}^{\Rightarrow} \otimes \mathrm{QCoh}(\mathbf{G}_m)$ is the natural one coming from the pre-existing action on $\mathcal{A}^{\Rightarrow}$, while in (2.2) the $\mathbf{G}_m$ action on $\mathcal{B}$ is trivial.

2.2.6. Note that shearing induces a commutative diagram in which the horizontal arrows are equivalences.

$$\begin{array}{ccc} (\mathcal{A}^{\Rightarrow} \otimes \mathrm{QCoh}(\mathbf{G}_m))\text{-}\mathbf{mod}(\mathbf{G}_m\text{-}\mathbf{mod}) & \xrightarrow{(-)^{\Leftarrow}} & (\mathcal{A} \otimes \mathrm{QCoh}(\mathbf{G}_m)^{\Leftarrow})\text{-}\mathbf{mod}(\mathbf{G}_m\text{-}\mathbf{mod}) \\ \downarrow{\scriptstyle \mathrm{oblv}} & & \downarrow{\scriptstyle \mathrm{oblv}} \\ \mathrm{QCoh}(\mathbf{G}_m)\text{-}\mathbf{mod}(\mathbf{G}_m\text{-}\mathbf{mod}) & \xrightarrow{(-)^{\Leftarrow}} & \mathrm{QCoh}(\mathbf{G}_m)^{\Leftarrow}\text{-}\mathbf{mod}(\mathbf{G}_m\text{-}\mathbf{mod}) \end{array}$$

Concatenating this with (2.3), we obtain the following commutative diagram where the lower horizontal equivalence is given by

$$\mathcal{M} \mapsto \mathcal{M} \otimes \mathrm{QCoh}(\mathbf{G}_m)^{\Leftarrow}.$$

$$\begin{array}{ccc} \mathcal{A}^{\Rightarrow}\text{-}\mathbf{mod} & \xrightarrow{\sim} & (\mathcal{A} \otimes \mathrm{QCoh}(\mathbf{G}_m)^{\Leftarrow})\text{-}\mathbf{mod}(\mathbf{G}_m\text{-}\mathbf{mod}) \\ \downarrow{\scriptstyle \mathrm{oblv}} & & \downarrow{\scriptstyle \mathrm{oblv}} \\ \mathrm{dgCat} & \xrightarrow{\sim} & \mathrm{QCoh}(\mathbf{G}_m)^{\Leftarrow}\text{-}\mathbf{mod}(\mathbf{G}_m\text{-}\mathbf{mod}) \end{array}$$

2.2.7. This is our answer to the question posed in § 2.2.2: giving an action of $\mathcal{A}^{\Rightarrow}$ on $\mathcal{M}$ is equivalent to giving a $\mathbf{G}_m$-equivariant action of $\mathcal{A}$ on

$$\mathcal{M} \otimes \mathrm{QCoh}(\mathbf{G}_m)^{\Leftarrow},$$

commuting with the natural $\mathbf{G}_m$-equivariant action of $\mathrm{QCoh}(\mathbf{G}_m)^{\Leftarrow}$ on the right factor.

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2.2.8 *Remark.* Since

$$\mathrm{QCoh}(\mathbf{G}_m)^{\Leftarrow} \simeq k[\beta^{\pm}]\text{-mod}$$

where $|\beta| = 2$, the category $\mathcal{M} \otimes \mathrm{QCoh}(\mathbf{G}_m)^{\Leftarrow}$ is a 2-periodization of $\mathcal{M}$. One may interpret the result of this subsection as a *deperiodization* statement, showing what additional data is needed to reconstruct an action of $\mathcal{A}^{\Rightarrow}$ on $\mathcal{M}$ from an action of $\mathcal{A}$ on the 2-periodization of $\mathcal{M}$.

2.3 Geometric incarnation of shearing

2.3.1. For the rest this section, we will let Z denote a fixed choice of quasi-smooth scheme. Instead of performing Construction 2.1 as is, we will consider its reformulation in the terms provided by § 2.2.

Construction 2.2. There is a natural lift of

$$\mathrm{IndCoh}(\mathrm{Z}) \otimes \mathrm{QCoh}(\mathbf{G}_m)^{\Leftarrow} \in \mathrm{QCoh}(\mathbf{G}_m)^{\Leftarrow}\textbf{-mod}(\mathbf{G}_m\textbf{-mod})$$

to an object of

$$(\mathrm{D}(\mathrm{Sing}(\mathrm{Z})) \otimes \mathrm{QCoh}(\mathbf{G}_m)^{\Leftarrow})\textbf{-mod}(\mathbf{G}_m\textbf{-mod}).$$

2.3.2. In fact, we will not perform Construction 2.2 directly either. Instead, we give a geometric incarnation of $\mathrm{QCoh}(\mathbf{G}_m)^{\Leftarrow}$, viewed as a symmetric monoidal category with its dilation action of $\mathbf{G}_m$. The resulting reformulation of Construction 2.2 will appear below as Construction 2.3.

Regard $\mathrm{IndCoh}(\Omega\mathbf{A}^1)$ as a symmetric monoidal category under convolution. Under the Koszul duality equivalence (1.4) for $\mathrm{V} = \mathbf{A}^1$, the monoidal localization

$$(j^*)^{\Rightarrow} : \mathrm{QCoh}(\mathbf{A}^1)^{\Rightarrow} \to \mathrm{QCoh}(\mathbf{G}_m)^{\Rightarrow}$$

goes over the monoidal localization

$$\mathrm{IndCoh}(\Omega\mathbf{A}^1) \to \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(\Omega\mathbf{A}^1).$$

Therefore, we obtain a $\mathbf{G}_m$-equivariant equivalence of symmetric monoidal categories

$$\mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(\Omega\mathbf{A}^1) \simeq \mathrm{QCoh}(\mathbf{G}_m)^{\Rightarrow}$$

intertwining the *inverse* dilation action on the left-hand side with the dilation action on the right-hand side. After inverting the $\mathbf{G}_m$ action on the right-hand side, we obtain a symmetric monoidal equivalence

$$\mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(\Omega\mathbf{A}^1) \simeq \mathrm{QCoh}(\mathbf{G}_m)^{\Leftarrow}$$

which is equivariant for the dilation actions on both sides.

It follows that Construction 2.2 admits the following reformulation.

Construction 2.3. There is a natural lift of

$$\mathrm{IndCoh}(\mathrm{Z}) \otimes \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(\Omega\mathbf{A}^1) \in \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(\Omega\mathbf{A}^1)\textbf{-mod}(\mathbf{G}_m\textbf{-mod}) \tag{2.4}$$

to an object of

$$(\mathrm{D}(\mathrm{Sing}(\mathrm{Z})) \otimes \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(\Omega\mathbf{A}^1))\textbf{-mod}(\mathbf{G}_m\textbf{-mod}).$$

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2.3.3. We now carry out Construction 2.3 by studying the product

$$\mathrm{Z} \times \Omega \mathbf{A}^1,$$

equipped with the action of $\mathbf{G}_m$ by dilation on the second factor.

To start off, $\mathbf{P}\mathrm{Sing}(\mathrm{Z} \times \Omega \mathbf{A}^1)$ inherits a $\mathbf{G}_m$ action by the naturality of $\mathrm{Sing}(-)$, and we claim that

$$\mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(\mathrm{Z} \times \Omega \mathbf{A}^1) \in \mathbf{G}_m\text{-}\mathbf{mod}$$

naturally lifts to an object

$$\mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(\mathrm{Z} \times \Omega \mathbf{A}^1) \in \mathrm{D}(\mathbf{P}\mathrm{Sing}(\mathrm{Z} \times \Omega \mathbf{A}^1))\text{-}\mathbf{mod}(\mathbf{G}_m\text{-}\mathbf{mod}). \tag{2.5}$$

In other words, we are saying that

$$\mathrm{Ind}\overset{\circ}{\mathrm{Coh}}((\mathrm{Z} \times \Omega \mathbf{A}^1)/\mathbf{G}_m) \simeq \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(\mathrm{Z} \times \Omega \mathbf{A}^1)^{\mathbf{G}_m} \in \mathrm{Rep}(\mathbf{G}_m)\text{-}\mathbf{mod} \tag{2.6}$$

admits a natural lift to

$$\begin{aligned}&\mathrm{ShCat}(\mathbf{P}\mathrm{Sing}(\mathrm{Z} \times \Omega \mathbf{A}^1)_{\mathrm{dR}}/\mathbf{G}_m)\\ &\quad \simeq \mathrm{D}(\mathbf{P}\mathrm{Sing}(\mathrm{Z} \times \Omega \mathbf{A}^1))^{\mathbf{G}_m}\text{-}\mathbf{mod}(\mathrm{Rep}(\mathbf{G}_m)\text{-}\mathbf{mod}).\end{aligned}$$

Indeed, applying [AG18, Proposition-Construction 1.5.2] to the quotient stack

$$(\mathrm{Z} \times \Omega \mathbf{A}^1)/\mathbf{G}_m$$

produces a lift of (2.6) to a sheaf of categories over

$$\mathbf{P}\mathrm{Sing}((\mathrm{Z} \times \Omega \mathbf{A}^1)/\mathbf{G}_m)_{\mathrm{dR}} \underset{((\mathrm{Z}\times\Omega\mathbf{A}^1)/\mathbf{G}_m)_{\mathrm{dR}}}{\times} (\mathrm{Z} \times \Omega \mathbf{A}^1)/\mathbf{G}_m,$$

and this last stack maps to

$$\begin{aligned}&\mathbf{P}\mathrm{Sing}((\mathrm{Z} \times \Omega \mathbf{A}^1)/\mathbf{G}_m)_{\mathrm{dR}} \underset{((\mathrm{Z}\times\Omega\mathbf{A}^1)/\mathbf{G}_m)_{\mathrm{dR}}}{\times} (\mathrm{Z} \times \Omega \mathbf{A}^1)_{\mathrm{dR}}/\mathbf{G}_m\\ &\quad \simeq \mathbf{P}\mathrm{Sing}(\mathrm{Z} \times \Omega \mathbf{A}^1)_{\mathrm{dR}}/\mathbf{G}_m.\end{aligned}$$

2.3.4. One has

$$\mathrm{Sing}(\mathrm{Z} \times \Omega \mathbf{A}^1)_{\mathrm{dR}} \simeq (\mathrm{Sing}(\mathrm{Z}) \oplus \mathbf{1})_{\mathrm{dR}},$$

where the latter expression denotes the de Rham prestack of $\mathrm{Sing}(\mathrm{Z}) \underset{\mathrm{Z}}{\times} \mathbf{A}^1_{\mathrm{Z}}$.

In terms of this identification, the action of $\mathbf{G}_m$ on $\mathrm{Sing}(\mathrm{Z} \times \Omega \mathbf{A}^1)$ obtained from the dilation action of $\mathbf{G}_m$ on $\Omega \mathbf{A}^1$ by the naturality of the $\mathrm{Sing}(-)$ construction matches with *inverse* dilation on $\mathbf{A}^1_{\mathrm{Z}}$.

Therefore, the open embedding

$$\mathrm{Sing}(\mathrm{Z}) \hookrightarrow \mathbf{P}(\mathrm{Sing}(\mathrm{Z}) \oplus \mathbf{1}) \simeq \mathbf{P}\mathrm{Sing}(\mathrm{Z} \times \Omega \mathbf{A}^1) \quad \xi \mapsto [\xi : 1] \tag{2.7}$$

intertwines the action of $\mathbf{G}_m$ on its source by dilation with the preceding action of $\mathbf{G}_m$ on its target, and restricting (2.5) along this map defines a lift of

$$\begin{aligned}&\mathrm{IndCoh}(\mathrm{Z}) \otimes \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(\Omega \mathbf{A}^1)\\ &\quad \simeq \mathrm{D}(\mathrm{Sing}(\mathrm{Z})) \underset{\mathrm{D}(\mathbf{P}(\mathrm{Sing}(\mathrm{Z})\oplus\mathbf{1}))}{\otimes} \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(\mathrm{Z} \times \Omega \mathbf{A}^1) \in \mathbf{G}_m\text{-}\mathbf{mod}\end{aligned}$$

to an object

$$\mathrm{IndCoh}(\mathrm{Z}) \otimes \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(\Omega \mathbf{A}^1) \in \mathrm{D}(\mathrm{Sing}(\mathrm{Z}))\text{-}\mathbf{mod}(\mathbf{G}_m\text{-}\mathbf{mod}). \tag{2.8}$$

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We claim the following.

Proposition 2.1. *The* $\mathbf{G}_m$*-equivariant action of* $\mathrm{D}(\mathbf{P}\mathrm{Sing}(\mathrm{Z}))$ *on*

$$\mathrm{IndCoh}(\mathrm{Z}) \otimes \mathrm{Ind}\overset{\circ}{\mathrm{C}}\mathrm{oh}(\Omega\mathbf{A}^1) \in \mathbf{G}_m\textbf{-mod}$$

encoded by (2.8) commutes with the $\mathbf{G}_m$*-equivariant action of* $\mathrm{Ind}\overset{\circ}{\mathrm{C}}\mathrm{oh}(\Omega\mathbf{A}^1)$ *encoded by (2.4).*

2.3.5. Observe that Construction 2.3 follows immediately from Proposition 2.1. However, it is not clear *a priori* that Proposition 2.1 is actually a proposition (as opposed to a construction). This is shown in § 2.4, and the proof of Proposition 2.1 appears in § 2.5.

2.4 Relative crystals of categories

2.4.1. In this subsection, we prove a general assertion about relative crystals of categories. A consequence of this result will be that the commutativity of the two actions appearing in Proposition 2.1 is merely a property and not additional data.

Proposition 2.2. *Let* $\mathcal{A} \to \mathcal{B}$ *be a map of monoidal categories. If the multiplication map*

$$\mu : \mathcal{B} \underset{\mathcal{A}}{\otimes} \mathcal{B} \to \mathcal{B}$$

admits a $(\mathcal{B}, \mathcal{B})$*-bilinear fully faithful left adjoint, then the functor*

$$\mathbf{res} : \mathcal{B}\textbf{-mod} \to \mathcal{A}\textbf{-mod}$$

is locally fully faithful.[5]

Proof of Proposition 2.2. A stronger assertion holds true: for every pair of $\mathcal{B}$-modules $\mathcal{M}$ and $\mathcal{M}'$, the functor

$$\begin{aligned}\mathrm{Fun}_{\mathcal{B}\textbf{-mod}}(\mathcal{M}, \mathcal{M}') &\to \mathrm{Fun}_{\mathcal{A}\textbf{-mod}}(\mathbf{res}(\mathcal{M}), \mathbf{res}(\mathcal{M}')) \\ &\simeq \mathrm{Fun}_{\mathcal{B}\textbf{-mod}}(\mathbf{ind} \circ \mathbf{res}(\mathcal{M}), \mathcal{M}')\end{aligned}$$

is a colocalization. Indeed, the counit

$$\mathbf{ind} \circ \mathbf{res} \to \mathbf{1}$$

identifies with the image of μ under the functor of 2-categories

$$(\mathcal{B}, \mathcal{B})\textbf{-bimod} \to \mathrm{End}(\mathcal{B}\textbf{-mod}) \quad \mathcal{N} \mapsto \mathcal{N} \otimes_{\mathcal{B}} -$$

and μ is assumed to admit a fully faithful left adjoint. □

2.4.2. Let $\mathcal{Z} \to \mathcal{Y}$ be a schematic morphism of prestacks, and write

$$p : \mathcal{Z}_{\mathrm{dR}/\mathcal{Y}} = \mathcal{Z}_{\mathrm{dR}} \underset{\mathcal{Y}_{\mathrm{dR}}}{\times} \mathcal{Y} \to \mathcal{Y}$$

for its formal completion.

Proposition 2.3. *The functor*

$$\mathbf{coind}_p : \mathrm{ShCat}(\mathcal{Z}_{\mathrm{dR}/\mathcal{Y}}) \to \mathrm{ShCat}(\mathcal{Y})$$

is locally fully faithful.

[5] A functor of $(\infty, 2)$-categories is said to be *locally fully faithful* if it induces fully faithful functors on mapping categories.

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Proof of Proposition 2.3. Note that it suffices to consider the case where $\mathcal{Y}$ is an affine scheme. Indeed, under the identifications

$$\mathrm{ShCat}(\mathcal{Z}_{\mathrm{dR}/\mathcal{Y}}) \simeq \lim_{\mathrm{S}} \mathrm{ShCat}((\mathcal{Z}_{\mathrm{S}})_{\mathrm{dR}/\mathrm{S}}), \quad \mathrm{ShCat}(\mathcal{Y}) = \lim_{\mathrm{S}} \mathrm{ShCat}(\mathrm{S}),$$

where the limits are taken over all test affine schemes mapping to $\mathcal{Y}$, the functor $\mathbf{coind}_p$ is the one induced by the functors

$$\mathbf{coind}_{p_{\mathrm{S}}} : \mathrm{ShCat}((\mathcal{Z}_{\mathrm{S}})_{\mathrm{dR}/\mathrm{S}}) \to \mathrm{ShCat}(\mathrm{S}).$$

Therefore, we may replace $\mathcal{Z} \to \mathcal{Y}$ with a map of schemes $\mathrm{Z} \to \mathrm{Y}$, with Y affine.

By Proposition 2.2, it suffices to check that the morphism of $\mathrm{QCoh}(\mathrm{Z}_{\mathrm{dR}/\mathrm{Y}})$-bimodules

$$\begin{aligned} &\mathrm{QCoh}(\mathrm{Z}_{\mathrm{dR}/\mathrm{Y}}) \underset{\mathrm{QCoh}(\mathrm{Y})}{\otimes} \mathrm{QCoh}(\mathrm{Z}_{\mathrm{dR}/\mathrm{Y}}) \\ &\quad \simeq \mathrm{QCoh}((\mathrm{Z} \times_{\mathrm{Y}} \mathrm{Z})_{\mathrm{dR}/\mathrm{Y}}) \xrightarrow{\Delta^*_{\mathrm{dR}/\mathrm{Y}}} \mathrm{QCoh}(\mathrm{Z}_{\mathrm{dR}/\mathrm{Y}}) \end{aligned} \tag{2.9}$$

admits a fully faithful $\mathrm{QCoh}(\mathrm{Z}_{\mathrm{dR}/\mathrm{Y}})$-bilinear left adjoint.

Now Kashiwara's lemma implies that the functor

$$\Delta^*_{\mathrm{dR}} : \mathrm{QCoh}((\mathrm{Z} \times_{\mathrm{Y}} \mathrm{Z})_{\mathrm{dR}}) \to \mathrm{QCoh}(\mathrm{Z}_{\mathrm{dR}})$$

admits a fully faithful $\mathrm{QCoh}((\mathrm{Z} \times_{\mathrm{Y}} \mathrm{Z})_{\mathrm{dR}})$-linear left adjoint. Tensoring up along

$$- \underset{\mathrm{QCoh}(\mathrm{Y}_{\mathrm{dR}})}{\otimes} \mathrm{QCoh}(\mathrm{Y})$$

shows that (2.9) admits a left adjoint with the desired properties. □

2.4.3 *Porism.* Let us record some consequences of Proposition 2.3 (or, more precisely, its proof).

Let $\mathcal{C}$ be a sheaf of categories over $\mathcal{Z}_{\mathrm{dR}/\mathcal{Y}}$. Then the data of $\mathcal{C}$ is encoded by the data of $\mathbf{coind}_p(\mathcal{C})$ along with compatible choices of colocalizing subcategories

$$\mathcal{C}_{\mathrm{S}} \hookrightarrow \mathbf{\Gamma}(\mathrm{S}; \mathbf{coind}_p(\mathcal{C})) \tag{2.10}$$

for every test affine scheme S mapping to $\mathcal{Z}_{\mathrm{dR}/\mathcal{Y}}$.

We note for future reference that the functor (2.10) and its right adjoint are $\mathrm{QCoh}(\mathrm{S})$-linear.

Let $\mathcal{D}$ be another sheaf of categories over $\mathcal{Z}_{\mathrm{dR}/\mathcal{Y}}$. Then a morphism $\mathcal{C} \to \mathcal{D}$ in $\mathrm{ShCat}(\mathcal{Z}_{\mathrm{dR}/\mathcal{Y}})$ amounts to a morphism

$$\mathrm{F} : \mathbf{coind}_p(\mathcal{C}) \to \mathbf{coind}_p(\mathcal{D})$$

in $\mathrm{ShCat}(\mathcal{Y})$ with the property: for every test affine scheme S mapping to $\mathcal{Z}_{\mathrm{dR}/\mathcal{Y}}$, the functor

$$\mathbf{\Gamma}(\mathrm{S}; \mathrm{F}) : \mathbf{\Gamma}(\mathrm{S}; \mathbf{coind}_p(\mathcal{C})) \to \mathbf{\Gamma}(\mathrm{S}; \mathbf{coind}_p(\mathcal{D}))$$

takes $\mathcal{C}_{\mathrm{S}}$ into $\mathcal{D}_{\mathrm{S}}$.

Let $\mathcal{A}$ be a sheaf of monoidal categories over $\mathcal{Y}$. Then an action of $\mathbf{cores}_p(\mathcal{A})$ on $\mathcal{C}$ amounts to an action of $\mathcal{A}$ on $\mathbf{coind}_p(\mathcal{C})$ with the property: for every test affine scheme S mapping to $\mathcal{Z}_{\mathrm{dR}/\mathcal{Y}}$, the action map

$$\begin{aligned} \mathbf{\Gamma}(\mathrm{S}; \mathbf{coind}_p(\mathcal{C})) \underset{\mathrm{QCoh}(\mathrm{S})}{\otimes} \mathbf{\Gamma}(\mathrm{S}; \mathcal{A}) \simeq \mathbf{\Gamma}(\mathrm{S}; \mathbf{coind}_p(\mathcal{C}) \underset{\mathcal{Y}}{\otimes} \mathcal{A}) \\ \xrightarrow{\mathbf{\Gamma}(\mathrm{S};\mathrm{act})} \mathbf{\Gamma}(\mathrm{S}; \mathbf{coind}_p(\mathcal{C})) \end{aligned}$$

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restricts to a functor

$$\mathcal{C}_S \underset{\mathrm{QCoh}(S)}{\otimes} \mathbf{\Gamma}(S;\mathcal{A}) \to \mathcal{C}_S.$$

2.5 Proof of Proposition 2.1

2.5.1. We apply the paradigm of § 2.4 in the case

$$\mathcal{Z} = \mathrm{Sing}(Z)/\mathbf{G}_m \subset \mathbf{P}(\mathrm{Sing}(Z) \oplus \mathbf{1})/\mathbf{G}_m, \quad \mathcal{Y} = B\mathbf{G}_m.$$

We interpret (2.8) as making

$$\mathcal{C}^{\mathbf{G}_m} = (\mathrm{IndCoh}(Z) \otimes \mathrm{Ind}\overset{\circ}{\mathrm{C}}\mathrm{oh}(\Omega\mathbf{A}^1))^{\mathbf{G}_m}$$

into a sheaf of categories over $\mathcal{Z}_{\mathrm{dR}/\mathcal{Y}}$, and (2.4) as encoding an action of

$$\mathcal{A}^{\mathbf{G}_m} = \mathrm{Ind}\overset{\circ}{\mathrm{C}}\mathrm{oh}(\Omega\mathbf{A}^1)^{\mathbf{G}_m},$$

viewed as a constant sheaf of symmetric monoidal categories over $\mathcal{Y}$, on $\mathbf{coind}_p(\mathcal{C}^{\mathbf{G}_m})$.

In these terms, the assertion of Proposition 2.1 says that the action of $\mathcal{A}^{\mathbf{G}_m}$ on $\mathbf{coind}_p(\mathcal{C}^{\mathbf{G}_m})$ lifts to an action of $\mathbf{cores}_p(\mathcal{A}^{\mathbf{G}_m})$ on $\mathcal{C}^{\mathbf{G}_m}$. Since $\mathcal{Z}$ and $\mathcal{Y}$ are 1-affine, this amounts to the assertion: for every test affine scheme S mapping to $\mathcal{Z}_{\mathrm{dR}/\mathcal{Y}}$, the full subcategory

$$\mathrm{QCoh}(S) \underset{\mathrm{QCoh}(\mathcal{Z}_{\mathrm{dR}/\mathcal{Y}})}{\otimes} \mathcal{C}^{\mathbf{G}_m} \subset \mathrm{QCoh}(S) \underset{\mathrm{QCoh}(\mathcal{Y})}{\otimes} \mathcal{C}^{\mathbf{G}_m} \tag{2.11}$$

is stable under the action of $\mathcal{A}^{\mathbf{G}_m}$.

Observe that the inclusion (2.11) rewrites as

$$(\mathrm{QCoh}(S') \underset{\mathrm{D}(\mathcal{Z}')}{\otimes} \mathcal{C})^{\mathbf{G}_m} \subset (\mathrm{QCoh}(S') \otimes \mathcal{C})^{\mathbf{G}_m},$$

where

$$\mathcal{Z}' = \mathcal{Z} \underset{B\mathbf{G}_m}{\times} \mathrm{pt} \quad S' = S \underset{B\mathbf{G}_m}{\times} \mathrm{pt}.$$

Therefore, it suffices to show: for every test affine scheme T mapping to $\mathcal{Z}'_{\mathrm{dR}}$, the full subcategory

$$\mathrm{QCoh}(T) \underset{\mathrm{D}(\mathcal{Z}')}{\otimes} \mathcal{C} \subset \mathrm{QCoh}(T) \otimes \mathcal{C} \tag{2.12}$$

is stable under the action of $\mathcal{A}$.

2.5.2. To proceed further, we need to recall the description of (2.12) from [AG18, § 2.4]. Write

$$\Gamma_f \hookrightarrow T_{\mathrm{red}} \times \mathrm{Sing}(Z) \subset T_{\mathrm{red}} \times \mathbf{P}\mathrm{Sing}(Z \times \Omega\mathbf{A}^1)$$

for the graph of the associated morphism

$$f : T_{\mathrm{red}} \to \mathrm{Sing}(Z) \subset \mathbf{P}(\mathrm{Sing}(Z) \oplus \mathbf{1}).$$

It corresponds to a closed conical subscheme

$$\mathrm{Cone}(\Gamma_f) \hookrightarrow T_{\mathrm{red}} \times \mathrm{Sing}(Z \times \Omega\mathbf{A}^1).$$

By definition, (2.12) is the quotient of the full subcategory

$$\begin{aligned}&(\mathrm{QCoh}(T) \otimes \mathrm{IndCoh}(Z \times \Omega\mathbf{A}^1))_{\mathrm{Cone}(\Gamma_f)} \\ &\quad \subset \mathrm{QCoh}(T) \otimes \mathrm{IndCoh}(Z \times \Omega\mathbf{A}^1)\end{aligned} \tag{2.13}$$

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by the full subcategory

$$\mathrm{QCoh}(\mathrm{T}) \otimes \mathrm{QCoh}(\mathrm{Z} \times \Omega\mathbf{A}^1).$$

It remains for us to check that (2.13) is stable under the action of $\mathrm{IndCoh}(\Omega\mathbf{A}^1)$.

2.5.3. Let us make the identification

$$\begin{aligned}(\mathrm{QCoh}(\mathrm{T}) \otimes \mathrm{IndCoh}(\mathrm{Z} \times \Omega\mathbf{A}^1))_{\mathrm{Cone}(\Gamma_f)} \otimes \mathrm{IndCoh}(\Omega\mathbf{A}^1) \\ \simeq (\mathrm{QCoh}(\mathrm{T}) \otimes \mathrm{IndCoh}(\mathrm{Z} \times \Omega\mathbf{A}^1 \times \Omega\mathbf{A}^1))_{\mathrm{Cone}(\Gamma_f)\oplus\mathbf{1}}\end{aligned} \tag{2.14}$$

as full subcategories of

$$\mathrm{QCoh}(\mathrm{T}) \otimes \mathrm{IndCoh}(\mathrm{Z} \times \Omega\mathbf{A}^1 \times \Omega\mathbf{A}^1). \tag{2.15}$$

In these terms, what remains to be seen is that the action map

$$\mathbf{1}_{\mathrm{QCoh}(\mathrm{T})} \otimes (\mathbf{1} \times \mathrm{add})_* : \mathrm{QCoh}(\mathrm{T}) \otimes \mathrm{IndCoh}(\mathrm{Z} \times \Omega\mathbf{A}^1 \times \Omega\mathbf{A}^1) \to \mathrm{QCoh}(\mathrm{T}) \otimes \mathrm{IndCoh}(\mathrm{Z} \times \Omega\mathbf{A}^1)$$

takes (2.14) into (2.13).

2.5.4. Consider the factorization of add into the composite

$$\Omega\mathbf{A}^1 \times \Omega\mathbf{A}^1 \xrightarrow{\sigma} \Omega\mathbf{A}^1 \times \Omega\mathbf{A}^1 \xrightarrow{\mathrm{pr}_1} \Omega\mathbf{A}^1 \tag{2.16}$$

of the automorphism σ defined by the formula

$$(s, t) \mapsto (s + t, t)$$

and the coordinate projection pr_1.

The automorphism of

$$\mathrm{S}_{\mathrm{red}} \times \mathrm{Sing}(\mathrm{Z} \times \Omega\mathbf{A}^1 \times \Omega\mathbf{A}^1)$$

induced by σ restricts to an automorphism of $\mathrm{Cone}(\Gamma_f) \oplus \mathbf{1}$, so

$$\mathbf{1}_{\mathrm{QCoh}(\mathrm{T})} \otimes (\mathbf{1} \times \sigma)_*$$

defines an automorphism of (2.15) restricting to an automorphism of (2.14).[6]

The functor

$$\mathbf{1}_{\mathrm{QCoh}(\mathrm{T})} \otimes (\mathbf{1} \times \mathrm{pr}_1)_* \simeq \mathbf{1}_{\mathrm{QCoh}(\mathrm{T})\otimes\mathrm{IndCoh}(\mathrm{Z}\times\Omega\mathbf{A}^1)} \otimes \Gamma_{\mathrm{IndCoh}}$$

takes (2.14) into (2.13).

3. Compatibilities

In this section, Z will denote a fixed choice of quasi-smooth scheme.

We show in Proposition 3.6 that the action of $\mathrm{D}(\mathrm{Sing}(\mathrm{Z}))^{\Rightarrow}$ on $\mathrm{IndCoh}(\mathrm{Z})$ constructed in § 2 has the expected backwards compatibility with the action of $\mathrm{D}(\mathbf{P}\mathrm{Sing}(\mathrm{Z}))$ on $\mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(\mathrm{Z})$ constructed in [AG18]. When there is any possibility of confusion, we will refer to the former as the *inhomogeneous action* and the latter as the *homogeneous action*.

[6] Recall that for a vector scheme V, $\mathrm{Sing}(\Omega\mathrm{V} \times \Omega\mathrm{V})^{\mathrm{cl}} \simeq \mathrm{V}^\vee \times \mathrm{V}^\vee$, and duality exchanges shearing on the first factor with shearing on the second factor.

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The outline of this section is as follows.

– In § 3.1, we prove a few straightforward results about the action of $\mathrm{D}(\mathrm{Sing}(\mathrm{Z}))^{\Rightarrow}$ on $\mathrm{IndCoh}(\mathrm{Z})$.
– In § 3.2, we record some results about the relation between IndCoh on formal schemes and the action of $\mathrm{D}(\mathrm{Sing}(\mathrm{Z}))^{\Rightarrow}$. In particular, we give a simple description of the category $\mathrm{IndCoh}_0(\mathrm{Y}^{\wedge}_{\mathrm{Z}})$ in terms of this action. The proofs of the results in this section are deferred to §§ 3.3 and 3.4.
– In § 3.3, we show how to reduce the statements in § 3.2 to Proposition 3.4, a statement about the Arinkin–Gaitsgory homogeneous action of $\mathrm{D}(\mathbf{P}\mathrm{Sing}(\mathrm{Z}))$ on the category of singularities, whose proof is deferred to § 3.4.
– In § 3.4, we prove Proposition 3.4.
– In § 3.5, we prove a natural compatibility statement relating the inhomogeneous action and the homogeneous action. (Our proof of this compatibility takes Proposition 3.4 as input, which is why this subsection doesn't appear earlier in this section.)
– In § 3.6, we give an alternative description of the inhomogeneous action when Z is presented as a global complete intersection scheme. This version of the action was considered previously in [Ber20, § 2.2], so this section gives a backwards compatibility between our construction and the construction of [Ber20, § 2.2].

3.1 The basics

3.1.1. In this subsection, we collect some features of the inhomogeneous action which are more or less immediate from its construction.

