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AN ANALYTICAL APPROACH FOR THE BREAKDOWN DISTRIBUTION OF A THIN OXIDE

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Dielectric breakdown in a thin oxide is presented in terms of an interacting particle system on a two-dimensional lattice. All edges in the system are initially assumed to be closed. An edge between two adjacent vertices will open according to an exponentially distributed random variable. Breakdown occurs at the time an open path connects the top layer of the lattice to the bottom layer. Using the extreme value theory, we show that the time until breakdown is asymptotically Weibull distributed.

Keywords: Weibull distribution, extreme value theory, interacting particle systems, dielectric breakdown

1. INTRODUCTION

There is strong empirical evidence, see, for example, Dumin [4], that suggests the time until dielectric breakdown of a thin oxide follows an extreme value distribution. Suñé et al. [13] suggest that the time until breakdown follows a gamma distribution, while Sahimi [12] provides reasons for time until breakdown to follow an extreme value distribution. In this paper, we model dielectric breakdown failure using an interacting particle system. The time until dielectric breakdown is the minimum of N dependent and nonidentically distributed random variables each having distribution similar to that proposed by Suñé et al. [13]. Using the results from the extreme value theory, we are able to show that the time to breakdown is asymptotically Weibull distributed.

Our particle system models the dielectric breakdown phenomenon as presented in Degraeve et al. [3] and Dumin [4]. We initially assume that the oxide is perfect containing no electron trap. As time progresses, electron traps are generated at random positions as voltage is applied to the oxide (see Figure 1(a)). When two electron traps are within a fixed distance, there may be conduction between them. Conduction may also occur if the electron trap lies within the fixed distance from the interface. As traps are generated during electrical stressing, conducting chains of neighboring traps are formed (see Figure 1(b)). Finally, a conducting path which connects the cathode and anode interface is formed (see Figure 1(c)). This will cause the oxide to collapse (dielectric breakdown). The phenomenon of dielectric breakdown in insulating films is of importance in semiconductor devices (see [5,13]). Their is great deal of data to suggest that the distribution of dielectric breakdown is either a Weibull or a Gumbel distributed.

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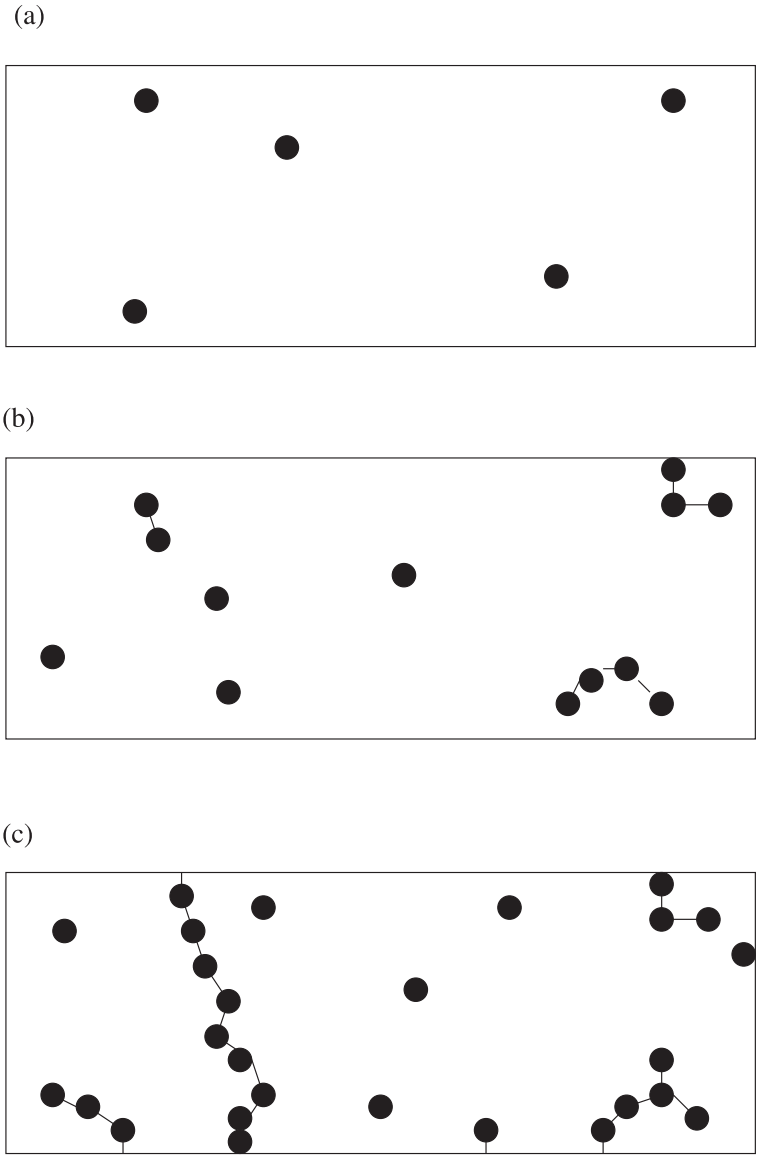


FIGURE **1.** The process of dielectric breakdown.

We model the thin oxide as an interacting particle system on a two-dimensional lattice $\{0,\dots,N\}\times\{0,\dots,h\}$ where $h \ll N$. The lattice points (i,j)'s are considered as the vertices of a graph in two dimensions. There is an edge between two vertices if the distance between them equals to 1. Edges can be in only one of these two states: open or closed. All edges are initially assumed to be closed. The model resembles an edge percolation. The main difference between a percolation and this model is our edges turn on (open) at random times. The length of time it takes an edge, e, to open is an exponentially distributed random variable, X_e, with mean λ^{-1}. We assume the random variables X_e's are independent. Conduction between two electron traps of the thin oxide is viewed as the process of opening an edge. Dielectric breakdown is defined as the time when there exists an open path from

