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A strong Frankel theorem for shrinkers

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Abstract

We prove a strong Frankel theorem for mean curvature flow shrinkers in all dimensions: Any two shrinkers in a sufficiently large ball must intersect. In particular, the shrinker itself must be connected in all large balls. The key to the proof is a strong Bernstein theorem for incomplete stable Gaussian surfaces.

1. Introduction

A shrinker $\Sigma^n \subset \mathbf{R}^N$ is a submanifold so that mean curvature flow (MCF) starting from Σ evolves by scaling and shrinks homothetically to become singular. The blow up at any MCF singularity is a shrinker; understanding them is crucial to understanding singularity formation [Hui90, Ilm95, Whi97]. Shrinkers also have a variational interpretation as Gaussian minimal submanifolds.

We will prove a strong Frankel theorem for shrinkers that holds even with boundary.

Theorem 1.1. *If $\Sigma^n_1, \Sigma^n_2 \subset \overline{B_R} \subset \mathbf{R}^{n+1}$ are compact embedded shrinkers, $\partial\Sigma_1, \partial\Sigma_2 \subset \partial B_R$, $R \geq R_n$, and $B_R \cap \Sigma_i \neq \emptyset$, then*

$$B_{R_n} \cap \Sigma_1 \cap \Sigma_2 \neq \emptyset\,. \tag{1.1}$$

The constant R_n only depends on the dimension n.

A classical theorem of Hadamard from 1897 proves that any two closed geodesics in a complete surface with positive curvature must intersect. In [Fra66], Frankel generalized this to compact minimal hypersurfaces in a manifold of positive Ricci curvature. Frankel called this the 'Generalized Hadamard Theorem,' but it is now known as the Frankel theorem. It has played an important role in minimal surface theory [Law70], free boundary problems [FL14], and MCF [Mra20, Mra24, MW20].

A consequence of Theorem 1.1 is that shrinkers are connected in all sufficiently large balls around the origin.

Corollary 1.2. *If $\Sigma^n \subset \mathbf{R}^{n+1}$ is a compact embedded shrinker with $R \geq R_n$ and*

$$\partial\Sigma \subset \partial B_R \quad \text{and} \quad B_R \cap \Sigma \neq \emptyset\,, \tag{1.2}$$

then $B_R \cap \Sigma$ is connected. In particular, if Σ is complete and $\partial\Sigma = \emptyset$, then $B_R \cap \Sigma$ is connected for every $R \geq R_n$.

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The simplest shrinkers are generalized cylinders $\mathbf{S}^k_{\sqrt{2k}} \times \mathbf{R}^{n-k}$, where the k-sphere is centered at 0 with radius $\sqrt{2k}$ and $\mathbf{R}^{n-k}$ contains 0. Here $k = 0, 1, \ldots, n$, where $k = 0$ gives a hyperplane, $k = n$ gives an n-sphere, and we allow all rotations of the Euclidean factor. Taking one of the shrinkers in Theorem 1.1 to be a generalized cylinder implies that any shrinker must intersect such a cylinder.

Corollary 1.3. *If $\Sigma^n \subset \mathbf{R}^{n+1}$ is a properly embedded shrinker, $R \geq R_n$,*

$$\partial\Sigma \subset \partial B_R, \quad \text{and} \quad B_R \cap \Sigma \neq \emptyset\,, \tag{1.3}$$

then $B_{R_n} \cap \Sigma$ intersects every $\mathbf{S}^k_{\sqrt{2k}} \times \mathbf{R}^{n-k}$ for $k = 0, 1, \ldots, n$.

There is also a corresponding rigidity theorem (see Theorem 3.3 for a stronger version).

Theorem 1.4. *If $\Sigma^n \subset \mathbf{R}^{n+1}$ is a complete properly embedded shrinker that lies on one side of $\mathbf{S}^k_{\sqrt{2k}} \times \mathbf{R}^{n-k}$ for some k, then $\Sigma = \mathbf{S}^k_{\sqrt{2k}} \times \mathbf{R}^{n-k}$.*

The case $k = 0$ of Theorem 1.4 is a strong half-space theorem for shrinkers. The strong half-space theorem for minimal surfaces, [HM90], states that any complete proper minimal surface in $\mathbf{R}^3$ that lies in a half-space must itself be a plane. Unlike the results here, the half-space theorem for minimal surfaces holds only in $\mathbf{R}^3$ and is a global result where it is essential that the minimal surface is complete. However, for embedded minimal disks, there is a local estimate that plays a key role in understanding embedded minimal surfaces [CM04].

Theorem 3.3 in [PR14] gave the first half-space-type theorem for complete proper shrinkers without boundary, cf. [CE16, Bre16, CHHW22] for other results in the complete case. Corollary 1.3 only requires a large enough piece of a shrinker and it holds for all generalized cylinders.