Proposition 3.1. *Let $\pi : \mathrm{Sing}(\mathrm{Z}) \to \mathrm{Z}$ denote the canonical projection. The restriction of the inhomogeneous action along the symmetric monoidal functor*

$$(\pi^!)^{\Rightarrow} : \mathrm{D}(\mathrm{Z}) \to \mathrm{D}(\mathrm{Sing}(\mathrm{Z}))^{\Rightarrow}$$

recovers the usual action of $\mathrm{D}(\mathrm{Z})$ *on* $\mathrm{IndCoh}(\mathrm{Z})$.

3.1.2 *Proof.* This follows from [AG18, Corollary 1.4.5(b)].

3.1.3. Now consider the zero section $\mathrm{Z} \to \mathrm{Sing}(\mathrm{Z})$ and view $\mathrm{D}(\mathrm{Z})$ as tensored over $\mathrm{D}(\mathrm{Sing}(\mathrm{Z}))$ using this map. We claim the following.

Proposition 3.2. *The full subcategory*

$$\mathrm{D}(\mathrm{Z}) \underset{\mathrm{D}(\mathrm{Sing}(\mathrm{Z}))^{\Rightarrow}}{\otimes} \mathrm{IndCoh}(\mathrm{Z}) \subset \mathrm{IndCoh}(\mathrm{Z})$$

is equal to

$$\mathrm{QCoh}(\mathrm{Z}) \subset \mathrm{IndCoh}(\mathrm{Z}).$$

Proof of Proposition 3.2. Recall the action of $\mathrm{D}(\mathrm{Sing}(\mathrm{Z}))$ on

$$\mathrm{IndCoh}(\mathrm{Z}) \otimes \mathrm{Ind}\mathring{\mathrm{Coh}}(\Omega\mathbf{A}^1)$$

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from (2.8). We need to show that the full subcategory

$$\mathrm{D}(\mathrm{Z}) \underset{\mathrm{D}(\mathrm{Sing}(\mathrm{Z}))}{\otimes} (\mathrm{IndCoh}(\mathrm{Z}) \otimes \mathrm{Ind}\mathring{\mathrm{C}}\mathrm{oh}(\Omega \mathbf{A}^1)) \subset \mathrm{IndCoh}(\mathrm{Z}) \otimes \mathrm{Ind}\mathring{\mathrm{C}}\mathrm{oh}(\Omega \mathbf{A}^1)$$

is equal to

$$\mathrm{QCoh}(\mathrm{Z}) \otimes \mathrm{Ind}\mathring{\mathrm{C}}\mathrm{oh}(\Omega \mathbf{A}^1) \subset \mathrm{IndCoh}(\mathrm{Z}) \otimes \mathrm{Ind}\mathring{\mathrm{C}}\mathrm{oh}(\Omega \mathbf{A}^1).$$

This is a consequence of the following assertion, Proposition 3.3. □

Proposition 3.3. *Let Z_1 and Z_2 be quasi-smooth schemes, and consider the closed embedding*

$$\mathbf{P}\mathrm{Sing}(\mathrm{Z}_1) \times \mathrm{Z}_2 \hookrightarrow \mathbf{P}\mathrm{Sing}(\mathrm{Z}_1 \times \mathrm{Z}_2) \quad \xi \mapsto [\xi : 0].$$

Then

$$\mathrm{D}(\mathbf{P}\mathrm{Sing}(\mathrm{Z}_1) \times \mathrm{Z}_2) \underset{\mathrm{D}(\mathbf{P}\mathrm{Sing}(\mathrm{Z}_1 \times \mathrm{Z}_2))}{\otimes} \mathrm{Ind}\mathring{\mathrm{C}}\mathrm{oh}(\mathrm{Z}_1 \times \mathrm{Z}_2) \simeq \mathrm{Ind}\mathring{\mathrm{C}}\mathrm{oh}(\mathrm{Z}_1) \otimes \mathrm{QCoh}(\mathrm{Z}_2)$$

as full subcategories of

$$\mathrm{Ind}\mathring{\mathrm{C}}\mathrm{oh}(\mathrm{Z}_1 \times \mathrm{Z}_2).$$

Proof of Proposition 3.3. This follows from [AG18, Theorem-Construction 1.4.2(a)] and [AG15, Lemma 4.6.4]. □

3.2 Relative D-modules

3.2.1. Let $f : \mathrm{Z} \to \mathrm{Y}$ be a morphism of quasi-smooth schemes. We are interested in the category

$$\mathrm{IndCoh}(\mathrm{Y}^{\wedge}_{\mathrm{Z}}).$$

By [AG18, Proposition 3.1.2], the functor of IndCoh !-pullback

$$f^! : \mathrm{IndCoh}(\mathrm{Y}) \to \mathrm{IndCoh}(\mathrm{Y}^{\wedge}_{\mathrm{Z}})$$

induces an equivalence of categories

$$\mathrm{D}(\mathrm{Z}) \underset{\mathrm{D}(\mathrm{Y})}{\otimes} \mathrm{IndCoh}(\mathrm{Y}) \xrightarrow{\sim} \mathrm{IndCoh}(\mathrm{Y}^{\wedge}_{\mathrm{Z}}). \tag{3.1}$$

In view of Proposition 3.1 and the relative Kunneth formula for D-modules on schemes (which follows from [Gai15, Proposition 3.1.9 and Theorem 2.6.3], cf. [Che23, Lemma A.4.5]), we may regard (3.1) as defining an equivalence of categories

$$\mathrm{D}(\mathrm{Sing}(\mathrm{Y})_{\mathrm{Z}})^{\Rightarrow} \underset{\mathrm{D}(\mathrm{Sing}(\mathrm{Y}))^{\Rightarrow}}{\otimes} \mathrm{IndCoh}(\mathrm{Y}) \xrightarrow{\sim} \mathrm{IndCoh}(\mathrm{Y}^{\wedge}_{\mathrm{Z}}), \tag{3.2}$$

where

$$\mathrm{Sing}(\mathrm{Y})_{\mathrm{Z}} = \mathrm{Z} \underset{\mathrm{Y}}{\times} \mathrm{Sing}(\mathrm{Y}).$$

In particular, this equips $\mathrm{IndCoh}(\mathrm{Y}^{\wedge}_{\mathrm{Z}})$ an action of $\mathrm{D}(\mathrm{Sing}(\mathrm{Y})_{\mathrm{Z}})^{\Rightarrow}$.

3.2.2. Now consider the natural map

$$\iota : \mathrm{Z} \to \mathrm{Y}^{\wedge}_{\mathrm{Z}}$$

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and the resulting adjunction

$$\iota_* : \mathrm{IndCoh}(\mathrm{Z}) \rightleftharpoons \mathrm{IndCoh}(\mathrm{Y}^{\wedge}_{\mathrm{Z}}) : \iota^!. \tag{3.3}$$

Note that $\mathrm{IndCoh}(\mathrm{Y}^{\wedge}_{\mathrm{Z}})$ is naturally tensored over $\mathrm{D}(\mathrm{Sing}(\mathrm{Z}))^{\Rightarrow}$ via the singular codifferential

$$\mathrm{Sing}(f) : \mathrm{Sing}(\mathrm{Y})_{\mathrm{Z}} \to \mathrm{Sing}(\mathrm{Z}).$$

We prove the following assertion over the course of §§ 3.3 and 3.4.

Theorem 3.1. *The* $(\iota_*, \iota^!)$ *adjunction is naturally* $\mathrm{D}(\mathrm{Sing}(\mathrm{Z}))^{\Rightarrow}$*-linear.*

3.2.3. Recall that

$$\mathrm{IndCoh}_0(\mathrm{Y}^{\wedge}_{\mathrm{Z}}) \subset \mathrm{IndCoh}(\mathrm{Y}^{\wedge}_{\mathrm{Z}})$$

denotes the full subcategory of objects whose images under the functor $\iota^!$ of (3.3) belong to

$$\mathrm{QCoh}(\mathrm{Z}) \subset \mathrm{IndCoh}(\mathrm{Z}).$$

Recall also the 'relative' scheme of singularities defined by the following fiber product.

$$\begin{array}{ccc} \mathrm{Sing}(\mathrm{Z}/\mathrm{Y}) & \longrightarrow & \mathrm{Z} \\ \downarrow & & \downarrow \\ \mathrm{Sing}(\mathrm{Y})_{\mathrm{Z}} & \xrightarrow{\mathrm{Sing}(f)} & \mathrm{Sing}(\mathrm{Z}) \end{array}$$

One consequence of Theorem 3.1 is the following.

Corollary 3.1. *The equivalence (3.2) restricts to an equivalence*

$$\mathrm{D}(\mathrm{Sing}(\mathrm{Z}/\mathrm{Y}))^{\Rightarrow} \underset{\mathrm{D}(\mathrm{Sing}(\mathrm{Y}))^{\Rightarrow}}{\otimes} \mathrm{IndCoh}(\mathrm{Y}) \xrightarrow{\sim} \mathrm{IndCoh}_0(\mathrm{Y}^{\wedge}_{\mathrm{Z}}).$$

3.3 Proof of Theorem 3.1

3.3.1. In this subsection, we will formulate Proposition 3.4, an analogue of Theorem 3.1 for the homogeneous action, and use it to deduce Theorem 3.1 itself. The proof of Proposition 3.4 will appear in § 3.4.

3.3.2. Recall from [AG18, Proposition 3.1.8] that (3.1) restricts to an equivalence

$$\mathrm{D}(\mathrm{Z}) \underset{\mathrm{D}(\mathrm{Y})}{\otimes} \overset{\circ}{\mathrm{IndCoh}}(\mathrm{Y}) \xrightarrow{\sim} \overset{\circ}{\mathrm{IndCoh}}(\mathrm{Y}^{\wedge}_{\mathrm{Z}}) \tag{3.4}$$

onto the full subcategory

$$\overset{\circ}{\mathrm{IndCoh}}(\mathrm{Y}^{\wedge}_{\mathrm{Z}}) \subset \mathrm{IndCoh}(\mathrm{Y}^{\wedge}_{\mathrm{Z}})$$

which is right-orthogonal to $\mathrm{QCoh}(\mathrm{Y}^{\wedge}_{\mathrm{Z}})$.

Let us view (3.4) as tensored over $\mathrm{D}(\mathbf{P}\mathrm{Sing}(\mathrm{Y})_{\mathrm{Z}})$ using the homogeneous action of $\mathrm{D}(\mathbf{P}\mathrm{Sing}(\mathrm{Y}))$ on $\overset{\circ}{\mathrm{IndCoh}}(\mathrm{Y})$.

Now, the singular codifferential does not induce a morphism from the entirety of $\mathbf{P}\mathrm{Sing}(\mathrm{Y})_{\mathrm{Z}}$ to $\mathbf{P}\mathrm{Sing}(\mathrm{Z})$. Instead, we only have a morphism

$$\mathbf{P}\mathrm{Sing}(f) : \mathbf{P}\mathrm{Sing}(\mathrm{Y})_{\overset{\circ}{\mathrm{Z}}} \to \mathbf{P}\mathrm{Sing}(\mathrm{Z}) \tag{3.5}$$

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out of the open subscheme

$$\mathbf{P}\mathrm{Sing}(\mathrm{Y})^{\circ}_{\mathrm{Z}} \hookrightarrow \mathbf{P}\mathrm{Sing}(\mathrm{Y})$$

defined via the following fiber product, where $\overset{\circ}{\mathrm{Sing}}(\mathrm{Z})$ is the complement of the zero section in $\mathrm{Sing}(\mathrm{Z})$.

$$\begin{array}{ccc} \mathrm{Sing}(\mathrm{Y})^{\circ}_{\mathrm{Z}} & \longrightarrow & \overset{\circ}{\mathrm{Sing}}(\mathrm{Z}) \\ \downarrow & & \downarrow \\ \mathrm{Sing}(\mathrm{Y})_{\mathrm{Z}} & \xrightarrow{\mathrm{Sing}(f)} & \mathrm{Sing}(\mathrm{Z}) \end{array}$$

According to [AG18, Theorem 3.2.9], there is a natural identification

$$\overset{\circ}{\mathrm{IndCoh}}_0(\mathrm{Y}^{\wedge}_{\mathrm{Z}}) = \mathrm{D}(\mathbf{P}(\mathrm{Sing}(f)^{-1}(0))) \underset{\mathrm{D}(\mathbf{P}\mathrm{Sing}(\mathrm{Y})_{\mathrm{Z}})}{\otimes} \overset{\circ}{\mathrm{IndCoh}}(\mathrm{Y}^{\wedge}_{\mathrm{Z}})$$

as full subcategories of $\overset{\circ}{\mathrm{IndCoh}}(\mathrm{Y}^{\wedge}_{\mathrm{Z}})$. Combined with [AG18, Proposition 3.2.3], this shows that (3.3) restricts to an adjunction

$$\overset{\circ}{\mathrm{IndCoh}}(\mathrm{Z}) \rightleftharpoons \mathrm{D}(\mathbf{P}\mathrm{Sing}(\mathrm{Y})^{\circ}_{\mathrm{Z}}) \underset{\mathrm{D}(\mathbf{P}\mathrm{Sing}(\mathrm{Y})_{\mathrm{Z}})}{\otimes} \overset{\circ}{\mathrm{IndCoh}}(\mathrm{Y}^{\wedge}_{\mathrm{Z}}). \tag{3.6}$$

Both categories appearing in this adjunction are naturally tensored over $\mathrm{D}(\mathbf{P}\mathrm{Sing}(\mathrm{Z}))$; the left-hand side via the homogeneous action and the right-hand side via the morphism (3.5). The analogue of Theorem 3.1 for the homogeneous action is then the following.

Proposition 3.4. *The adjunction (3.6) is* $\mathrm{D}(\mathbf{P}\mathrm{Sing}(\mathrm{Z}))$*-linear.*

Proof of Theorem 3.1, assuming Proposition 3.4. Consider the adjunction

$$\begin{aligned} \iota_* \otimes \mathbf{1} : \mathrm{IndCoh}(\mathrm{Z}) \otimes \mathrm{QCoh}(\mathbf{G}_m)^{\Leftarrow} \\ \rightleftharpoons \mathrm{IndCoh}(\mathrm{Y}^{\wedge}_{\mathrm{Z}}) \otimes \mathrm{QCoh}(\mathbf{G}_m)^{\Leftarrow} : \iota^! \otimes \mathbf{1}, \end{aligned} \tag{3.7}$$

taking place in

$$\mathrm{QCoh}(\mathbf{G}_m)^{\Leftarrow}\textbf{-mod}(\mathbf{G}_m\textbf{-mod}).$$

Both categories appearing in this adjunction are equipped with lifts to

$$(\mathrm{D}(\mathrm{Sing}(\mathrm{Z})) \otimes \mathrm{QCoh}(\mathbf{G}_m)^{\Leftarrow})\textbf{-mod}(\mathbf{G}_m\textbf{-mod}),$$

and we are claiming that the adjunction itself lifts to this category. Note that Proposition 2.3 implies that this is a property and not additional structure, and that it suffices to check that (3.7) is $\mathrm{D}(\mathrm{Sing}(\mathrm{Z}))$-linear as an adjunction of plain dg categories.

Consider the adjunction (3.6) for the map

$$\mathrm{Z} \times \Omega\mathbf{A}^1 \to \mathrm{Y} \times \Omega\mathbf{A}^1.$$

By Proposition 3.4, it respects the homogeneous action of

$$\mathrm{D}(\mathbf{P}(\mathrm{Sing}(\mathrm{Z}) \oplus \mathbf{1})),$$

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so after restriction along the open embedding (2.7), we obtain that the adjunction

$$\mathrm{IndCoh}(\mathrm{Z}) \otimes \mathring{\mathrm{IndCoh}}(\Omega \mathbf{A}^1) \rightleftharpoons \mathrm{IndCoh}(\mathrm{Y}^{\wedge}_{\mathrm{Z}}) \otimes \mathring{\mathrm{IndCoh}}(\Omega \mathbf{A}^1)$$

is linear over $\mathrm{D}(\mathrm{Sing}(\mathrm{Z}))$.[7] □

3.4 Proof of Proposition 3.4

3.4.1. Let Y be a scheme and let $\mathcal{C}$ be a category tensored over D(Y). So we are in the framework of § 2.4, but in the absolute case. In particular, for every test affine scheme S mapping to Y, the colocalization (2.10) becomes a QCoh(S)-linear colocalization

$$\mathcal{C}_{\mathrm{S}} = \mathrm{QCoh}(\mathrm{S}) \underset{\mathrm{D}(\mathrm{Y})}{\otimes} \mathcal{C} \rightleftharpoons \mathrm{QCoh}(\mathrm{S}) \otimes \mathcal{C}. \tag{3.8}$$

Proposition 3.5. *Suppose that* S *is eventually coconnective. Let* c *be an object of* $\mathrm{QCoh}(\mathrm{S}) \otimes \mathcal{C}$. *Then the following are equivalent:*

(a) c *belongs to* $\mathcal{C}_{\mathrm{S}}$;

(b) *for every geometric point* $i : s \to \mathrm{S}$, *the functor*

$$i^* \otimes \mathbf{1} : \mathrm{QCoh}(\mathrm{S}) \otimes \mathcal{C} \to \mathrm{QCoh}(s) \otimes \mathcal{C}$$

takes c *into* $\mathcal{C}_s$.

Proof of Proposition 3.5. Note that c belongs to $\mathcal{C}_{\mathrm{S}}$ if and only if the counit of (3.8) is an isomorphism when evaluated on c. Therefore, the result follows from the fact that the collection of functors $\iota^* \otimes \mathbf{1}$ appearing in (b) are jointly conservative (for a proof of this assertion, see [AGK+20, Lemma 21.4.6]). □

Corollary 3.2. *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be categories tensored over* D(Y), *and let*

$$\mathrm{F} : \mathcal{C} \to \mathcal{D}$$

be a functor of plain dg categories. The following are equivalent:

(a) F *is* D(Y)*-linear;*

(b) *for every geometric point* $\eta \to \mathrm{Y}$, *the functor*

$$\mathbf{1} \otimes \mathrm{F} : \mathrm{QCoh}(\eta) \otimes \mathcal{C} \to \mathrm{QCoh}(\eta) \otimes \mathcal{D}$$

takes $\mathcal{C}_\eta$ *into* $\mathcal{D}_\eta$.

Proof of Corollary 3.2. This follows from Proposition 3.5 and the fact that Y_{dR} is right Kan extended from a functor on classical schemes. □

Proof of Proposition 3.4. Let η be an arbitrary geometric point of $\mathbf{P}\mathrm{Sing}(\mathrm{Z})$. By Corollary 3.2, it suffices to check that the adjunction

$$\mathbf{1}_{\mathrm{QCoh}(\eta)} \otimes (3.6)$$

restricts to an adjunction

$$\begin{aligned}&(\mathrm{QCoh}(\eta) \otimes \mathring{\mathrm{IndCoh}}(\mathrm{Z}))_\eta \\ &\quad \rightleftharpoons (\mathrm{QCoh}(\eta) \otimes \mathrm{D}(\mathbf{P}\mathrm{Sing}(\mathrm{Y})^{\circ}_{\mathrm{Z}}) \underset{\mathrm{D}(\mathbf{P}\mathrm{Sing}(\mathrm{Y})_{\mathrm{Z}})}{\otimes} \mathring{\mathrm{IndCoh}}(\mathrm{Y}^{\wedge}_{\mathrm{Z}}))_\eta.\end{aligned}$$

[7] Let us point out that the open subset $\mathrm{Sing}(\mathrm{Y}) \subset \mathbf{P}(\mathrm{Sing}(\mathrm{Y}) \oplus \mathbf{1})$ is always contained in $\mathbf{P}\mathrm{Sing}(\mathrm{Y} \times \Omega \mathbf{A}^1)_{\mathrm{Z} \times \mathring{\Omega} \mathbf{A}^1}$.

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After extending the base field, we may assume that η is a closed point. Then it remains to check that (3.6) restricts to an adjunction

$$\begin{aligned}&\mathrm{Vect}\underset{\mathrm{D}(\mathbf{P}\mathrm{Sing}(\mathrm{Z}))}{\otimes}\overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(\mathrm{Z})\\&\rightleftharpoons\mathrm{Vect}\underset{\mathrm{D}(\mathbf{P}\mathrm{Sing}(\mathrm{Z}))}{\otimes}\mathrm{D}(\mathbf{P}\mathrm{Sing}(\mathrm{Y})\underset{\mathrm{Z}}{\overset{\circ}{}})\underset{\mathrm{D}(\mathbf{P}\mathrm{Sing}(\mathrm{Y})_{\mathrm{Z}})}{\otimes}\overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(\mathrm{Y}^{\wedge}_{\mathrm{Z}}).\end{aligned}$$

This follows from [AG18, Theorem 3.2.9] and [AG18, Proposition 3.2.3]. □

3.5 Relation to the action of Arinkin–Gaitsgory

3.5.1. To formulate the backwards compatibility between the inhomogeneous and homogeneous actions, consider the full subcategory

$$\mathrm{D}(\overset{\circ}{\mathrm{Sing}}(\mathrm{Z}))^{\Rightarrow}\underset{\mathrm{D}(\mathrm{Sing}(\mathrm{Z}))^{\Rightarrow}}{\otimes}\mathrm{IndCoh}(\mathrm{Z})\subset\mathrm{IndCoh}(\mathrm{Z}).$$

By Proposition 3.2, it is equal to

$$\overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(\mathrm{Z})\subset\mathrm{IndCoh}(\mathrm{Z}).$$

Therefore, we obtain an action of $\mathrm{D}(\overset{\circ}{\mathrm{Sing}}(\mathrm{Z}))^{\Rightarrow}$ on $\overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(\mathrm{Z})$, and we may consider its restriction along the symmetric monoidal functor[8]

$$\mathrm{D}(\mathbf{P}\mathrm{Sing}(\mathrm{Z}))\to\mathrm{D}(\overset{\circ}{\mathrm{Sing}}(\mathrm{Z}))^{\mathbf{G}_m}\simeq(\mathrm{D}(\overset{\circ}{\mathrm{Sing}}(\mathrm{Z}))^{\Rightarrow})^{\mathbf{G}_m}\to\mathrm{D}(\overset{\circ}{\mathrm{Sing}}(\mathrm{Z}))^{\Rightarrow}.$$

We claim the following.

Proposition 3.6. *This action agrees with the homogeneous action.*

Proof of Proposition 3.6. Consider the action of $\mathrm{D}(\overset{\circ}{\mathrm{Sing}}(\mathrm{Z}))$ on the full subcategory

$$\begin{aligned}&\mathrm{D}(\overset{\circ}{\mathrm{Sing}}(\mathrm{Z}))\underset{\mathrm{D}(\mathrm{Sing}(\mathrm{Z}))}{\otimes}(\mathrm{IndCoh}(\mathrm{Z})\otimes\overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(\Omega\mathbf{A}^1))\\&\subset\mathrm{IndCoh}(\mathrm{Z})\otimes\overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(\Omega\mathbf{A}^1).\end{aligned}$$

After taking $\mathbf{G}_m$-invariants, we obtain an action of $\mathrm{D}(\overset{\circ}{\mathrm{Sing}}(\mathrm{Z}))^{\mathbf{G}_m}$ on

$$\begin{aligned}&(\overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(\mathrm{Z})\otimes\overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(\Omega\mathbf{A}^1))^{\mathbf{G}_m}\\&\simeq(\overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(\mathrm{Z})\otimes\mathrm{QCoh}(\mathbf{G}_m)^{\Leftarrow})^{\mathbf{G}_m}\simeq\overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(\mathrm{Z}).\end{aligned}\tag{3.9}$$

This is the action obtained by restricting the action of $\mathrm{D}(\overset{\circ}{\mathrm{Sing}}(\mathrm{Z}))^{\Rightarrow}$ on $\overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(\mathrm{Z})$ along the functor

$$\mathrm{D}(\overset{\circ}{\mathrm{Sing}}(\mathrm{Z}))^{\mathbf{G}_m}\simeq(\mathrm{D}(\overset{\circ}{\mathrm{Sing}}(\mathrm{Z}))^{\Rightarrow})^{\mathbf{G}_m}\to\mathrm{D}(\overset{\circ}{\mathrm{Sing}}(\mathrm{Z}))^{\Rightarrow}.$$

Therefore, it suffices to show that the functor

$$\begin{aligned}&\overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(\mathrm{Z})\otimes\overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(\Omega\mathbf{A}^1)\\&\to(\overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(\mathrm{Z})\otimes\overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(\Omega\mathbf{A}^1))_{\mathbf{G}_m}\\&\simeq(\overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(\mathrm{Z})\otimes\overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(\Omega\mathbf{A}^1))^{\mathbf{G}_m}\xrightarrow{(3.9)}\overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(\mathrm{Z})\end{aligned}\tag{3.10}$$

[8] We remind the reader that in this paper, the notation $(-)^{\mathbf{G}_m}$ will always refer to *weak* $\mathbf{G}_m$-invariants.

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intertwines the action of $\mathrm{D}(\overset{\circ}{\mathrm{Sing}}(\mathrm{Z}))$ on its source with the homogeneous action of $\mathrm{D}(\mathbf{P}\mathrm{Sing}(\mathrm{Z}))$ on its target, via the quotient map

$$\overset{\circ}{\mathrm{Sing}}(\mathrm{Z}) \to \mathbf{P}\mathrm{Sing}(\mathrm{Z}). \tag{3.11}$$

Note that the functor

$$\mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(\Omega\mathbf{A}^1) \to \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(\Omega\mathbf{A}^1)_{\mathbf{G}_m} \xrightarrow{\sim} \mathrm{Vect}$$

identifies with the composite

$$\mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(\Omega\mathbf{A}^1) \hookrightarrow \mathrm{IndCoh}(\Omega\mathbf{A}^1) \xrightarrow{i^!} \mathrm{Vect},$$

where $i : \mathrm{pt} \to \Omega\mathbf{A}^1$ is the inclusion of zero into the vector scheme $\Omega\mathbf{A}^1$. Therefore, we may reinterpret (3.10) as the composite

$$\mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(\mathrm{Z}) \otimes \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(\Omega\mathbf{A}^1) \hookrightarrow \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(\mathrm{Z} \times \Omega\mathbf{A}^1) \xrightarrow{(\mathbf{1}\times i)^!} \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(\mathrm{Z}).$$

Now observe that i is a nil-isomorphism, so the natural map

$$(\mathrm{Z} \times \Omega\mathbf{A}^1)^{\wedge}_{\mathrm{Z}} \to \mathrm{Z} \times \Omega\mathbf{A}^1$$

is an isomorphism. Therefore, Proposition 3.4 implies that $(\mathbf{1} \times i)^!$ intertwines the action of $\mathrm{D}(\mathbf{P}\mathrm{Sing}(\mathrm{Z} \times \Omega\mathbf{A}^1)_{\overset{\circ}{\mathrm{Z}}})$ on $\mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(\mathrm{Z} \times \Omega\mathbf{A}^1)$ with the action of $\mathrm{D}(\mathbf{P}\mathrm{Sing}(\mathrm{Z}))$ on $\mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(\mathrm{Z})$, via the map

$$\mathbf{P}\mathrm{Sing}(\mathbf{1} \times i) : \mathbf{P}\mathrm{Sing}(\mathrm{Z} \times \Omega\mathbf{A}^1)_{\overset{\circ}{\mathrm{Z}}} \to \mathbf{P}\mathrm{Sing}(\mathrm{Z}).$$

The result follows because the restriction of $\mathbf{P}\mathrm{Sing}(\mathbf{1} \times i)$ to the open subscheme

$$\overset{\circ}{\mathrm{Sing}}(\mathrm{Z}) \subset \mathbf{P}\mathrm{Sing}(\mathrm{Z} \times \Omega\mathbf{A}^1)_{\overset{\circ}{\mathrm{Z}}}$$

is the quotient map (3.11). □

Proposition 3.7. *For every closed conical subset $\mathcal{N} \subset \mathrm{Sing}(\mathrm{Z})$, the full subcategory*

$$\mathrm{D}(\mathcal{N})^{\Rightarrow} \underset{\mathrm{D}(\mathrm{Sing}(\mathrm{Z}))^{\Rightarrow}}{\otimes} \mathrm{IndCoh}(\mathrm{Z}) \subset \mathrm{IndCoh}(\mathrm{Z})$$

is equal to

$$\mathrm{IndCoh}_{\mathcal{N}}(\mathrm{Z}) \subset \mathrm{IndCoh}(\mathrm{Z}).$$

Proof of Proposition 3.7. By Proposition 3.2, it suffices to show that the full subcategory

$$\mathrm{D}(\mathbf{P}\mathcal{N})^{\Rightarrow} \underset{\mathrm{D}(\mathbf{P}\mathrm{Sing}(\mathrm{Z}))^{\Rightarrow}}{\otimes} \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(\mathrm{Z}) \subset \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(\mathrm{Z}) \tag{3.12}$$

is equal to

$$\mathrm{Ind}\overset{\circ}{\mathrm{Coh}}_{\mathcal{N}}(\mathrm{Z}) \subset \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(\mathrm{Z}). \tag{3.13}$$

Although the tensor product in (3.12) is formed using the inhomogeneous action, Proposition 3.6 implies that we may replace it with the homogeneous action. Now, the equality of (3.12) and (3.13) is [AG18, Theorem-Construction 1.4.2(a)]. □

3.6 Case of a global complete intersection

3.6.1. Suppose that Z has been presented as a global complete intersection; that is, it has been written as a fiber product where U is smooth and $\mathrm{pt} \to \mathrm{V}$ is the inclusion of the zero vector

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into a vector space.

$$\begin{array}{ccc} Z & \longrightarrow & U \\ \downarrow & & \downarrow \\ pt & \longrightarrow & V \end{array} \tag{3.14}$$

In this situation, it is possible to give a description of the action of $D(\mathrm{Sing}(Z))^{\Rightarrow}$ on $\mathrm{IndCoh}(Z)$ in the spirit of [AG15, § 5.4].

Construction 3.1. For every choice of presentation (3.14), one obtains an action of $D(U \times V^{\vee})^{\Rightarrow}$ on $\mathrm{IndCoh}(Z)$.

3.6.2. Recall that we may regard $Z \times_U Z$ as a groupoid over Z, so that

$$\mathrm{IndCoh}(Z \times_U Z)$$

acts on $\mathrm{IndCoh}(Z)$ by convolution.

If we let f denote the given map from U to V, then we obtain a Cartesian diagram where the upper horizontal arrow identifies with the given map $Z \to U$.

$$\begin{array}{ccc} Z & \longrightarrow & U \\ \downarrow & & \downarrow{\scriptstyle \Gamma_f} \\ U & \xrightarrow[\Gamma_0]{} & U \times V \end{array}$$

We therefore obtain a map of groupoids

$$\epsilon : Z \times_U Z \to U \times_{(U \times V)} U \simeq U \times \Omega V. \tag{3.15}$$

To obtain Construction 3.1, we precompose the convolution action of $\mathrm{IndCoh}(Z \times_U Z)$ on $\mathrm{IndCoh}(Z)$ with the monoidal functor

$$\begin{aligned} D(U) \otimes D(V^{\vee})^{\Rightarrow} &\xrightarrow{\mathrm{oblv} \otimes \mathrm{oblv}^{\Rightarrow}} \mathrm{IndCoh}(U) \otimes \mathrm{QCoh}(V^{\vee})^{\Rightarrow} \\ &\simeq \mathrm{IndCoh}(U) \otimes \mathrm{IndCoh}(\Omega V) \xrightarrow{\epsilon^!} \mathrm{IndCoh}(Z \times_U Z). \end{aligned} \tag{3.16}$$

3.6.3. Since U is smooth, the morphism

$$\mathbf{L}_Z \to \mathbf{L}_{Z/U} \simeq \mathcal{O}_Z \otimes V^{\vee}[1]$$

induces a closed embedding

$$\delta : \mathrm{Sing}(Z) \hookrightarrow Z \times V^{\vee} \hookrightarrow U \times V^{\vee}. \tag{3.17}$$

We claim the following.

Proposition 3.8. *The action of Construction 3.1 agrees with composition of the inhomogeneous action with the monoidal colocalization*

$$(\delta^!)^{\Rightarrow} : D(U \times V^{\vee})^{\Rightarrow} \to D(\mathrm{Sing}(Z))^{\Rightarrow}.$$

3.6.4 *A tautology.* Let $\mathcal{A}$ be a monoidal category for which the multiplication map

$$\mathrm{mult} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$$

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admits a fully faithful $(\mathcal{A},\mathcal{A})$-bilinear left adjoint. According to Proposition 2.2, this implies that the forgetful functor

$$\mathcal{A}\textbf{-mod}\to\textbf{dgCat}$$

is locally fully faithful.