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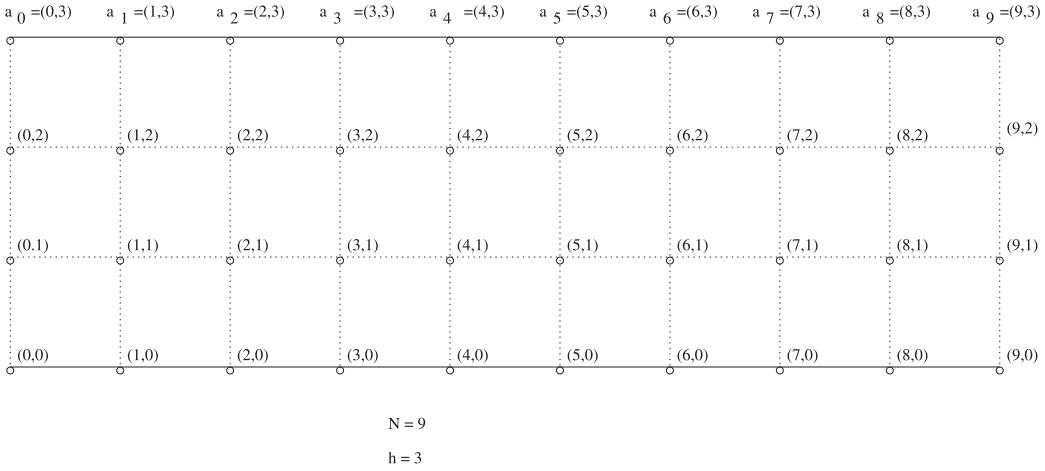


FIGURE **2.** The system $\mathbb{S}_N$ where $N = 9$ and $h = 3$.

one layer to the opposite layer. Thus, the lifetime of the system is the amount of time it takes for at least one open path connecting the top to the bottom layer. We show that for sufficiently large N, the lifetime distribution is approximately Weibull. More precisely, the Weibull distribution is a normalized limit of the lifetime distributions as $N \to \infty$.

The proof that the limit distribution is Weibull is an application of extreme value theory (see [9,11] for background on the extreme value theory). To each lattice point (i, h) on the top layer, we associate a random variable $\tau_{N,i}$ which is the length of time it takes for a path of open edges connecting the vertex (i, h) to the bottom layer. The lifetime of the system is then the minimum of the variables, $\tau_{N,i}$'s. If the $\tau_{N,i}$'s are independent, which they are not, the lifetime distribution must be one of the classical extreme value distributions. The first step is to show that each of the $\tau_{N,i}$'s belongs to the domain of attraction of the Weibull. This can be done by showing that distribution of each $\tau_{N,i}$ is regularly varying at h (see [1,11]). The remainder of the proof is to show that the independent assumption can be relaxed. This can be accomplished by checking the modified versions of the mixing conditions $D(u_N)$ and $D'(u_N)$ given in Leadbetter et al. [9].

2. PROBLEM DESCRIPTION

In this section, we give a precise description of the lattice and introduce some notation that will be needed in the remainder of the paper. Let $\mathbb{Z}^2$ be the set of all vectors with integral coordinates in two dimensions. The distance $\delta(\mathbf{x}, \mathbf{y})$ from $\mathbf{x} = (x_1, x_2)$ to $\mathbf{y} = (y_1, y_2)$ is defined by

$$\delta(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{2} |x_i - y_i|.$$

There is an edge connecting two vertices in the system if the distance between them equals to 1. The particle model is constructed on the graph $\mathbb{S}_N = (\mathbb{V}_N, \mathbb{E}_N)$, where the set of vertices $\mathbb{V}_N = \{(x_1, x_2) : x_1 = 0, \ldots, N \text{ and } x_2 = 0, \ldots, h\}$ and $\mathbb{E}_N$ is the set of all edges. Figure 2 describes the system $\mathbb{S}_N$ where $N = 9$ and $h = 3$.

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Let $\{X_e, e \in \mathbb{E}_N\}$ be a sequence of independent exponentially distributed random variables each having mean λ^{-1}. Construct the process $Y = \{Y_e(t), t \geq 0, e \in \mathbb{E}_N\}$ by setting

$$Y_e(t) = \begin{cases} 0, & \text{if } t < X_e; \\ 1, & \text{if } t \geq X_e. \end{cases}$$

The random vector $Y(t) = \{Y_e(t), e \in \mathbb{E}_N\}$ can be viewed as a subgraph of $\mathbb{S}_N$ where the vertices are $\mathbb{V}_N$ and the edges are those $e \in \mathbb{E}_N$ for which $Y_e(t) = 1$. We will call the path $(\mathbf{x}_0, \ldots, \mathbf{x}_N)$ open at time t if $e_i = (\mathbf{x}_{i-1}, \mathbf{x}_i) \in \mathbb{E}_N$ for $i = 1, \ldots, N$ and $Y_{e_i}(t) = 1$.

The stochastic process Y is a Markov jump process that evolves by adding edges to a graph one at a time. Let T_n be the time of the nth jump and $X_n = Y(T_n)$. Observe that we can characterize X_n by specifying which edges are open. Thus, we consider $X_n \subseteq \mathbb{E}_N$ to be those e with $Y_e(T_n) = 1$. Then,

$$\begin{aligned} &P(T_{n+1} - T_n > t, X_{n+1} = A \cup \{j\} | X_0, T_0, \ldots, X_n = A, T_n) \\ &\quad = \begin{cases} \dfrac{1}{|\mathbb{E}_N| - |A|} e^{-(|\mathbb{E}_N| - |A|)\lambda t}, & \text{if } j \notin A; \\ 0, & \text{if } j \in A, \end{cases} \end{aligned}$$

where $|A|$ denotes the number of elements in the set A. The above equation follows since given the open edges at time T_n is A, the time from T_n to T_{n+1} is the minimum of $|\mathbb{E}_N| - |A|$ independent exponentially distributed random variables each having mean λ^{-1}. Since these random variables are also identically distributed, any element of $\mathbb{E}_N \setminus A$ has an equal chance of being the minimum.

We can also construct Y from a sequence $\{\tilde{X}_n, \tilde{T}_n; n = 0, 1, \ldots\}$ as follows. Set

$$P(\tilde{T}_{n+1} - \tilde{T}_n > t, \tilde{X}_{n+1} = A \cup \{j\} | \tilde{X}_0, \tilde{T}_0, \ldots, \tilde{X}_n = A, \tilde{T}_n) = \frac{1}{|\mathbb{E}_N|} e^{-|\mathbb{E}_N|\lambda t},$$

where we allow $j \in A$. Thus, $\{\tilde{X}_n, \tilde{T}_n; n = 0, 1, \ldots\}$ permits jumps from a state to itself and is usually called the uniformization of the process Y. We observe that the above comments can be applied to any subgraph $(\mathbb{V}'_N, \mathbb{E}'_N)$ of $\mathbb{S}_N$. The process $\{Y_e(t), t \geq 0, e \in \mathbb{E}'_N\}$ is a Markov jump process which can be uniformized in a manner similar to Y (see [2, pp. 259–261]).