The proof of the classical Frankel theorem [Fra66] used the second variation for geodesics. This argument was extended to closed shrinkers, and more generally weighted closed minimal surfaces, in [WW09]. The paper [IPR21] proved a Frankel theorem for complete noncompact shrinkers without boundary under some assumptions at infinity, this time using a Reilly formula and potential theory; see also [BW17, Wan23]. In the complete case, Frankel theorems for varifold shrinkers were proven in [CCMS24, CHHW22].

The approach here is different, relying on a barrier argument and strong Bernstein theorem for Gaussian stable shrinkers, possibly with boundary. It follows from [CM12a] that every properly embedded complete Gaussian minimal hypersurface $\Sigma^n \subset \mathbf{R}^{n+1}$ must be Gaussian unstable (cf. [CM12b, Theorem 0.5]). The next theorem gives a strong generalization of this, allowing for boundary outside of a fixed ball (see Theorem 3.1 for a stronger result).

Theorem 1.5. *If $\Sigma^n \subset \mathbf{R}^N$ is a properly embedded shrinker, $R \geq R_n$, $B_R \cap \Sigma \neq \emptyset$, and $B_R \cap \partial\Sigma = \emptyset$, then Σ is Gaussian unstable.*

There are three contrasts with the classical Euclidean Bernstein theorems. First, we do not require completeness; instead, it is already possible to rule out any sufficient large piece. Second, the theorem applies in all dimensions, instead of holding just up to $\mathbf{R}^7$. Finally, the theorem applies in all codimensions, while the classical Bernstein theorems do not hold already in codimension 2 since complex submanifolds are area-minimizing [Fed69].

2. Gaussian area

We will next recall the variational characterization of shrinkers as Gaussian minimal submanifolds. It will be useful to work with more general rectifiable varifolds, rather than restricting

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to smooth submanifolds. As in [CM12a], define the Gaussian area, or F functional, of an n-dimensional rectifiable varifold $\Sigma^n \subset \mathbf{R}^N$ by

$$F(\Sigma) = (4\,\pi)^{-n/2} \int_{\Sigma} \mathrm{e}^{-|x|^2/4}\,. \tag{2.1}$$

The constant $(4\,\pi)^{-n/2}$ makes the Gaussian area of an n-plane through the origin equal to one. Critical points of F are said to be F-stationary (see [CM12a], § 12). If Σ is an immersed submanifold, then F-stationary is equivalent to the shrinker equation

$$\mathbf{H} = \tfrac{1}{2}\, x^{\perp}\,, \tag{2.2}$$

where $\mathbf{H}$ is the mean curvature vector and $x^{\perp}$ is the perpendicular part of the position vector field.[1] More generally, F-stationary varifolds are weak solutions of (2.2).

The simplest shrinkers are an n-plane through the origin, where both $\mathbf{H}$ and $x^{\perp}$ vanish identically, and an n-sphere of radius $\sqrt{2n}$ centered at the origin. These are the extreme cases of the generalized cylinders $\mathbf{S}^k_{\sqrt{2k}} \times \mathbf{R}^{n-k}$, for $k = 0, \ldots, n$.

2.1 Volume bounds for F-stationary rectifiable varifolds

The next proposition gives local bounds on volume growth for shrinkers; the argument is adapted from global results [CZ13, CZ10] for complete smooth shrinkers.

Proposition 2.1. *If $\Sigma^n \subset B_{\bar{R}} \subset \mathbf{R}^N$ is a proper F-stationary rectifiable varifold and $\sqrt{4+2n} < r_1 < r_2 < \bar{R}$, then*

$$\left(1 - \frac{2n}{r_2^2}\right) \frac{\mathrm{Vol}(B_{r_2} \cap \Sigma)}{r_2^n} \leq \frac{\mathrm{Vol}(B_{r_1} \cap \Sigma)}{r_1^n}\,, \tag{2.3}$$

$$\int_{B_r \cap \Sigma} |x^{\perp}|^2 \leq 2\,n\,\mathrm{Vol}(B_r \cap \Sigma)\,. \tag{2.4}$$

A rectifiable integral varifold Σ has a well-defined tangent plane, and thus also an orthonormal tangent frame $\{e_i\}$, almost everywhere. Thus, given an ambient vector field Y, we can define its tangential and normal parts Y^T and $Y^{\perp}$ almost everywhere and, moreover, we can also define its divergence on Σ,

$$\mathrm{div}_{\Sigma}(Y) = \sum_i \langle \nabla_{e_i} Y, e_i \rangle\,. \tag{2.5}$$

The varifold is said to have mean curvature $\mathbf{H}$ if, for every Lipschitz Y with compact support, we have that[2]

$$\int_{\Sigma} \mathrm{div}_{\Sigma}\, Y = \int_{\Sigma} \langle Y, \mathbf{H} \rangle\,. \tag{2.6}$$

It is F-stationary (shrinker) when $\mathbf{H} = \frac{1}{2}\, x^{\perp}$ in (2.6).