Suppose now that a category $\mathcal{M}$ has been given two actions of $\mathcal{A}$, to be denoted

$$\mathrm{act}_1:\mathcal{A}\otimes\mathcal{M}\to\mathcal{M}\quad \mathrm{act}_2:\mathcal{A}\otimes\mathcal{M}\to\mathcal{M}.$$

Tensoring up the $(\mathrm{mult}^{\mathrm{L}},\mathrm{mult})$ adjunction shows that act_1 and act_2 admit fully faithful $\mathcal{A}$-linear left adjoints realizing $\mathcal{M}$ as a full subcategory of $\mathcal{A}\otimes\mathcal{M}$ stable under the action of $\mathcal{A}$. Therefore, the two actions of $\mathcal{A}$ on $\mathcal{M}$ coincide if and only if act_1 and act_2 are isomorphic as functors of plain dg categories.

We claim that act_1 and act_2 are isomorphic as functors once act_1 is $\mathcal{A}$-linear for the tautological action of $\mathcal{A}$ on $\mathcal{A}\otimes\mathcal{M}$ and the second action of $\mathcal{A}$ on $\mathcal{M}$. Indeed, this follows from precomposing the following commutative diagram with the map

$$\mathbf{1}\otimes\mathrm{unit}\otimes\mathbf{1}:\mathcal{A}\otimes\mathcal{M}\to\mathcal{A}\otimes\mathcal{A}\otimes\mathcal{M}.$$

$$\begin{array}{ccc}\mathcal{A}\otimes\mathcal{A}\otimes\mathcal{M} & \xrightarrow{\mathbf{1}\otimes\mathrm{act}_1} & \mathcal{A}\otimes\mathcal{M}\\ \downarrow{\scriptstyle \mathrm{mult}\otimes\mathbf{1}} & & \downarrow{\scriptstyle \mathrm{act}_2}\\ \mathcal{A}\otimes\mathcal{M} & \xrightarrow{\mathrm{act}_1} & \mathcal{M}\end{array}$$

Proof of Proposition 3.8. We apply the setup of § 3.6.4 to the data

$$\mathcal{A}=\mathrm{D}(\mathrm{U}\times\mathrm{V}^\vee)^{\Rightarrow}\quad \mathcal{M}=\mathrm{IndCoh}(\mathrm{Z}),$$

with act_1 being the action of Construction 3.1 and act_2 being the composite of the inhomogeneous action with $(\delta^!)^{\Rightarrow}$.

To check that the action map

$$\mathrm{act}_1:\mathcal{A}\otimes\mathcal{M}\to\mathcal{M}$$

is $\mathrm{D}(\mathrm{U}\times\mathrm{V}^\vee)^{\Rightarrow}$-linear, it suffices to check separately that it is linear for $\mathrm{D}(\mathrm{U})$ and for $\mathrm{D}(\mathrm{V}^\vee)^{\Rightarrow}$.

Since the action of Construction 3.1 is $\mathrm{D}(\mathrm{U})$-linear, the linearity of act_1 for $\mathrm{D}(\mathrm{U})$ follows from Proposition 3.1. Therefore, it remains to show that act_1 is $\mathrm{D}(\mathrm{V}^\vee)^{\Rightarrow}$-linear.

Recall from [AG15, § 5.3.1] that the group scheme $\Omega\mathrm{V}$ acts on Z, and there is an isomorphism $\mathrm{Z}\times\Omega\mathrm{V}\simeq\mathrm{Z}\times_{\mathrm{U}}\mathrm{Z}$ making the following diagram commute.

$$\begin{array}{ccc}\mathrm{Z}\times\Omega\mathrm{V} & \xrightarrow{\sim} & \mathrm{Z}\times_{\mathrm{U}}\mathrm{Z}\\ & {\scriptstyle \mathrm{act}}\searrow \quad \swarrow{\scriptstyle \pi_2} & \\ & \mathrm{Z} & \end{array}$$

In these terms, the action of $\mathrm{D}(\mathrm{V}^\vee)^{\Rightarrow}$ on $\mathrm{IndCoh}(\mathrm{Z})$ from Construction 3.1 identifies with the composite

$$\mathrm{D}(\mathrm{V}^\vee)^{\Rightarrow}\otimes\mathrm{IndCoh}(\mathrm{Z})\xrightarrow{\mathrm{oblv}\otimes\mathbf{1}}\mathrm{IndCoh}(\Omega\mathrm{V})\otimes\mathrm{IndCoh}(\mathrm{Z})\xrightarrow{\mathrm{act}_*}\mathrm{IndCoh}(\mathrm{Z}).$$

We claim that the functor

$$\mathrm{act}_*:\mathrm{IndCoh}(\mathrm{Z}\times\Omega\mathrm{V})\to\mathrm{IndCoh}(\mathrm{Z})\tag{3.18}$$

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is already $\mathrm{D}(\mathrm{V}^\vee)^{\Rightarrow}$-linear in the appropriate sense. It is enough to show that (3.18) intertwines the action of $\mathrm{D}(\mathrm{V}^\vee)^{\Rightarrow} \otimes \mathrm{D}(\mathrm{V}^\vee)^{\Rightarrow}$ on its source with the action of $\mathrm{D}(\mathrm{V}^\vee)^{\Rightarrow}$ on its target, via the map

$$\Delta : \mathrm{V}^\vee \to \mathrm{V}^\vee \times \mathrm{V}^\vee.$$

Indeed, this follows from Theorem 3.1 for the map act, together with the following commutative diagram.

$$\begin{array}{ccc} \mathrm{Sing}(\mathrm{Z}) & \xrightarrow{\mathrm{Sing}(\mathrm{act})} & \mathrm{Sing}(\mathrm{Z}) \times \mathrm{V}^\vee \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta \times \mathbf{1}} \\ \mathrm{V}^\vee & \xrightarrow{\Delta} & \mathrm{V}^\vee \times \mathrm{V}^\vee \end{array}$$

□

4. Morita equivalence

In this section we formulate and prove the second main result of our paper, Theorem 4.1, which says that $\mathrm{IndCoh}(\mathrm{Z})$ defines a Morita equivalence between $\mathrm{D}(\mathrm{Sing}(\mathrm{Z}))^{\Rightarrow}$ and the monoidal category $\mathbf{H}(\mathrm{Z})$.

We then study the interaction of this equivalence with a morphism of quasi-smooth schemes $\mathrm{Z} \to \mathrm{Y}$.

The outline of this section is as follows.

- In § 4.1, we describe how the action of $\mathrm{D}(\mathrm{Sing}(\mathrm{Z}))^{\Rightarrow}$ on $\mathrm{IndCoh}(\mathrm{Z})$ behaves under duality, and as a consequence we show that $\mathrm{IndCoh}(\mathrm{Z})$ upgrades to a $(\mathrm{D}(\mathrm{Sing}(\mathrm{Z}))^{\Rightarrow}, \mathbf{H}(\mathrm{Z}))$-bimodule.
- In § 4.2, we give a new proof of a theorem of Ben-Zvi, Gunningham, and Orem on Morita bimodules. Their result [BGO20, Theorem 1.3] appears as our Proposition 4.2 and Corollary 4.1.
- In § 4.3, we show that in our situation the hypotheses of Proposition 4.2 are satisfied.
- In § 4.4, we show that the additional hypothesis of Corollary 4.1 is satisfied, proving Theorem 4.1.
- In § 4.5, we calculate the image of the transfer bimodule $\mathbf{H}_{\mathrm{Z}\to\mathrm{Y}}$ under our Morita equivalence. (Recall from [Ber19] that $\mathbf{H}_{\mathrm{Z}\to\mathrm{Y}}$ characterizes the functoriality of the category $\mathbf{H}(\mathrm{Z})$ as Z varies.)
- In § 4.6, we calculate the image of the composition map

$$\mathbf{H}_{\mathrm{Z}\leftarrow\mathrm{Y}} \underset{\mathbf{H}(\mathrm{Y})}{\otimes} \mathbf{H}_{\mathrm{Y}\leftarrow\mathrm{X}} \to \mathbf{H}_{\mathrm{Z}\leftarrow\mathrm{X}}$$

under our Morita equivalence.

4.1 Bimodule structure on the category of ind-coherent sheaves

4.1.1. Let $\mathcal{M}$ be a module over

$$\mathcal{A} = \mathrm{D}(\mathrm{Sing}(\mathrm{Z}))^{\Rightarrow},$$

and suppose that $\mathcal{M}$ admits a dual $\mathcal{M}^\vee$ as a plain dg category. By [Che23, § A.3.9], the composite

$$\mathrm{Vect} \to \mathcal{M} \otimes \mathcal{M}^\vee \to \mathcal{M} \otimes_{\mathcal{A}} \mathcal{M}^\vee$$

forms the unit of an $\mathcal{A}$-module duality.[9]

[9]Strictly speaking, [Che23, § A.3.9] only considers categories of the form $\mathrm{D}(\mathrm{Z})$, without shearing. However, the proof in [Che23, § A.3.9] only relies on the category-theoretic properties of $\mathrm{D}(\mathrm{Z})$ as an object of dgCat. Since

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The counit of the duality

$$\mathrm{ev} : \mathcal{M}^\vee \otimes \mathcal{M} \to \mathcal{A}$$

is given by the composite

$$\mathcal{M}^\vee \otimes \mathcal{M} \xrightarrow{\mathbf{1}\otimes \mathrm{coact}} \mathcal{M}^\vee \otimes \mathcal{M} \otimes \mathcal{A} \to \mathcal{A},$$

where coact is the morphism

$$\mathrm{coact} : \mathcal{M} \to \mathcal{M} \otimes \mathcal{A}^\vee \xrightarrow{\sim} \mathcal{M} \otimes \mathcal{A}$$

obtained by transposing the action map and applying Verdier duality.

We observe, for future reference, that the functor

$$\mathcal{A} \simeq \mathcal{A}^\vee \xrightarrow{\mathrm{ev}^\vee} (\mathcal{M}^\vee \otimes \mathcal{M})^\vee \simeq \mathcal{M}^\vee \otimes \mathcal{M} \simeq \mathrm{End}(\mathcal{M})$$

is the map of $(\mathcal{A}, \mathcal{A})$-bimodules classifying the action of $\mathcal{A}$ on $\mathcal{M}$.

4.1.2. Now let us consider the module $\mathcal{M} = \mathrm{IndCoh}(\mathrm{Z})$. In this case, Serre duality provides a canonical equivalence of dg categories

$$\mathrm{IndCoh}(\mathrm{Z})^\vee \simeq \mathrm{IndCoh}(\mathrm{Z}). \tag{4.1}$$

An irksome feature of the theory is that this equivalence *does not* intertwine the $\mathcal{A}$-module structure on the left-hand side from § 4.1.1 with the pre-existing $\mathcal{A}$-module structure on the right-hand side.

To correct this, let

$$\mathrm{IndCoh}(\mathrm{Z})^\tau$$

denote the category $\mathrm{IndCoh}(\mathrm{Z})$ equipped with the action of $\mathcal{A}$ obtained by precomposing the standard action with the involution of $\mathcal{A}$ induced by the negation map:

$$\tau : \mathrm{Sing}(\mathrm{Z}) \to \mathrm{Sing}(\mathrm{Z}) \quad \xi \mapsto -\xi.$$

Proposition 4.1. *The equivalence of plain dg categories (4.1) lifts to an equivalence of* $\mathrm{D}(\mathrm{Sing}(\mathrm{Z}))^{\Rightarrow}$*-modules*

$$\mathrm{IndCoh}(\mathrm{Z})^\vee \simeq \mathrm{IndCoh}(\mathrm{Z})^\tau.$$

Proof of Proposition 4.1. We need to lift the map of left $\mathrm{D}(\mathrm{Sing}(\mathrm{Z}))^{\Rightarrow}$-modules

$$\begin{aligned} \mathrm{D}(\mathrm{Sing}(\mathrm{Z}))^{\Rightarrow} &\xrightarrow{\mathrm{coev}^\vee} \mathrm{IndCoh}(\mathrm{Z}) \otimes \mathrm{IndCoh}(\mathrm{Z})^\vee \\ &\simeq \mathrm{IndCoh}(\mathrm{Z}) \otimes \mathrm{IndCoh}(\mathrm{Z})^\tau \end{aligned} \tag{4.2}$$

to a map of bimodules. It suffices to show that the functor (4.2) factors through the full subcategory

$$\begin{aligned} &\mathrm{D}(\mathrm{Sing}(\mathrm{Z}))^{\Rightarrow} \underset{\mathrm{D}(\mathrm{Sing}(\mathrm{Z}))^{\Rightarrow} \otimes \mathrm{D}(\mathrm{Sing}(\mathrm{Z}))^{\Rightarrow}}{\otimes} (\mathrm{IndCoh}(\mathrm{Z}) \otimes \mathrm{IndCoh}(\mathrm{Z})^\tau) \\ &\quad \subset \mathrm{IndCoh}(\mathrm{Z}) \otimes \mathrm{IndCoh}(\mathrm{Z})^\tau, \end{aligned} \tag{4.3}$$

where $\mathrm{D}(\mathrm{Sing}(\mathrm{Z}))^{\Rightarrow} \otimes \mathrm{D}(\mathrm{Sing}(\mathrm{Z}))^{\Rightarrow}$ acts on $\mathrm{D}(\mathrm{Sing}(\mathrm{Z}))^{\Rightarrow}$ via the diagonal embedding

$$\mathrm{Sing}(\mathrm{Z}) \hookrightarrow \mathrm{Sing}(\mathrm{Z}) \times \mathrm{Sing}(\mathrm{Z}).$$

shearing is a symmetric-monoidal automorphism of $\mathbf{G}_m$**-mod**, the assumptions listed in [Che23, § A.3.9] are also satisfied in our situation.

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Observe that (4.2) is given by acting on the object

$$\Delta_*(\omega_Z) \in \mathrm{IndCoh}(Z \times Z),$$

and (4.3) is a submodule inclusion. Therefore, it suffices to show that $\Delta_*(\omega_Z)$ belongs to the submodule (4.3).

Let us rewrite the full subcategory (4.3) as

$$\begin{aligned} &\mathrm{D}(\mathrm{Sing}(Z))^{\Rightarrow} \underset{\mathrm{D}(\mathrm{Sing}(Z\times Z))^{\Rightarrow}}{\otimes} \mathrm{IndCoh}(Z \times Z) \\ &\quad \subset \mathrm{IndCoh}(Z \times Z), \end{aligned} \tag{4.4}$$

where $\mathrm{D}(\mathrm{Sing}(Z \times Z))^{\Rightarrow}$ now acts on $\mathrm{IndCoh}(Z \times Z)$ in the usual way and on $\mathrm{D}(\mathrm{Sing}(Z))^{\Rightarrow}$ via the closed embedding

$$\mathbf{1} \times \tau : \mathrm{Sing}(Z) \to \mathrm{Sing}(Z) \times \mathrm{Sing}(Z) \simeq \mathrm{Sing}(Z \times Z).$$

This embedding identifies with the morphism

$$\mathrm{Sing}(Z/Z \times Z) \to \mathrm{Sing}(Z \times Z),$$

so Corollary 3.1 identifies (4.4) with the full subcategory

$$\mathrm{IndCoh}_0((Z \times Z)^{\wedge}_Z) \subset \mathrm{IndCoh}(Z \times Z) \simeq \mathrm{IndCoh}(Z) \otimes \mathrm{IndCoh}(Z)^{\tau}. \qquad \square$$

4.1.3 *Porism.* Since the inclusion (4.3) identifies with

$$\mathrm{End}_{\mathrm{D}(\mathrm{Sing}(Z))^{\Rightarrow}}(\mathrm{IndCoh}(Z)) \subset \mathrm{End}(\mathrm{IndCoh}(Z)),$$

the proof of Proposition 4.1 shows that

$$\mathrm{End}_{\mathrm{D}(\mathrm{Sing}(Z))^{\Rightarrow}}(\mathrm{IndCoh}(Z)) \simeq \mathrm{IndCoh}_0((Z \times Z)^{\wedge}_Z)$$

as full monoidal subcategories of

$$\mathrm{End}(\mathrm{IndCoh}(Z)) \simeq \mathrm{IndCoh}(Z \times Z),$$

acting on the *right* on $\mathrm{IndCoh}(Z)$.

By definition,

$$\mathbf{H}(Z) = \mathrm{IndCoh}_0((Z \times Z)^{\wedge}_Z),$$

so we have upgraded $\mathrm{IndCoh}(Z)$ to a bimodule

$$\mathrm{IndCoh}(Z) \in (\mathrm{D}(\mathrm{Sing}(Z))^{\Rightarrow}, \mathbf{H}(Z))\textbf{-bimod}.$$

Theorem 4.1. *Let* Z *be a quasi-smooth scheme. The bimodule* $\mathrm{IndCoh}(Z)$ *defines a Morita equivalence*

$$\mathbf{H}(Z)\textbf{-mod} \xrightarrow{\sim} \mathrm{D}(\mathrm{Sing}(Z))^{\Rightarrow}\textbf{-mod}.$$

4.1.4. The symmetric monoidal structure on $\mathrm{D}(\mathrm{Sing}(Z))^{\Rightarrow}$ gives a canonical identification of its categories of left and right modules. We use this identification freely.

The monoidal involution

$$\sigma : \mathbf{H}(Z)^{\mathrm{rev}} \xrightarrow{\sim} \mathbf{H}(Z)$$

induced by the swap map on $Z \times Z$ gives a canonical identification between the categories of left and right modules over $\mathbf{H}(Z)$. This identification exchanges $\mathrm{IndCoh}(Z)$, viewed as a right $\mathbf{H}(Z)$-module under convolution, with $\mathrm{IndCoh}(Z)$, viewed as a left $\mathbf{H}(Z)$-module under convolution. We will use this identification freely.

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In particular, we may regard $\mathrm{IndCoh}(\mathrm{Z})^\tau$ as a bimodule

$$\mathrm{IndCoh}(\mathrm{Z})^\tau \in (\mathbf{H}(\mathrm{Z}), \mathrm{D}(\mathrm{Sing}(\mathrm{Z}))^{\Rightarrow})\textbf{-bimod}. \tag{4.5}$$

4.1.5. By definition, the equivalence of Theorem 4.1 fits into the following commutative diagrams.

$$\begin{array}{ccc} \mathbf{H}(\mathrm{Z})\textbf{-mod} & \xrightarrow{\sim} & \mathrm{D}(\mathrm{Sing}(\mathrm{Z}))^{\Rightarrow}\textbf{-mod} \\ & {\scriptstyle \mathrm{IndCoh}(\mathrm{Z})}\searrow \quad \swarrow{\scriptstyle \mathbf{oblv}} & \\ & \mathbf{dgCat} & \end{array} \qquad \begin{array}{ccc} \mathbf{H}(\mathrm{Z})\textbf{-mod} & \xrightarrow{\sim} & \mathrm{D}(\mathrm{Sing}(\mathrm{Z}))^{\Rightarrow}\textbf{-mod} \\ & {\scriptstyle \mathbf{ind}}\nwarrow \quad \nearrow{\scriptstyle \mathrm{IndCoh}(\mathrm{Z})} & \\ & \mathbf{dgCat} & \end{array}$$

After passing to left (respectively right) adjoints, we obtain the following diagrams.

$$\begin{array}{ccc} \mathbf{H}(\mathrm{Z})\textbf{-mod} & \xrightarrow{\sim} & \mathrm{D}(\mathrm{Sing}(\mathrm{Z}))^{\Rightarrow}\textbf{-mod} \\ & {\scriptstyle \mathrm{IndCoh}(\mathrm{Z})}\nwarrow \quad \nearrow{\scriptstyle \mathbf{ind}} & \\ & \mathbf{dgCat} & \end{array} \qquad \begin{array}{ccc} \mathbf{H}(\mathrm{Z})\textbf{-mod} & \xrightarrow{\quad\sim\quad} & \mathrm{D}(\mathrm{Sing}(\mathrm{Z}))^{\Rightarrow}\textbf{-mod} \\ & {\scriptstyle \mathbf{oblv}}\searrow \quad \swarrow{\scriptstyle \mathrm{IndCoh}(\mathrm{Z})^\tau} & \\ & \mathbf{dgCat} & \end{array}$$

This shows, in particular, that the inverse to the Morita equivalence of Theorem 4.1 is given by the bimodule (4.5).

4.2 Categorical framework for the argument

4.2.1. The assertion of Theorem 4.1 takes place in the context of the Morita $(\infty, 2)$-category of algebras and bimodules. For the convenience of the reader, we recall the basic features of this theory which are relevant to us.

4.2.2. Let $\mathcal{A}$ and $\mathcal{B}$ be monoidal categories, and let $\mathcal{M}$ be a $(\mathcal{B}, \mathcal{A})$-bimodule. Suppose that $\mathcal{M}$ admits a left dual $\mathcal{M}^\vee$, i.e. an $(\mathcal{A}, \mathcal{B})$-bimodule equipped with a pair of bimodule morphisms

$$\mathrm{coev} : \mathcal{B} \to \mathcal{M} \otimes_{\mathcal{A}} \mathcal{M}^\vee, \quad \mathrm{ev} : \mathcal{M}^\vee \otimes_{\mathcal{B}} \mathcal{M} \to \mathcal{A} \tag{4.6}$$

satisfying the triangle identities.

4.2.3. Suppose that the morphisms appearing in the duality data (4.6) admit continuous right adjoints, which are moreover linear for the apparent bimodule structures

$$\mathcal{B} \leftarrow \mathcal{M} \otimes_{\mathcal{A}} \mathcal{M}^\vee : \mathrm{coev}^{\mathrm{R}}, \quad \mathcal{M}^\vee \otimes_{\mathcal{B}} \mathcal{M} \leftarrow \mathcal{A} : \mathrm{ev}^{\mathrm{R}}.$$

Since passing to adjoints is functorial, these right adjoints identify $\mathcal{M}^\vee$ as a right dual of $\mathcal{M}$.

Proposition 4.2. *Suppose, in the situation of § 4.2.3, that* coev *is an equivalence. Then* ev *is fully faithful.*

Proof of Proposition 4.2. It suffices to show that the morphism

$$\mathcal{M}^\vee \otimes_{\mathcal{B}} \mathcal{M} \xrightarrow{\mathrm{ev}} \mathcal{A} \xrightarrow{\mathrm{ev}^{\mathrm{R}}} \mathcal{M}^\vee \otimes_{\mathcal{B}} \mathcal{M}$$

is isomorphic to the identity[10]

$$\mathcal{M}^\vee \otimes_{\mathcal{B}} \mathcal{M} \xrightarrow{\mathbf{1}} \mathcal{M}^\vee \otimes_{\mathcal{B}} \mathcal{M}.$$

[10]The unit of any algebra structure on the monoidal unit of a monoidal category $\mathcal{A}$ is an isomorphism. Indeed, unitality says that $\mathrm{mult} : \mathbf{1} \simeq \mathbf{1} \otimes \mathbf{1} \to \mathbf{1}$ and $\mathrm{unit} : \mathbf{1} \to \mathbf{1}$ satisfy $\mathrm{mult} \circ \mathrm{unit} = \mathrm{id}$. Since $\mathrm{End}_{\mathcal{A}}(\mathbf{1})$ is an $\mathbf{E}_2$-algebra, this implies unit and mult are mutually inverse.

Viewing the monad $\mathrm{ev}^{\mathrm{R}} \circ \mathrm{ev}$ as an algebra in the endofunctor category of $\mathcal{M}^\vee \otimes_{\mathcal{B}} \mathcal{M}$ shows that the unit of the $(\mathrm{ev}, \mathrm{ev}^{\mathrm{R}})$ adjunction is an equivalence once we know that $\mathrm{ev}^{\mathrm{R}} \circ \mathrm{ev}$ happens to be isomorphic to $\mathbf{1}$.

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After transposition, we see that it is equivalent to compare the morphisms

$$\mathbf{1} \otimes \mathrm{ev}^{\mathrm{R}} : \mathcal{M} \to \mathcal{M} \otimes_{\mathcal{A}} \mathcal{M}^{\vee} \otimes_{\mathcal{B}} \mathcal{M}$$

and

$$\mathrm{coev} \otimes \mathbf{1} : \mathcal{M} \to \mathcal{M} \otimes_{\mathcal{A}} \mathcal{M}^{\vee} \otimes_{\mathcal{B}} \mathcal{M}.$$

Since coev is an equivalence, so is $\mathrm{coev}^{\mathrm{R}}$. Therefore, it suffices to see that both morphisms are the identity after composition with $\mathrm{coev}^{\mathrm{R}} \otimes \mathbf{1}$. For the first morphism, this is a triangular identity. For the second morphism, we use $\mathrm{coev}^{\mathrm{R}} \simeq \mathrm{coev}^{-1}$. □

Note that Proposition 4.2 provides a more direct proof of [BGO20, Theorem 1.3].

Corollary 4.1. *Suppose, in the situation of § 4.2.3, that* coev *is an equivalence. Then* $\mathcal{M}$ *defines a Morita equivalence if and only if* ev^{R} *is conservative.*

4.3 Our situation fits into the framework

4.3.1. The goal of this subsection is to show that the hypothesis of Corollary 4.1 is applicable to the data

$$\mathcal{A} = \mathrm{D}(\mathrm{Sing}(\mathrm{Z}))^{\Rightarrow}, \quad \mathcal{B} = \mathbf{H}(\mathrm{Z}), \quad \mathcal{M} = \mathrm{IndCoh}(\mathrm{Z}). \tag{4.7}$$

That is:

- $\mathcal{M}$ admits a left dual as a $(\mathcal{B}, \mathcal{A})$-bimodule (Proposition 4.3);
- the coevaluation map for this duality is an equivalence (see § 4.1.3); and
- the evaluation map admits an $(\mathcal{A}, \mathcal{A})$-bilinear right adjoint (Proposition 4.4).

Proposition 4.3. *In the situation of (4.7),* $\mathcal{M}$ *admits a left dual as a* $(\mathcal{B}, \mathcal{A})$*-bimodule.*

Proof of Proposition 4.3. We need to show that the functor

$$\mathcal{A}\textbf{-mod} \to \mathcal{B}\textbf{-mod}$$

admits a left adjoint. Indeed, its composite with the forgetful functor

$$\mathrm{oblv} : \mathcal{B}\textbf{-mod} \to \mathrm{dgCat}$$

admits a left adjoint because $\mathcal{M}$ is dualizable as an $\mathcal{A}$-module, and $\mathcal{B}$**-mod** is generated under colimits by the image of ind. □

4.3.2. Note that the proof of Proposition 4.3 identifies the dg category underlying the left dual of $\mathcal{M}$ with its dual $\mathcal{M}^{\vee}$ as a plain dg category. In these terms, the resulting duality data

$$\mathrm{coev} : \mathcal{B} \to \mathcal{M} \otimes_{\mathcal{A}} \mathcal{M}^{\vee}, \quad \mathrm{ev} : \mathcal{M}^{\vee} \otimes_{\mathcal{B}} \mathcal{M} \to \mathcal{A}$$

can be described as follows.

(i) The composite

$$\mathcal{B} \xrightarrow{\mathrm{coev}} \mathcal{M} \otimes_{\mathcal{A}} \mathcal{M}^{\vee} \simeq \mathrm{End}_{\mathcal{A}}(\mathcal{M})$$

is the action map. In the case at hand, we saw in § 4.1.3 that this map in equivalence.

(ii) The composite

$$\mathcal{M}^{\vee} \otimes \mathcal{M} \to \mathcal{M}^{\vee} \otimes_{\mathcal{B}} \mathcal{M} \xrightarrow{\mathrm{ev}} \mathcal{A} \tag{4.8}$$

is the counit of the $\mathcal{A}$-module duality between $\mathcal{M}^{\vee}$ and $\mathcal{M}$.

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Proposition 4.4. *In the situation of (4.7),* ev *admits a continuous right adjoint, which is moreover* $(\mathcal{A},\mathcal{A})$*-bilinear.*

Proof of Proposition 4.4. Since $\mathcal{B}$ is rigid, the functor

$$\mu : \mathcal{M}^\vee \otimes \mathcal{M} \to \mathcal{M}^\vee \otimes_{\mathcal{B}} \mathcal{M}$$

admits a continuous right adjoint given by the tensor product

$$\mathrm{mult}^{\mathrm{R}} \otimes \mathbf{1} : \mathcal{B} \underset{\mathcal{B}\otimes\mathcal{B}^{\mathrm{rev}}}{\otimes} (\mathcal{M}^\vee \otimes \mathcal{M}) \to (\mathcal{B}\otimes\mathcal{B}^{\mathrm{rev}}) \underset{\mathcal{B}\otimes\mathcal{B}^{\mathrm{rev}}}{\otimes} (\mathcal{M}^\vee \otimes \mathcal{M}).$$

Now μ^{R} is visibly $(\mathcal{A},\mathcal{A})$-bilinear, and it is conservative because the image of μ generates its target. Therefore, it suffices to check that the composite (4.8) admits a continuous $(\mathcal{A},\mathcal{A})$-bilinear right adjoint.

Recall from § 4.1.1 that the composite (4.8) identifies with the functor dual to the action map

$$\mathcal{A} \to \mathcal{M} \otimes \mathcal{M}^\vee. \tag{4.9}$$

Therefore, it suffices to show that (4.9) admits an $(\mathcal{A},\mathcal{A})$-bilinear *left* adjoint.

After Zariski localization, we may assume that Z is presented as a global complete intersection (3.14). Recall the closed immersion (3.17) and the morphism (3.15)

$$\delta : \mathrm{Sing}(\mathrm{Z}) \hookrightarrow \mathrm{Z}\times\mathrm{V}^\vee, \quad \epsilon : \mathrm{Z}\times_{\mathrm{U}}\mathrm{Z} \to \mathrm{U}\times\Omega\mathrm{V}$$

from § 3.6.

Since the $(\delta_!,\delta^!)$ adjunction is a colocalization, it suffices to check that the composite of monoidal functors

$$\mathrm{D}(\mathrm{U}\times\mathrm{V}^\vee)^{\Rightarrow} \xrightarrow{(\delta^!)^{\Rightarrow}} \mathrm{D}(\mathrm{Sing}(\mathrm{Z}))^{\Rightarrow} \xrightarrow{(4.9)} \mathrm{IndCoh}(\mathrm{Z}\times\mathrm{Z}) \tag{4.10}$$

admits a $\mathrm{D}(\mathrm{U}\times\mathrm{V}^\vee)^{\Rightarrow}$-bilinear left adjoint.

By Proposition 3.8, we may replace (4.10) with the sequence of monoidal functors

$$\begin{aligned}\mathrm{D}(\mathrm{U})\otimes\mathrm{D}(\mathrm{V}^\vee)^{\Rightarrow} &\xrightarrow{\mathrm{oblv}\otimes\mathrm{oblv}^{\Rightarrow}} \mathrm{IndCoh}(\mathrm{U})\otimes\mathrm{QCoh}(\mathrm{V}^\vee)^{\Rightarrow}\\ &\xrightarrow{\sim} \mathrm{IndCoh}(\mathrm{U})\otimes\mathrm{IndCoh}(\Omega\mathrm{V}) \xrightarrow{\epsilon^!} \mathrm{IndCoh}(\mathrm{Z}\times_{\mathrm{U}}\mathrm{Z}) \xrightarrow{(\Delta_{\mathrm{Z/U}})_*} \mathrm{IndCoh}(\mathrm{Z}\times\mathrm{Z}),\end{aligned}$$

where the last functor is that of IndCoh $*$-pushforward along the relative diagonal

$$\Delta_{\mathrm{Z/U}} : \mathrm{Z}\times_{\mathrm{U}}\mathrm{Z} \to \mathrm{Z}\times\mathrm{Z}.$$

The left adjoint

$$\mathrm{ind}\otimes\mathrm{ind}^{\Rightarrow} : \mathrm{IndCoh}(\mathrm{U})\otimes\mathrm{QCoh}(\mathrm{V}^\vee)^{\Rightarrow} \to \mathrm{D}(\mathrm{U})\otimes\mathrm{D}(\mathrm{V}^\vee)^{\Rightarrow}$$

of $\mathrm{oblv}\otimes\mathrm{oblv}^{\Rightarrow}$ is $\mathrm{D}(\mathrm{U})\otimes\mathrm{D}(\mathrm{V}^\vee)^{\Rightarrow}$-bilinear.

Since U is smooth, the monoidal category

$$\mathrm{IndCoh}(\mathrm{U})\otimes\mathrm{QCoh}(\mathrm{V}^\vee)^{\Rightarrow} \simeq \mathrm{QCoh}(\mathrm{U})\otimes\mathrm{QCoh}(\mathrm{V}^\vee)^{\Rightarrow} \tag{4.11}$$

is rigid. This implies that any left adjoint to the monoidal functor $(\Delta_{\mathrm{Z/U}})_*\circ\epsilon^!$ is automatically bilinear for (4.11) and, hence, for $\mathrm{D}(\mathrm{U})\otimes\mathrm{D}(\mathrm{V}^\vee)^{\Rightarrow}$. Therefore, it remains to show that both $\epsilon^!$ and $(\Delta_{\mathrm{Z/U}})_*$ are left adjointable.

The morphism ϵ is proper because it is a base change of the graph

$$\mathrm{U} \to \mathrm{U}\times\mathrm{V},$$

so $\epsilon^!$ identifies with the right adjoint of ϵ_*.