Define A_N and B_N to be the set of vertices for the top and bottom layers,

$$A_N = \{(x_1, h) : 0 \leq x_1 \leq N\}, \quad B_N = \{(x_1, 0) : 0 \leq x_1 \leq N\}.$$

We write "$A_N \overset{t}{\leftrightarrow} B_N$" to be the event that at time t, there exists an open path connecting some vertex in A_N to some vertex in B_N. Let τ_N be the lifetime of the system which is the minimum over all t until the event $A_N \overset{t}{\leftrightarrow} B_N$ occurs. Denote the vertices of A_N by

$$\mathbf{a}_0 = (0, h), \mathbf{a}_1 = (1, h), \ldots, \mathbf{a}_N = (N, h).$$

Let $\tau_{N,j}$ be the length of time until the event $\mathbf{a}_j \overset{t}{\leftrightarrow} B_N$ occurs. For each N,

$$\tau_N = \min_j \tau_{N,j}.$$

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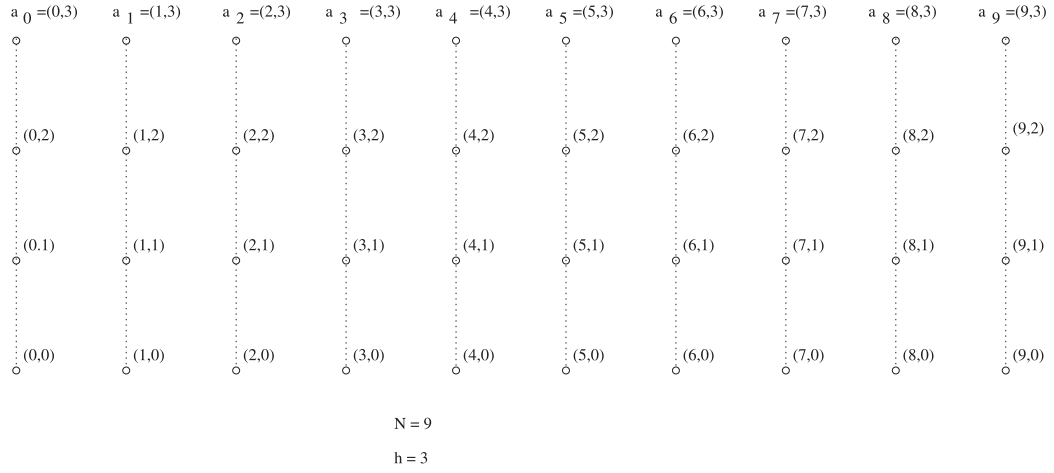


FIGURE **3.** The modified system where $N = 9$ and $h = 3$.

THEOREM 2.1: *Let τ_N denote the lifetime of the system $\mathbb{S}_N$ where $\lambda = 1$ and set $u_N = (1/N)^{1/h}$. Then, for each $x > 0$,*

$$\lim_{N\to\infty} P(\tau_N > x u_N) = e^{-x^h}.$$

The limit distribution is Weibull (see [10, p. 205]), and that it is an extreme value distribution (see [7, p. 628]).

The following modified version of the problem, which reduces the problem to the classical extreme value problem, is motivation for the proof of Theorem 2.1 (see [9, Chapter 1] or [11, Chapter 1] for a proof when the random variables are independent and identically distributed). We first remove all the horizontal edges from the graph. Thus, there are only $N + 1$ disjoint paths that connect a top to a bottom vertex (see Figure 3).

Since the paths are disjoint, the $\tau_{N,i}$'s are independent and identically distributed. Each $\tau_{N,j}$ is the sum of h independent exponentially distributed random variables having means $1/\lambda, \ldots, h/\lambda$. Therefore, the limit distribution of τ_N, if it exists, must be an extreme value distribution. It is easy to show that each $\tau_{N,j}$ is regular varying at 0 with index h. Thus, there exists a sequence $u_N \to 0$ such that τ_N/u_N converges in distribution to a Weibull distributed random variable.

Adding the horizontal edges back to the graph complicates the system. First, the number of possible paths change significantly. The most interesting change is the $\tau_{N,i}$'s are neither independent nor identically distributed. Theorem 2.1 states that the extreme value result still holds when the horizontal edges are added.

There will be two steps to prove Theorem 2.1. First is to show that $\tau_{N,i}$ is regular varying at 0 with an index independent of i. This will be accomplished by dividing the lattice into two sets. On the one hand, we will show that $\tau_{N,i}$ is regular varying with index h, and on the other hand, $\tau_{N,i}$ is essentially 0. The second is to show that slightly modified versions of conditions $D(u_N)$ and $D'(u_N)$, given in Leadbetter et al. [9], hold. We will show that the condition $D(u_N)$ holds by showing that if $|i - j|$ is large, then $\tau_{N,i}$ and $\tau_{N,j}$ are nearly independent. Condition $D'(u_N)$ follows in a manner similar to Kiessler and Nimitkiatklai [8].

The following corollary is more useful for many applications.

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COROLLARY 2.2: *Let u_N be the rate of exponential random variables X_e's where u_N is defined as in Theorem 2.1. Then,*

$$\lim_{N\to\infty} P(\tau_N > x) = e^{-x^h}.$$

3. PROOF OF THE THEOREM

The proof is rather lengthy which can be divided into four parts. The first part, which consists of Lemmas 3.1 and 3.2, shows that the $\tau_{N,j}$s are regular varying with index h. The remainder of the proof is similar in spirit to the derivation in Chapter 3 of Leadbetter et al. [9], for extremes in stationary sequences. The idea is divide the graph $\mathbb{S}_N$ into blocks which alternate between being big and small. As $N \to \infty$ both the big blocks and small blocks will get infinitely wide, the big blocks growing at a faster rate than the small ones. We want the small blocks to grow large enough, so that the time it takes to form a path from the top of a big block to the bottom of the graph to be independent for distinct big blocks. This is accomplished in the second part of the proof via Lemma 3.3, which gives the mixing condition $D(u_N)$, and Lemmas 3.4 and 3.5 which demonstrate the asymptotic independence. We also want the big blocks to grow slow enough, so that asymptotically for only one path in a big block, $\tau_{N,j}$ can be below a prescribed threshold. This is the third part of the proof and is subject of Lemma 3.6. The final part, the proof of the theorem, use the first and third parts to show that the lifetime of a big block is regular varying with index h, and the second part to demonstrate that lifetimes of the big blocks are nearly independent. The proof can then be completed as in the case where the lifetimes are independent and identically distributed.

We analyze the distribution of $\tau_{N,j}$ by using the fact that the sum of h independent and identically distributed exponential random variables has a distribution which is regular varying with index h and the shortest path for the event $\mathbf{a}_j \overset{t}{\leftrightarrow} B_N$ can be done in exactly h transitions.