Lemma 2.2. *If $\gamma : \mathbf{R} \to \mathbf{R}$ is a bounded Lipschitz function with $\gamma(s) = 0$ for all large enough s, then*

$$\int_{\Sigma} \left(n\,\gamma(|x|) + |x|\,\gamma'(|x|)\right) = \int_{\Sigma} \left(\frac{1}{2}\,\gamma(|x|)\,|x^{\perp}|^2 + \gamma'(|x|)\,\frac{|x^{\perp}|^2}{|x|}\right)\,. \tag{2.7}$$

[1] See [CM12a, Proposition 3.6] for hypersurfaces, cf. [AHW14, AS13, LL15].

[2] Note that our convention is that the classical mean curvature is minus the trace of A. As a result, we have the opposite sign in (2.6) compared with [Sim83, Definition 16.5].

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If in addition $\gamma' \leq 0$, *then*

$$\int_\Sigma \gamma(|x|)\, |x^\perp|^2 \leq 2n \int_\Sigma \gamma(|x|)\,. \tag{2.8}$$

Proof. Note that $Y = \gamma(|x|)\, x$ is bounded, Lipschitz, and has compact support, and

$$\langle Y, \mathbf{H}\rangle = \tfrac{1}{2}\, \gamma(|x|)\, |x^\perp|^2\,. \tag{2.9}$$

The divergence of Y is given by

$$\begin{aligned} \operatorname{div}_\Sigma Y &= \gamma(|x|)\operatorname{div}_\Sigma x + \gamma'(|x|)\, \langle \nabla |x|, e_i\rangle \langle x, e_i\rangle = n\,\gamma(|x|) + \gamma'(|x|)\, \langle \nabla^T |x|, x^T\rangle \\ &= n\,\gamma(|x|) + \gamma'(|x|)\, \frac{|x^T|^2}{|x|} = n\,\gamma(|x|) + \gamma'(|x|)\, \left(|x| - \frac{|x^\perp|^2}{|x|}\right)\,. \end{aligned} \tag{2.10}$$

The first claim follows from this and (2.6). The second claim follows from the first using that $\gamma' \leq 0$ and $|x| \geq |x^\perp|^2/|x|$. □

Proof of Proposition 2.1. Let $\gamma(s) = \phi(s/R)$, where $\phi : \mathbf{R} \to \mathbf{R}$ is a Lipschitz function with $\phi(s) = 1$ for $s \in [0, 1/2]$, $\phi' \leq 0$, and $\phi(s) = 0$ for all $s \geq 1$. Define[3] $V(R)$ and $T(R)$ by

$$V(R) = \int_\Sigma \phi(|x|/R)\,, \tag{2.11}$$

$$T(R) = \int_\Sigma |x^\perp|^2\, \phi(|x|/R)\,. \tag{2.12}$$

In this notation, the second claim in Lemma 2.2 gives that

$$T(R) \leq 2\, n\, V(R)\,. \tag{2.13}$$

Taking the limit as ϕ goes to the characteristic function on $[0, 1]$ gives (2.4).

Since $\gamma(s) = \phi(s/R)$ and $\partial_R \phi(s/R) = -(s/R^2)\, \phi'(s/R) = -(s/R)\, \gamma'(s)$, we have that[4]

$$V'(R) = -R^{-1} \int_\Sigma |x|\, \gamma'(|x|)\,, \tag{2.14}$$

$$T'(R) = -R^{-1} \int_\Sigma |x^\perp|^2\, |x|\, \gamma'(|x|)\,. \tag{2.15}$$

Thus, we have that

$$R\, V'(R) - n\, V(R) = -\frac{1}{2}\, T(R) + \int_\Sigma (-\gamma'(|x|))\, \frac{|x^\perp|^2}{|x|}\,. \tag{2.16}$$

Let $\rho \in [1/2, 1]$ be the supremum of $\{s \,|\, \phi(s) = 1\}$. It follows that the support of γ' is on $[\rho\, R, R]$ and, consequently,

$$\int_\Sigma (-\gamma'(|x|))\, \frac{|x^\perp|^2}{|x|} \leq \frac{1}{\rho^2\, R^2} \int_\Sigma (-\gamma'(|x|))\, |x^\perp|^2\, |x| = \frac{1}{\rho^2\, R}\, T'(R)\,. \tag{2.17}$$

Using this in (2.16) gives that

$$R\, V'(R) - n\, V(R) \leq -\frac{1}{2}\, T(R) + \rho^{-2}\, \frac{T'(R)}{R}\,. \tag{2.18}$$

[3]Compare [Sim83, § 17] where a similar approach is used to prove monotonicity.

[4]The quantity V is a smoothed out version of the volume in B_R and the derivative V' is a smoothed version of the boundary integral. These smoothed out versions work well for varifolds. In the classical case, one can instead use the co-area formula to differentiate the volume; cf. [CM25, Lemma 3.14].