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The morphism $\Delta_{Z/U}$ is quasi-smooth because it is a base change of the diagonal

$$U \to U \times U,$$

so it has finite tor amplitude and admits an IndCoh $*$-pullback. □

4.4 Proof of Theorem 4.1

4.4.1. By Corollary 4.1, it remains to check that the functor

$$\mathrm{ev}^{R} : \mathcal{A} \to \mathcal{M}^{\vee} \otimes_{\mathcal{B}} \mathcal{M}$$

is conservative. It suffices to prove that the composite

$$\mathcal{A} \xrightarrow{\mathrm{ev}^{R}} \mathcal{M}^{\vee} \otimes_{\mathcal{B}} \mathcal{M} \xrightarrow{\mu^{R}} \mathcal{M}^{\vee} \otimes \mathcal{M}$$

is conservative. As before, we may Zariski localize and assume that Z is presented as a global complete intersection.

4.4.2. The description of

$$(\mu \circ \mathrm{ev})^{R,\vee} \simeq (\mu \circ \mathrm{ev})^{\vee,L}$$

from the proof of Proposition 4.4 identifies $\mu^{R} \circ \mathrm{ev}^{R}$ with

$$\begin{aligned} D(\mathrm{Sing}(Z))^{\Rightarrow} &\xrightarrow{(\delta^{!})^{\Rightarrow,\vee}} D(U) \otimes D(V^{\vee})^{\Rightarrow} \xrightarrow{\mathrm{ind}^{\vee} \otimes (\mathrm{ind}^{\Rightarrow})^{\vee}} \mathrm{IndCoh}(U) \otimes \mathrm{QCoh}(V^{\vee})^{\Rightarrow} \\ &\xrightarrow{\sim} \mathrm{IndCoh}(U) \otimes \mathrm{IndCoh}(\Omega V) \xrightarrow{(\epsilon_{*})^{\vee}} \mathrm{IndCoh}(Z \times_{U} Z) \xrightarrow{(\Delta^{*}_{Z/U})^{\vee}} \mathrm{IndCoh}(Z \times Z). \end{aligned}$$

Now, $\Delta^{*}_{Z/U}$ differs from $\Delta^{!}_{Z/U}$ by a shifted line bundle because $\Delta_{Z/U}$ is quasi-smooth and hence Gorenstein. Therefore, it is equivalent to show that composition of the functor

$$\begin{aligned} D(\mathrm{Sing}(Z))^{\Rightarrow} &\xrightarrow{(\delta_{*})^{\Rightarrow}} D(U) \otimes D(V^{\vee})^{\Rightarrow} \xrightarrow{\mathrm{oblv} \otimes \mathrm{oblv}^{\Rightarrow}} \mathrm{IndCoh}(U) \otimes \mathrm{QCoh}(V^{\vee})^{\Rightarrow} \\ &\xrightarrow{\sim} \mathrm{IndCoh}(U \times \Omega V) \xrightarrow{\epsilon^{!}} \mathrm{IndCoh}(Z \times_{U} Z) \end{aligned} \tag{4.12}$$

with the functor of IndCoh $*$-pushforward along the map $\Delta_{Z/U}$ is conservative.

4.4.3. Both of the functors $(\delta_{*})^{\Rightarrow}$ and $\mathrm{oblv} \otimes \mathrm{oblv}^{\Rightarrow}$ are conservative, and their composition takes values in the full subcategory of $\mathrm{IndCoh}(U \times \Omega V)$ consisting of objects whose singular support is contained in the image of δ. Since the functor $\epsilon^{!}$ is conservative when restricted to objects that are set-theoretically supported on the image of ϵ, it follows that the functor (4.12) is conservative. Furthermore, [AG15, Proposition 7.1.3(a)] implies that the image of (4.12) is contained in the full subcategory consisting of objects whose singular support is contained in the image of the map

$$\mathrm{Sing}(Z) \hookrightarrow Z \times V^{\vee} \hookrightarrow \mathrm{Sing}(Z \times_{U} Z), \tag{4.13}$$

where the first arrow is the one appearing in the definition (3.17) of δ and the second is the one induced by the singular codifferential of ϵ.

By [AG15, Proposition 7.3.10], the functor of IndCoh $*$-pushforward along $\Delta_{Z/U}$ is conservative when restricted to those objects whose singular support is contained in the image of the singular codifferential

$$\mathrm{Sing}(\Delta_{Z/U}) : \mathrm{Sing}(Z \times Z)_{Z} \hookrightarrow \mathrm{Sing}(Z \times_{U} Z).$$

We observe that this locus contains the image of (4.13).

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4.5 Transfer bimodules

4.5.1. Let $f : \mathrm{Z} \to \mathrm{Y}$ be a morphism of quasi-smooth schemes. We may form the category

$$\mathbf{H}_{\mathrm{Z}\to\mathrm{Y}} = \mathrm{IndCoh}_0((\mathrm{Z}\times\mathrm{Y})^{\wedge}_{\mathrm{Z}})$$

associated to the graph of f by the procedure of § 3.2.3, regarded as an $(\mathbf{H}(\mathrm{Z}), \mathbf{H}(\mathrm{Y}))$-bimodule under convolution. According to [Ber19, Proposition 5.2.8], the bimodule $\mathbf{H}_{\mathrm{Z}\to\mathrm{Y}}$ admits a right dual (and also a left dual) canonically identified with

$$\mathbf{H}_{\mathrm{Y}\leftarrow\mathrm{Z}} = \mathrm{IndCoh}_0((\mathrm{Y}\times\mathrm{Z})^{\wedge}_{\mathrm{Z}}),$$

viewed as an $(\mathbf{H}(\mathrm{Y}), \mathbf{H}(\mathrm{Z}))$-bimodule.[11]

In particular, we obtain a pair of adjoint functors

$$\mathbf{H}_{\mathrm{Z}\to\mathrm{Y}} : \mathbf{H}(\mathrm{Y})\textbf{-mod} \rightleftharpoons \mathbf{H}(\mathrm{Z})\textbf{-mod} : \mathbf{H}_{\mathrm{Y}\leftarrow\mathrm{Z}}.$$

The purpose of this subsection is to describe the adjunction

$$\mathrm{D}(\mathrm{Sing}(\mathrm{Y}))^{\Rightarrow} \rightleftharpoons \mathrm{D}(\mathrm{Sing}(\mathrm{Z}))^{\Rightarrow}$$

corresponding to this one under the Morita equivalence of Theorem 4.1.

4.5.2. Consider the following correspondence induced by the singular codifferential of f.

$$\begin{array}{ccc} \mathrm{Sing}(\mathrm{Y})_{\mathrm{Z}} & \xrightarrow{\mathrm{Sing}(f)} & \mathrm{Sing}(\mathrm{Z}) \\ \downarrow & & \\ \mathrm{Sing}(\mathrm{Y}) & & \end{array} \tag{4.14}$$

It defines a bimodule

$$\mathrm{D}(\mathrm{Sing}(\mathrm{Y})_{\mathrm{Z}})^{\Rightarrow} \in (\mathrm{D}(\mathrm{Sing}(\mathrm{Z}))^{\Rightarrow}, \mathrm{D}(\mathrm{Sing}(\mathrm{Y}))^{\Rightarrow})\textbf{-bimod}. \tag{4.15}$$

We claim the following.

Construction 4.1. The following diagram commutes.

$$\begin{array}{ccc} \mathbf{H}(\mathrm{Y})\textbf{-mod} & \xrightarrow{\mathrm{IndCoh}(\mathrm{Y})} & \mathrm{D}(\mathrm{Sing}(\mathrm{Y}))^{\Rightarrow}\textbf{-mod} \\ {\scriptstyle \mathbf{H}_{\mathrm{Z}\to\mathrm{Y}}}\downarrow & & \downarrow{\scriptstyle \mathrm{D}(\mathrm{Sing}(\mathrm{Y})_{\mathrm{Z}})^{\Rightarrow}} \\ \mathbf{H}(\mathrm{Z})\textbf{-mod} & \xrightarrow[\mathrm{IndCoh}(\mathrm{Z})]{} & \mathrm{D}(\mathrm{Sing}(\mathrm{Z}))^{\Rightarrow}\textbf{-mod} \end{array} \tag{4.16}$$

4.5.3. Note that we may also view the correspondence (4.14) as defining a bimodule

$$\mathrm{D}(\mathrm{Sing}(\mathrm{Y})_{\mathrm{Z}})^{\Rightarrow} \in (\mathrm{D}(\mathrm{Sing}(\mathrm{Y}))^{\Rightarrow}, \mathrm{D}(\mathrm{Sing}(\mathrm{Z}))^{\Rightarrow})\textbf{-bimod}. \tag{4.17}$$

[11] Strictly speaking, the material in this subsection is logically independent of this citation, and the results of this section can be used to give a new proof of [Ber19, Proposition 5.2.8] for quasi-smooth schemes. Namely, one can recover the ambidextrous duality between $\mathbf{H}_{\mathrm{Z}\to\mathrm{Y}}$ and $\mathbf{H}_{\mathrm{Y}\leftarrow\mathrm{Z}}$ from Construction 4.1 and the ambidextrous duality between (4.15) and (4.17).

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This bimodule is canonically right dual (and also left dual) to (4.15), so after taking right adjoints of the vertical arrows, we obtain from (4.16) the following commutative diagram.

$$\begin{array}{ccc} \mathbf{H}(\mathrm{Z})\text{-}\mathbf{mod} & \xrightarrow{\mathrm{IndCoh}(\mathrm{Z})} & \mathrm{D}(\mathrm{Sing}(\mathrm{Z}))^{\Rightarrow}\text{-}\mathbf{mod} \\ {\scriptstyle \mathbf{H}_{\mathrm{Y}\leftarrow\mathrm{Z}}}\downarrow & & \downarrow{\scriptstyle \mathrm{D}(\mathrm{Sing}(\mathrm{Y})_{\mathrm{Z}})^{\Rightarrow}} \\ \mathbf{H}(\mathrm{Y})\text{-}\mathbf{mod} & \xrightarrow[\mathrm{IndCoh}(\mathrm{Y})]{} & \mathrm{D}(\mathrm{Sing}(\mathrm{Y}))^{\Rightarrow}\text{-}\mathbf{mod} \end{array} \tag{4.18}$$

4.5.4 *Execution of Construction 4.1.* We need to construct an isomorphism

$$\mathrm{IndCoh}(\mathrm{Z}) \underset{\mathbf{H}(\mathrm{Z})}{\otimes} \mathbf{H}_{\mathrm{Z}\to\mathrm{Y}} \simeq \mathrm{D}(\mathrm{Sing}(\mathrm{Y})_{\mathrm{Z}})^{\Rightarrow} \underset{\mathrm{D}(\mathrm{Sing}(\mathrm{Y}))^{\Rightarrow}}{\otimes} \mathrm{IndCoh}(\mathrm{Y})$$

of $(\mathbf{H}(\mathrm{Z}), \mathrm{D}(\mathrm{Sing}(\mathrm{Y}))^{\Rightarrow})$-bimodules.

In view of § 4.1.5, it is equivalent to produce an isomorphism

$$\mathbf{H}_{\mathrm{Z}\to\mathrm{Y}} \simeq \mathrm{IndCoh}(\mathrm{Z})^{\tau} \underset{\mathrm{D}(\mathrm{Sing}(\mathrm{Z}))^{\Rightarrow}}{\otimes} \mathrm{D}(\mathrm{Sing}(\mathrm{Y})_{\mathrm{Z}})^{\Rightarrow} \underset{\mathrm{D}(\mathrm{Sing}(\mathrm{Y}))^{\Rightarrow}}{\otimes} \mathrm{IndCoh}(\mathrm{Y}) \tag{4.19}$$

of $(\mathbf{H}(\mathrm{Z}), \mathbf{H}(\mathrm{Y}))$-bimodules.

This amounts to an isomorphism of $\mathbf{H}(\mathrm{Z}\times\mathrm{Y})$-modules

$$\mathrm{IndCoh}_0((\mathrm{Z}\times\mathrm{Y})^{\wedge}_{\mathrm{Z}}) \simeq \mathrm{IndCoh}(\mathrm{Z}\times\mathrm{Y}) \underset{\mathrm{D}(\mathrm{Sing}(\mathrm{Z}\times\mathrm{Y}))^{\Rightarrow}}{\otimes} \mathrm{D}(\mathrm{Sing}(\mathrm{Y})_{\mathrm{Z}})^{\Rightarrow} \tag{4.20}$$

where $\mathrm{D}(\mathrm{Sing}(\mathrm{Y})_{\mathrm{Z}})^{\Rightarrow}$ is regarded as a module over $\mathrm{D}(\mathrm{Sing}(\mathrm{Z}\times\mathrm{Y}))^{\Rightarrow}$ via the map

$$\mathrm{Sing}(\mathrm{Y})_{\mathrm{Z}} \xrightarrow{(4.14)} \mathrm{Sing}(\mathrm{Z})\times\mathrm{Sing}(\mathrm{Y}) \xrightarrow{\tau\times\mathbf{1}} \mathrm{Sing}(\mathrm{Z})\times\mathrm{Sing}(\mathrm{Y}). \tag{4.21}$$

Both sides of (4.20) are naturally full subcategories of

$$\mathrm{IndCoh}(\mathrm{Z}\times\mathrm{Y})$$

stable under the action of $\mathbf{H}(\mathrm{Z}\times\mathrm{Y})$, and we claim that they are equal as such. Indeed, (4.21) identifies with the closed embedding

$$\mathrm{Sing}(\mathrm{Z}/\mathrm{Z}\times\mathrm{Y}) \hookrightarrow \mathrm{Sing}(\mathrm{Z}\times\mathrm{Y})$$

so the claim follows from Corollary 3.1.

4.6 Composition of transfer bimodules

4.6.1. Let us record for future use the sense in which the formation of the diagrams (4.18) is compatible with composition.

Suppose that

$$\mathrm{Z}\to\mathrm{Y}\to\mathrm{X}$$

are morphisms of quasi-smooth schemes. According to [Ber19, Theorem 4.3.4], the convolution functor

$$\mathbf{H}_{\mathrm{Z}\leftarrow\mathrm{Y}} \underset{\mathbf{H}(\mathrm{Y})}{\otimes} \mathbf{H}_{\mathrm{Y}\leftarrow\mathrm{X}} \to \mathbf{H}_{\mathrm{Z}\leftarrow\mathrm{X}} \tag{4.22}$$

is an isomorphism of $(\mathbf{H}(\mathrm{Z}), \mathbf{H}(\mathrm{X}))$-bimodules.[12]

[12]Strictly speaking, the material in this subsection is logically independent of this citation. Namely, the diagram (4.24) with its left face removed implies that there exists *some* isomorphism of the form (4.22), which Construction 4.2 identifies with the convolution map.

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On the other hand, the following cartesian diagram defines an isomorphism

$$\mathrm{D}(\mathrm{Sing}(\mathrm{Y})_{\mathrm{Z}})^{\Rightarrow} \underset{\mathrm{D}(\mathrm{Sing}(\mathrm{Y}))^{\Rightarrow}}{\otimes} \mathrm{D}(\mathrm{Sing}(\mathrm{X})_{\mathrm{Y}})^{\Rightarrow} \simeq \mathrm{D}(\mathrm{Sing}(\mathrm{X})_{\mathrm{Y}})^{\Rightarrow} \tag{4.23}$$

of $(\mathrm{D}(\mathrm{Sing}(\mathrm{Z}))^{\Rightarrow}, \mathrm{D}(\mathrm{Sing}(\mathrm{X}))^{\Rightarrow})$-bimodules.

$$\begin{array}{ccccc} \mathrm{Sing}(\mathrm{X})_{\mathrm{Z}} & \longrightarrow & \mathrm{Sing}(\mathrm{Y})_{\mathrm{Z}} & \longrightarrow & \mathrm{Sing}(\mathrm{Z}) \\ \downarrow & & \downarrow & & \\ \mathrm{Sing}(\mathrm{X})_{\mathrm{Y}} & \longrightarrow & \mathrm{Sing}(\mathrm{Y}) & & \\ \downarrow & & & & \\ \mathrm{Sing}(\mathrm{X}) & & & & \end{array}$$

The situation at this point is that we have obtained commutativity data for every face of the diagram below.

$$\begin{array}{ccccc} \mathbf{H}(\mathrm{Z})\text{-}\mathbf{mod} & \longrightarrow & \mathrm{D}(\mathrm{Sing}(\mathrm{Z}))^{\Rightarrow}\text{-}\mathbf{mod} & & \\ \downarrow & \searrow & \downarrow & \searrow & \\ \mathbf{H}(\mathrm{Y})\text{-}\mathbf{mod} & \longrightarrow & \mathrm{D}(\mathrm{Sing}(\mathrm{Y}))^{\Rightarrow}\text{-}\mathbf{mod} & & \\ & \searrow & & \searrow & \\ & & \mathbf{H}(\mathrm{X})\text{-}\mathbf{mod} & \longrightarrow & \mathrm{D}(\mathrm{Sing}(\mathrm{X}))^{\Rightarrow}\text{-}\mathbf{mod} \end{array} \tag{4.24}$$

Namely, the left and right faces are given by the isomorphisms (4.22) and (4.23), respectively, while the remaining faces are given by (4.18).

The goal of this subsection is to construct a 3-morphism representing the interior of (4.24).

Construction 4.2. The diagram (4.24) commutes.

4.6.2 *Execution of Construction 4.2.* We need to show that the isomorphisms (4.19) used to perform Construction 4.1 identify (4.22) with the composite

$$\begin{aligned} &\mathrm{IndCoh}(\mathrm{X})^{\tau} \underset{\mathrm{D}(\mathrm{Sing}(\mathrm{X}))^{\Rightarrow}}{\otimes} \mathrm{D}(\mathrm{Sing}(\mathrm{X})_{\mathrm{Y}})^{\Rightarrow} \underset{\mathrm{D}(\mathrm{Sing}(\mathrm{Y}))^{\Rightarrow}}{\otimes} \mathrm{IndCoh}(\mathrm{Y}) \\ &\quad \underset{\mathbf{H}(\mathrm{Y})}{\otimes} \mathrm{IndCoh}(\mathrm{Y})^{\tau} \underset{\mathrm{D}(\mathrm{Sing}(\mathrm{Y}))^{\Rightarrow}}{\otimes} \mathrm{D}(\mathrm{Sing}(\mathrm{Y})_{\mathrm{Z}})^{\Rightarrow} \underset{\mathrm{D}(\mathrm{Sing}(\mathrm{Z}))^{\Rightarrow}}{\otimes} \mathrm{IndCoh}(\mathrm{Z}) \\ &\xrightarrow{\sim} \mathrm{IndCoh}(\mathrm{X})^{\tau} \underset{\mathrm{D}(\mathrm{Sing}(\mathrm{X}))^{\Rightarrow}}{\otimes} \mathrm{D}(\mathrm{Sing}(\mathrm{X})_{\mathrm{Y}})^{\Rightarrow} \underset{\mathrm{D}(\mathrm{Sing}(\mathrm{Y}))^{\Rightarrow}}{\otimes} \mathrm{D}(\mathrm{Sing}(\mathrm{Y}))^{\Rightarrow} \\ &\qquad \underset{\mathrm{D}(\mathrm{Sing}(\mathrm{Y}))^{\Rightarrow}}{\otimes} \mathrm{D}(\mathrm{Sing}(\mathrm{Y})_{\mathrm{Z}})^{\Rightarrow} \underset{\mathrm{D}(\mathrm{Sing}(\mathrm{Z}))^{\Rightarrow}}{\otimes} \mathrm{IndCoh}(\mathrm{Z}) \\ &\xrightarrow{\sim} \mathrm{IndCoh}(\mathrm{X})^{\tau} \underset{\mathrm{D}(\mathrm{Sing}(\mathrm{X}))^{\Rightarrow}}{\otimes} \mathrm{D}(\mathrm{Sing}(\mathrm{X})_{\mathrm{Y}})^{\Rightarrow} \\ &\qquad \underset{\mathrm{D}(\mathrm{Sing}(\mathrm{Y}))^{\Rightarrow}}{\otimes} \mathrm{D}(\mathrm{Sing}(\mathrm{Y})_{\mathrm{Z}})^{\Rightarrow} \underset{\mathrm{D}(\mathrm{Sing}(\mathrm{Z}))^{\Rightarrow}}{\otimes} \mathrm{IndCoh}(\mathrm{Z}) \\ &\xrightarrow{(4.23)} \mathrm{IndCoh}(\mathrm{X})^{\tau} \underset{\mathrm{D}(\mathrm{Sing}(\mathrm{X}))^{\Rightarrow}}{\otimes} \mathrm{D}(\mathrm{Sing}(\mathrm{X})_{\mathrm{Z}})^{\Rightarrow} \underset{\mathrm{D}(\mathrm{Sing}(\mathrm{Z}))^{\Rightarrow}}{\otimes} \mathrm{IndCoh}(\mathrm{Z}). \end{aligned} \tag{4.25}$$

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4.6.3. Recall that the identification of Construction 4.1 matches the targets of (4.22) and (4.25) as full subcategories of

$$\mathrm{IndCoh}(\mathrm{X}) \otimes \mathrm{IndCoh}(\mathrm{Z}). \tag{4.26}$$

The sources of (4.22) and (4.25) are likewise identified as full subcategories of

$$\mathrm{IndCoh}(\mathrm{X}) \otimes \mathrm{IndCoh}(\mathrm{Y}) \underset{\mathbf{H}(\mathrm{Y})}{\otimes} \mathrm{IndCoh}(\mathrm{Y}) \otimes \mathrm{IndCoh}(\mathrm{Z}). \tag{4.27}$$

Furthermore, under these embeddings, both (4.22) and (4.25) are induced by functors

$$(4.27) \to (4.26)$$

between the ambient categories. Namely, the ambient functor for (4.22) is the one obtained from the pairing

$$\mathrm{IndCoh}(\mathrm{Y}) \underset{\mathbf{H}(\mathrm{Y})}{\otimes} \mathrm{IndCoh}(\mathrm{Y}) \to \mathrm{Vect} \tag{4.28}$$

and the ambient functor for (4.25) is the one obtained from the composite

$$\mathrm{IndCoh}(\mathrm{Y}) \underset{\mathbf{H}(\mathrm{Y})}{\otimes} \mathrm{IndCoh}(\mathrm{Y})^{\tau} \xrightarrow{\sim} \mathrm{D}(\mathrm{Sing}(\mathrm{Y}))^{\Rightarrow} \xrightarrow{(\Gamma_{\mathrm{dR}})^{\Rightarrow}} \mathrm{Vect}. \tag{4.29}$$

So, it remains to see that the two functors (4.28) and (4.29) are canonically isomorphic.

4.6.4. Let us use the notation of (4.7) for convenience. Then, we are trying to identify the composites

$$\mathcal{M}^{\vee} \otimes_{\mathcal{B}} \mathcal{M} \xrightarrow{\mathrm{ev}} \mathcal{A} \xrightarrow{(\Gamma_{\mathrm{dR}})^{\Rightarrow}} \mathrm{Vect} \quad \text{and} \quad \mathcal{M}^{\vee} \otimes_{\mathcal{B}} \mathcal{M} \to \mathrm{Vect},$$

where the second map is the pairing on $\mathcal{M}$ as a left-dualizable $\mathcal{B}$-module.

According to § 4.1.1, the left composite is given by

$$\mathcal{M}^{\vee} \otimes_{\mathcal{B}} \mathcal{M} \xrightarrow{\mathbf{1} \otimes \mathrm{coact}} \mathcal{M}^{\vee} \otimes_{\mathcal{B}} \mathcal{M} \otimes \mathcal{A} \to \mathcal{A} \xrightarrow{(\Gamma_{\mathrm{dR}})^{\Rightarrow}} \mathrm{Vect},$$

which rewrites as

$$\mathcal{M}^{\vee} \otimes_{\mathcal{B}} \mathcal{M} \xrightarrow{\mathbf{1} \otimes \mathrm{coact}} \mathcal{M}^{\vee} \otimes_{\mathcal{B}} \mathcal{M} \otimes \mathcal{A} \xrightarrow{\mathbf{1} \otimes (\Gamma_{\mathrm{dR}})^{\Rightarrow}} \mathcal{M}^{\vee} \otimes_{\mathcal{B}} \mathcal{M} \to \mathrm{Vect}.$$

Therefore, it is enough to show that the composite

$$\mathcal{M} \xrightarrow{\mathrm{coact}} \mathcal{M} \otimes \mathcal{A} \xrightarrow{(\Gamma_{\mathrm{dR}})^{\Rightarrow}} \mathcal{M} \tag{4.30}$$

is the identity.

For this, it suffices to consider the universal case $\mathcal{M} = \mathcal{A}$, where the assertion is clear because (4.30) becomes the composite

$$\mathcal{A} \xrightarrow{(\Delta_*)^{\Rightarrow}} \mathcal{A} \otimes \mathcal{A} \xrightarrow{(\mathrm{pr}_{2,*})^{\Rightarrow}} \mathcal{A}.$$

5. Extension to stacks

In this section, we will establish a version of the Morita equivalence of Theorem 4.1 when the quasi-smooth scheme Z is replaced with a quasi-smooth stack $\mathcal{Y}$.

Let us point out that although we do have a natural bimodule structure

$$\mathrm{IndCoh}(\mathcal{Y}) \in (\mathrm{D}(\mathrm{Sing}(\mathcal{Y}))^{\Rightarrow}, \mathbf{H}(\mathcal{Y}))\textbf{-bimod},$$

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the resulting functor

$$\mathbf{H}(\mathcal{Y})\text{-}\mathbf{mod} \to \mathrm{D}(\mathrm{Sing}(\mathcal{Y}))^{\Rightarrow}\text{-}\mathbf{mod} \tag{5.1}$$

will usually not be an equivalence. This is essentially due to the failure of 1-affineness for the de Rham space of an algebraic stack. Because of this, the correct generalization of Theorem 4.1 to the setting of stacks will involve sheaves of $(\infty, 2)$-categories.

We will then return to the category $\mathrm{D}(\mathrm{Sing}(\mathcal{Y}))^{\Rightarrow}$-**mod** and its relation to $\mathbf{H}(\mathcal{Y})$ and show that (5.1) is a colocalization in the $(\infty, 2)$-category **2-dgCat** of stable $(\infty, 2)$-categories tensored over **dgCat**.

The outline of this section is as follows.

- In § 5.1, we describe a category $\mathrm{ShCat}_{\mathbf{S}^{\Rightarrow}}(\mathcal{Y})$, associated to any quasi-smooth stack $\mathcal{Y}$, which is related to $\mathrm{D}(\mathrm{Sing}(\mathcal{Y}))^{\Rightarrow}$-**mod** in the same way that crystals of categories are related to $\mathrm{D}(\mathcal{Y})$-**mod**.
- In § 5.2, we prove that $\mathrm{ShCat}_{\mathbf{S}^{\Rightarrow}}(\mathcal{Y})$ is equivalent to the category $\mathrm{ShCat}_{\mathbf{H}}(\mathcal{Y})$ defined in [Ber19]. This amounts to showing that our Morita equivalence Theorem 4.1 is sufficiently functorial as the scheme Z varies.
- In § 5.3, we show that the result of § 4.5 on the image of transfer bimodules under our Morita equivalence extends to stacks, conditional on a construction involving relative D-modules Construction 5.4.
- In § 5.4, we carry out Construction 5.4.
- In § 5.5, we apply the results of the previous subsections to study the relationship between $\mathrm{D}(\mathrm{Sing}(\mathcal{Y}))^{\Rightarrow}$-**mod** and $\mathrm{ShCat}_{\mathbf{S}^{\Rightarrow}}(\mathcal{Y})$.

5.1 A sheaf version of the Morita equivalence

5.1.1. Let $\mathcal{Y}$ be a quasi-smooth algebraic stack, and let $\mathrm{Sm}_{/\mathcal{Y}}$ denote the category whose objects are schemes S equipped with a smooth morphism $\mathrm{S} \to \mathcal{Y}$ and whose morphisms are composites $\mathrm{T} \to \mathrm{S} \to \mathcal{Y}$ with $\mathrm{T} \to \mathrm{S}$ smooth.

5.1.2. Recall that [Ber19] assembles the assignment

$$\mathrm{S} \rightsquigarrow \mathbf{H}(\mathrm{S})\text{-}\mathbf{mod} \quad (\mathrm{T} \to \mathrm{S}) \rightsquigarrow \mathbf{H}_{\mathrm{T}\to\mathrm{S}} \otimes_{\mathbf{H}(\mathrm{S})} (-) \tag{5.2}$$

into a functor

$$\mathbf{H}\text{-}\mathbf{mod} : (\mathrm{Sm}_{/\mathcal{Y}})^{\mathrm{op}} \to \mathbf{2\text{-}dgCat}.$$

Now let

$$\mathrm{ShCat}_{\mathbf{H}}(\mathcal{Y}) = \lim_{\mathrm{S}\to\mathcal{Y}} \mathbf{H}(\mathrm{S})\text{-}\mathbf{mod}$$

denote the limit of the functor **H-mod**.[13] The **H**-affineness result of [Ber19] exhibits an adjoint equivalence

$$\mathbf{Loc}_{\mathbf{H}} : \mathbf{H}(\mathcal{Y})\text{-}\mathbf{mod} \rightleftharpoons \mathrm{ShCat}_{\mathbf{H}}(\mathcal{Y}) : \mathbf{\Gamma}_{\mathbf{H}}.$$

In particular, viewing $\mathrm{IndCoh}(\mathcal{Y})$ a module over $\mathbf{H}(\mathcal{Y})$, we obtain a canonical object

$$\mathbf{Loc}_{\mathbf{H}}(\mathrm{IndCoh}(\mathcal{Y})) \in \mathrm{ShCat}_{\mathbf{H}}(\mathcal{Y}). \tag{5.3}$$

[13] Smooth descent for **H** [Ber19, Theorem 6.1.2] implies that this limit agrees with the notion of sheaves of categories with **H**-coefficients considered in [Ber19].

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5.1.3. According to [Ber19, Lemma 6.6.1], the value of (5.3) at an object $\mathrm{S} \in \mathrm{Sm}_{/\mathcal{Y}}$ is

$$\mathbf{Loc}_{\mathbf{H}}(\mathrm{IndCoh}(\mathcal{Y}))_{\mathrm{S}} \simeq \mathrm{IndCoh}(\mathrm{S})$$

and its value at a morphism $\mathrm{T} \to \mathrm{S}$ is given by the convolution map

$$\mathbf{H}_{\mathrm{T}\to\mathrm{S}} \underset{\mathbf{H}(\mathrm{S})}{\otimes} \mathrm{IndCoh}(\mathrm{S}) \xrightarrow{\sim} \mathrm{IndCoh}(\mathrm{T}).$$

5.1.4. Now let us describe the counterparts of **H-mod** and $\mathrm{ShCat}_{\mathbf{H}}(\mathcal{Y})$ that will appear on the other side of the Morita equivalence.