LEMMA 3.1: *For $t > 0$,*

$$P(\tau_{N,j} \le t) = t^h + o(t^h).$$

PROOF: We first construct a block $\mathbf{U}_j$ as follows. A block $\mathbf{U}_j$ is a subgraph of $\mathbb{S}_N$ where the vertices of $\mathbf{U}_j$ form the set $\{(x_1, x_2) \in \mathbb{V}_N : j - h \le x_1 \le j + h, 0 \le x_2 \le h\}$. The block $\mathbf{U}_j$ is shown in Figure 4 where $j = 4$ and $h = 3$. Let $B_{\mathbf{U}_j}$ be the set of vertices on the bottom layer for the block $\mathbf{U}_j$. Conditioning on whether the open path is formed in $\mathbf{U}_j$,

$$\begin{aligned} P(\tau_{N,j} \le t) = {} & P(\tau_{N,j} \le t, \mathbf{a}_j \overset{t}{\leftrightarrow} B_{\mathbf{U}_j}) \\ & + P(\tau_{N,j} \le t, \mathbf{a}_j \overset{t}{\nleftrightarrow} B_{\mathbf{U}_j}). \end{aligned} \tag{1}$$

We next construct a uniformized process $X_{\mathbf{U}_j} = \{X_{\mathbf{U}_j}(t), t \ge 0\}$. Since there are at most $K = 2h(h+1) + (2h+1)h = 4h^2 + 3h$ and at least $2h^2 + 2h$ edges in the block $\mathbf{U}_j$, we set the rate of the Poisson process equal to K. Let $|\mathbf{U}_j|$ be the number of edges in $\mathbf{U}_j$ $(K/2 < |\mathbf{U}_j| \le K)$. The underlying Markov chain adds a given edge at the nth jump with probability $1/K$ and makes a dummy jump with probability $(K - |\mathbf{U}_j| + |X_{\mathbf{U}_j}(t_{n-1})|)/K$.

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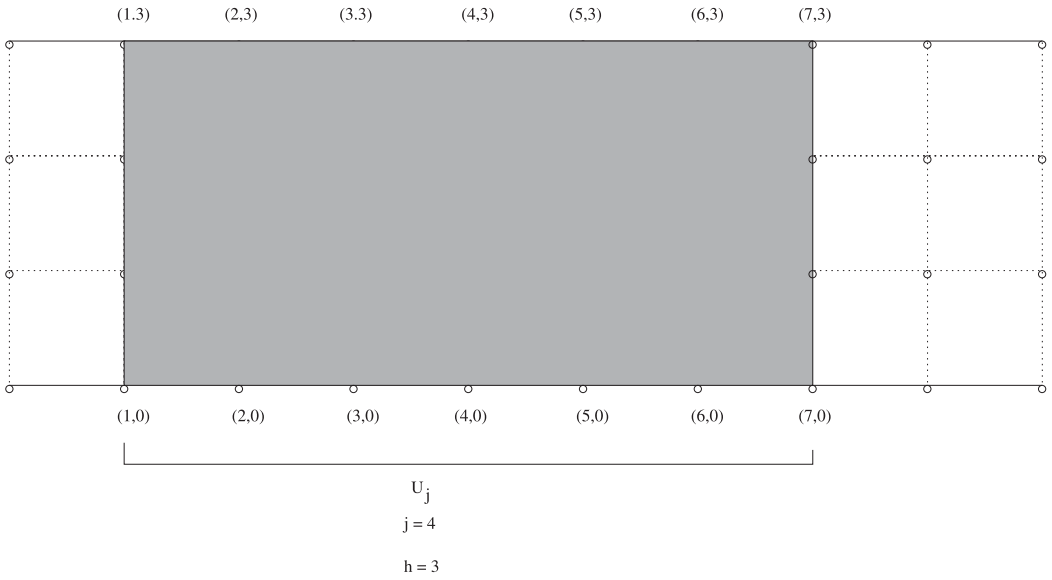


FIGURE **4.** The block $\mathbf{U}_j$ where $j = 4$ and $h = 3$.

For the first term in the right-hand side of (1),

$$\begin{aligned} P(\tau_{N,j} \le t, \mathbf{a}_j \overset{t}{\leftrightarrow} B_{\mathbf{U}_j}) &= P(\mathbf{a}_j \overset{t}{\leftrightarrow} B_{\mathbf{U}_j}) \\ &= \sum_{n=h}^{\infty} P(\text{path occurs in } n \text{ steps of } X_{\mathbf{U}_j} \text{ prior to time } t) \\ &= P(\text{path occurs in } h \text{ steps of } X_{\mathbf{U}_j} \text{ prior to time } t) \\ &\quad + \sum_{n=h+1}^{\infty} P(\text{path occurs in } n \text{ steps of } X_{\mathbf{U}_j} \text{ prior to time } t). \end{aligned} \tag{2}$$

We write the first term on the right-hand side of (2) as follows:

$$P_{a_j}^h(B_{U_j}) \frac{(Kt)^h}{h!} e^{-Kt},$$

where $P_{a_j}^h(B_{U_j})$ is the probability an open path from $\mathbf{a}_j$ to $B_{\mathbf{U}_j}$ occurs in exactly h steps. Since there are $h!$ ways to form a path in $\mathbf{U}_j$ by using only h open edges, and there are K^h ways to choose h edges from the block $\mathbf{U}_j$,

$$P_{a_j}^h(B_{U_j}) = \frac{h!}{K^h}.$$

The second term on the right-hand side of (2) is bounded above by

$$\sum_{n=h+1}^{\infty} \frac{(Kt)^n}{n!} e^{-Kt} \le \frac{(Kt)^{h+1}}{(h+1)!}. \tag{3}$$

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Hence,

$$P(\tau_{N,j} \le t, \mathbf{a}_j \overset{t}{\leftrightarrow} B_{\mathbf{U}_j}) = P^h_{a_j}(B_{U_j})\frac{(Kt)^h}{h!}e^{-Kt} + o(t^h)$$
$$= t^h e^{-Kt} + o(t^h). \tag{4}$$

In order for the events $\tau_{N,j} \le t$ and $\mathbf{a}_j \overset{t}{\nleftrightarrow} B_{\mathbf{U}_j}$ to occur, at least $h+1$ jumps of $X_{\mathbf{U}_j}$ must occur prior to time t. Thus, the second term on the right-hand side of Eq. (1) is also bounded above by

$$\sum_{n=h+1}^{\infty} P(n \text{ open edges of } X_{\mathbf{U}_j}) \le \sum_{n=h+1}^{\infty} \frac{(Kt)^n}{n!}e^{-Kt}$$
$$\le \frac{(Kt)^{h+1}}{(h+1)!}. \tag{5}$$

Since the right-hand side of (5) is $o(t^h)$, so is the left-hand side.