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Thus, we get that

$$(R^{-n}\,V(R))' = R^{-n-1}\,\left(R\,V'(R) - n\,V(R)\right) \leq \left(-\frac{1}{2}\,T(R) + \rho^{-2}\,\frac{T'(R)}{R}\right)\,R^{-n-1}\,. \tag{2.19}$$

Integrating this from r_1 to r_2 gives that

$$\frac{V(r_2)}{r_2^n} - \frac{V(r_1)}{r_1^n} \leq \int_{r_1}^{r_2} \left(\rho^{-2}\,T'(s)\,s^{-n-2} - \frac{1}{2}\,T(s)\,s^{-n-1}\right)\,ds\,. \tag{2.20}$$

Integrating by parts on the $T'(s)$ term gives

$$\begin{aligned}\int_{r_1}^{r_2} T'(s)\,s^{-n-2}\,ds &= \frac{T(r_2)}{r_2^{n+2}} - \frac{T(r_1)}{r_1^{n+2}} + (n+2)\,\int_{r_1}^{r_2}\frac{T(s)}{s^{n+3}}\,ds \\ &\leq \frac{2n\,V(r_2)}{r_2^{n+2}} + (n+2)\,\int_{r_1}^{r_2}\frac{T(s)}{s^{n+3}}\,ds\,,\end{aligned} \tag{2.21}$$

where the last inequality used that $T(s) \geq 0$ and $T(s) \leq 2n\,V(s)$ by (2.13). Using this in (2.20) gives that

$$\frac{V(r_2)}{r_2^n} - \frac{V(r_1)}{r_1^n} \leq \rho^{-2}\,\frac{2n\,V(r_2)}{r_2^{n+2}} + \rho^{-2}\,(n+2)\,\int_{r_1}^{r_2}\frac{T(s)}{s^{n+3}}\,ds - \frac{1}{2}\,\int_{r_1}^{r_2}\frac{T(s)}{s^{n+1}}\,ds\,. \tag{2.22}$$

Since $r_1^2 > 2\,n+4$, we can take $\rho < 1$ large enough that

$$\frac{\rho^{-2}\,(n+2)}{s^2} \leq \frac{1}{2}\,, \tag{2.23}$$

so that the $T(s)$ integrand on the right-hand in (2.22) is non-positive. It follows that

$$\frac{V(r_2)}{r_2^n}\left(1 - \rho^{-2}\,\frac{2n}{r_2^2}\right) \leq \frac{V(r_1)}{r_1^n}\,. \tag{2.24}$$

Taking the limit of this as ϕ goes to the characteristic function of $[0,1]$ and $\rho \to 1$ gives the first claim. □

2.2 Singularities

An F-stationary rectifiable varifold Σ can be decomposed into an open set $\mathrm{Reg}(\Sigma)$ of regular points, where Σ is an embedded submanifold, together with a closed singular set $\mathrm{Sing}(\Sigma) = \Sigma \setminus \mathrm{Reg}(\Sigma)$.

If $\Sigma^n \subset \mathbf{R}^{n+1}$ is F-minimizing, then it is also area-minimizing for the Gaussian metric (see, e.g., [CM12a, p. 770]). Therefore, the regularity theory for area-minimizers ([SS81, Theorem 4] or [Wic14, § 18]) implies that

$$\dim_H \mathrm{Sing}(\Sigma) \leq n-7\,, \tag{2.25}$$

where $\dim_H$ is Hausdorff dimension. By [CN13, Theorem 5.8], the same bound holds for Minkowski dimension $\dim_M$. Recall that the ρ-covering number $\mathrm{Cov}_\rho(S)$ of a set S is the number of closed balls of radius $\rho > 0$ in a minimal covering[5] of S. A set $S \subset \mathbf{R}^N$ has Minkowski dimension less than k if

$$\lim_{\rho\to 0}\,\rho^k\,\mathrm{Cov}_\rho(S) = 0\,. \tag{2.26}$$

[5]Every ball in the covering has the same radius ρ; this differs from coverings used to compute Hausdorff measure and dimension, where the balls can have different radii.

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Since each ball has Euclidean volume comparable to ρ^N, an immediate consequence of (2.26) is that

$$\lim_{\rho\to 0} \rho^{k-N} \operatorname{Vol}_{\mathbf{R}^N}(T_\rho(S)) = 0\,, \tag{2.27}$$

where $T_\rho(S)$ is the Euclidean tubular neighborhood of radius $\rho > 0$ of S. A bound on $\dim_M$ is stronger than a bound on the Hausdorff dimension $\dim_H$ and, in particular, implies capacity estimates and removability of singularities for elliptic equations.

The next lemma constructs good cutoff functions on the regular part when the Minkowski dimension of the singular set is less than $n-2$.