Let $f : \mathrm{T} \to \mathrm{S}$ be any morphism in $\mathrm{Sm}_{/\mathcal{Y}}$. Since f is required to be smooth, its singular codifferential is an isomorphism

$$\mathrm{Sing}(\mathrm{S})_{\mathrm{T}} \xrightarrow{\sim} \mathrm{Sing}(\mathrm{T})$$

and the correspondence (4.14) induces a map

$$\mathrm{Sing}(\mathrm{T}) \to \mathrm{Sing}(\mathrm{S}). \tag{5.4}$$

In particular, we may consider the adjunction

$$\mathbf{ind} : \mathrm{D}(\mathrm{Sing}(\mathrm{S}))^{\Rightarrow}\textbf{-mod} \rightleftharpoons \mathrm{D}(\mathrm{Sing}(\mathrm{T}))^{\Rightarrow}\textbf{-mod} : \mathbf{oblv}.$$

The left adjoints assemble into a functor

$$\mathbf{S}^{\Rightarrow}\textbf{-mod} : (\mathrm{Sm}_{/\mathcal{Y}})^{\mathrm{op}} \to \textbf{2-dgCat},$$

given at the level of objects and morphisms by the assignment

$$(\mathrm{T} \to \mathrm{S}) \rightsquigarrow (\mathrm{D}(\mathrm{Sing}(\mathrm{S}))^{\Rightarrow}\textbf{-mod} \xrightarrow{\mathbf{ind}} \mathrm{D}(\mathrm{Sing}(\mathrm{T}))^{\Rightarrow}\textbf{-mod}).$$

Let

$$\mathrm{ShCat}_{\mathbf{S}^{\Rightarrow}}(\mathcal{Y}) = \lim_{\mathrm{S}\to\mathcal{Y}} \mathrm{D}(\mathrm{Sing}(\mathrm{S}))^{\Rightarrow}\textbf{-mod} \tag{5.5}$$

denote the limit of $\mathbf{S}^{\Rightarrow}$**-mod**, viewed as a functor. It is equipped with tautological evaluation functors

$$\mathbf{\Gamma}(\mathrm{S}; -) : \mathrm{ShCat}_{\mathbf{S}^{\Rightarrow}}(\mathcal{Y}) \to \mathrm{D}(\mathrm{Sing}(\mathrm{S}))^{\Rightarrow}\textbf{-mod}.$$

For any stack $\mathcal{W}$ equipped with a $\mathbf{G}_m$-equivariant morphism

$$\mathcal{W} \to \mathrm{Sing}(\mathcal{Y}),$$

we obtain an object

$$\underline{\mathrm{D}}(\mathcal{W})^{\Rightarrow} \in \mathrm{ShCat}_{\mathbf{S}^{\Rightarrow}}(\mathcal{Y})$$

with

$$\mathbf{\Gamma}(\mathrm{S}; \underline{\mathrm{D}}(\mathcal{W})) \simeq \mathrm{D}(\mathcal{W}_{\mathrm{S}})^{\Rightarrow} \quad \mathcal{W}_{\mathrm{S}} = \mathrm{S} \times_{\mathcal{Y}} \mathcal{W}.$$

Construction 5.1. There is an equivalence of categories

$$\mathrm{ShCat}_{\mathbf{H}}(\mathcal{Y}) \simeq \mathrm{ShCat}_{\mathbf{S}^{\Rightarrow}}(\mathcal{Y}) \tag{5.6}$$

exchanging the objects

$$\mathbf{Loc}_{\mathbf{H}}(\mathrm{IndCoh}(\mathcal{Y})) \quad \text{and} \quad \underline{\mathrm{D}}(\mathrm{Sing}(\mathcal{Y}))^{\Rightarrow}.$$

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5.1.5. We will obtain Construction 5.1 from a slightly stronger result, Construction 5.2, whose statement we will now explain.

Let **dgCat** denote the constant functor

$$(\mathrm{Sm}_{/\mathcal{Y}})^{\mathrm{op}} \to \mathbf{2\text{-}dgCat} \tag{5.7}$$

with value **dgCat**. Then, we may regard $\mathbf{Loc}_{\mathrm{H}}(\mathrm{IndCoh}(\mathcal{Y}))$ as a natural transformation

$$\mathbf{dgCat} \to \mathbf{H\text{-}mod} \tag{5.8}$$

of functors of the shape (5.7), whose value at an object $\mathrm{S} \in \mathrm{Sm}_{/\mathcal{Y}}$ is

$$\mathrm{IndCoh}(\mathrm{S}) : \mathbf{dgCat} \to \mathbf{H}(\mathrm{S})\text{-}\mathbf{mod}. \tag{5.9}$$

Similarly, we may regard $\underline{\mathrm{D}}(\mathrm{Sing}(\mathcal{Y}))^{\Rightarrow}$ as a natural transformation

$$\mathbf{dgCat} \to \mathbf{S}^{\Rightarrow}\text{-}\mathbf{mod}, \tag{5.10}$$

whose value at an object $\mathrm{S} \in \mathrm{Sm}_{/\mathcal{Y}}$ is

$$\mathbf{ind} : \mathbf{dgCat} \to \mathrm{D}(\mathrm{Sing}(\mathrm{S}))^{\Rightarrow}\text{-}\mathbf{mod}. \tag{5.11}$$

To perform Construction 5.1, it suffices to perform the following and pass to the limit.

Construction 5.2. There is a commutative diagram of functors of the shape (5.7).

$$\begin{array}{ccc} \mathbf{H\text{-}mod} & \xrightarrow{\sim} & \mathbf{S}^{\Rightarrow}\text{-}\mathbf{mod} \\ & {\scriptstyle (5.8)}\nwarrow \quad \nearrow{\scriptstyle (5.10)} & \\ & \mathbf{dgCat} & \end{array}$$

5.1.6. The value of the putative natural isomorphism

$$\mathbf{H\text{-}mod} \simeq \mathbf{S}^{\Rightarrow}\text{-}\mathbf{mod}$$

at an individual morphism $\mathrm{T} \to \mathrm{S}$ is the following diagram provided by Construction 4.1.

$$\begin{array}{ccc} \mathbf{H}(\mathrm{S})\text{-}\mathbf{mod} & \xrightarrow{\mathrm{IndCoh}(\mathrm{S})} & \mathrm{D}(\mathrm{Sing}(\mathrm{S}))^{\Rightarrow}\text{-}\mathbf{mod} \\ {\scriptstyle \mathbf{H}_{\mathrm{T}\to\mathrm{S}}}\downarrow & & \downarrow{\scriptstyle \mathbf{ind}} \\ \mathbf{H}(\mathrm{T})\text{-}\mathbf{mod} & \xrightarrow{\mathrm{IndCoh}(\mathrm{T})} & \mathrm{D}(\mathrm{Sing}(\mathrm{T}))^{\Rightarrow}\text{-}\mathbf{mod} \end{array} \tag{5.12}$$

Therefore, the content of Construction 5.2 is that the diagram (5.12) can be assembled into a natural transformation. This will be done in § 5.2.

5.2 Execution of Construction 5.2

5.2.1. It turns out that Construction 5.2 as stated is difficult to approach directly, so we will have to provide a more tractable reformulation. This will appear below as Construction 5.3.

5.2.2. Recall from § 4.5.1 that all of the bimodules $\mathbf{H}_{\mathrm{T}\to\mathrm{S}}$ admit right duals. Therefore, by passing to right adjoints in **2-dgCat**, we obtain from **H-mod** a functor

$$\mathbf{H\text{-}mod}^{\mathrm{R}} : \mathrm{Sm}_{/\mathcal{Y}} \to \mathbf{2\text{-}dgCat}$$

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given at the level of objects and morphisms by the assignment

$$\mathrm{S} \rightsquigarrow \mathbf{H}(\mathrm{S})\textbf{-mod}, \quad (\mathrm{T} \to \mathrm{S}) \rightsquigarrow \mathbf{H}_{\mathrm{S} \leftarrow \mathrm{T}} \underset{\mathbf{H}(\mathrm{T})}{\otimes} (-).$$

Applying the same procedure to $\mathbf{S}^{\Rightarrow}$**-mod**, we obtain a functor

$$\mathbf{S}^{\Rightarrow}\textbf{-mod}^{\mathrm{R}} : \mathrm{Sm}_{/\mathcal{Y}} \to \textbf{2-dgCat}$$

given at the level of objects and morphisms by the assignment

$$(\mathrm{T} \to \mathrm{S}) \rightsquigarrow (\mathrm{D}(\mathrm{Sing}(\mathrm{T}))^{\Rightarrow}\textbf{-mod} \xrightarrow{\textbf{oblv}} \mathrm{D}(\mathrm{Sing}(\mathrm{S}))^{\Rightarrow}\textbf{-mod}).$$

Now let $\mathbf{dgCat}^{\mathrm{R}}$ denote the constant functor

$$\mathrm{Sm}_{/\mathcal{Y}} \to \textbf{2-dgCat} \tag{5.13}$$

with value **dgCat** (alternatively, it is obtained from the functor **dgCat** by passing to right adjoints). Since IndCoh(S) is self-dual as a module over $\mathbf{H}(\mathrm{S})$, all of the functors (5.9) admit right adjoints in **2-dgCat**. Therefore, by passing to right adjoints, we obtain from (5.8) a natural transformation

$$\textbf{H-mod}^{\mathrm{R}} \to \mathbf{dgCat}^{\mathrm{R}} \tag{5.14}$$

of functors of the shape (5.13). Similarly, by virtue of the (**ind**, **oblv**) adjunction, we obtain from (5.10) a natural transformation

$$\mathbf{S}^{\Rightarrow}\textbf{-mod}^{\mathrm{R}} \to \mathbf{dgCat}^{\mathrm{R}} \tag{5.15}$$

whose value at $\mathrm{S} \in \mathrm{Sm}_{/\mathcal{Y}}$ is the forgetful functor

$$\textbf{oblv} : \mathrm{D}(\mathrm{Sing}(\mathrm{S}))^{\Rightarrow}\textbf{-mod} \to \mathbf{dgCat}. \tag{5.16}$$

Construction 5.2 is equivalent to the following.

Construction 5.3. There is a commutative diagram of functors of the shape (5.13).

$$\begin{array}{ccc} \textbf{H-mod}^{\mathrm{R}} & \xrightarrow{\sim} & \mathbf{S}^{\Rightarrow}\textbf{-mod}^{\mathrm{R}} \\ & {\scriptstyle (5.14)}\searrow \quad \swarrow{\scriptstyle (5.15)} & \\ & \mathbf{dgCat}^{\mathrm{R}} & \end{array}$$

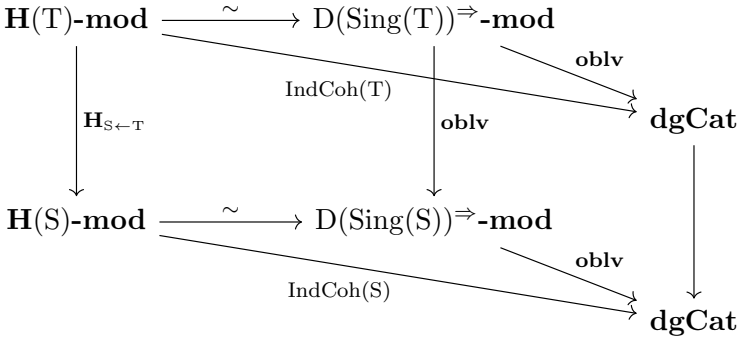
5.2.3. The feature of Construction 5.3 which makes it manageable is the fact that the forgetful functors (5.16) are locally fully faithful (this follows from Proposition 2.2). This means (see Proposition A.5) that, in order to perform Construction 5.3, it is equivalent to do the following.

(i) For every object $\mathrm{S} \in \mathrm{Sm}_{/\mathcal{Y}}$, specify a commutative diagram.

$$\begin{array}{ccc} \mathbf{H}(\mathrm{S})\textbf{-mod} & \xrightarrow{\sim} & \mathrm{D}(\mathrm{Sing}(\mathrm{S}))^{\Rightarrow}\textbf{-mod} \\ & {\scriptstyle \mathrm{IndCoh}(\mathrm{S})}\searrow \quad \swarrow{\scriptstyle \textbf{oblv}} & \\ & \mathbf{dgCat} & \end{array}$$

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(ii) For every morphism $\mathrm{T} \to \mathrm{S}$ in $\mathrm{Sm}_{/\mathcal{Y}}$, verify that the resulting diagram (in which the back face is empty) can be completed to a commutative prism.



(5.17)

Note that the data required by (i) is provided by Theorem 4.1, so it remains to verify the assertion of (ii). This occupies the remainder of the subsection.

5.2.4. The front face of (5.17) encodes the isomorphism of right $\mathbf{H}(\mathrm{T})$-modules

$$\mathrm{IndCoh}(\mathrm{S}) \underset{\mathbf{H}(\mathrm{S})}{\otimes} \mathbf{H}_{\mathrm{S}\leftarrow\mathrm{T}} \to \mathrm{IndCoh}(\mathrm{T}) \tag{5.18}$$

given by convolution. We need to check that (5.18) is linear for the pre-existing left actions of $\mathrm{D}(\mathrm{Sing}(\mathrm{S}))^{\Rightarrow}$ on its source and target.

Indeed, let us write (5.18) as the composite

$$\begin{aligned} &\mathrm{IndCoh}(\mathrm{S}) \underset{\mathbf{H}(\mathrm{S})}{\otimes} \mathbf{H}_{\mathrm{S}\leftarrow\mathrm{T}} \xrightarrow{\sim} \mathrm{IndCoh}(\mathrm{S}) \underset{\mathbf{H}(\mathrm{S})}{\otimes} \mathrm{IndCoh}(\mathrm{S})^{\tau} \underset{\mathrm{D}(\mathrm{Sing}(\mathrm{S}))^{\Rightarrow}}{\otimes} \mathrm{IndCoh}(\mathrm{T}) \\ &\hookrightarrow \mathrm{IndCoh}(\mathrm{S}) \underset{\mathbf{H}(\mathrm{S})}{\otimes} \mathrm{IndCoh}(\mathrm{S})^{\tau} \otimes \mathrm{IndCoh}(\mathrm{T}) \xrightarrow{\mathrm{ev}\otimes\mathbf{1}} \mathrm{IndCoh}(\mathrm{T}). \end{aligned} \tag{5.19}$$

By § 4.6.4, we may further rewrite (5.19) as

$$\begin{aligned} &\mathrm{IndCoh}(\mathrm{S}) \underset{\mathbf{H}(\mathrm{S})}{\otimes} \mathbf{H}_{\mathrm{S}\leftarrow\mathrm{T}} \xrightarrow{\sim} \mathrm{IndCoh}(\mathrm{S}) \underset{\mathbf{H}(\mathrm{S})}{\otimes} \mathrm{IndCoh}(\mathrm{S})^{\tau} \underset{\mathrm{D}(\mathrm{Sing}(\mathrm{S}))^{\Rightarrow}}{\otimes} \mathrm{IndCoh}(\mathrm{T}) \\ &\xrightarrow{\sim} \mathrm{D}(\mathrm{Sing}(\mathrm{S}))^{\Rightarrow} \underset{\mathrm{D}(\mathrm{Sing}(\mathrm{S}))^{\Rightarrow}}{\otimes} \mathrm{IndCoh}(\mathrm{T}) \hookrightarrow \mathrm{D}(\mathrm{Sing}(\mathrm{S}))^{\Rightarrow} \otimes \mathrm{IndCoh}(\mathrm{T}) \\ &\xrightarrow{(\Gamma_{\mathrm{dR}})^{\Rightarrow}\otimes\mathbf{1}} \mathrm{IndCoh}(\mathrm{T}). \end{aligned}$$

This last expression identifies with

$$\begin{aligned} &\mathrm{IndCoh}(\mathrm{S}) \underset{\mathbf{H}(\mathrm{S})}{\otimes} \mathbf{H}_{\mathrm{S}\leftarrow\mathrm{T}} \xrightarrow{\sim} \mathrm{IndCoh}(\mathrm{S}) \underset{\mathbf{H}(\mathrm{S})}{\otimes} \mathrm{IndCoh}(\mathrm{S})^{\tau} \underset{\mathrm{D}(\mathrm{Sing}(\mathrm{S}))^{\Rightarrow}}{\otimes} \mathrm{IndCoh}(\mathrm{T}) \\ &\xrightarrow{\sim} \mathrm{D}(\mathrm{Sing}(\mathrm{S}))^{\Rightarrow} \underset{\mathrm{D}(\mathrm{Sing}(\mathrm{S}))^{\Rightarrow}}{\otimes} \mathrm{IndCoh}(\mathrm{T}) \xrightarrow{\sim} \mathrm{IndCoh}(\mathrm{T}). \end{aligned}$$

5.3 Relative D-modules over stacks

5.3.1. Let $\mathcal{Z} \to \mathcal{Y}$ be a schematic morphism of quasi-smooth stacks. Then $\mathrm{IndCoh}(\mathcal{Y}^{\wedge}_{\mathcal{Z}})$ is naturally acted on by $\mathbf{H}(\mathcal{Y})$, and we may form the object

$$\mathbf{Loc}(\mathrm{IndCoh}(\mathcal{Y}^{\wedge}_{\mathcal{Z}})) \in \mathrm{ShCat}_{\mathbf{H}}(\mathcal{Y}).$$

On the other hand, from the morphism

$$\mathrm{Sing}(\mathcal{Y})_{\mathcal{Z}} \to \mathrm{Sing}(\mathcal{Y}),$$

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we obtain an object

$$\underline{\mathrm{D}}(\mathrm{Sing}(\mathcal{Y})_{\mathcal{Z}})^{\Rightarrow} \in \mathrm{ShCat}_{\mathbf{S}^{\Rightarrow}}(\mathcal{Y}).$$

We note for future reference that for $\mathrm{S} \in \mathrm{Sm}_{/\mathcal{Y}}$, we have

$$\mathrm{Sing}(\mathrm{S})_{\mathcal{Z}_{\mathrm{S}}} \simeq (\mathrm{Sing}(\mathcal{Y})_{\mathcal{Z}})_{\mathrm{S}}. \tag{5.20}$$

Construction 5.4. The Morita equivalence (5.6) exchanges the objects

$$\underline{\mathrm{D}}(\mathrm{Sing}(\mathcal{Y})_{\mathcal{Z}})^{\Rightarrow} \quad \text{and} \quad \mathbf{Loc}(\mathrm{IndCoh}(\mathcal{Y}^{\wedge}_{\mathcal{Z}})).$$

5.3.2. Let us explain why Construction 5.4 is non-trivial. Consider the object

$$\mathcal{D} \in \mathrm{ShCat}_{\mathbf{S}^{\Rightarrow}}(\mathcal{Y})$$

corresponding to $\mathbf{Loc}(\mathrm{IndCoh}(\mathcal{Y}^{\wedge}_{\mathcal{Z}}))$ under the Morita equivalence (5.6).

The value of $\mathbf{Loc}(\mathrm{IndCoh}(\mathcal{Y}^{\wedge}_{\mathcal{Z}}))$ at an object $\mathrm{S} \in \mathrm{Sm}_{/\mathcal{Y}}$ identifies with

$$\mathbf{Loc}(\mathrm{IndCoh}(\mathcal{Y}^{\wedge}_{\mathcal{Z}}))_{\mathrm{S}} \simeq \mathbf{H}_{\mathrm{S}\to\mathcal{Y}} \underset{\mathbf{H}(\mathcal{Y})}{\otimes} \mathrm{IndCoh}(\mathcal{Y}^{\wedge}_{\mathcal{Z}}) \simeq \mathrm{IndCoh}(\mathrm{S}^{\wedge}_{\mathcal{Z}_{\mathrm{S}}}),$$

with its natural action of $\mathbf{H}(\mathrm{S})$. Therefore, the equivalence (3.2), combined with the Morita equivalence for schemes, produces an equivalence of categories

$$\mathrm{D}(\mathrm{Sing}(\mathrm{S})_{\mathcal{Z}_{\mathrm{S}}})^{\Rightarrow} \simeq \mathcal{D}_{\mathrm{S}} \tag{5.21}$$

linear over $\mathrm{D}(\mathrm{Sing}(\mathrm{S}))^{\Rightarrow}$. However, for the purposes of Construction 5.4, this observation by itself is insufficient. It is not obvious how to supply the functoriality required to promote the data of (5.21) to an equivalence in $\mathrm{ShCat}_{\mathbf{S}^{\Rightarrow}}(\mathcal{Y})$.

So the challenge is to produce a morphism

$$\underline{\mathrm{D}}(\mathrm{Sing}(\mathcal{Y})_{\mathcal{Z}})^{\Rightarrow} \to \mathcal{D} \tag{5.22}$$

in $\mathrm{ShCat}_{\mathbf{S}^{\Rightarrow}}(\mathcal{Y})$. Once this has been done, it will be straightforward to verify that the value of (5.22) at every object of $\mathrm{Sm}_{/\mathcal{Y}}$ agrees with the equivalence (5.21), and hence that (5.22) itself is an equivalence.

We shall delay the execution of Construction 5.4 to § 5.4, and use the remainder of this subsection to explain some of its consequences.

5.3.3. The colocalization

$$\mathrm{IndCoh}_0(\mathcal{Y}^{\wedge}_{\mathcal{Z}}) \rightleftharpoons \mathrm{IndCoh}(\mathcal{Y}^{\wedge}_{\mathcal{Z}})$$

of $\mathbf{H}(\mathcal{Y})$-modules induces a colocalization

$$\mathbf{Loc}(\mathrm{IndCoh}_0(\mathcal{Y}^{\wedge}_{\mathcal{Z}})) \rightleftharpoons \mathbf{Loc}(\mathrm{IndCoh}(\mathcal{Y}^{\wedge}_{\mathcal{Z}})). \tag{5.23}$$

On the other hand, observe that we may identify

$$\mathrm{Sing}(\mathcal{Z}_{\mathrm{S}}/\mathrm{S}) \simeq \mathrm{S} \underset{\mathcal{Y}}{\times} \mathrm{Sing}(\mathcal{Z}/\mathcal{Y})$$

as closed subschemes of (5.20). In particular, we obtain a colocalization

$$\underline{\mathrm{D}}(\mathrm{Sing}(\mathcal{Z}/\mathcal{Y}))^{\Rightarrow} \rightleftharpoons \underline{\mathrm{D}}(\mathrm{Sing}(\mathcal{Y})_{\mathcal{Z}})^{\Rightarrow}$$

in $\mathrm{ShCat}_{\mathbf{S}^{\Rightarrow}}(\mathcal{Y})$ whose value at an object $\mathrm{S} \in \mathrm{Sm}_{/\mathcal{Y}}$ is given by the colocalization

$$\mathrm{D}(\mathrm{Sing}(\mathcal{Z}_{\mathrm{S}}/\mathrm{S}))^{\Rightarrow} \rightleftharpoons \mathrm{D}(\mathrm{Sing}(\mathrm{S})_{\mathcal{Z}_{\mathrm{S}}})^{\Rightarrow}.$$

Proposition 5.1. *Construction 5.4 exchanges the colocalizations*

$$\underline{\mathrm{D}}(\mathrm{Sing}(\mathcal{Z}/\mathcal{Y}))^{\Rightarrow} \quad \textit{and} \quad \mathbf{Loc}(\mathrm{IndCoh}_0(\mathcal{Y}^{\wedge}_{\mathcal{Z}})).$$

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5.3.4. Under our running assumptions on stacks (see § 1.5.6), for every schematic morphism $f : \mathcal{Z} \to \mathcal{Y}$ of quasi-smooth stacks, the graph $\Gamma_f : \mathcal{Z} \to \mathcal{Z} \times \mathcal{Y}$ is also schematic. Recall that one may consider the $(\mathbf{H}(\mathcal{Z}), \mathbf{H}(\mathcal{Y}))$-bimodule

$$\mathbf{H}_{\mathcal{Z}\to\mathcal{Y}} = \mathrm{IndCoh}_0((\mathcal{Z}\times\mathcal{Y})^{\wedge}_{\mathcal{Z}}).$$

Recall also the correspondence of stacks induced by the singular codifferential of f.

$$\begin{array}{ccc} \mathrm{Sing}(\mathcal{Y})_{\mathcal{Z}} & \longrightarrow & \mathrm{Sing}(\mathcal{Z}) \\ \downarrow & & \\ \mathrm{Sing}(\mathcal{Y}) & & \end{array} \tag{5.24}$$

Let us form the object

$$\underline{\mathrm{D}}(\mathrm{Sing}(\mathcal{Y})_{\mathcal{Z}})^{\Rightarrow} \in \mathrm{ShCat}_{\mathbf{S}^{\Rightarrow}}(\mathcal{Z}\times\mathcal{Y})$$

obtained from the morphism

$$\mathrm{Sing}(\mathcal{Y})_{\mathcal{Z}} \overset{(5.24)}{\longrightarrow} \mathrm{Sing}(\mathcal{Z}) \times \mathrm{Sing}(\mathcal{Y}) \overset{\tau\times\mathbf{1}}{\longrightarrow} \mathrm{Sing}(\mathcal{Z}) \times \mathrm{Sing}(\mathcal{Y}). \tag{5.25}$$

Since (5.25) identifies with the map

$$\mathrm{Sing}(\mathcal{Z}/\mathcal{Z}\times\mathcal{Y}) \to \mathrm{Sing}(\mathcal{Z}\times\mathcal{Y})_{\mathcal{Z}} \tag{5.26}$$

associated to the graph of f

$$\mathcal{Z} \to \mathcal{Z}\times\mathcal{Y},$$

we obtain the following consequence of Proposition 5.1.

Construction 5.5. The Morita equivalence (5.6) for $\mathcal{Z}\times\mathcal{Y}$ exchanges the objects

$$\underline{\mathrm{D}}(\mathrm{Sing}(\mathcal{Y})_{\mathcal{Z}})^{\Rightarrow} \quad \text{and} \quad \mathbf{Loc}(\mathbf{H}_{\mathcal{Z}\to\mathcal{Y}}).$$

5.4 Execution of Construction 5.4

5.4.1. As indicated in § 5.3.2, the task at hand is more or less to construct the morphism (5.22). This will be accomplished by subjecting the functor of IndCoh !-pullback

$$\mathrm{IndCoh}(\mathcal{Y}) \to \mathrm{IndCoh}(\mathcal{Y}^{\wedge}_{\mathcal{Z}}), \tag{5.27}$$

viewed as a morphism of $\mathbf{H}(\mathcal{Y})$-modules, to a series of transformations.

First we apply **Loc** to (5.27), and obtain a morphism

$$\mathbf{Loc}(\mathrm{IndCoh}(\mathcal{Y})) \to \mathbf{Loc}(\mathrm{IndCoh}(\mathcal{Y}^{\wedge}_{\mathcal{Z}})) \tag{5.28}$$

in $\mathrm{ShCat}_{\mathbf{H}}(\mathcal{Y})$.

Then, (5.28) may be viewed as a morphism

$$\underline{\mathrm{D}}(\mathrm{Sing}(\mathcal{Y}))^{\Rightarrow} \to \mathcal{D} \tag{5.29}$$

in $\mathrm{ShCat}_{\mathbf{S}^{\Rightarrow}}(\mathcal{Y})$.

5.4.2 *Digression.* To proceed further, we need to introduce the category

$$\mathrm{ShCat}_{\mathbf{S}^{\Rightarrow}}(\mathcal{Y})_{\mathcal{Z}} = \lim_{\mathrm{S}\to\mathcal{Y}} \mathrm{D}(\mathrm{Sing}(\mathrm{S})_{\mathcal{Z}_{\mathrm{S}}})^{\Rightarrow}\text{-}\mathbf{mod},$$

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where the limit is taken over $(\mathrm{Sm}_{/\mathcal{Y}})^{\mathrm{op}}$ with respect to the transition functors of induction:

$$\mathbf{ind} : \mathrm{D}(\mathrm{Sing}(\mathrm{S})_{\mathcal{Z}_{\mathrm{S}}})^{\Rightarrow}\text{-}\mathbf{mod} \to \mathrm{D}(\mathrm{Sing}(\mathrm{T})_{\mathcal{Z}_{\mathrm{T}}})^{\Rightarrow}\text{-}\mathbf{mod}.$$

Consider the adjunctions

$$\mathbf{ind}_{\mathrm{S}} : \mathrm{D}(\mathrm{Sing}(\mathrm{S}))^{\Rightarrow}\text{-}\mathbf{mod} \rightleftharpoons \mathrm{D}(\mathrm{Sing}(\mathrm{S})_{\mathcal{Z}_{\mathrm{S}}})^{\Rightarrow}\text{-}\mathbf{mod} : \mathbf{oblv}_{\mathrm{S}}.$$

Observe that the left adjoints $\mathbf{ind}_{\mathrm{S}}$ assemble tautologically into a functor

$$\mathbf{ind} : \mathrm{ShCat}_{\mathbf{S}^{\Rightarrow}}(\mathcal{Y}) \to \mathrm{ShCat}_{\mathbf{S}^{\Rightarrow}}(\mathcal{Y})_{\mathcal{Z}} \tag{5.30}$$

making the following diagram commute.

$$\begin{array}{ccc} \mathrm{ShCat}_{\mathbf{S}^{\Rightarrow}}(\mathcal{Y}) & \xrightarrow{\mathbf{ind}} & \mathrm{ShCat}_{\mathbf{S}^{\Rightarrow}}(\mathcal{Y})_{\mathcal{Z}} \\ \downarrow & & \downarrow \\ \mathrm{D}(\mathrm{Sing}(\mathrm{S}))^{\Rightarrow}\text{-}\mathbf{mod} & \xrightarrow{\mathbf{ind}_{\mathrm{S}}} & \mathrm{D}(\mathrm{Sing}(\mathrm{S})_{\mathcal{Z}_{\mathrm{S}}})^{\Rightarrow}\text{-}\mathbf{mod} \end{array}$$

Since the map $\mathcal{Z} \to \mathcal{Y}$ is schematic, the Beck–Chevalley transformations are isomorphisms.

$$\begin{array}{ccc} \mathrm{D}(\mathrm{Sing}(\mathrm{S}))^{\Rightarrow}\text{-}\mathbf{mod} & \xleftarrow{\mathbf{oblv}_{\mathrm{S}}} & \mathrm{D}(\mathrm{Sing}(\mathrm{S})_{\mathcal{Z}_{\mathrm{S}}})^{\Rightarrow}\text{-}\mathbf{mod} \\ {\scriptstyle \mathbf{ind}_{\mathrm{T}\to\mathrm{S}}}\downarrow & & \downarrow{\scriptstyle \mathbf{ind}_{\mathrm{T}\to\mathrm{S}}} \\ \mathrm{D}(\mathrm{Sing}(\mathrm{T}))^{\Rightarrow}\text{-}\mathbf{mod} & \xleftarrow[\mathbf{oblv}]{} & \mathrm{D}(\mathrm{Sing}(\mathrm{T})_{\mathcal{Z}_{\mathrm{T}}})^{\Rightarrow}\text{-}\mathbf{mod} \end{array} \tag{5.31}$$

Therefore, by passing to the limit, we obtain that (5.30) admits a right adjoint

$$\mathbf{oblv} : \mathrm{ShCat}_{\mathbf{S}^{\Rightarrow}}(\mathcal{Y})_{\mathcal{Z}} \to \mathrm{ShCat}_{\mathbf{S}^{\Rightarrow}}(\mathcal{Y})$$

making the following diagram commute.

$$\begin{array}{ccc} \mathrm{ShCat}_{\mathbf{S}^{\Rightarrow}}(\mathcal{Y}) & \xleftarrow{\mathbf{oblv}} & \mathrm{ShCat}_{\mathbf{S}^{\Rightarrow}}(\mathcal{Y})_{\mathcal{Z}} \\ \downarrow & & \downarrow \\ \mathrm{D}(\mathrm{Sing}(\mathrm{S}))^{\Rightarrow}\text{-}\mathbf{mod} & \xleftarrow[\mathbf{oblv}_{\mathrm{S}}]{} & \mathrm{D}(\mathrm{Sing}(\mathrm{S})_{\mathcal{Z}_{\mathrm{S}}})^{\Rightarrow}\text{-}\mathbf{mod} \end{array}$$

Note that we may identify

$$\mathbf{oblv} \circ \mathbf{ind}(\underline{\mathrm{D}}(\mathrm{Sing}(\mathcal{Y}))^{\Rightarrow}) \simeq \underline{\mathrm{D}}(\mathrm{Sing}(\mathcal{Y})_{\mathcal{Z}})^{\Rightarrow}. \tag{5.32}$$

5.4.3. Note that the value of (5.28) at an object $\mathrm{S} \in \mathrm{Sm}_{/\mathcal{Y}}$ is given by the functor of IndCoh !-pullback

$$\mathrm{IndCoh}(\mathrm{S}) \to \mathrm{IndCoh}(\mathrm{S}^{\wedge}_{\mathcal{Z}_{\mathrm{S}}}), \tag{5.33}$$

regarded as a morphism of $\mathbf{H}(\mathrm{S})$-modules.

We may view (5.33) as the functor

$$\mathrm{IndCoh}(\mathrm{S}) \to \mathrm{IndCoh}(\mathrm{S}) \underset{\mathrm{D}(\mathrm{Sing}(\mathrm{S}))^{\Rightarrow}}{\otimes} \mathrm{D}(\mathrm{Sing}(\mathrm{S})_{\mathcal{Z}_{\mathrm{S}}})^{\Rightarrow},$$

induced by the D-module !-pullback

$$\mathrm{D}(\mathrm{Sing}(\mathrm{S}))^{\Rightarrow} \to \mathrm{D}(\mathrm{Sing}(\mathrm{S})_{\mathcal{Z}_{\mathrm{S}}})^{\Rightarrow}. \tag{5.34}$$

In other words, the value of (5.29) at S identifies with the morphism (5.34).

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In particular, this lifts $\mathcal{D}_S$ to an object

$$\tilde{\mathcal{D}}_S \in \mathrm{D}(\mathrm{Sing}(S)_{\mathcal{Z}_S})^{\Rightarrow}\text{-}\mathbf{mod}.$$

We claim the following.