From $e^{-Kt} = 1 + o(1)$, we have

$$P(\tau_{N,j} \le t) = t^h(1 + o(1)) + o(t^h) + o(t^h) = t^h + o(t^h),$$

which completes the proof. ■

Lemma 3.2: *For $x > 0$,*

$$P(\tau_{N,j} \le xu_N) = \frac{x^h}{N} + o\left(\frac{1}{N}\right).$$

Proof: Recall that $u_N = (1/N)^{1/h}$. From Lemma 3.1,

$$P(\tau_{N,j} \le xu_N) = (xu_N)^h + o((u_N)^h) = \frac{x^h}{N} + o\left(\frac{1}{N}\right).$$

■

The above lemma shows that the $\tau_{N,j}$ are regularly varying at 0 with index h [11]. This property is typically expressed as follows:

$$\lim_{N\to\infty} \frac{P(\tau_{N,j} \le xu_N)}{P(\tau_{N,j} \le u_N)} = x^h,$$

which follows directly from Lemma 3.2.

We develop a mixing condition analogous to the condition $D(u_N)$ as given in Leadbetter et al. [9]. Until the end of the proof of Theorem 2.1, $x > 0$ will be fixed. Recall that the set of vertices for the system $\mathbb{S}_N$ is $\{(x_1, x_2) : 0 \le x_1 \le N, 0 \le x_2 \le h\}$. Define a box R_i to be a subgraph of $\mathbb{S}_N$ whose vertices are

$$V(R_i) = \{(x_1, x_2) : i - 1 \le x_1 \le i, 0 \le x_2 \le h\}.$$

It can be seen in Figure 5 that the system $\mathbb{S}_N$ is the union of the R_i's, $i = 1, \ldots, N$.

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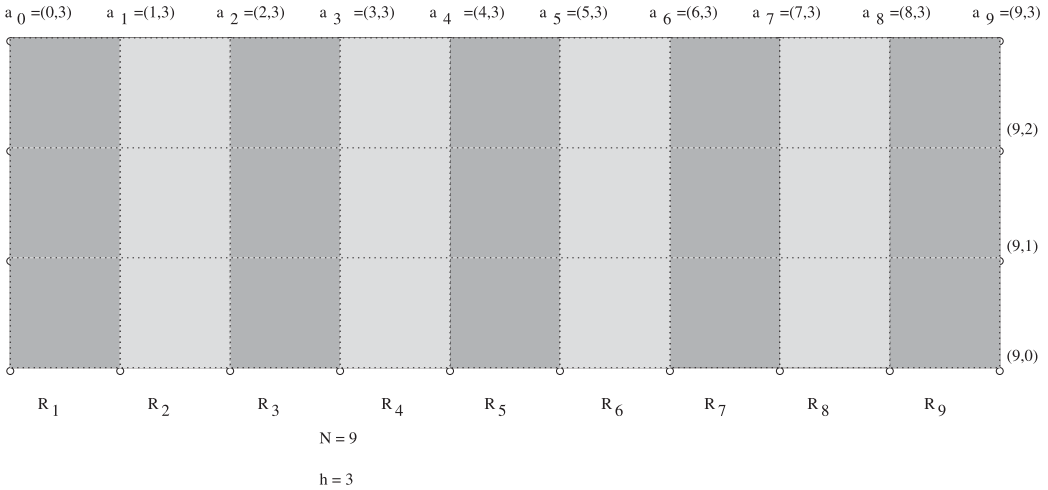


FIGURE **5.** The system $\mathbb{S}_N$ as the union of R_i's.

For $1 \le r \le N - i$, let a block $\mathbf{S}$ be a collection of r adjacent boxes in $\mathbb{S}_N$, that is,

$$\mathbf{S} = R_{i+1} \cup \ldots \cup R_{i+r}.$$

Therefore, $\mathbf{S}$ is a subgraph of $\mathbb{S}_N$ where the set of vertices for $\mathbf{S}$ is $\{(x_1, x_2) : i \le x_1 \le i + r, 0 \le x_2 \le h\}$. $\mathbf{S}$ is said to have width r if it consists of r boxes.

Let $\tilde{\mathbf{S}}$ be the union of $\mathbf{S}$ and all boxes, R_k, that are within $[m/2]$ from $\mathbf{S}$, where $[x]$ is the greatest integer less than or equal to x and $m \ge 2$. Let e be an arbitrary edge that does not belong to the block $\tilde{\mathbf{S}}$ and let p be the probability that e is an edge in an open path connecting a vertex in $\mathbf{S}$ that forms prior to xu_N. Then, there must be boxes $R_i, \ldots, R_{i+[m/2]}$ such that at least one horizontal edge in each R_j must be open prior to ux_N. The probability that a horizontal edge in R_j opens prior to xu_N is $1 - e^{-(h+1)xu_N}$. The probability that a horizontal edge in each of the boxes $R_i, \ldots, R_{i+[m/2]}$ opens prior to xu_N is then $(1 - e^{-(h+1)xu_N})^{[m/2]}$. Since $xu_N \to 0$ as $N \to \infty$, there is a constant K_1 such that

$$\sup_N (h+1)xu_N \le K_1.$$

This gives $(1 - e^{-(h+1)xu_N})^{[m/2]} \le (1 - e^{-K_1})^{[m/2]}$. Since the system consists of $N(h+1) + (N+1)h = 2Nh + N + h \le 4Nh$ edges, we obtain

$$p \le K_2 N (1 - e^{-K_1})^{[m/2]}$$

where $K_2 = 4h$.

The next three lemmas demonstrate how to obtain the asymptotic independence. $D(u_N)$ is the mixing condition that Lemma 3.3 states is satisfied.