Lemma 2.3. *Let $\Sigma^n \subset \mathbf{R}^N$ be a proper F-stationary rectifiable varifold in B_R with*

$$\dim_M \operatorname{Sing}(\Sigma) < n-2\,. \tag{2.28}$$

Given $\epsilon > 0$, there is a Lipschitz function $\phi : \Sigma \to [0,1]$ that is zero on $\overline{B_{R-1}} \cap \operatorname{Sing}(\Sigma)$ and

$$\int_\Sigma |\nabla\phi|^2\, \mathrm{e}^{-|x|^2/4} + \int_\Sigma (1-\phi^2)\, \mathrm{e}^{-|x|^2/4} \leq \epsilon\,. \tag{2.29}$$

Proof. Set $S = \overline{B_{R-1}} \cap \operatorname{Sing}(\Sigma)$. Let $\delta > 0$ be a small constant to be chosen below. The assumption (2.28) implies that there exists $\rho < 1/3R$ so that

$$\operatorname{Cov}_\rho(S) < \delta\, \rho^{2-n}\,. \tag{2.30}$$

Let $\{B_\rho(x_i)\}$, $i = 1, \ldots, m$ with $m < \delta\, \rho^{2-n}$, be a minimal covering of S by balls of radius ρ. In particular, we have that every $x_i \in \overline{B_{R-1+R^{-1}}}$. Let $d_S(x)$ be the Euclidean distance from x to S and define the function ϕ by

$$\phi(x) = \begin{cases} 0 & \text{if } d_S(x) \leq \rho, \\ \rho^{-1}\, d_S(x) - 1 & \text{if } \rho < d_S(x) < 2\rho, \\ 1 & \text{if } 2\,\rho \leq d_S(x). \end{cases} \tag{2.31}$$

Note that ϕ maps to $[0,1]$ and vanishes on S. Moreover, both $(1-\phi^2)$ and $|\nabla\phi|$ have support in the tubular neighborhood $T_{2\rho}(S)$ of radius 2ρ about S. Since $|\nabla\phi| \leq \rho^{-1}$, the exponential weight is at most one, and $T_{2\rho}(S)$ is contained in the union of the balls $B_{3\rho}(x_i)$, it follows that

$$\begin{aligned} \int |\nabla\phi|^2\, \mathrm{e}^{-|x|^2/4} + \int (1-\phi^2)\, \mathrm{e}^{-|x|^2/4} &\leq \left(1+\rho^{-2}\right)\, \operatorname{Vol}\left(T_{2\rho}(S) \cap \Sigma\right) \\ &\leq 2\,\rho^{-2} \sum_i \operatorname{Vol}\left(B_{3\rho}(x_i) \cap \Sigma\right) \\ &< 2\,\delta\,\rho^{-n} \max_i \operatorname{Vol}\left(B_{3\rho}(x_i) \cap \Sigma\right)\,. \end{aligned} \tag{2.32}$$

The shrinker equation implies that Σ has mean curvature weakly in L^∞ and bounded by $R/2$ on B_R. Therefore, since $3\rho \leq 1/R$ and $B_{1/R}(x_i) \subset B_R$ for every i, monotonicity of volume with bounded mean curvature ([CM11, Lemma 1.18]; cf. [Sim83, § 18]) gives a constant c so that

$$\operatorname{Vol}\left(B_{3\rho}(x_i) \cap \Sigma\right) \leq c\, \left(\frac{\rho}{R^{-1}}\right)^n\, \operatorname{Vol}(B_{1/R}(x_i) \cap \Sigma) \leq c\,\rho^n\, R^n \operatorname{Vol}(B_R \cap \Sigma)\,. \tag{2.33}$$

Using this in (2.32) gives

$$\int |\nabla\phi|^2\, \mathrm{e}^{-|x|^2/4} + \int (1-\phi^2)\, \mathrm{e}^{-|x|^2/4} < 2\,c\,\delta R^n \operatorname{Vol}(B_R \cap \Sigma)\,. \tag{2.34}$$

Since R here is fixed and Σ is proper, the right-hand side of (2.34) is bounded by $C'\,\delta$ for a fixed constant C' independent of δ. The lemma follows by choosing $\delta > 0$ sufficiently small depending on ϵ and C'. □

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3. A strong Bernstein theorem for shrinkers

The classical Bernstein theorem for minimal surfaces gives that an entire minimal graph in $\mathbf{R}^3$ must be flat. This was extended to area-minimizing hypersurfaces up to $\mathbf{R}^7$ and shown to be false in $\mathbf{R}^8$ (see, e.g., [Sim83, CM11]). The second variation of Gaussian area was introduced in [CM12a] and was used to show that there are no complete Gaussian stable hypersurfaces.[6] Wang used the second variation and a calibration argument to prove a Bernstein theorem for entire graphical shrinkers in [Wan11].