Proposition 5.2. *The data of $\tilde{\mathcal{D}}_S$ for $S \in \mathrm{Sm}_{/\mathcal{Y}}$ determine a lift of $\mathcal{D}$ to an object*

$$\tilde{\mathcal{D}} \in \mathrm{ShCat}_{\mathbf{S}^{\Rightarrow}}(\mathcal{Y})_{\mathcal{Z}}.$$

5.4.4. Note that this assertion is a proposition and not a construction, because Proposition 2.3 (which is applicable by the running assumption that $\mathcal{Z} \to \mathcal{Y}$ is schematic) implies that the functors $\mathbf{oblv}_S$ are locally fully faithful. Namely, it amounts to the assertion that for every arrow $T \to S$ in $\mathrm{Sm}_{/\mathcal{Y}}$, the map of $\mathrm{D}(\mathrm{Sing}(T))^{\Rightarrow}$-modules

$$\mathbf{oblv}_T \circ \mathbf{ind}_{T\to S}(\tilde{\mathcal{D}}_S) \simeq \mathbf{ind}_{T\to S} \circ \mathbf{oblv}_S(\tilde{\mathcal{D}}_S) \simeq \mathbf{ind}_{T\to S}(\mathcal{D}_S) \xrightarrow{\sim} \mathcal{D}_T \simeq \mathbf{oblv}_T(\tilde{\mathcal{D}}_T) \tag{5.35}$$

is actually linear over $\mathrm{D}(\mathrm{Sing}(T)_{\mathcal{Z}_T})^{\Rightarrow}$.

5.4.5. We now take Proposition 5.2 as given and resume the construction of (5.22). Its proofs appear at the end of this section, see § 5.4.6.

By Proposition 5.2, we may perceive (5.29) as a morphism

$$\underline{\mathrm{D}}(\mathrm{Sing}(\mathcal{Y}))^{\Rightarrow} \to \mathbf{oblv}(\tilde{\mathcal{D}}) \tag{5.36}$$

in $\mathrm{ShCat}_{\mathbf{S}^{\Rightarrow}}(\mathcal{Y})$.

The transpose of (5.36) under the $(\mathbf{ind}, \mathbf{oblv})$ adjunction is a morphism

$$\mathbf{ind}(\underline{\mathrm{D}}(\mathrm{Sing}(\mathcal{Y}))^{\Rightarrow}) \to \tilde{\mathcal{D}} \tag{5.37}$$

in $\mathrm{ShCat}_{\mathbf{S}^{\Rightarrow}}(\mathcal{Y})_{\mathcal{Z}}$.

After applying $\mathbf{oblv}$ to (5.37), we obtain a morphism

$$\mathbf{oblv} \circ \mathbf{ind}(\underline{\mathrm{D}}(\mathrm{Sing}(\mathcal{Y}))^{\Rightarrow}) \to \mathcal{D} \tag{5.38}$$

in $\mathrm{ShCat}_{\mathbf{S}^{\Rightarrow}}(\mathcal{Y})$.

Finally, after making the identification (5.32), we find that (5.38) defines the sought-after morphism

$$\underline{\mathrm{D}}(\mathrm{Sing}(\mathcal{Y})_{\mathcal{Z}})^{\Rightarrow} \to \mathcal{D} \tag{5.39}$$

in $\mathrm{ShCat}_{\mathbf{S}^{\Rightarrow}}(\mathcal{Y})$.

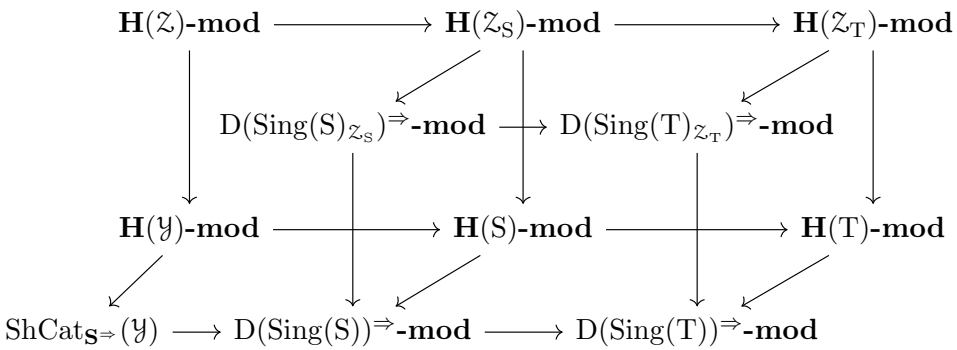
5.4.6. To finish Construction 5.4, it remains to verify that (5.39) is an isomorphism. Indeed, by the description of (5.29) given in § 5.4.3, the value of (5.39) at an object $S \in \mathrm{Sm}_{/\mathcal{Y}}$ is the canonical arrow

$$\mathrm{D}(\mathcal{Z}_S) \underset{\mathrm{D}(S)}{\otimes} \mathrm{D}(\mathrm{Sing}(S))^{\Rightarrow} \xrightarrow{\sim} \mathrm{D}(\mathrm{Sing}(S)_{\mathcal{Z}_S})^{\Rightarrow}.$$

Proof of Proposition 5.2. Consider the following diagram, where the upper back-to-front maps are the composites.

$$\mathbf{H}(\mathcal{Z}_S)\text{-}\mathbf{mod} \xrightarrow{\sim} \mathrm{D}(\mathrm{Sing}(\mathcal{Z}_S))^{\Rightarrow}\text{-}\mathbf{mod} \xrightarrow{\mathbf{ind}} \mathrm{D}(\mathrm{Sing}(S)_{\mathcal{Z}_S})^{\Rightarrow}\text{-}\mathbf{mod}$$

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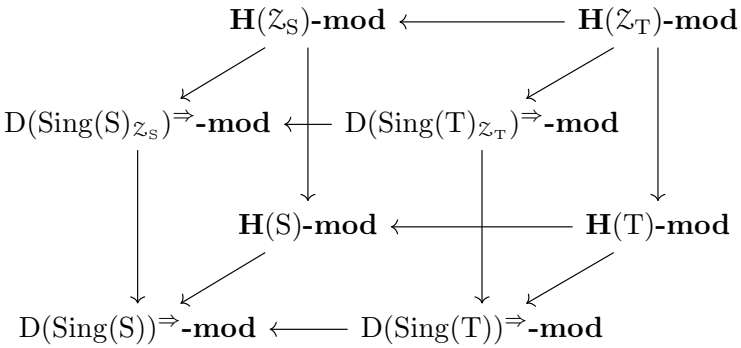
The two back faces are commutative by the functoriality of **H**, the front face is commutative by the Beck–Chevalley transformation (5.31), the two lower faces are commutative by the construction of the Morita equivalence (5.6), and the two vertical faces are commutative by Construction 4.1.

At the moment, we have *not* filled in the upper face of the cube. Observe that the isomorphism (5.35) is encoded by the commutativity of the preceding diagram, when evaluated on the object

$$\mathrm{IndCoh}(\mathcal{Z}) \in \mathbf{H}(\mathcal{Z})\text{-}\mathbf{mod},$$

and that a completion of the right half of the diagram into a commutative cube would lift (5.35) to an isomorphism in $\mathrm{D}(\mathrm{Sing}(\mathrm{T})_{\mathcal{Z}_\mathrm{T}})^{\Rightarrow}$. Let us construct this completion.

All five faces of the diagram obtained by taking right adjoints of the horizontal arrows remain commutative, so it is equivalent to complete this new diagram into a commutative cube.



To do so, apply Construction 4.2 to the composites

$$\mathcal{Z}_\mathrm{T} \to \mathcal{Z}_\mathrm{S} \to \mathrm{S}, \quad \mathcal{Z}_\mathrm{T} \to \mathrm{T} \to \mathrm{S}$$

and paste together the two resulting prisms. □

5.4.7 *Porism.* Construction 5.4 comes with an additional compatibility: the Morita equivalence (5.6) exchanges the morphism

$$\underline{\mathrm{D}}(\mathrm{Sing}(\mathcal{Y}))^{\Rightarrow} \to \underline{\mathrm{D}}(\mathrm{Sing}(\mathcal{Y})_{\mathcal{Z}})^{\Rightarrow}$$

induced by D-module !-pullback with the morphism

$$\mathbf{Loc}(\mathrm{IndCoh}(\mathcal{Y}) \to \mathrm{IndCoh}(\mathcal{Y}^{\wedge}_{\mathcal{Z}}))$$

induced by IndCoh !-pullback.

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Combining this with Proposition 5.1, we see that Construction 5.5 comes with an additional compatibility: the Morita equivalence (5.6) for $\mathcal{Z} \times \mathcal{Y}$ exchanges the morphism

$$\underline{\mathrm{D}}(\mathrm{Sing}(\mathcal{Z} \times \mathcal{Y}))^{\Rightarrow} \to \underline{\mathrm{D}}(\mathrm{Sing}(\mathcal{Z}/\mathcal{Z} \times \mathcal{Y}))^{\Rightarrow}$$

induced by D-module !-pullback with the morphism

$$\mathbf{Loc}(\mathrm{IndCoh}(\mathcal{Z} \times \mathcal{Y}) \to \mathrm{IndCoh}((\mathcal{Z} \times \mathcal{Y})^{\wedge}_{\mathcal{Z}}) \to \mathbf{H}_{\mathcal{Z} \to \mathcal{Y}})$$

obtained by composing the functor of IndCoh !-pullback with the right adjoint of the colocalization defining $\mathbf{H}_{\mathcal{Z} \to \mathcal{Y}}$.

5.5 Sheared crystals of categories on the stack of singularities

5.5.1. The goal of this somewhat digressive subsection is to study the relationship between $\mathrm{D}(\mathrm{Sing}(\mathcal{Y}))^{\Rightarrow}$**-mod** and $\mathrm{ShCat}_{\mathbf{S}^{\Rightarrow}}(\mathcal{Y})$. We show that they are connected by a pair of adjoint functors

$$\mathbf{Loc} : \mathrm{D}(\mathrm{Sing}(\mathcal{Y}))^{\Rightarrow}\text{-}\mathbf{mod} \rightleftharpoons \mathrm{ShCat}_{\mathbf{S}^{\Rightarrow}}(\mathcal{Y}) : \mathbf{\Gamma}$$

in **2-dgCat**, and that this adjunction is in fact a colocalization. Applying the Morita equivalence (5.6) and the H-affineness equivalence, we obtain that $\mathrm{IndCoh}(\mathcal{Y})$ is naturally a $(\mathbf{H}(\mathcal{Y}), \mathrm{D}(\mathrm{Sing}(\mathcal{Y}))^{\Rightarrow})$-bimodule defining a colocalization

$$\mathrm{D}(\mathrm{Sing}(\mathcal{Y}))^{\Rightarrow}\text{-}\mathbf{mod} \hookrightarrow \mathbf{H}(\mathcal{Y})\text{-}\mathbf{mod}.$$

5.5.2. Observe that the family of functors

$$\mathbf{ind} : \mathrm{D}(\mathrm{Sing}(\mathcal{Y}))^{\Rightarrow}\text{-}\mathbf{mod} \to \mathrm{D}(\mathrm{Sing}(\mathrm{S}))^{\Rightarrow}\text{-}\mathbf{mod}$$

assemble into a functor

$$\mathbf{Loc} : \mathrm{D}(\mathrm{Sing}(\mathcal{Y}))^{\Rightarrow}\text{-}\mathbf{mod} \to \mathrm{ShCat}_{\mathbf{S}^{\Rightarrow}}(\mathcal{Y}).$$

We claim the following.

Proposition 5.3. *The functor* **Loc** *admits a right adjoint in* ***2-dgCat****.*

5.5.3. It is easy to see that **Loc** admits a right adjoint

$$\mathbf{\Gamma} : \mathrm{ShCat}_{\mathbf{S}^{\Rightarrow}}(\mathcal{Y}) \to \mathrm{D}(\mathrm{Sing}(\mathcal{Y}))^{\Rightarrow}\text{-}\mathbf{mod},$$

computed by the formula

$$\mathcal{C} \mapsto \lim_{\mathrm{S} \to \mathcal{Y}} \mathbf{\Gamma}(\mathrm{S}; \mathcal{C}),$$

where the limit is taken over $\mathrm{Sm}_{/\mathcal{Y}}$. The content of Proposition 5.3 is that this functor is continuous and commutes with the operation of tensoring with a plain dg category.

Proof of Proposition 5.3. In light of the Morita equivalence (5.6) and the H-affineness theorem, it is equivalent to show that the functor

$$\mathrm{D}(\mathrm{Sing}(\mathcal{Y}))^{\Rightarrow}\text{-}\mathbf{mod} \xrightarrow{\mathbf{Loc}} \mathrm{ShCat}_{\mathbf{S}^{\Rightarrow}}(\mathcal{Y}) \xrightarrow{\sim} \mathrm{ShCat}_{\mathbf{H}}(\mathcal{Y}) \xrightarrow{\sim} \mathbf{H}(\mathcal{Y})\text{-}\mathbf{mod}$$

admits a right adjoint in **2-dgCat**.

Now observe that we have the adjunction

$$\mathbf{ind} : \mathbf{dgCat} \rightleftharpoons \mathrm{D}(\mathrm{Sing}(\mathcal{Y}))^{\Rightarrow}\text{-}\mathbf{mod} : \mathbf{oblv}$$

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in **2-dgCat**, where the right adjoint **oblv** is conservative. Therefore, it is enough to show that the composite

$$\mathbf{dgCat} \xrightarrow{\mathbf{ind}} \mathrm{D}(\mathrm{Sing}(\mathcal{Y}))^{\Rightarrow}\text{-}\mathbf{mod} \to \mathbf{H}(\mathcal{Y})\text{-}\mathbf{mod}$$

admits a continuous right adjoint.

This functor is the one defined by $\mathrm{IndCoh}(\mathcal{Y})$, as an $\mathbf{H}(\mathcal{Y})$-module. Therefore, it is equivalent to show that $\mathrm{IndCoh}(\mathcal{Y})$ is dualizable as an $\mathbf{H}(\mathcal{Y})$-module. This follows from the fact that $\mathrm{IndCoh}(\mathcal{Y})$ is dualizable as a plain dg category and that $\mathbf{H}(\mathcal{Y})$ is rigid. □

Proposition 5.4. *The functor* **Loc** *is fully faithful.*

Proof of Proposition 5.4. Since $(\mathbf{Loc}, \mathbf{\Gamma})$ is an adjunction in **2-dgCat**, it suffices to check that the functor

$$\mathrm{D}(\mathrm{Sing}(\mathcal{Y}))^{\Rightarrow} \to \mathrm{End}_{\mathrm{ShCat}_{\mathbf{S}^{\Rightarrow}(\mathcal{Y})}}(\mathbf{Loc}(\mathrm{D}(\mathrm{Sing}(\mathcal{Y}))^{\Rightarrow}))$$

is an equivalence. But this functor rewrites as

$$\mathrm{D}(\mathrm{Sing}(\mathcal{Y}))^{\Rightarrow} \to \lim_{\mathrm{S}\to\mathcal{Y}} \mathrm{D}(\mathrm{Sing}(\mathrm{S}))^{\Rightarrow},$$

where the limit is taken over $\mathrm{Sm}_{/\mathcal{Y}}$, so the result follows from smooth descent for D-modules. □

5.5.4 *Remark.* Note that we have a commutative diagram in which the vertical arrows are fully faithful.

$$\begin{array}{ccc} \mathrm{D}(\mathcal{Y})\text{-}\mathbf{mod} & \xrightarrow{\mathbf{Loc}} & \mathrm{ShCat}(\mathcal{Y}_{\mathrm{dR}}) \\ \downarrow{\scriptstyle \mathbf{res}} & & \downarrow \\ \mathrm{D}(\mathrm{Sing}(\mathcal{Y}))^{\Rightarrow}\text{-}\mathbf{mod} & \xrightarrow{\mathbf{Loc}} & \mathrm{ShCat}_{\mathbf{S}^{\Rightarrow}}(\mathcal{Y}) \end{array}$$

Therefore, Proposition 5.4 confirms [Gai15, Conjecture 2.6.6] for stacks which are nil-isomorphic to a quasi-smooth stack.

6. Trace computations

The outline of this section is as follows.

- In § 6.1, we review the basic formalism of categorical traces and prove some simple results about the behavior of traces with respect to duality. The results of this section are a categorification of the fact that the trace of a matrix equals the trace of its transpose.
- In § 6.2, we use Theorem 4.1 to calculate the trace of $\mathbf{H}(\mathcal{Y})$-**mod** with respect to the bimodule $\mathbf{H}_{\phi:\mathcal{Y}\to\mathcal{Y}}$ for any schematic endomorphism of a quasi-smooth stack.
- In § 6.3, we relate the Chern class of $\mathrm{IndCoh}(\mathcal{Y})$ as a 'Weil-object' of $\mathbf{H}(\mathcal{Y})$-**mod** with the enhanced trace of [GKR$^+$22].
- In § 6.4, we calculate how taking traces interacts with the singular support filtration on $\mathrm{IndCoh}(\mathcal{Y})$.

6.1 Review of categorical trace

6.1.1. Recall that for any endomorphism $\phi : \mathcal{C} \to \mathcal{C}$ of a dualizable object in an ambient symmetric monoidal category $\mathbf{O}$, it makes sense to form the trace

$$\mathrm{tr}(\phi; \mathcal{C}) \in \mathrm{End}(\mathbf{1}_{\mathbf{O}}),$$

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given definitionally by the composite

$$\mathbf{1_O} \xrightarrow{\text{coev}} \mathcal{C} \otimes \mathcal{C}^\vee \xrightarrow{\phi \otimes \mathbf{1}} \mathcal{C} \otimes \mathcal{C}^\vee \xrightarrow{\text{ev}} \mathbf{1_O}.$$

Under the identification $\mathbf{1_O} \simeq \mathbf{1_O^\vee}$, the following diagram always naturally commutes for any ψ in $\mathrm{End}(\mathbf{1_O})$.

$$\begin{array}{ccc} \mathbf{1_O} & \xrightarrow{\psi} & \mathbf{1_O} \\ \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\ \mathbf{1_O^\vee} & \xrightarrow{\psi^\vee} & \mathbf{1_O^\vee} \end{array}$$

This gives rise to a canonical identification

$$\mathrm{tr}(\phi; \mathcal{C}) \simeq \mathrm{tr}(\phi^\vee; \mathcal{C}^\vee). \tag{6.1}$$

6.1.2. Suppose now that $\mathbf{O}$ is a symmetric monoidal $(\infty, 2)$-category. Then a natural transformation $\alpha : \phi \to \psi$ of endomorphisms induces a morphism

$$\mathrm{tr}(\phi; \mathcal{C}) \to \mathrm{tr}(\psi; \mathcal{C}).$$

The formation of this arrow is compatible with duality in the sense that the morphism

$$\mathrm{tr}(\phi^\vee; \mathcal{C}^\vee) \to \mathrm{tr}(\psi^\vee; \mathcal{C}^\vee)$$

induced by the natural transformation $\alpha^\vee : \phi^\vee \to \psi^\vee$ fits into a commutative diagram.

$$\begin{array}{ccc} \mathrm{tr}(\phi; \mathcal{C}) & \longrightarrow & \mathrm{tr}(\psi; \mathcal{C}) \\ \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\ \mathrm{tr}(\phi^\vee; \mathcal{C}^\vee) & \longrightarrow & \mathrm{tr}(\psi^\vee; \mathcal{C}^\vee) \end{array} \tag{6.2}$$

6.1.3. More generally, suppose that $\mathcal{D}$ is another dualizable object equipped with an endomorphism, which we shall denote by ψ. Take any right adjointable functor $\mathrm{F} : \mathcal{C} \to \mathcal{D}$ equipped with a natural transformation as follows.

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\phi} & \mathcal{C} \\ {\scriptstyle \mathrm{F}}\downarrow & \swarrow & \downarrow{\scriptstyle \mathrm{F}} \\ \mathcal{D} & \xrightarrow[\psi]{} & \mathcal{D} \end{array} \qquad \alpha : \mathrm{F} \circ \phi \to \psi \circ \mathrm{F} \tag{6.3}$$

We then obtain a morphism

$$\mathrm{cl}(\mathrm{F}, \alpha) : \mathrm{tr}(\phi; \mathcal{C}) \to \mathrm{tr}(\psi; \mathcal{D}), \tag{6.4}$$

given definitionally by the composite

$$\begin{aligned} \mathrm{tr}(\phi; \mathcal{C}) &\xrightarrow{\text{unit}} \mathrm{tr}(\phi \circ \mathrm{F}^{\mathrm{R}} \circ \mathrm{F}; \mathcal{C}) \\ &\simeq \mathrm{tr}(\mathrm{F} \circ \phi \circ \mathrm{F}^{\mathrm{R}}; \mathcal{D}) \xrightarrow{\alpha} \mathrm{tr}(\psi \circ \mathrm{F} \circ \mathrm{F}^{\mathrm{R}}; \mathcal{D}) \xrightarrow{\text{counit}} \mathrm{tr}(\psi; \mathcal{D}), \end{aligned}$$

where middle equivalence is the cyclicity of trace (see e.g. [BN21, Definition 3.24] [GKR$^+$22, § 0.2.4]), and the arrow labeled α comes from the construction of § 6.1.2. In fact, it is straightforward to see that the arrow constructed in § 6.1.2 identifies with $\mathrm{cl}(\mathbf{1}_{\mathcal{C}}, \alpha)$.

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6.1.4. The 2-categorical functoriality of traces enjoys a further compatibility with duality. Namely, consider the natural transformation

$$\alpha^\vee : \phi^\vee \circ \mathrm{F}^\vee \to \mathrm{F}^\vee \circ \psi^\vee$$

obtained from α by passing to dual functors. Since F is right adjointable, $\mathrm{F}^\vee$ is left adjointable with $(\mathrm{F}^\vee)^{\mathrm{L}} \simeq (\mathrm{F}^{\mathrm{R}})^\vee$. Therefore, we obtain from $\alpha^\vee$ a natural transformation

$$\beta : (\mathrm{F}^\vee)^{\mathrm{L}} \circ \phi^\vee \to \psi^\vee \circ (\mathrm{F}^\vee)^{\mathrm{L}}.$$

We will obtain the following compatibility by a routine diagram chase.

Computation 6.1. There is a commutative diagram as follows.

$$\begin{array}{ccc} \mathrm{tr}(\phi; \mathcal{C}) & \xrightarrow{\mathrm{cl}(\mathrm{F},\alpha)} & \mathrm{tr}(\psi; \mathcal{D}) \\ \downarrow \sim & & \downarrow \sim \\ \mathrm{tr}(\phi^\vee; \mathcal{C}^\vee) & \xrightarrow{\mathrm{cl}((\mathrm{F}^\vee)^{\mathrm{L}},\beta)} & \mathrm{tr}(\psi^\vee; \mathcal{D}^\vee) \end{array}$$

6.1.5 *Execution of Computation 6.1.* Applying (6.2) to every arrow appearing in the definition of $\mathrm{cl}((\mathrm{F}^\vee)^{\mathrm{L}}, \beta)$ identifies the lower circuit of this diagram with the composite

$$\begin{aligned} &\mathrm{tr}(\phi; \mathcal{C}) \xrightarrow{\mathrm{unit}} \mathrm{tr}(\mathrm{F}^{\mathrm{R}} \circ \mathrm{F} \circ \phi; \mathcal{C}) \simeq \mathrm{tr}(\mathrm{F} \circ \phi \circ \mathrm{F}^{\mathrm{R}}; \mathcal{D}) \\ &\xrightarrow{\beta^\vee} \mathrm{tr}(\mathrm{F} \circ \mathrm{F}^{\mathrm{R}} \circ \psi; \mathcal{D}) \xrightarrow{\mathrm{counit}} \mathrm{tr}(\psi; \mathcal{D}) \simeq \mathrm{tr}(\psi^\vee; \mathcal{D}^\vee), \end{aligned} \tag{6.5}$$

where the second arrow is induced by the natural transformation

$$\beta^\vee : \phi \circ \mathrm{F}^{\mathrm{R}} \to \mathrm{F}^{\mathrm{R}} \circ \psi.$$

Using the cyclic property of the trace, we rewrite the first line of (6.5) to replace it with

$$\begin{aligned} &\mathrm{tr}(\phi; \mathcal{C}) \xrightarrow{\mathrm{unit}} \mathrm{tr}(\phi \circ \mathrm{F}^{\mathrm{R}} \circ \mathrm{F}; \mathcal{C}) \simeq \mathrm{tr}(\mathrm{F} \circ \phi \circ \mathrm{F}^{\mathrm{R}}; \mathcal{D}) \\ &\xrightarrow{\beta^\vee} \mathrm{tr}(\mathrm{F} \circ \mathrm{F}^{\mathrm{R}} \circ \psi; \mathcal{D}) \xrightarrow{\mathrm{counit}} \mathrm{tr}(\psi; \mathcal{D}) \simeq \mathrm{tr}(\psi^\vee; \mathcal{D}^\vee). \end{aligned} \tag{6.6}$$

Comparing this expression with the definition of $\mathrm{cl}(\alpha; \phi)$, we see that it will suffice to produce a commutative diagram.

$$\begin{array}{ccc} \mathrm{F} \circ \phi \circ \mathrm{F}^{\mathrm{R}} & \xrightarrow{\beta^\vee} & \mathrm{F} \circ \mathrm{F}^{\mathrm{R}} \circ \psi \\ \downarrow \alpha & & \downarrow \mathrm{counit} \\ \psi \circ \mathrm{F} \circ \mathrm{F}^{\mathrm{R}} & \xrightarrow{\mathrm{counit}} & \psi \end{array} \tag{6.7}$$

By definition, the upper circuit of (6.7) is given by the upper circuit of the following.

$$\begin{array}{ccccc} & & \mathrm{F} \circ \phi \circ \mathrm{F}^{\mathrm{R}} & & \\ & & \downarrow \mathrm{unit} & & \\ \mathrm{F} \circ \mathrm{F}^{\mathrm{R}} \circ \mathrm{F} \circ \phi \circ \mathrm{F}^{\mathrm{R}} & \xrightarrow{\alpha} & \mathrm{F} \circ \mathrm{F}^{\mathrm{R}} \circ \psi \circ \mathrm{F} \circ \mathrm{F}^{\mathrm{R}} & \xrightarrow{\mathrm{counit}} & \mathrm{F} \circ \mathrm{F}^{\mathrm{R}} \circ \psi \\ \downarrow \mathrm{counit} & & \downarrow \mathrm{counit} & & \downarrow \mathrm{counit} \\ \mathrm{F} \circ \phi \circ \mathrm{F}^{\mathrm{R}} & \xrightarrow{\alpha} & \psi \circ \mathrm{F} \circ \mathrm{F}^{\mathrm{R}} & \xrightarrow{\mathrm{counit}} & \psi \end{array} \tag{6.8}$$

The triangular identity implies that the left vertical composite of (6.8) is the identity, so its lower circuit agrees with the lower circuit of (6.7). This completes Computation 6.1.

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6.1.6. Let $\mathcal{A}$ be a monoidal category and let Φ be an $(\mathcal{A},\mathcal{A})$-bimodule, perceived as an endofunctor

$$\Phi : \mathcal{A}\textbf{-mod} \to \mathcal{A}\textbf{-mod} \quad \mathcal{M} \mapsto \Phi \otimes_{\mathcal{A}} \mathcal{M}.$$

It makes sense to take the trace of this endofunctor when Φ is right dualizable; this is the dg category

$$\mathrm{tr}(\Phi; \mathcal{A}\textbf{-mod}) = \mathcal{A} \underset{\mathcal{A}\otimes\mathcal{A}^{\mathrm{rev}}}{\otimes} \Phi.$$

Now let $(\mathcal{M},\alpha)$ be an $\mathcal{A}$-module equipped with lax equivariance data for Φ; that is, $\mathcal{M}$ is an $\mathcal{A}$-module and α is a morphism of $\mathcal{A}$-modules

$$\alpha : \mathcal{M} \to \Phi \otimes_{\mathcal{A}} \mathcal{M}.$$

If $\mathcal{M}$ admits a right dual as an $\mathcal{A}$-module, then one may form the Chern class

$$\mathrm{cl}(\mathcal{M},\alpha) \in \mathrm{tr}(\Phi; \mathcal{A}\textbf{-mod}),$$

given by the composite

$$\begin{aligned} \mathbf{1} &\xrightarrow{\mathrm{coev}} \mathcal{M}^\vee \otimes_{\mathcal{A}} \mathcal{M} \xrightarrow{\mathbf{1}\otimes\alpha} \mathcal{M}^\vee \otimes_{\mathcal{A}} (\Phi \otimes_{\mathcal{A}} \mathcal{M}) \\ &\simeq (\mathcal{M}\otimes\mathcal{M}^\vee) \underset{\mathcal{A}\otimes\mathcal{A}^{\mathrm{rev}}}{\otimes} \Phi \xrightarrow{\mathrm{ev}\otimes\mathbf{1}} \mathcal{A} \underset{\mathcal{A}\otimes\mathcal{A}^{\mathrm{rev}}}{\otimes} \Phi. \end{aligned} \tag{6.9}$$

Let us note for future reference that this composite admits an alternative description. Since $\mathcal{M}$ is dualizable, we may view α as a morphism of $(\mathcal{A},\mathcal{A})$-bimodules

$$\alpha' : \mathcal{M}\otimes\mathcal{M}^\vee \to \Phi.$$

Then (6.9) identifies with the composite

$$\begin{aligned} \mathbf{1} &\xrightarrow{\mathrm{coev}} \mathcal{M}^\vee \otimes_{\mathcal{A}} \mathcal{M} \\ &\simeq \mathcal{A} \underset{\mathcal{A}\otimes\mathcal{A}^{\mathrm{rev}}}{\otimes} (\mathcal{M}\otimes\mathcal{M}^\vee) \to \mathcal{A} \underset{\mathcal{A}\otimes\mathcal{A}^{\mathrm{rev}}}{\otimes} \Phi. \end{aligned} \tag{6.10}$$

6.2 Carré du champ

6.2.1. Let $\phi : \mathcal{Y} \to \mathcal{Y}$ be a schematic endomorphism of a quasi-smooth stack. Consider the $(\mathbf{H}(\mathcal{Y}),\mathbf{H}(\mathcal{Y}))$-bimodule

$$\mathbf{H}_\phi = \mathbf{H}_{\mathcal{Y}\to\mathcal{Y}}$$

associated to the morphism ϕ by § 5.3.4. We will use the same symbol to denote the corresponding endomorphism:

$$\mathbf{H}_\phi : \mathbf{H}(\mathcal{Y})\textbf{-mod} \to \mathbf{H}(\mathcal{Y})\textbf{-mod}.$$

Observe that $\mathrm{IndCoh}(\mathcal{Y})$ is naturally equipped with a lax equivariance datum

$$\alpha : \mathrm{IndCoh}(\mathcal{Y}) \to \mathbf{H}_\phi \underset{\mathbf{H}(\mathcal{Y})}{\otimes} \mathrm{IndCoh}(\mathcal{Y})$$

for $\mathbf{H}_\phi$, corresponding under the self-duality of $\mathrm{IndCoh}(\mathcal{Y})$ as an $\mathbf{H}(\mathcal{Y})$-module to the composite

$$\mathrm{IndCoh}(\mathcal{Y}\times\mathcal{Y}) \to \mathrm{IndCoh}((\mathcal{Y}\times\mathcal{Y})^\wedge_{\mathcal{Y}}) \to \mathbf{H}_\phi \tag{6.11}$$

of IndCoh !-pullback along the formal completion of the graph of ϕ with the right adjoint of the colocalization defining $\mathbf{H}_\phi$. The goal of this subsection is to give a description of the object

$$\mathrm{cl}(\mathrm{IndCoh}(\mathcal{Y}),\alpha) \in \mathrm{tr}(\mathbf{H}_\phi; \mathbf{H}(\mathcal{Y})\textbf{-mod}).$$

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6.2.2. Recall that a stack $\mathcal{X}$ is said to be quasi-quasi-smooth if its cotangent complex has tor amplitude $[2,-1]$. When this happens, $\mathbf{L}_{\mathcal{X}}[-2]$ is a perfect complex of tor amplitude $[0,-3]$ and we may form the algebraic stack

$$\mathrm{Sing}_2(\mathcal{X}) = \mathrm{Spec}_{\mathcal{X}}(\mathrm{Sym}(\mathbf{L}_{\mathcal{X}}[-2])^{\vee}).$$

We think of $\mathrm{Sing}_2(\mathcal{X})$ as measuring the extent to which $\mathcal{X}$ fails to be quasi-smooth. More precisely, the projection map

$$\mathrm{Sing}_2(\mathcal{X}) \to \mathcal{X}$$

will induce an isomorphism at the level of underlying classical stacks if and only if $\mathcal{X}$ is quasi-smooth.