LEMMA 3.3: *For i= 1,2, let $\mathbf{S}_i$ be the union of k_i disjoint blocks, that is,*

$$\mathbf{S}_i = \mathbf{S}_{i,1} \cup \ldots \cup \mathbf{S}_{i,k_i}.$$

Let $A_{\mathbf{S}_i}$ be the set of vertices on the top layer for $\mathbf{S}_i$. The lifetime, $\tau_{\mathbf{S}_i}$, is the length of time for the event $A_{\mathbf{S}_i} \overset{t}{\leftrightarrow} B_N$ to occur. Assume that the minimum number of open edges needed

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to form a path from $\mathbf{S}_1$ *to* $\mathbf{S}_2$ *is* m *where* $m \geq 2$. *Then,*

$$|P(\tau_{\mathbf{S}_1} \leq xu_N, \tau_{\mathbf{S}_2} \leq xu_N) - P(\tau_{\mathbf{S}_1} \leq xu_N)P(\tau_{\mathbf{S}_2} \leq xu_N)| \leq \alpha_{N,m}$$

where $\alpha_{N,m} = \bar{K}N(1 - e^{-K_1})^{[m/2]}$ *and* $\bar{K} = 16h$.

Proof: For each i, let $\tilde{\mathbf{S}}_i$ be the union of all the sub-blocks $\mathbf{S}_{i,j}$'s in $\mathbf{S}_i$ and all the boxes R_i's that are within $[m/2]$ from some $\mathbf{S}_{i,j}$. For $i = 1,2$, define $\mathcal{A}_i$ to be the event that no edge which belongs to $\tilde{\mathbf{S}}_i^c$ is in an open path that forms prior to xu_N and contains a vertex in $\mathbf{S}_i$. Set $\mathcal{A}_3$ to be the event that no edge which belongs to $\tilde{\mathbf{S}}_1^c \cap \tilde{\mathbf{S}}_2^c$ is in an open path that forms prior to u_N and contains a vertex in $\mathbf{S}_1 \cup \mathbf{S}_2$. Let $c = K_2N(1 - e^{-K_1})^{[m/2]}$. By the discussion preceding this lemma, for $i = 1,2$

$$P(\mathcal{A}_i^c) \leq c,$$

and since $\mathcal{A}_3^c \subset \mathcal{A}_1^c \cup \mathcal{A}_2^c$,

$$P(\mathcal{A}_3^c) \leq 2c.$$

Color the edges in $\tilde{\mathbf{S}}_1$ white, those in $\tilde{\mathbf{S}}_2$ orange, and the remaining edges purple. Let $\tilde{\tau}_{\mathbf{S}_1}$, respectively, $\tilde{\tau}_{\mathbf{S}_2}$ be the length of time to form an open path from $A_{\mathbf{S}_1}$ to B_N, respectively, $A_{\mathbf{S}_2}$ to B_N by using only edges that are white, respectively, and orange. Then, $\{\tilde{\tau}_{\mathbf{S}_1} \leq xu_N\}, \{\tilde{\tau}_{\mathbf{S}_2} \leq xu_N\}$ and $\mathcal{A}_3$ are independent events and $\{\tau_{\mathbf{S}_1} \leq xu_N, \tau_{\mathbf{S}_2} \leq xu_N, \mathcal{A}_3\} = \{\tilde{\tau}_{\mathbf{S}_1} \leq xu_N, \tilde{\tau}_{\mathbf{S}_2} \leq xu_N, \mathcal{A}_3\}$.

Thus,

$$\begin{aligned}
&|P(\tau_{\mathbf{S}_1} \leq xu_N, \tau_{\mathbf{S}_2} \leq xu_N) - P(\tau_{\mathbf{S}_1} \leq xu_N)P(\tau_{\mathbf{S}_2} \leq xu_N)| \\
&\quad = |P(\tau_{\mathbf{S}_1} \leq xu_N, \tau_{\mathbf{S}_2} \leq xu_N, \mathcal{A}_3) - P(\tau_{\mathbf{S}_1} \leq xu_N)P(\tau_{\mathbf{S}_2} \leq xu_N)P(\mathcal{A}_3)| \\
&\qquad + |P(\tau_{\mathbf{S}_1} \leq xu_N, \tau_{\mathbf{S}_2} \leq xu_N, \mathcal{A}_3^c) - P(\tau_{\mathbf{S}_1} \leq xu_N)P(\tau_{\mathbf{S}_2} \leq xu_N)P(\mathcal{A}_3^c)| \\
&\quad = |P(\tilde{\tau}_{\mathbf{S}_1} \leq xu_N)P(\tilde{\tau}_{\mathbf{S}_2} \leq xu_N) - P(\tau_{\mathbf{S}_1} \leq xu_N)P(\tau_{\mathbf{S}_2} \leq xu_N)| + 2c.
\end{aligned}$$

Since $|a_1a_2 - b_1b_2| \leq |a_1 - b_1| + |a_2 - b_2|$, when the magnitude of each term is less than or equal to 1,

$$\begin{aligned}
&|P(\tilde{\tau}_{\mathbf{S}_1} \leq xu_N)P(\tilde{\tau}_{\mathbf{S}_2} \leq xu_N) - P(\tau_{\mathbf{S}_1} \leq xu_N)P(\tau_{\mathbf{S}_2} \leq xu_N)| \\
&\quad \leq |P(\tilde{\tau}_{\mathbf{S}_1} \leq xu_N) - P(\tau_{\mathbf{S}_1} \leq xu_N)| + |P(\tilde{\tau}_{\mathbf{S}_2} \leq xu_N) - P(\tau_{\mathbf{S}_2} \leq xu_N)|.
\end{aligned}$$

From $\{\tilde{\tau}_{\mathbf{S}_1} \leq xu_N, \mathcal{A}_1\} = \{\tau_{\mathbf{S}_1} \leq xu_N, \mathcal{A}_1\}$,

$$\begin{aligned}
&|P(\tilde{\tau}_{\mathbf{S}_1} \leq xu_N) - P(\tau_{\mathbf{S}_1} \leq xu_N)| \\
&\quad = |P(\tilde{\tau}_{\mathbf{S}_1} \leq xu_N, \mathcal{A}_1) - P(\tau_{\mathbf{S}_1} \leq ux_N, \mathcal{A}_1)| \\
&\qquad + |P(\tilde{\tau}_{\mathbf{S}_1} \leq xu_N, \mathcal{A}_1^c) - P(\tau_{\mathbf{S}_1} \leq xu_N, \mathcal{A}_1^c)| \\
&\quad \leq c.
\end{aligned}$$

Similarly,

$$|P(\tilde{\tau}_{\mathbf{S}_2} \leq xu_N) - P(\tau_{\mathbf{S}_2} \leq xu_N)| \leq c.$$

Therefore,

$$\begin{aligned}
|P(\tau_{\mathbf{S}_1} \leq xu_N, \tau_{\mathbf{S}_2} \leq xu_N) - P(\tau_{\mathbf{S}_1} \leq xu_N)P(\tau_{\mathbf{S}_2} \leq xu_N)| &\leq c + c + 2c \\
&= 4c \\
&= \bar{K}N(1 - e^{-K_1})^{[m/2]} \\
&= \alpha_{N,m}.
\end{aligned}$$