In this section, we will prove the strong Bernstein theorem, Theorem 1.5. In fact, we show the following more general theorem that allows boundary and a small set of singularities.

THEOREM 3.1. *If $\Sigma^n \subset B_R \subset \mathbf{R}^N$ is a proper F-stationary rectifiable varifold with $R \geq R_n$ and $\dim_M \mathrm{Sing}(\Sigma) < (n-2)$, then Σ is Gaussian unstable.*

Theorem 3.1 has Theorem 1.5 as an immediate consequence when the singular set is empty. We will need this more general version to prove the strong Frankel theorem in high dimensions where Gaussian minimizers can have singularities.

3.1 Second variation and stability

The second variation operator L for Gaussian area from [CM12a] ([CM12a, equation (4.13)] for hypersurfaces, [AHW14, Theorem 4.1], cf. [AS13, LL15, CM20, CM23] for higher codimensions) is given on normal vector fields v by

$$L\, v = \mathcal{L}\, v + \langle A_{ij}, v\rangle\, A_{ij} + \tfrac{1}{2}\, v\,. \tag{3.1}$$

Here $\mathcal{L}\, v = \Delta^{\perp} v - \frac{1}{2}\, \nabla^{\perp}_{x^T} v$, $\perp$ denotes the component normal to Σ, A is the second fundamental form, and $A_{ij} = A(f_i, f_j)$, where f_i is an orthonormal basis for the tangent space of Σ at the given point. When Σ is a hypersurface with unit normal $\mathbf{n}$, then

$$L = \mathcal{L} + |A|^2 + \tfrac{1}{2}\,. \tag{3.2}$$

An F-stationary rectifiable varifold $\Sigma^n \subset \mathbf{R}^N$ is said to be Gaussian stable if

$$\int_{\Sigma} \langle \eta,\, L\, \eta\rangle\, \mathrm{e}^{-|x|^2/4} \leq 0 \tag{3.3}$$

for every normal vector field η with compact support on $\mathrm{Reg}(\Sigma)$. The inequality (3.3) is known as the stability inequality.

The next proposition gives a uniform Poincaré inequality on the regular part for Gaussian stable F-stationary rectifiable varifolds.

PROPOSITION 3.2. *If $\Sigma^n \subset \mathbf{R}^N$ is a proper Gaussian stable F-stationary rectifiable varifold, then every function w with compact support on $\mathrm{Reg}(\Sigma)$ satisfies*

$$\int_{\Sigma} w^2\, \mathrm{e}^{-|x|^2/4} \leq 2 \int_{\Sigma} |\nabla w|^2 \mathrm{e}^{-|x|^2/4}\,. \tag{3.4}$$

Proof. When Σ is a smooth embedded hypersurface, then (3.4) follows immediately from the stability inequality with $\eta = w\,\mathbf{n}$ (where $\mathbf{n}$ is a unit normal) and integration by parts. We will show that (3.4) holds in arbitrary codimension and without smoothness or embeddedness.

[6] This requires that there is at most polynomial volume growth; by [DX13], this holds when Σ is proper.

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Let $e_1, \dots, e_N \in \mathbf{R}^N$ be a Euclidean orthonormal basis. For each i, define the normal vector field v_i by taking the perpendicular part of the constant vector e_i,

$$v_i = e_i^{\perp} \, . \tag{3.5}$$

Since the rank of the normal bundle is the codimension $N-n$, we have that

$$\sum_i |v_i|^2 = N - n \, . \tag{3.6}$$

By [CM12a, Theorem 5.2] (see[AHW14, 5.1] for the higher codimension case), each v_i satisfies[7]

$$L \, v_i = \tfrac{1}{2} \, v_i \, . \tag{3.7}$$

Now let w have compact support on $\mathrm{Reg}(\Sigma)$ and observe that

$$L \, (w \, v_i) = (\mathcal{L} \, w) \, v_i + \tfrac{1}{2} \, w \, v_i + 2 \, \nabla^{\perp}_{\nabla w} v_i \, . \tag{3.8}$$

Using $w \, v_i$ in the stability inequality (3.3) gives

$$\begin{aligned} \frac{1}{2} \int w^2 \, |v_i|^2 \, \mathrm{e}^{-|x|^2/4} &\leq - \int_{\Sigma} \left(w \, (\mathcal{L} \, w) \, |v_i|^2 + 2 \, w \, \langle v_i, \nabla^{\perp}_{\nabla w} v_i \rangle \right) \, \mathrm{e}^{-|x|^2/4} \\ &= - \int_{\Sigma} \left(w \, (\mathcal{L} \, w) \, |v_i|^2 + w \, \langle \nabla w, \nabla |v_i|^2 \rangle \right) \, \mathrm{e}^{-|x|^2/4} \, . \end{aligned} \tag{3.9}$$

Summing this over i and using (3.6) gives

$$\frac{N-n}{2} \int w^2 \, \mathrm{e}^{-|x|^2/4} \leq -(N-n) \int_{\Sigma} w \, (\mathcal{L} \, w) \, \mathrm{e}^{-|x|^2/4} = (N-n) \int_{\Sigma} |\nabla w|^2 \, \mathrm{e}^{-|x|^2/4} \, , \tag{3.10}$$

where the last equality used the divergence theorem. □

Proof of Theorem 3.1. Let $R > \sqrt{4 + 2\,n}$ be a radius to be chosen large enough below. Since

$$\mathcal{L} \, |x|^2 = 2\,n - |x|^2 \tag{3.11}$$

by [CM12a, Lemma 3.20], the maximum principle guarantees that Σ intersects the closure of $B_{\sqrt{2\,n}}$.