6.2.3. The reason we need to consider quasi-quasi-smooth stacks is that the stack of fixed points $\mathcal{Y}^{\phi}$ is not usually quasi-smooth. Instead, it is quasi-quasi-smooth, being a fiber product of quasi-smooth stacks as follows.

$$\begin{array}{ccc} \mathcal{Y}^{\phi} & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow{\scriptstyle \mathbf{1}\times\phi} \\ \mathcal{Y} & \xrightarrow{\Delta} & \mathcal{Y}\times\mathcal{Y} \end{array}$$

One has that

$$\mathrm{Sing}_2(\mathcal{Y}^{\phi}) \simeq \mathrm{Sing}(\mathcal{Y}/\mathcal{Y}\times\mathcal{Y}) \underset{\mathrm{Sing}(\mathcal{Y}\times\mathcal{Y})}{\times} \mathrm{Sing}_{\phi}(\mathcal{Y}/\mathcal{Y}\times\mathcal{Y}),$$

where

$$\mathrm{Sing}_{\phi}(\mathcal{Y}/\mathcal{Y}\times\mathcal{Y}) \to \mathrm{Sing}(\mathcal{Y}) \times \mathrm{Sing}(\mathcal{Y})$$

is the map (5.26) for the morphism ϕ.

Computation 6.2. We have

$$\mathrm{tr}(\mathbf{H}_{\phi}(\mathcal{Y}); \mathbf{H}(\mathcal{Y})\textbf{-mod}) \simeq \mathrm{D}(\mathrm{Sing}_2(\mathcal{Y}^{\phi}))^{\Rightarrow}, \tag{6.12}$$

and this identification takes the object

$$\mathrm{cl}(\mathrm{IndCoh}(\mathcal{Y}), \alpha) \in \mathrm{tr}(\mathbf{H}_{\phi}; \mathbf{H}(\mathcal{Y})\textbf{-mod}) \tag{6.13}$$

to the object

$$(\omega_{\mathrm{Sing}_2(\mathcal{Y}^{\phi})})^{\Rightarrow} \in \mathrm{D}(\mathrm{Sing}_2(\mathcal{Y}^{\phi}))^{\Rightarrow}.$$

6.2.4 *Execution of Computation 6.2.* Let us begin with the identification

$$\begin{aligned} \mathrm{tr}(\mathbf{H}_{\phi}; \mathbf{H}(\mathcal{Y})\textbf{-mod}) &\simeq \mathbf{H}(\mathcal{Y}) \underset{\mathbf{H}(\mathcal{Y})\otimes\mathbf{H}(\mathcal{Y})^{\mathrm{rev}}}{\otimes} \mathbf{H}_{\phi} \\ &\simeq \mathrm{Maps}_{(\mathbf{H}(\mathcal{Y}),\mathbf{H}(\mathcal{Y}))}(\mathbf{H}(\mathcal{Y}), \mathbf{H}_{\phi}). \end{aligned} \tag{6.14}$$

Since the coevaluation

$$\mathrm{Vect} \to \mathrm{IndCoh}(\mathcal{Y}) \underset{\mathbf{H}(\mathcal{Y})}{\otimes} \mathrm{IndCoh}(\mathcal{Y}) \simeq \mathrm{Maps}_{(\mathbf{H}(\mathcal{Y}),\mathbf{H}(\mathcal{Y}))}(\mathbf{H}(\mathcal{Y}), \mathrm{IndCoh}(\mathcal{Y}\times\mathcal{Y}))$$

classifies the map of $(\mathbf{H}(\mathcal{Y}), \mathbf{H}(\mathcal{Y}))$-bimodules

$$\mathbf{H}(\mathcal{Y}) \hookrightarrow \mathrm{IndCoh}((\mathcal{Y}\times\mathcal{Y})^{\wedge}_{\mathcal{Y}}) \xrightarrow{(-)_*} \mathrm{IndCoh}(\mathcal{Y}\times\mathcal{Y}),$$

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the description (6.10) of Chern classes implies that the isomorphism (6.14) takes the class of IndCoh (6.13) to the object of

$$\mathrm{Maps}_{(\mathbf{H}(\mathcal{Y}),\mathbf{H}(\mathcal{Y}))}(\mathbf{H}(\mathcal{Y}),\mathbf{H}_\phi)$$

given by the composite

$$\mathbf{H}(\mathcal{Y}) \to \mathrm{IndCoh}(\mathcal{Y}\times\mathcal{Y}) \to \mathbf{H}_\phi. \tag{6.15}$$

After applying the equivalence

$$(\mathbf{H}(\mathcal{Y}),\mathbf{H}(\mathcal{Y}))\textbf{-bimod} \simeq \mathbf{H}(\mathcal{Y}\times\mathcal{Y})\textbf{-mod} \simeq \mathrm{ShCat}_{\mathbf{H}}(\mathcal{Y}\times\mathcal{Y}) \xrightarrow{(5.6)} \mathrm{ShCat}_{\mathbf{S}^\Rightarrow}(\mathcal{Y}\times\mathcal{Y}),$$

we find that (6.14) identifies with

$$\mathrm{Maps}_{\mathrm{ShCat}_{\mathbf{S}^\Rightarrow}(\mathcal{Y}\times\mathcal{Y})}(\underline{\mathrm{D}}(\mathrm{Sing}(\mathcal{Y}/\mathcal{Y}\times\mathcal{Y}))^\Rightarrow, \underline{\mathrm{D}}(\mathrm{Sing}_\phi(\mathcal{Y}/\mathcal{Y}\times\mathcal{Y}))^\Rightarrow). \tag{6.16}$$

By § 5.4.7, this identification sends (6.15) to the composite

$$\underline{\mathrm{D}}(\mathrm{Sing}(\mathcal{Y}/\mathcal{Y}\times\mathcal{Y}))^\Rightarrow \to \underline{\mathrm{D}}(\mathrm{Sing}(\mathcal{Y}\times\mathcal{Y}))^\Rightarrow \to \underline{\mathrm{D}}(\mathrm{Sing}_\phi(\mathcal{Y}/\mathcal{Y}\times\mathcal{Y}))^\Rightarrow \tag{6.17}$$

in $\mathrm{ShCat}_{\mathbf{S}^\Rightarrow}(\mathcal{Y}\times\mathcal{Y})$.

We finish the computation by writing (6.16) as the limit of the categories

$$\begin{aligned}&\lim_{\mathrm{S}\to\mathcal{Y}\times\mathcal{Y}} \mathrm{Maps}_{\mathrm{D}(\mathrm{Sing}(\mathrm{S}))^\Rightarrow}(\mathrm{D}(\mathrm{Sing}(\mathcal{Y}/\mathcal{Y}\times\mathcal{Y})_{\mathrm{S}})^\Rightarrow, \mathrm{D}(\mathrm{Sing}_\phi(\mathcal{Y}/\mathcal{Y}\times\mathcal{Y})_{\mathrm{S}})^\Rightarrow)\\ &\simeq \lim_{\mathrm{S}\to\mathcal{Y}\times\mathcal{Y}} \mathrm{D}(\mathrm{Sing}_2(\mathcal{Y}^\phi)_{\mathrm{S}})^\Rightarrow \simeq \mathrm{D}(\mathrm{Sing}_2(\mathcal{Y}^\phi))^\Rightarrow\end{aligned} \tag{6.18}$$

taken over $\mathrm{Sm}_{/\mathcal{Y}\times\mathcal{Y}}$. By base change for the following Cartesian diagram, the composite arrow (6.17) is sent to $(\omega_{\mathrm{Sing}_2(\mathcal{Y}^\phi)})^\Rightarrow$ under (6.18).

$$\begin{array}{ccc}\mathrm{Sing}_2(\mathcal{Y}^\phi) & \longrightarrow & \mathrm{Sing}(\mathcal{Y}/\mathcal{Y}\times\mathcal{Y})\\ \downarrow & & \downarrow\\ \mathrm{Sing}_\phi(\mathcal{Y}/\mathcal{Y}\times\mathcal{Y}) & \longrightarrow & \mathrm{Sing}(\mathcal{Y}\times\mathcal{Y})\end{array}$$

6.2.5 *Remark.* In the case where ϕ is the identity morphism, this computation produces an equivalence of $\mathbf{E}_2$-categories:

$$\mathcal{Z}_{\mathbf{E}_2}(\mathbf{H}(\mathcal{Y})) = \mathrm{End}_{\mathbf{H}(\mathcal{Y})\textbf{-bimod}}(\mathbf{H}(\mathcal{Y})) \simeq \mathrm{D}(\mathrm{Sing}_2(\mathrm{L}\mathcal{Y}))^\Rightarrow.$$

On the other hand, the first author has exhibited (see [Ber21]) an equivalence of plain dg categories

$$\mathcal{Z}_{\mathbf{E}_2}(\mathbf{H}(\mathcal{Y})) \simeq \mathrm{D}^{\mathrm{der}}(\mathrm{L}\mathcal{Y}).$$

The corresponding $\mathbf{E}_2$-monoidal structure on the dg category $\mathrm{D}^{\mathrm{der}}(\mathrm{L}\mathcal{Y})$ is expected to be the one induced by the 'pair of pants' construction for $\mathrm{L}\mathcal{Y}$. To confirm this expectation, one needs to establish the functoriality of derived D-modules.

In view of this coincidence, we expect that there is an equivalence of categories

$$\mathrm{D}(\mathrm{Sing}_2(\mathcal{Z}))^\Rightarrow \simeq \mathrm{D}^{\mathrm{der}}(\mathcal{Z}) \tag{6.19}$$

for every quasi-quasi-smooth stack $\mathcal{Z}$, exchanging $(\omega_{\mathrm{Sing}_2(\mathcal{Z})})^\Rightarrow$ with an object of $\mathrm{D}^{\mathrm{der}}(\mathcal{Z})$ whose underlying quasi-coherent sheaf is

$$\det(\mathbf{T}_{\mathcal{Z}}) \otimes \Psi(\omega_{\mathcal{Z}}) \in \mathrm{QCoh}(\mathcal{Z}).$$

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6.3 Relation to the enhanced trace

6.3.1. In this subsection, we will explain the relation between the object

$$\mathrm{cl}(\mathrm{IndCoh}(\mathcal{Y}),\alpha)\in \mathrm{tr}(\mathbf{H}_\phi;\mathbf{H}(\mathcal{Y})\textbf{-mod})\simeq \mathrm{D}(\mathrm{Sing}_2(\mathcal{Y}^\phi))^{\Rightarrow} \tag{6.20}$$

and the so-called *enhanced trace*

$$\mathrm{cl}(\mathrm{IndCoh}(\mathcal{Y}),\phi^!)\in \mathrm{tr}(\mathbf{res}_\phi;\mathrm{QCoh}(\mathcal{Y})\textbf{-mod})\simeq \mathrm{QCoh}(\mathcal{Y}^\phi) \tag{6.21}$$

of [GKR+22].

Let us recall the definition of the latter object. Consider the natural action of $\mathrm{QCoh}(\mathcal{Y})$ on $\mathrm{IndCoh}(\mathcal{Y})$. Then the functor of IndCoh !-pullback

$$\phi^!:\mathrm{IndCoh}(\mathcal{Y})\to\mathrm{IndCoh}(\mathcal{Y})$$

defines a lax equivariance datum

$$\mathrm{IndCoh}(\mathcal{Y})\to \mathbf{res}_\phi(\mathrm{IndCoh}(\mathcal{Y})) \tag{6.22}$$

for the endofunctor

$$\mathbf{res}_\phi:\mathrm{QCoh}(\mathcal{Y})\to\mathrm{QCoh}(\mathcal{Y}),$$

whose class is the aforementioned enhanced trace.

6.3.2. The monoidal functor

$$\mathrm{QCoh}(\mathcal{Y})\xrightarrow{\Upsilon}\mathrm{IndCoh}(\mathcal{Y})\xrightarrow{\Delta^*}\mathrm{IndCoh}((\mathcal{Y}\times\mathcal{Y})^\wedge_{\mathcal{Y}})$$

factors through a monoidal functor

$$\mathrm{QCoh}(\mathcal{Y})\to\mathbf{H}(\mathcal{Y}). \tag{6.23}$$

As we shall see, the relation between the objects (6.20) and (6.21) is mediated by the functor of restriction

$$\mathbf{oblv}:\mathbf{H}(\mathcal{Y})\textbf{-mod}\to\mathrm{QCoh}(\mathcal{Y})\textbf{-mod}$$

along (6.23).

The aforementioned restriction functor enjoys the following compatibility with respect to a morphism $\mathcal{Z}\to\mathcal{Y}$ of quasi-smooth stacks.

$$\begin{array}{ccc}\mathbf{H}(\mathcal{Y})\textbf{-mod} & \xrightarrow{\mathbf{oblv}} & \mathrm{QCoh}(\mathcal{Y})\textbf{-mod}\\ {\scriptstyle \mathbf{H}_{\mathcal{Z}\to\mathcal{Y}}}\downarrow & & \downarrow{\scriptstyle \mathbf{ind}}\\ \mathbf{H}(\mathcal{Z})\textbf{-mod} & \xrightarrow{\mathbf{oblv}} & \mathrm{QCoh}(\mathcal{Z})\textbf{-mod}\end{array} \tag{6.24}$$

This is verified by a straightforward monad computation (see [Ber21, Proposition 3.4.1] for a more general assertion).

6.3.3. Let us apply (6.24) to our endomorphism $\phi:\mathcal{Y}\to\mathcal{Y}$. We claim that the functor

$$\mathbf{oblv}:\mathbf{H}(\mathcal{Y})\textbf{-mod}\to\mathrm{QCoh}(\mathcal{Y})\textbf{-mod}$$

is right adjointable, and hence the following diagram induces the functor below.

$$\begin{array}{ccc}\mathbf{H}(\mathcal{Y})\textbf{-mod} & \xrightarrow{\mathbf{H}_\phi} & \mathbf{H}(\mathcal{Y})\textbf{-mod}\\ \downarrow{\scriptstyle \mathbf{oblv}} & & \downarrow{\scriptstyle \mathbf{oblv}}\\ \mathrm{QCoh}(\mathcal{Y})\textbf{-mod} & \xrightarrow{\mathbf{ind}_\phi} & \mathrm{QCoh}(\mathcal{Y})\textbf{-mod}\end{array} \tag{6.25}$$

$$\mathrm{tr}(\mathbf{H}_\phi;\mathbf{H}(\mathcal{Y})\textbf{-mod})\to\mathrm{tr}(\mathbf{ind}_\phi;\mathrm{QCoh}(\mathcal{Y})\textbf{-mod}) \tag{6.26}$$

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Indeed, the functor **oblv** is given by the bimodule

$$\mathbf{H}(\mathcal{Y}) \in (\mathrm{QCoh}(\mathcal{Y}), \mathbf{H}(\mathcal{Y})),$$

and $\mathbf{H}(\mathcal{Y})$ is dualizable as a $\mathrm{QCoh}(\mathcal{Y})$-module because it is dualizable as a plain dg category. Therefore, the existence of the claimed right adjoint is a consequence of the following general assertion.

Proposition 6.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be monoidal categories, and let $\mathcal{M}$ be an $(\mathcal{A}, \mathcal{B})$-bimodule. Suppose that $\mathcal{M}$ admits a right dual $\mathcal{M}^\vee$ as an $\mathcal{A}$-module. Then $\mathcal{M}^\vee$ is naturally a right dual of $\mathcal{M}$ as an $(\mathcal{A}, \mathcal{B})$-bimodule.*

Proof of Proposition 6.1. We need to show that the adjunction

$$\mathcal{M} \otimes_{\mathcal{B}} - : \mathcal{B}\textbf{-mod} \rightleftharpoons \mathcal{A}\textbf{-mod} : \mathrm{Maps}_{\mathcal{A}}(\mathcal{M}, -)$$

actually takes place in **2-dgCat**. That is, we have to see that the right adjoint is continuous and commutes with the operation of tensoring with a plain dg category.

It suffices to check this condition after composing with the forgetful functor

$$\mathbf{dgCat} \leftarrow \mathcal{B}\textbf{-mod}. : \mathbf{oblv}$$

Our assumption on $\mathcal{M}$ says that this composite participates in an adjunction in **2-dgCat**

$$\mathcal{M} : \mathbf{dgCat} \rightleftharpoons \mathcal{A}\textbf{-mod} \rightleftharpoons \mathcal{B}\textbf{-mod} : \mathcal{M}^\vee,$$

so the result follows. □

6.3.4. Now observe that $\mathrm{QCoh}(\mathcal{Y})$**-mod** is canonically self-dual because $\mathrm{QCoh}(\mathcal{Y})$ is symmetric monoidal, and that this duality identifies the endofunctor $(\mathbf{ind}_\phi)^\vee$ with $\mathbf{res}_\phi$. Therefore, by (6.1), we obtain an equivalence

$$\mathrm{tr}(\mathbf{ind}_\phi; \mathrm{QCoh}(\mathcal{Y})\textbf{-mod}) \simeq \mathrm{tr}(\mathbf{res}_\phi; \mathrm{QCoh}(\mathcal{Y})\textbf{-mod}). \tag{6.27}$$

We claim the following.

Computation 6.3. The composite

$$\begin{aligned} \mathrm{tr}(\mathbf{H}_\phi; \mathbf{H}(\mathcal{Y})\textbf{-mod}) &\xrightarrow{(6.26)} \mathrm{tr}(\mathbf{ind}_\phi; \mathrm{QCoh}(\mathcal{Y})\textbf{-mod}) \\ &\simeq \mathrm{tr}(\mathbf{res}_\phi; \mathrm{QCoh}(\mathcal{Y})\textbf{-mod}) \end{aligned} \tag{6.28}$$

maps $\mathrm{cl}(\mathrm{IndCoh}(\mathcal{Y}), \alpha)$ to $\mathrm{cl}(\mathrm{IndCoh}(\mathcal{Y}), \phi^!)$.

6.3.5. Note that, after composing with (6.25), α defines a lax equivariance datum in $\mathrm{QCoh}(\mathcal{Y})$**-mod**, which we shall denote by the same symbol:

$$\alpha : \mathrm{IndCoh}(\mathcal{Y}) \to \mathbf{ind}_\phi(\mathrm{IndCoh}(\mathcal{Y})).$$

The functor (6.26) takes

$$\mathrm{cl}(\mathrm{IndCoh}(\mathcal{Y}), \alpha) \in \mathrm{tr}(\mathbf{H}_\phi; \mathbf{H}(\mathcal{Y})\textbf{-mod})$$

to

$$\mathrm{cl}(\mathrm{IndCoh}(\mathcal{Y}), \alpha) \in \mathrm{tr}(\mathbf{ind}_\phi; \mathrm{QCoh}(\mathcal{Y})\textbf{-mod}).$$

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Let us apply Computation 6.1 to α, viewed as a natural transformation.

$$\begin{array}{ccc} \mathbf{dgCat} & \xrightarrow{\mathbf{1}} & \mathbf{dgCat} \\ {\scriptstyle \mathrm{IndCoh}(\mathcal{Y})}\downarrow & \swarrow & \downarrow{\scriptstyle \mathrm{IndCoh}(\mathcal{Y})} \\ \mathrm{QCoh}(\mathcal{Y})\textbf{-mod} & \xrightarrow[\mathbf{ind}_\phi]{} & \mathrm{QCoh}(\mathcal{Y})\textbf{-mod} \end{array}$$

The self-duality of $\mathrm{IndCoh}(\mathcal{Y})$ as a module over $\mathrm{QCoh}(\mathcal{Y})$ identifies the functor

$$\mathrm{IndCoh}(\mathcal{Y}) : \mathbf{dgCat} \to \mathrm{QCoh}(\mathcal{Y})$$

with the left adjoint of its dual. We obtain that the isomorphism (6.27) takes $\mathrm{cl}(\mathrm{IndCoh}(\mathcal{Y}), \alpha)$ to the class of the lax equivariance datum

$$\beta : \mathrm{IndCoh}(\mathcal{Y}) \to \mathbf{res}_\phi(\mathrm{IndCoh}(\mathcal{Y}))$$

obtained by adjunction from the map

$$\begin{aligned} \mathbf{ind}_\phi(\mathrm{IndCoh}(\mathcal{Y})) &\simeq \mathbf{ind}_\phi(\mathrm{IndCoh}(\mathcal{Y})^\vee) \\ &\simeq \mathbf{ind}_\phi(\mathrm{IndCoh}(\mathcal{Y}))^\vee \xrightarrow{\alpha^\vee} \mathrm{IndCoh}(\mathcal{Y})^\vee \simeq \mathrm{IndCoh}(\mathcal{Y}). \end{aligned} \tag{6.29}$$

It remains to see that β is canonically identified with (6.22).

It is equivalent to identify (6.29) with the transpose

$$\mathbf{ind}_\phi(\mathrm{IndCoh}(\mathcal{Y})) \to \mathrm{IndCoh}(\mathcal{Y}) \tag{6.30}$$

of (6.22). This is a formal consequence of the commutative diagram of $(\mathrm{QCoh}(\mathcal{Y}), \mathbf{H}(\mathcal{Y}))$-bimodules.

$$\begin{array}{ccc} \mathbf{ind}_\phi(\mathrm{IndCoh}(\mathcal{Y})) \otimes \mathrm{IndCoh}(\mathcal{Y}) & \xrightarrow{(6.11)} & \mathbf{ind}_\phi(\mathbf{H}(\mathcal{Y})) \\ \downarrow{\scriptstyle (6.30)\otimes\mathbf{1}} & & \downarrow{\scriptstyle (6.25)} \\ \mathrm{IndCoh}(\mathcal{Y}) \otimes \mathrm{IndCoh}(\mathcal{Y}) & \xrightarrow{(\mathbf{ind}_\phi\otimes\mathbf{1})\circ(6.11)} & \mathbf{H}_\phi \end{array}$$

6.3.6 *Remark.* We do not currently have a good description of the composite appearing in Computation 6.3 as a map

$$\mathrm{D}(\mathrm{Sing}_2(\mathcal{Y}^\phi))^{\Rightarrow} \to \mathrm{QCoh}(\mathcal{Y}^\phi).$$

However, we expect that such a description would contain interesting geometric content.

6.4 Variant with singular support

6.4.1. Consider a closed conical subset $\mathcal{N}$ of $\mathrm{Sing}(\mathcal{Y})$. When we regard $\mathcal{N}$ as living over $\mathrm{Sing}(\mathcal{Y}) \times \mathrm{Sing}(\mathcal{Y})$, we will do so via the map

$$\mathcal{N} \hookrightarrow \mathrm{Sing}(\mathcal{Y}) \simeq \mathrm{Sing}(\mathcal{Y}/\mathcal{Y} \times \mathcal{Y}) \to \mathrm{Sing}(\mathcal{Y} \times \mathcal{Y}).$$

6.4.2. The category $\mathrm{ShCat}_{\mathbf{S}^\Rightarrow}(\mathcal{Y})$, being a limit of symmetric monoidal categories under symmetric monoidal functors, carries a natural symmetric monoidal structure. We let

$$\mathrm{pr}_{\mathcal{N}} : \mathrm{ShCat}_{\mathbf{S}^\Rightarrow}(\mathcal{Y}) \to \mathrm{ShCat}_{\mathbf{S}^\Rightarrow}(\mathcal{Y})$$

denote the idempotent endofunctor of $\mathrm{ShCat}_{\mathbf{S}^\Rightarrow}(\mathcal{Y})$ given by tensoring with $\underline{\mathrm{D}}(\mathcal{N})^{\Rightarrow}$. One has

$$\mathbf{\Gamma}(\mathrm{S}; \mathrm{pr}_{\mathcal{N}}(\mathcal{C})) \simeq \mathrm{D}(\mathcal{N}_{\mathrm{S}})^{\Rightarrow} \underset{\mathrm{D}(\mathrm{Sing}(\mathrm{S}))^{\Rightarrow}}{\otimes} \mathbf{\Gamma}(\mathrm{S}; \mathcal{C}).$$

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Since $\underline{\mathrm{D}}(\mathcal{N})^{\Rightarrow}$ is a colocalization of $\underline{\mathrm{D}}(\mathrm{Sing}(\mathcal{Y}))^{\Rightarrow}$, we obtain that $\mathrm{pr}_{\mathcal{N}}$ is a colocalization of the identity endofunctor of $\mathrm{ShCat}_{\mathbf{S}^{\Rightarrow}}(\mathcal{Y})$

$$\mathrm{pr}_{\mathcal{N}} \rightleftharpoons \mathbf{1}. \tag{6.31}$$

Since $\mathrm{pr}_{\mathcal{N}}$ is idempotent, this implies $\mathrm{pr}_{\mathcal{N}}$ is canonically adjoint to itself, with unit and counit

$$\eta : \mathrm{pr}_{\mathcal{N}} \circ \mathrm{pr}_{\mathcal{N}} \simeq \mathrm{pr}_{\mathcal{N}} \rightleftharpoons \mathbf{1} : \epsilon.$$

In particular, $\mathrm{pr}_{\mathcal{N}}$ admits a right adjoint.

Proposition 6.2. *The Morita equivalence (5.6) exchanges the colocalizations*

$$\underline{\mathrm{D}}(\mathcal{N})^{\Rightarrow} \quad \textit{and} \quad \mathbf{Loc}(\mathrm{IndCoh}_{\mathcal{N}}(\mathcal{Y})).$$

Proof of Proposition 6.2. Consider the image of the colocalization

$$\underline{\mathrm{D}}(\mathcal{N})^{\Rightarrow} \rightleftharpoons \underline{\mathrm{D}}(\mathrm{Sing}(\mathcal{Y}))^{\Rightarrow}$$

under the Morita equivalence (5.6). According to Proposition 3.7, its value at an object $\mathrm{S} \in \mathrm{Sm}_{/\mathcal{Y}}$ is the colocalization

$$\mathrm{IndCoh}_{\mathcal{N}_{\mathrm{S}}}(\mathrm{S}) \rightleftharpoons \mathrm{IndCoh}(\mathrm{S}), \tag{6.32}$$

so its image under $\mathbf{\Gamma}$ is the colocalization of $\mathrm{IndCoh}(\mathcal{Y})$ obtained from (6.32) by passing to the limit over $\mathrm{Sm}_{/\mathcal{Y}}$. This is definitionally the colocalization

$$\mathrm{IndCoh}_{\mathcal{N}}(\mathcal{Y}) \rightleftharpoons \mathrm{IndCoh}(\mathcal{Y}),$$

so the result now follows from H-affineness. □

6.4.3. After applying the Morita equivalence (5.6) and H-affineness, we obtain from $\mathrm{pr}_{\mathcal{N}}$ an endomorphism

$$\mathrm{pr}_{\mathcal{N}} : \mathbf{H}(\mathcal{Y})\textbf{-mod} \to \mathbf{H}(\mathcal{Y})\textbf{-mod},$$

to be denoted by the same symbol.

Let

$$\mathbf{H}_{\mathcal{N}} \in (\mathbf{H}(\mathcal{Y}), \mathbf{H}(\mathcal{Y}))\textbf{-bimod}$$

denote the bimodule responsible for the endofunctor $\mathrm{pr}_{\mathcal{N}}$. In these terms, the colocalization (6.31) manifests as a colocalization

$$\mathbf{H}_{\mathcal{N}} \rightleftharpoons \mathbf{H}(\mathcal{Y})$$

of $(\mathbf{H}(\mathcal{Y}), \mathbf{H}(\mathcal{Y}))$-bimodules. The idempotence of $\mathrm{pr}_{\mathcal{N}}$ says that

$$\mathbf{H}_{\mathcal{N}} \underset{\mathbf{H}(\mathcal{Y})}{\otimes} \mathbf{H}_{\mathcal{N}} \simeq \mathbf{H}_{\mathcal{N}} \tag{6.33}$$

as full subcategories of $\mathbf{H}(\mathcal{Y})$ stable under the left and right actions of $\mathbf{H}(\mathcal{Y})$.

6.4.4. We obtain a map

$$\mathrm{pr}_{\mathcal{N}} \circ \mathbf{H}_{\phi} \to \mathrm{pr}_{\mathcal{N}} \circ \mathbf{H}_{\phi} \circ \mathrm{pr}_{\mathcal{N}} \to \mathbf{H}_{\phi} \circ \mathrm{pr}_{\mathcal{N}}$$

encoding the following diagram.

$$\begin{array}{ccc} \mathbf{H}(\mathcal{Y})\textbf{-mod} & \xrightarrow{\mathbf{H}_{\phi}} & \mathbf{H}(\mathcal{Y})\textbf{-mod} \\ \downarrow{\scriptstyle \mathrm{pr}_{\mathcal{N}}} & \swarrow\!\!\swarrow & \downarrow{\scriptstyle \mathrm{pr}_{\mathcal{N}}} \\ \mathbf{H}(\mathcal{Y})\textbf{-mod} & \xrightarrow{\mathbf{H}_{\phi}} & \mathbf{H}(\mathcal{Y})\textbf{-mod} \end{array} \tag{6.34}$$

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Therefore, $\mathrm{pr}_{\mathcal{N}}$ induces an endomorphism of the trace:

$$\mathrm{tr}(\mathbf{H}_\phi; \mathbf{H}(\mathcal{Y})\text{-}\mathbf{mod}) \to \mathrm{tr}(\mathbf{H}_\phi; \mathbf{H}(\mathcal{Y})\text{-}\mathbf{mod}). \tag{6.35}$$

We claim the following.

Computation 6.4. Under the identification (6.12), this endomorphism identifies with the operation

$$(\omega_{\mathcal{N}^\phi})^{\Rightarrow} \otimes (-) : \mathrm{D}(\mathrm{Sing}_2(\mathcal{Y}^\phi))^{\Rightarrow} \to \mathrm{D}(\mathrm{Sing}_2(\mathcal{Y}^\phi))^{\Rightarrow},$$

where

$$\mathcal{N}^\phi = \mathrm{Sing}_\phi(\mathcal{Y}/\mathcal{Y}\times\mathcal{Y}) \underset{\mathrm{Sing}(\mathcal{Y}\times\mathcal{Y})}{\times} \mathcal{N}.$$

6.4.5. By the idempotence (6.33) of $\mathbf{H}_{\mathcal{N}}$ as a bimodule, (6.35) is obtained from the endomorphism

$$\mathbf{H}(\mathcal{Y}) \to \mathbf{H}_{\mathcal{N}} \to \mathbf{H}(\mathcal{Y})$$

by tensoring up along

$$- \underset{\mathbf{H}(\mathcal{Y})\otimes\mathbf{H}(\mathcal{Y})^{\mathrm{rev}}}{\otimes} \mathbf{H}_\phi.$$

Now observe that

$$\mathbf{H}_{\mathcal{N}} \simeq \mathrm{pr}_{\mathcal{N}\times\mathcal{N}}(\mathbf{H}(\mathcal{Y}))$$

as objects of

$$(\mathbf{H}(\mathcal{Y}), \mathbf{H}(\mathcal{Y}))\text{-}\mathbf{bimod} \simeq \mathbf{H}(\mathcal{Y}\times\mathcal{Y})\text{-}\mathbf{mod}.$$

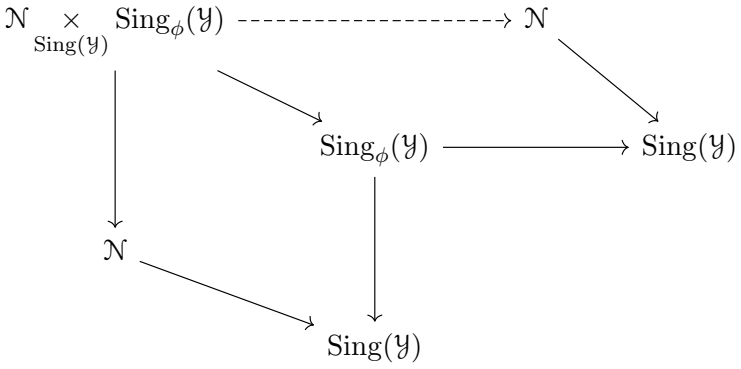
Therefore, it follows from Proposition 5.1 that (6.35) corresponds to the endomorphism of (6.16) induced by the composite

$$\underline{\mathrm{D}}(\mathrm{Sing}(\mathcal{Y}/\mathcal{Y}\times\mathcal{Y}))^{\Rightarrow} \to \underline{\mathrm{D}}(\mathcal{N})^{\Rightarrow} \to \underline{\mathrm{D}}(\mathrm{Sing}(\mathcal{Y}/\mathcal{Y}\times\mathcal{Y}).$$

6.4.6. Consider the pullback of $\mathcal{N}$ along the projection $\mathrm{Sing}_\phi(\mathcal{Y}) \to \mathrm{Sing}(\mathcal{Y})$, to be denoted $\mathcal{N}_\phi$. We impose the following assumption.

$$\text{The image of } \mathcal{N} \underset{\mathrm{Sing}(\mathcal{Y})}{\times} \mathrm{Sing}_\phi(\mathcal{Y}) \text{ under } \mathrm{Sing}(\phi) \text{ is contained in } \mathcal{N}. \tag{$*$}$$

In other words, we require the existence of the dashed arrow in the following diagram.



The reason for imposing the condition ($*$) is so that the following assertion holds true.

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Proposition 6.3. *Under the assumption (∗), the map*

$$\mathrm{pr}_{\mathcal{N}} \circ \mathbf{H}_\phi \circ \mathrm{pr}_{\mathcal{N}} \to \mathbf{H}_\phi \circ \mathrm{pr}_{\mathcal{N}}$$

is an isomorphism.