■

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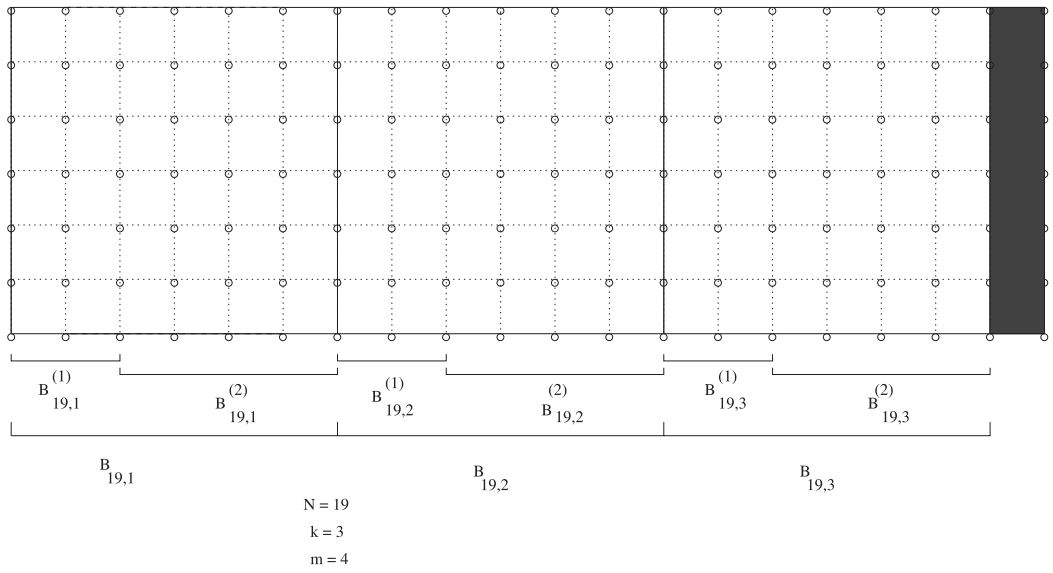


FIGURE **6.** Comparing the blocks $B_{N,i}, B^{(1)}_{N,i}$, and $B^{(2)}_{N,i}$ where $N = 19$, $k = 3$, and $m = 4$.

Fix a positive integer k, define $N' = [N/k]$. For $i = 1, \ldots, k$ and $k < m < N'$, set

$$B_{N,i} = R_{(i-1)N'+1} \cup \ldots \cup R_{iN'}$$
$$B^{(1)}_{N,i} = R_{(i-1)N'+1} \cup \ldots \cup R_{iN'-m}$$
$$B^{(2)}_{N,i} = R_{iN'-m+1} \cup \ldots \cup R_{iN'}.$$

See Figure 6 for an example of the blocks $B_{N,i}, B^{(1)}_{N,i}$, and $B^{(2)}_{N,i}$ when $i \leq k$.

For $i = k+1$,

$$B^{(1)}_{N,k+1} = R_{(k-1)N'+m+1} \cup \ldots \cup R_{kN'}$$
$$B^{(2)}_{N,k+1} = R_{kN'+1} \cup \ldots \cup R_{kN'+m}.$$

Let $A_{B_{N,i}}, A_{B^{(1)}_{N,i}}, A_{B^{(2)}_{N,i}}$ be the set of vertices on the top of the blocks $B_{N,i}, B^{(1)}_{N,i}, B^{(2)}_{N,i}$, respectively. Define

$\tau_{B_{N,i}}$ be the length of time for the event $A_{B_{N,i}} \overset{t}{\leftrightarrow} B_N$,

$\tau_{B^{(1)}_{N,i}}$ be the length of time for the event $A_{B^{(1)}_{N,i}} \overset{t}{\leftrightarrow} B_N$,

$\tau_{B^{(2)}_{N,i}}$ be the length of time for the event $A_{B^{(2)}_{N,i}} \overset{t}{\leftrightarrow} B_N$.

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LEMMA 3.4: *Let* $I = \{1, \ldots, k\}$ *and* $I' = I \cup \{k+1\}$, *then*

$$(i) \quad 0 \le P\left(\bigcap_{i\in I}\{\tau_{B^{(1)}_{N,i}} > xu_N\}\right) - P(\tau_N > xu_N) \le \sum_{i\in I'} P(\tau_{B^{(1)}_{N,i}} > xu_N \ge \tau_{B^{(2)}_{N,i}}),$$

$$(ii) \quad |P\left(\bigcap_{i\in I}\{\tau_{B^{(1)}_{N,i}} > xu_N\}\right) - \prod_{i\in I} P(\tau_{B^{(1)}_{N,i}} > xu_N)| \le (k-1)\alpha_{N,m},$$

$$(iii) \quad |\prod_{i\in I} P(\tau_{B^{(1)}_{N,i}} > xu_N) - \prod_{i\in I} P(\tau_{B_{N,i}} > xu_N)| \le \sum_{i\in I} P(\tau_{B^{(1)}_{N,i}} > xu_N \ge \tau_{B^{(2)}_{N,i}}).$$

By combining (i), (ii), and (iii),

$$\left|P(\tau_N > xu_N) - \prod_{i\in I} P(\tau_{B_{N,i}} > xu_N)\right| \le (2k+1)\sup_{i\in I'} P\{\tau_{B^{(1)}_{N,i}} > xu_N \ge \tau_{B^{(2)}_{N,i}}\} + (k-1)\alpha_{N,m}.$$

PROOF: The result (i) follows since $\{\tau_N > xu_N\} \subset \bigcap_{i\in I}\{\tau_{B^{(1)}_{N,i}} > xu_N\}$, the difference implies $\tau_{B^{(1)}_{N,j}} > xu_N \ge \tau_{B^{(2)}_{N,j}}$ for some $j \le k$, or otherwise $\tau_{N,j} \le xu_N$ for some $j = kN' + 1, \ldots, k(N'+1)$ which implies $\tau_{B^{(1)}_{N,k+1}} > xu_N \ge \tau_{B^{(2)}_{N,k+1}}$.

The result (ii) follows from Lemma 3.3.

The result (iii) follows form the inequalities

$$0 \le P(\tau_{B^{(1)}_{N,i}} > xu_N) - P(\tau_{B_{N,i}} > xu_N) \le P(\tau_{B^{(1)}_{N,i}} > xu_N \ge \tau_{B^{(2)}_{N,i}})$$

and

$$0 \le \left|\prod_{i=1}^{k} y_i - \prod_{i=1}^{k} x_i\right| \le \sum_{i=1}^{k} |y_i - x_i|,$$

when $|x_i| \le 1$ and $|y_i| \le 1$.