Let η be a radial (i.e. a function of $|x|$) cutoff function on Σ that is identically one on $B_{r_2 - 1}$ and cuts off linearly to zero from $r_2 - 1$ to r_2. Given $\epsilon > 0$, Lemma 2.3 gives a function $\phi = \phi_\epsilon$ that vanishes on $B_{r_2} \cap \mathrm{Sing}(\Sigma)$, has $0 \leq \phi \leq 1$, and

$$\int_{\Sigma} |\nabla \phi|^2 \, \mathrm{e}^{-|x|^2/4} \leq \epsilon \, , \tag{3.12}$$

$$\int_{\Sigma} (1 - \phi^2) \, \mathrm{e}^{-|x|^2/4} \leq \epsilon \, . \tag{3.13}$$

It follows that $w = \eta \, \phi$ has compact support in the regular part of $B_{r_2} \cap \Sigma$. Since η, ϕ map to $[0,1]$, the squared triangle inequality gives that

$$|\nabla w|^2 \leq 2 \, |\nabla \eta|^2 + 2 \, |\nabla \phi|^2 \, . \tag{3.14}$$

Integrating this and using that $|\nabla \eta| \leq 1$ vanishes inside $B_{r_2 - 1}$ gives

$$\begin{aligned} \int |\nabla w|^2 \, \mathrm{e}^{-|x|^2/4} &\leq 2 \int |\nabla \phi|^2 \, \mathrm{e}^{-|x|^2/4} + 2 \, \mathrm{e}^{-(r_2 - 1)^2/4} \, \mathrm{Vol}(B_{r_2} \cap \Sigma) \\ &\leq 2 \, \epsilon + 2 \, \mathrm{e}^{-(r_2 - 1)^2/4} \, \mathrm{Vol}(B_{r_2} \cap \Sigma) \, , \end{aligned} \tag{3.15}$$

[7] Equation (3.7) is special to shrinkers, as opposed to holding more generally for weighted minimal submanifolds in an ambient space with strictly positive Bakry–Emery Ricci curvature.

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where all integrals are on Σ and the last inequality used (3.12). On the other hand, taking $r_2 > r_1 + 1$ so that $\eta \equiv 1$ on B_{r_1}, we have that

$$\begin{aligned}\int w^2 \, \mathrm{e}^{-|x|^2/4} &\geq \int_{B_{r_1} \cap \Sigma} w^2 \, \mathrm{e}^{-|x|^2/4} = \int_{B_{r_1} \cap \Sigma} \phi^2 \, \mathrm{e}^{-|x|^2/4} \\ &\geq \mathrm{e}^{-r_1^2/4} \operatorname{Vol}(B_{r_1} \cap \Sigma) - \int (1 - \phi^2) \, \mathrm{e}^{-|x|^2/4} \geq \mathrm{e}^{-r_1^2/4} \operatorname{Vol}(B_{r_1} \cap \Sigma) - \epsilon \,, \end{aligned} \tag{3.16}$$

where the last inequality used (3.13). If Σ was Gaussian stable, then Proposition 3.2 would give that

$$\int w^2 \, \mathrm{e}^{-|x|^2/4} \leq 2 \int |\nabla w|^2 \, \mathrm{e}^{-|x|^2/4} \,. \tag{3.17}$$

Using the last two bounds on w and ∇w, this would imply that

$$\mathrm{e}^{-r_1^2/4} \operatorname{Vol}(B_{r_1} \cap \Sigma) - \epsilon \leq 4\,\epsilon + 4 \, \mathrm{e}^{-(r_2 - 1)^2/4} \operatorname{Vol}(B_{r_2} \cap \Sigma) \,. \tag{3.18}$$

Since $\epsilon > 0$ is arbitrary, this gives

$$\mathrm{e}^{-r_1^2/4} \operatorname{Vol}(B_{r_1} \cap \Sigma) \leq 4 \, \mathrm{e}^{-(r_2 - 1)^2/4} \operatorname{Vol}(B_{r_2} \cap \Sigma) \,. \tag{3.19}$$

However, Proposition 2.1 gives that

$$\left(1 - \frac{2n}{r_2^2}\right) \frac{\operatorname{Vol}(B_{r_2} \cap \Sigma)}{r_2^n} \leq \frac{\operatorname{Vol}(B_{r_1} \cap \Sigma)}{r_1^n} \,. \tag{3.20}$$

The last two inequalities are contradictory for r_1 fixed and r_2 large enough depending on the dimension n, so Σ cannot be Gaussian stable. □

3.2 Proof of the strong Frankel theorem

The idea for the strong Frankel theorem is that if there are two distinct shrinkers in a large ball, then a barrier argument produces an F-minimizer between them, violating the strong Bernstein theorem. In this subsection, we restrict to shrinkers in codimension one.