Proof of Proposition 6.3. Note that Proposition 6.3 asserts that the essential image of $\mathbf{H}_\phi \circ \mathrm{pr}_{\mathcal{N}}$ is contained in that of $\mathrm{pr}_{\mathcal{N}}$. In other words we need to show that, for every object $\mathrm{S} \in \mathrm{Sm}_{/\mathcal{Y}}$, the essential image of the composite

$$\begin{aligned} &\mathbf{H}(\mathcal{Y})\text{-}\mathbf{mod} \xrightarrow{\mathrm{pr}_{\mathcal{N}}} \mathbf{H}(\mathcal{Y})\text{-}\mathbf{mod} \xrightarrow{\mathbf{H}_\phi} \mathbf{H}(\mathcal{Y})\text{-}\mathbf{mod} \\ &\xrightarrow{\boldsymbol{\Gamma}(\mathcal{Y}_{\mathrm{S}};-)} \mathbf{H}(\mathcal{Y}_{\mathrm{S}})\text{-}\mathbf{mod} \xrightarrow{\mathrm{IndCoh}(\mathcal{Y}_{\mathrm{S}})^\tau} \mathrm{D}(\mathrm{Sing}(\mathcal{Y}_{\mathrm{S}}))^{\Rightarrow}\text{-}\mathbf{mod} \end{aligned} \tag{6.36}$$

is contained in

$$\mathrm{D}(\mathcal{N}_{\mathcal{Y}_{\mathrm{S}}})^{\Rightarrow}\text{-}\mathbf{mod} \subset \mathrm{D}(\mathrm{Sing}(\mathcal{Y}_{\mathrm{S}}))^{\Rightarrow}\text{-}\mathbf{mod}.$$

By the functoriality of the **H**-theory, (6.36) identifies with

$$\begin{aligned} &\mathbf{H}(\mathcal{Y})\text{-}\mathbf{mod} \xrightarrow{\mathrm{pr}_{\mathcal{N}}} \mathbf{H}(\mathcal{Y})\text{-}\mathbf{mod} \xrightarrow{\boldsymbol{\Gamma}(\mathrm{S};-)} \mathbf{H}(\mathrm{S})\text{-}\mathbf{mod} \\ &\xrightarrow{\mathbf{H}_{\mathcal{Y}_{\mathrm{S}} \to \mathrm{S}}} \mathbf{H}(\mathcal{Y}_{\mathrm{S}})\text{-}\mathbf{mod} \xrightarrow{\mathrm{IndCoh}(\mathcal{Y}_{\mathrm{S}})} \mathrm{D}(\mathrm{Sing}(\mathcal{Y}_{\mathrm{S}}))^{\Rightarrow}\text{-}\mathbf{mod}. \end{aligned} \tag{6.37}$$

Applying (4.16) rewrites this composite as

$$\begin{aligned} &\mathbf{H}(\mathcal{Y})\text{-}\mathbf{mod} \xrightarrow{\mathrm{pr}_{\mathcal{N}}} \mathbf{H}(\mathcal{Y})\text{-}\mathbf{mod} \xrightarrow{\boldsymbol{\Gamma}(\mathrm{S};-)} \mathbf{H}(\mathrm{S})\text{-}\mathbf{mod} \\ &\xrightarrow{\mathrm{IndCoh}(\mathrm{S})} \mathrm{D}(\mathrm{Sing}(\mathrm{S}))^{\Rightarrow}\text{-}\mathbf{mod} \xrightarrow{\mathrm{D}(\mathrm{Sing}(\mathrm{S})_{\mathcal{Y}_{\mathrm{S}}})^{\Rightarrow}} \mathrm{D}(\mathrm{Sing}(\mathcal{Y}_{\mathrm{S}}))^{\Rightarrow}\text{-}\mathbf{mod}, \end{aligned} \tag{6.38}$$

and applying the definition of $\mathrm{pr}_{\mathcal{N}}$ further rewrites our functor as

$$\begin{aligned} &\mathbf{H}(\mathcal{Y})\text{-}\mathbf{mod} \xrightarrow{\boldsymbol{\Gamma}(\mathrm{S};-)} \mathbf{H}(\mathrm{S})\text{-}\mathbf{mod} \xrightarrow{\mathrm{IndCoh}(\mathrm{S})} \mathrm{D}(\mathrm{Sing}(\mathrm{S}))^{\Rightarrow}\text{-}\mathbf{mod} \\ &\xrightarrow{\mathrm{D}(\mathcal{N}_{\mathrm{S}})^{\Rightarrow}} \mathrm{D}(\mathrm{Sing}(\mathrm{S}))^{\Rightarrow}\text{-}\mathbf{mod} \xrightarrow{\mathrm{D}(\mathrm{Sing}(\mathrm{S})_{\mathcal{Y}_{\mathrm{S}}})^{\Rightarrow}} \mathrm{D}(\mathrm{Sing}(\mathcal{Y}_{\mathrm{S}}))^{\Rightarrow}\text{-}\mathbf{mod}. \end{aligned} \tag{6.39}$$

The desired containment now follows from (∗), base changed to S. □

6.4.7. By Propositions 6.2 and 6.3, α descends to a lax equivariance datum $\alpha_{\mathcal{N}}$ fitting into the following diagram.

$$\begin{array}{ccc} \mathrm{IndCoh}(\mathcal{Y}) & \xrightarrow{\alpha} & \mathbf{H}_\phi \underset{\mathbf{H}(\mathcal{Y})}{\otimes} \mathrm{IndCoh}(\mathcal{Y}) \\ \downarrow & & \downarrow \\ \mathrm{IndCoh}_{\mathcal{N}}(\mathcal{Y}) & \xrightarrow{\alpha_{\mathcal{N}}} & \mathbf{H}_\phi \underset{\mathbf{H}(\mathcal{Y})}{\otimes} \mathrm{IndCoh}_{\mathcal{N}}(\mathcal{Y}) \end{array} \tag{6.40}$$

Observe that $\alpha_{\mathcal{N}}$ encodes the concatenation of the lax equivariance datum of α with (6.34).

$$\begin{array}{ccc} \mathbf{dgCat} & \longrightarrow & \mathbf{dgCat} \\ {\scriptstyle \mathrm{IndCoh}(\mathcal{Y})}\downarrow & \swarrow & \downarrow{\scriptstyle \mathrm{IndCoh}(\mathcal{Y})} \\ \mathbf{H}(\mathcal{Y})\text{-}\mathbf{mod} & \xrightarrow[\mathbf{H}_\phi]{} & \mathbf{H}(\mathcal{Y})\text{-}\mathbf{mod} \end{array}$$

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Therefore, the combination of Computations 6.2 and 6.4 shows that the equivalence (6.12) takes

$$\mathrm{cl}(\mathrm{IndCoh}_{\mathcal{N}}(\mathcal{Y}), \alpha_{\mathcal{N}}) \in \mathrm{tr}(\mathbf{H}_{\phi}; \mathbf{H}(\mathcal{Y})\textbf{-mod}) \tag{6.41}$$

to the object

$$(\omega_{\mathcal{N}^{\phi}})^{\Rightarrow} \in \mathrm{D}(\mathrm{Sing}_2(\mathcal{Y}^{\phi}))^{\Rightarrow}.$$

We obtain from (6.40) that (6.24) restricts to a lax equivariance datum $(\phi^!)_{\mathcal{N}}$ fitting into the following diagram.

$$\begin{array}{ccc} \mathrm{IndCoh}_{\mathcal{N}}(\mathcal{Y}) & \xrightarrow{(\phi^!)_{\mathcal{N}}} & \mathbf{res}_{\phi}(\mathrm{IndCoh}_{\mathcal{N}}(\mathcal{Y})) \\ \downarrow & & \downarrow \\ \mathrm{IndCoh}(\mathcal{Y}) & \xrightarrow{\phi^!} & \mathbf{res}_{\phi}(\mathrm{IndCoh}(\mathcal{Y})) \end{array}$$

As a result, Computation 6.3 shows that (6.28) maps (6.41) to

$$\mathrm{cl}(\mathrm{IndCoh}_{\mathcal{N}}(\mathcal{Y}), (\phi^!)_{\mathcal{N}}) \in \mathrm{tr}(\mathbf{res}_{\phi}; \mathrm{QCoh}(\mathcal{Y})\textbf{-mod}).$$

6.4.8 *Remark.* This computation implies [AGK+20, Conjecture 24.6.9].

Appendix A. Locally fully faithful functors

In this appendix, we will formulate and prove a lifting criterion Proposition A.5 for locally fully faithful functors.

A.0.1. Let us begin with a reminder that we use the word '2-category' to refer to a general $(\infty, 2)$-category, we use the word 'category' to refer to a general $(\infty, 1)$-category, and we use the word 'groupoid' to refer to a general ∞-groupoid.

Recall that a functor of 2-categories

$$\mathrm{F} : \mathcal{C} \to \mathcal{D}$$

is said to be *locally fully faithful* if, for every pair of objects $c, c' \in \mathcal{C}$, the functor

$$\mathrm{Maps}_{\mathcal{C}}(c, c') \to \mathrm{Maps}_{\mathcal{D}}(\mathrm{F}(c), \mathrm{F}(c'))$$

is fully faithful.

Recall that if $\mathcal{C}$ is a 2-category and I is a 1-category, we obtain a map of 2-categories

$$\mathrm{Fun}(*, \mathcal{C}) \to \mathrm{Fun}(\mathrm{I}, \mathcal{C})$$

by precomposition with the tautological functor $\mathrm{I} \to *$.[14] The composite of the isomorphism $\mathcal{C} \simeq \mathrm{Fun}(*, \mathcal{C})$ with this arrow is called

$$\mathrm{const} : \mathcal{C} \to \mathrm{Fun}(\mathrm{I}, \mathcal{C}).$$

We write $\lim_{\mathrm{I}}$ for the right adjoint of this arrow (provided that it exists).

We do not consider lax limits in this paper, nor do we consider lax functors. In general, our conventions regarding $(\infty, 2)$-category theory follow [GR17a].

[14] There is no need, mathematically, to require I to be a 1-category rather than a 2-category. However, this is the only case we will need to consider in this paper, and perhaps it is psychologically comforting to make this restriction.

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A.0.2. Recall that, for any category $\mathcal{C}$, one has the associated twisted arrows category $\mathrm{Tw}(\mathcal{C})$, whose objects are arrows $c \to c'$ in $\mathcal{C}$ and whose morphisms from $c \to c'$ to $d \to d'$ are commutative diagrams.[15]

$$\begin{array}{ccc} c & \longleftarrow & d \\ \downarrow & & \downarrow \\ c' & \longrightarrow & d' \end{array}$$

The formation of the twisted arrows category is natural in $\mathcal{C}$; every functor $\mathrm{F} : \mathcal{C} \to \mathcal{D}$ induces a functor $\mathrm{Tw}(\mathrm{F}) : \mathrm{Tw}(\mathcal{C}) \to \mathrm{Tw}(\mathcal{D})$.

If $\mathcal{C}$ is any category and $\mathcal{D}$ is any 2-category, one has the twisted arrows formula for the space of natural transformations between a pair of functors $\mathcal{C} \to \mathcal{D}$:

$$\mathrm{Maps}(\mathrm{F}, \mathrm{G}) \simeq \lim_{\mathrm{Tw}(\mathcal{C})} \mathrm{Maps}_{\mathcal{D}}(\mathrm{F}(c), \mathrm{G}(c')), \tag{A.1}$$

see [AS23, Theorem 0.2].

A.1 Lax surjective functors

A.1.1. We say that a functor $\mathrm{F} : \mathcal{C} \to \mathcal{D}$ is *lax surjective* if, for every object $d \in \mathcal{D}$, there is an object $c \in \mathcal{C}$ and a map $\mathrm{F}(c) \to d$.

Proposition A.1. *If $\mathrm{F} : \mathcal{C} \to \mathcal{D}$ is essentially surjective, then $\mathrm{Tw}(\mathrm{F}) : \mathrm{Tw}(\mathcal{C}) \to \mathrm{Tw}(\mathcal{D})$ is lax surjective.*

Proof of Proposition A.1. Let $f : d \to d'$ be an object of $\mathrm{Tw}(\mathcal{D})$. Since F is essentially surjective, we may choose $c \in \mathcal{C}$ along with an isomorphism $\alpha : \mathrm{F}(c) \xrightarrow{\sim} d$. Then the following diagram defines a map from $\mathrm{F}(c \to c)$ to $d \to d'$ in $\mathrm{Tw}(\mathcal{D})$.

$$\begin{array}{ccc} \mathrm{F}(c) & \xleftarrow{\alpha^{-1}} & d \\ \downarrow{\scriptstyle \mathbf{1}} & & \downarrow{\scriptstyle f} \\ \mathrm{F}(c) & \xrightarrow{f \circ \alpha} & d' \end{array}$$

□

A.1.2. Let Cat denote the $(\infty, 2)$-category of $(\infty, 1)$-categories. According to our conventions (see § A.0.1), we also call this the 2-category of categories.

Let $\mathrm{Prop} \subset \mathrm{Cat}$ denote the full sub-2-category containing only the initial and terminal object. Note that this 2-category is a poset, with only one non-identity arrow $\varnothing \to *$. We call the objects of this category *propositions*. Note that if we use $\top$ to denote $*$ and $\bot$ to denote $\varnothing$, then if p and q are in Prop, there is an arrow from p to q if and only if p implies q in the sense of propositional logic.

We also note that Prop is closed under the formation of limits in Cat, and in the notation of the propositional calculus, the limit of a diagram

$$\mathrm{F} : \mathrm{I} \to \mathrm{Prop}$$

is equal to

$$\lim_{i \in \mathrm{I}} \mathrm{F}(i) = (\forall i \in \mathrm{I})(\mathrm{F}(i)).$$

[15] *Warning*: the literature does not appear to be consistent as to whether the 'twisted arrows category' is what we have denoted $\mathrm{Tw}(\mathcal{C})$ or rather its opposite, $\mathrm{Tw}(\mathcal{C})^{\mathrm{op}}$.

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Proposition A.2. *Let* $\mathrm{F}:\mathrm{J}\to\mathrm{Prop}$ *be a diagram. For any lax surjective functor* $\alpha:\mathrm{I}\to\mathrm{J}$*, the natural morphism*

$$\lim_{j\in\mathrm{J}}\mathrm{F}(j)\to\lim_{i\in\mathrm{I}}\mathrm{F}(\alpha(i))$$

is an isomorphism.

Proof of Proposition A.2. We need to establish the reverse implication:

$$(\forall i\in\mathrm{I})(\mathrm{F}(\alpha(i)))\to(\forall j\in\mathrm{J})(\mathrm{F}(j)).$$

Since α is lax surjective, we have

$$(\forall j\in\mathrm{J})(\exists i\in\mathrm{I})(\mathrm{F}(\alpha(i))\to\mathrm{F}(j)),$$

so the conclusion follows. □

A.1.3. Let $\mathrm{Poset}\subset\mathrm{Cat}$ denote the full sub-2-category of partially ordered sets. Observe that this full subcategory is closed under limits, and that a category $\mathcal{C}$ is a poset if and only if the functor $\mathcal{C}\to *$ is locally fully faithful.

Proposition A.3. *Let* $\mathrm{F}:\mathrm{J}\to\mathrm{Poset}$ *be a diagram of posets. For any lax surjective functor* $\alpha:\mathrm{I}\to\mathrm{J}$*, the natural morphism*

$$\lim_{j\in\mathrm{J}}\mathrm{F}(j)\to\lim_{i\in\mathrm{I}}\mathrm{F}(\alpha(i)) \tag{A.2}$$

is fully faithful. Furthermore, the essential image of this morphism consists of objects $(x_i : i\in\mathrm{I})$ *satisfying the following property.*

For every tuple (j,i,i',ϕ,ϕ') *determining a diagram of the following form, we have the following equality.*

$$\begin{array}{ccc}\alpha(i) & & \alpha(i')\\ & {\scriptstyle\phi}\searrow\quad\swarrow{\scriptstyle\phi'} & \\ & j & \end{array} \tag{$*$}$$

$$\mathrm{F}_\phi(x_i)=\mathrm{F}_{\phi'}(x_{i'}).$$

Proof of Proposition A.3. The mapping space between two objects of a poset is always a proposition, so the assertion about fully faithfulness follows from Proposition A.2.

It is clear that every object in the image of the morphism (A.2) satisfies $(*)$. It remains to check that every object (x_i) of $\lim\mathrm{F}(\alpha(i))$ satisfying $(*)$ admits a lift (x_j) to $\lim\mathrm{F}(j)$. For $j\in\mathrm{J}$, set

$$x_j=\mathrm{F}_\phi(x_i)$$

where ϕ is any morphism $\alpha(i)\to j$. Such a morphism exists because α is lax surjective, and $(*)$ says precisely that the value of $\mathrm{F}_\phi(x_i)$ is independent of the choice of ϕ. □

A.1.4. Let us remark that the first conclusion of A.3 does *not* follow from the fact that (A.2) is a morphism of posets. A map of posets need not be fully faithful. For example, the map from a totally disconnected poset of two elements to the singleton poset is not fully faithful.

A.2 Lifting along locally fully faithful functors

A.2.1. If $\mathrm{F}:\mathcal{C}\to\mathcal{D}$ is a map of categories, we will say that F is *locally fully faithful* if it is locally fully faithful when we view $\mathcal{C}$ and $\mathcal{D}$ as 2-categories with the property that every

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2-morphism is invertible. Similarly, we will say that a map $\mathcal{C} \to \mathcal{D}$ of groupoids is *locally fully faithful* if it is locally fully faithful when viewed as a map of categories where every 1-morphism is invertible.

For example, if G is a finite group, and H and K are two subgroups of G, then the natural map

$$\mathrm{BH} \sqcup \mathrm{BK} \to \mathrm{BG}$$

is a locally fully faithful map of groupoids.

Recall from [Lur25, Tag 01XA] that if $\mathcal{C}$ is a 2-category, then its 'pith' $\mathrm{Pith}(\mathcal{C})$ is the category obtained by removing all non-invertible 2-morphisms. We also call $\mathrm{Pith}(\mathcal{C})$ the 1-core of $\mathcal{C}$. For reference, in [GR17a] the 1-core of $\mathcal{C}$ is denoted $\mathcal{C}^{1\text{-Cat}}$.

We note that if $\mathrm{F} : \mathcal{C} \to \mathcal{D}$ is a locally fully faithful map of 2-categories, then the induced map of 1-cores $\mathrm{Pith}(\mathrm{F}) : \mathrm{Pith}(\mathcal{C}) \to \mathrm{Pith}(\mathcal{D})$ is locally fully faithful as well. Similarly, if $\mathrm{F} : \mathcal{C} \to \mathcal{D}$ is a locally fully faithful map of categories, then the induced map $\mathrm{F}^{\simeq} : \mathcal{C}^{\simeq} \to \mathcal{D}^{\simeq}$ is locally fully faithful.

A.2.2. As remarked in [GR17a, § 10.2.3.2], a functor $\mathrm{F} : \mathcal{C} \to \mathcal{D}$ between 2-categories is locally fully faithful if and only if the natural map

$$\mathrm{Fun}(\Delta^1, \mathcal{C}) \to (\mathcal{C} \times \mathcal{C}) \underset{\mathcal{D} \times \mathcal{D}}{\times} \mathrm{Fun}(\Delta^1, \mathcal{D}) \tag{A.3}$$

is fully faithful. For completeness, we will supply a proof of this fact: note that (A.3) is a morphism over $\mathcal{C} \times \mathcal{C}$, and if we take the fiber at $(x, y) : * \to \mathcal{C} \times \mathcal{C}$, we obtain the map

$$\mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{D}}(\mathrm{F}x, \mathrm{F}y).$$

If (A.3) is fully faithful then F is evidently locally fully faithful. To prove the converse, we need to show that for every $\mathrm{G}, \mathrm{H} \in \mathrm{Fun}(\Delta^1, \mathcal{C})$, the following diagram is Cartesian.

$$\begin{array}{ccc} \mathrm{Nat}(\mathrm{G}, \mathrm{H}) & \longrightarrow & \mathrm{Nat}(\mathrm{FG}, \mathrm{FH}) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{C}}(\mathrm{G}_0, \mathrm{H}_0) \times \mathrm{Hom}_{\mathcal{C}}(\mathrm{G}_1, \mathrm{H}_1) & \longrightarrow & \mathrm{Hom}_{\mathcal{D}}(\mathrm{FG}_0, \mathrm{FH}_0) \times \mathrm{Hom}_{\mathcal{D}}(\mathrm{FG}_1, \mathrm{FH}_1) \end{array}$$

(We are using the notation G_0 to denote the image of the 0-vertex in Δ^1 under G.) After invoking the twisted arrows formula (A.1), it suffices to know that the diagonal map

$$\mathrm{Hom}_{\mathcal{C}}(\mathrm{G}_0, \mathrm{H}_1) \to \mathrm{Hom}_{\mathcal{C}}(\mathrm{G}_0, \mathrm{H}_1) \underset{\mathrm{Hom}_{\mathcal{D}}(\mathrm{FG}_0, \mathrm{FH}_1)}{\times} \mathrm{Hom}_{\mathcal{C}}(\mathrm{G}_0, \mathrm{H}_1)$$

is an isomorphism, which follows from the assumption that F is locally fully faithful.

A.2.3. If $\mathrm{F} : \mathcal{D} \to \mathcal{E}$ is a locally fully faithful map of 2-categories, we claim that for any 2-category $\mathcal{C}$, the postcomposition functor

$$\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{E})$$

is locally fully faithful as well. To prove this, we need to show that the following commutative square induces a fully faithful functor from $\mathrm{Fun}(\Delta^1, \mathrm{Fun}(\mathcal{C}, \mathcal{D}))$ to the fiber product of the lower

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right wedge.

$$\begin{array}{ccc} \mathrm{Fun}(\Delta^1, \mathrm{Fun}(\mathcal{C}, \mathcal{D})) & \longrightarrow & \mathrm{Fun}(\Delta^1, \mathrm{Fun}(\mathcal{C}, \mathcal{E})) \\ \downarrow & & \downarrow \\ \mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\times 2} & \longrightarrow & \mathrm{Fun}(\mathcal{C}, \mathcal{E})^{\times 2} \end{array} \tag{A.4}$$

However, the commutative square (A.4) identifies with the following, so the result follows from the fact that postcomposition with a fully faithful functor is fully faithful.

$$\begin{array}{ccc} \mathrm{Fun}(\mathcal{C}, \mathrm{Fun}(\Delta^1, \mathcal{D})) & \longrightarrow & \mathrm{Fun}(\mathcal{C}, \mathrm{Fun}(\Delta^1, \mathcal{E})) \\ \downarrow & & \downarrow \\ \mathrm{Fun}(\mathcal{C}, \mathcal{D}^{\times 2}) & \longrightarrow & \mathrm{Fun}(\mathcal{C}, \mathcal{E}^{\times 2}) \end{array}$$

A variant of this argument, taking into account dgCat-linear structures, shows that if $\mathrm{F} : \mathcal{D} \to \mathcal{E}$ is a map of 2-dg categories which is locally fully faithful as a map of plain 2-categories, then postcomposition with F induces a locally fully faithful functor between internal mapping objects in **2-dgCat**.

A.2.4. If $\mathcal{C}$ is a category, we say that a map $x \to y$ in $\mathcal{C}$ is *locally fully faithful* if, for every $z \in \mathcal{C}$, the map

$$\mathrm{Hom}_{\mathcal{C}}(z, x) \to \mathrm{Hom}_{\mathcal{C}}(z, y)$$

is locally fully faithful (as a map of groupoids).

We note that if $\mathrm{F} : \mathcal{D} \to \mathcal{E}$ is a locally fully faithful map of 2-dg categories, then § A.2.3 shows that for every 2-dg category $\mathcal{C}$, the map

$$\mathrm{Fun}_{\mathbf{2\text{-}dgCat}}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}_{\mathbf{2\text{-}dgCat}}(\mathcal{C}, \mathcal{E})$$

is locally fully faithful as well. Since

$$\mathrm{Fun}_{\mathbf{2\text{-}dgCat}}(\mathcal{C}, \mathcal{D}_1)^{\simeq} \to \mathrm{Fun}_{\mathbf{2\text{-}dgCat}}(\mathcal{C}, \mathcal{D}_2)^{\simeq}$$

is then automatically fully faithful, we see that every locally fully faithful map of 2-dg categories induces a locally fully faithful morphism in $\mathrm{Pith}(\mathbf{2\text{-}dgCat})$.

A.2.5. For any category $\mathcal{C}$, we let

$$\mathcal{C}^{\mathrm{obj}} = \pi_0(\mathcal{C}^{\simeq})$$

denote the category with the same objects as $\mathcal{C}$ but where the only maps are the identity endomorphisms. So $\mathcal{C}^{\mathrm{obj}}$ is a set and we may identify $\mathrm{Tw}(\mathcal{C}^{\mathrm{obj}}) \simeq \mathcal{C}^{\mathrm{obj}}$.

We choose a base point for every connected component of $\mathcal{C}^{\simeq}$. This defines a (non-canonical) functor:

$$\eta : \mathcal{C}^{\mathrm{obj}} \to \mathcal{C}^{\simeq} \to \mathcal{C}.$$

In particular, we obtain a map

$$\lim_{\mathrm{Tw}(\mathcal{C})} \mathrm{F}(c \to c') \to \lim_{\mathrm{Tw}(\mathcal{C}^{\mathrm{obj}})} \mathrm{F}(\eta(d \xrightarrow{\mathbf{1}} d)) \simeq \prod_{d \in \mathcal{C}^{\mathrm{obj}}} \mathrm{F}(d), \tag{A.5}$$

where by a slight abuse of notation we have denoted

$$\mathrm{F}(d) = \mathrm{F}(\eta(d \xrightarrow{\mathbf{1}} d)).$$

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Proposition A.4. *Let* $\mathrm{F}:\mathrm{Tw}(\mathcal{C})\to\mathrm{Poset}$ *be a diagram of posets. The map (A.5) is fully faithful. Furthermore, its essential image consists of those objects* (x_d) *with the following property.*

$$\text{For every arrow } c\to c' \text{ in } \mathcal{C}\text{, the images of } x_c \text{ and } x_{c'} \text{ in } \mathrm{F}(c\to c') \text{ are equal.} \tag{$**$}$$

Proof of Proposition A.4. The functor $\mathrm{Tw}(\mathcal{C}^{\mathrm{obj}})\to\mathrm{Tw}(\mathcal{C})$ is lax surjective by Proposition A.1, so we may apply Proposition A.3. We need to show, for any pair of diagrams as follows, that the images of x_d and $x_{d'}$ in $\mathrm{F}(c\to c')$ are equal.

$$\begin{array}{ccc} d & \longleftarrow & c \\ \downarrow{\scriptstyle \mathbf{1}} & & \downarrow{\scriptstyle f} \\ d & \longrightarrow & c' \end{array} \qquad \begin{array}{ccc} d' & \longleftarrow & c \\ \downarrow{\scriptstyle \mathbf{1}} & & \downarrow{\scriptstyle f} \\ d' & \longrightarrow & c' \end{array} \tag{A.6}$$

We are free to assume that $d'=c'$ and that the lower horizontal arrow in the right square of (A.6) is the identity. Then, the two squares appearing in (A.6) may be rewritten as the following composites, and the condition $(**)$ implies that the images of x_d and $x_{c'}$ in $\mathrm{F}(d\to c')$ are equal.

$$\begin{array}{ccccc} d & \longleftarrow & d & \longleftarrow & c \\ \downarrow{\scriptstyle \mathbf{1}} & & \downarrow & & \downarrow{\scriptstyle f} \\ d & \longrightarrow & c' & \longrightarrow & c' \end{array} \qquad \begin{array}{ccccc} c' & \longleftarrow & d & \longleftarrow & c \\ \downarrow{\scriptstyle \mathbf{1}} & & \downarrow & & \downarrow{\scriptstyle f} \\ c' & \longrightarrow & c' & \longrightarrow & c' \end{array}$$

□

Proposition A.5. *Let* $\mathcal{S}$ *and* $\mathcal{C}$ *be categories, and let the following be a diagram in* $\mathrm{Fun}(\mathcal{S},\mathcal{C})$.

$$\begin{array}{ccc} & & \mathcal{Z} \\ & & \downarrow{\scriptstyle \mathrm{G}} \\ \mathcal{X} & \xrightarrow{\mathrm{F}} & \mathcal{Y} \end{array}$$

Suppose that G *is pointwise locally fully faithful, meaning that for every object* s *of* $\mathcal{S}$*, the functor*

$$\mathrm{G}(s):\mathcal{Z}(s)\to\mathcal{Y}(s)$$

is locally fully faithful as a map in $\mathcal{C}$ *(in the sense of § A.2.4). Then, the functor*

$$\begin{aligned} &\mathrm{Maps}(\mathcal{X},\mathcal{Z})\underset{\mathrm{Maps}(\mathcal{X},\mathcal{Y})}{\times}\{\mathrm{F}\} \\ &\quad\to \prod_{s\in\mathcal{S}^{\mathrm{obj}}}(\mathrm{Fun}(\mathcal{X}(s),\mathcal{Z}(s))\underset{\mathrm{Fun}(\mathcal{X}(s),\mathcal{Y}(s))}{\times}\{\mathrm{F}(s)\}) \end{aligned} \tag{A.7}$$

is fully faithful, and a collection $(\tilde{\mathrm{F}}_s)$ *belongs to its essential image if and only if, for every arrow* $s\to s'$ *in* $\mathcal{S}$*, the images of*

$$\tilde{\mathrm{F}}_s\in\mathrm{Fun}(\mathcal{X}(s),\mathcal{Z}(s))\underset{\mathrm{Fun}(\mathcal{X}(s),\mathcal{Y}(s))}{\times}\{\mathrm{F}(s)\}$$

and

$$\tilde{\mathrm{F}}_{s'}\in\mathrm{Fun}(\mathcal{X}(s'),\mathcal{Z}(s'))\underset{\mathrm{Fun}(\mathcal{X}(s'),\mathcal{Y}(s'))}{\times}\{\mathrm{F}(s')\}$$

in

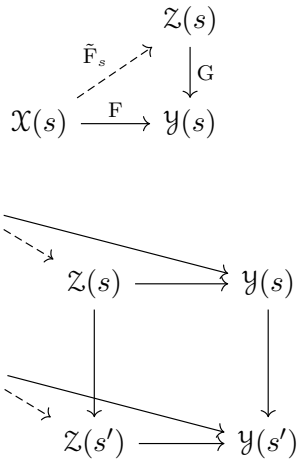
$$\mathrm{Fun}(\mathcal{X}(s),\mathcal{Z}(s'))\underset{\mathrm{Fun}(\mathcal{X}(s),\mathcal{Y}(s'))}{\times}\{\mathrm{F}\}$$

are equal.

A.2.6 *Remark.* This result says that a lift of F to a functor $\tilde{\mathrm{F}}:\mathcal{X}\to\mathcal{Z}$ amounts to a collection of lifts as follows for $s\in\mathcal{S}^{\mathrm{obj}}$ with the property that, for every arrow $s\to s'$ in $\mathcal{S}$, the following

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diagram commutes.



The assumption that G is locally fully faithful guarantees that the commutativity of this diagram is a property and not additional data.

Proof of Proposition A.5. By the twisted arrows formula (A.1), we may perceive (A.7) as the restriction morphism

$$\lim_{\mathrm{Tw}(\mathcal{S})} (\mathrm{Fun}(\mathcal{X}(s), \mathcal{Z}(s')) \underset{\mathrm{Fun}(\mathcal{X}(s),\mathcal{Y}(s'))}{\times} \{\mathrm{F}\}) \\ \to \prod_{s\in\mathcal{S}^{\mathrm{obj}}} (\mathrm{Fun}(\mathcal{X}(s), \mathcal{Z}(s)) \underset{\mathrm{Fun}(\mathcal{X}(s),\mathcal{Y}(s))}{\times} \{\mathrm{F}(s)\}). \tag{A.8}$$

The functors

$$\mathrm{G}(s') \circ (-) : \mathrm{Fun}(\mathcal{X}(s), \mathcal{Z}(s')) \to \mathrm{Fun}(\mathcal{X}(s), \mathcal{Y}(s'))$$

are locally fully faithful because $\mathrm{G}(s')$ is locally fully faithful, so the left-hand side of (A.8) is a limit of posets and the result follows from Proposition A.4. □

A.2.7 *Remark.* This proposition is only used in § 5.2, where $\mathcal{S}$ is $\mathrm{Sm}_{/\mathcal{Y}}$ and $\mathcal{C}$ is the $(\infty, 1)$-categorical version of **2-dgCat**, or in other words, Pith(**2-dgCat**).

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Conflicts of interest

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Coherent sheaves, sheared D-modules, and Hochschild cochains

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