Proof of Theorem 1.1. Applying the maximum principle to $|x|^2$, it follows that

$$\overline{B_{\sqrt{2n}}} \cap \Sigma_i \neq \emptyset \quad \text{for } i = 1, 2 \,. \tag{3.21}$$

Let R_n be from Theorem 3.1. We will argue by contradiction, so assume that

$$B_{R_n} \cap \Sigma_1 \cap \Sigma_2 = \emptyset \,. \tag{3.22}$$

In particular, we can find a line segment

$$\sigma \subset \overline{B_{\sqrt{2n}}} \tag{3.23}$$

that starts at a point in Σ_1, ends at a point in Σ_2, and is otherwise disjoint from $\Sigma_1 \cup \Sigma_2$. Let Ω denote the component of $B_{R_n} \setminus (\Sigma_1 \cup \Sigma_2)$ that contains the interior of σ. It follows that σ has non-zero linking number with $\partial \Sigma_1$ in Ω.

Next, we solve for the F-minimizer $\Gamma \subset \overline{\Omega}$ with $\partial \Gamma = \partial (B_{R_n} \cap \Sigma_1)$ (see, e.g., [Fed69, 5.1.6]). The part $B_{R_n} \cap \Gamma$ of Γ disjoint from ∂B_{R_n} must be F-stationary, but Γ might contain pieces in ∂B_{R_n} that do not satisfy the shrinker equation. However, the linking argument implies that $\Gamma \cap \sigma \neq \emptyset$. Thus, there is a connected component Γ_1 of $B_{R_n} \cap \Gamma$ with

$$\Gamma_1 \cap \sigma \neq \emptyset \,. \tag{3.24}$$

For $n \leq 6$, the interior regularity theory for area-minimizers, [SS81, Theorem 4] or [Wic14, § 18], gives that Γ_1 is smooth, embedded, and stable. This contradicts Theorem 1.5, completing the

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proof when $n \leq 6$. For $n \geq 7$, Γ_1 can have singularities, but [SS81] and [CN13, Theorem 5.8] give that $\mathrm{Reg}(\Gamma_1)$ is stable and

$$\dim_M \mathrm{Sing}(\Gamma_1) \leq n-7\,. \tag{3.25}$$

This contradicts Theorem 3.1, so we conclude that

$$B_{R_n} \cap \Sigma_1 \cap \Sigma_2 \neq \emptyset\,. \tag{3.26}$$

□

Proof of Corollary 1.2. Connectedness follows immediately from Theorem 1.1 by regarding the possible connected components of $B_R \cap \Sigma$ as distinct shrinkers with boundary in ∂B_R. □

Proof of Corollary 1.3. This follows from Theorem 1.1 with $\Sigma_1 = \Sigma$ and Σ_2 equal to a generalized cylinder $\mathbf{S}^k_{\sqrt{2k}} \times \mathbf{R}^{n-k}$. □

Theorem 1.4 follows immediately from the following stronger effective version.

Theorem 3.3. *Let Ω^+ and Ω^- be the components of the complement of $\mathbf{S}^k_{\sqrt{2k}} \times \mathbf{R}^{n-k}$ for some $k \in \{0, 1, \ldots, n\}$. If $\Sigma^n \subset \mathbf{R}^{n+1}$ is a properly embedded shrinker, $R \geq R_n$,*

$$\partial\Sigma \subset \partial B_R, \quad B_R \cap \Sigma \neq \emptyset, \quad \text{and} \quad B_R \cap \Sigma \quad \text{intersects at most one of } \Omega^{\pm}\,, \tag{3.27}$$

then $B_R \cap \Sigma \subset \mathbf{S}^k_{\sqrt{2k}} \times \mathbf{R}^{n-k}$.

Proof. By assumption, $B_R \cap \Sigma$ lies on one side of $\mathbf{S}^k_{\sqrt{2k}} \times \mathbf{R}^{n-k}$. Since Corollary 1.3 gives that $B_R \cap \Sigma$ is connected and must intersect $\mathbf{S}^k_{\sqrt{2k}} \times \mathbf{R}^{n-k}$, the strong maximum principle (for minimal hypersurfaces in a Riemannian manifold) gives that they must agree identically. □

Conflicts of interest

None.

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