Therefore,

$$\begin{aligned}&\left|P(\tau_N > xu_N) - \prod_{i\in I} P(\tau_{B_{N,i}} > xu_N)\right| \\ &= \Bigg|P(\tau_N > xu_N) - P\left(\bigcap_{i\in I}\{\tau_{B^{(1)}_{N,i}} > xu_N\}\right) + P\left(\bigcap_{i\in I}\{\tau_{B^{(1)}_{N,i}} > xu_N\}\right) \\ &\quad - \prod_{i\in I} P(\tau_{B^{(1)}_{N,i}} > xu_N) + \prod_{i\in I} P(\tau_{B^{(1)}_{N,i}} > xu_N) - \prod_{i\in I} P(\tau_{B_{N,i}} > xu_N)\Bigg|\end{aligned}$$

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$$\leq \left| P(\tau_N > xu_N) - P\left(\bigcap_{i\in I}\{\tau_{B^{(1)}_{N,i}} > xu_N\}\right)\right|$$
$$+ \left| P\left(\bigcap_{i\in I}\{\tau_{B^{(1)}_{N,i}} > xu_N\}\right) - \prod_{i=1}^{k} P\{\tau_{B^{(1)}_{N,i}} > xu_N)\right|$$
$$+ \left|\prod_{i\in I} P(\tau_{B^{(1)}_{N,i}} > xu_N) - \prod_{i\in I} P(\tau_{B_{N,i}} > xu_N)\right|$$
$$\leq \sum_{i\in I'} P(\tau_{B^{(1)}_{N,i}} > xu_N \geq \tau_{B^{(2)}_{N,i}}) + (k-1)\alpha_{N,m} + \sum_{i\in I} P(\tau_{B^{(1)}_{N,i}} > xu_N \geq \tau_{B^{(2)}_{N,i}})$$
$$\leq (2k+1)\sup_{i\in I'} P\{\tau_{B^{(1)}_{N,i}} > xu_N \geq \tau_{B^{(2)}_{N,i}}\} + (k-1)\alpha_{N,m}.$$

■

LEMMA 3.5: *Set* $m = \sqrt{N}$, *then*

$$\sup_{i\in I'} P(\tau_{B^{(1)}_{N,i}} > xu_N \geq \tau_{B^{(2)}_{N,i}}) \to 0 \quad \text{as } N \to \infty.$$

PROOF: From Lemma 3.2, $P(\tau_{N,i} \leq xu_N) = x^h/N + o(1/N)$ for fix integer N and i. We have

$$\sup_{i\in I'} P(\tau_{N,i} \leq xu_N) = \frac{x^h}{N} + o\left(\frac{1}{N}\right).$$

As $N \to \infty$,

$$\sup_{i\in I'} P(\tau_{N,i} \leq xu_N) \to 0.$$

Since there are at most m of R_i's in each $B^{(2)}_{N,i}$,

$$\sup_{i\in I'} P(\tau_{B^{(1)}_{N,i}} > xu_N \geq \tau_{B^{(2)}_{N,i}}) \leq (\sqrt{N}+1)\sup_{i\in I'} P(\tau_{N,i} \leq xu_N) \to 0 \quad \text{as } N \to \infty.$$

■

In Lemma 3.3, $\alpha_{N,m}$ was defined to be $\bar{K}N(1-e^{-K_1})^{[m/2]}$. It plays the role of the constants $\alpha_{n,m}$ in Leadbetter et al. [9]. Since $N(1-e^{-K_1})^{\sqrt{N}} \to 0$ as $N \to \infty$

$$\begin{aligned}\alpha_{N,\sqrt{N}} &= \bar{K}N(1-e^{-K_1})^{[\sqrt{N}/2]} \\ &\leq \bar{K}\sqrt{1-e^{-K_1}}N(1-e^{-K_1})^{\sqrt{N}} \\ &\to 0 \quad \text{as } N \to \infty.\end{aligned}$$

Our final lemma shows that asymptotically there is at most one path in a big block for which $\tau_{N,j} \leq xu_N$. To prove this result, we use the BK inequality from the percolation theory (see [6]). To apply this result, we require that the event $\{\tau_{N,j} \leq t\}$ is increasing. To state this property, in the context of our problem, let ω be an outcome an set $Z(\omega,t) = \cup_e\{Y(\omega,t) = 1\}$, the collection of edges that open prior to t. The event $\{\tau_{N,j} \leq t\}$ is increasing if whenever $\omega \in \{\tau_{N,j} \leq t\}$ and ω' satisfies $Z(\omega,t) \subset Z(\omega',t)$, then $\omega' \in \{\tau_{N,j} \leq t\}$. But this is clearly

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true. Our modified version of condition $D'(u_N)$ is the condition that Lemma 3.6 below states is satisfied.

LEMMA 3.6: *Let $J = \{1, \dots, N'\}$. As $k \to \infty$,*

$$\limsup_N N \sup_{i \in I} \sup_{j_1 \in J} \sum_{j_2 \in J \setminus \{j_1\}} P(\tau_{N,(i-1)N'+j_1} \le x u_N, \tau_{N,(i-1)N'+j_2} \le x u_N) \to 0.$$

PROOF: For block $B_{N,1}$,

$$\begin{aligned} P(\tau_{N,j_1} \le x u_N, \tau_{N,j_2} \le x u_N) &= P(\tau_{N,j_1} \le x u_N, \tau_{N,j_2} \le x u_N, \text{ paths are disjoint}) \\ &\quad + P(\tau_{N,j_1} \le x u_N, \tau_{N,j_2} \le x u_N, \text{ paths intersect}). \end{aligned}$$

We apply the BK inequality (see [6, p. 37]) to the first term to obtain

$$P(\tau_{N,j_1} \le x u_N, \tau_{N,j_2} \le x u_N) \le P(\tau_{N,j_1} \le x u_N) P(\tau_{N,j_2} \le x u_N) + P(\mathcal{A}),$$

where $\mathcal{A}$ is the event of $\{\tau_{N,j_1} \le x u_N, \tau_{N,j_2} \le x u_N, \text{paths intersect}\}$.

When $j_1 \ne j_2$, if $\mathcal{A}$ occurs, at least $h+1$ edges must open prior to $x u_N$. From the proof of Lemma 3.1, starting after (2) and ending at (5),

$$\begin{aligned} P(\mathcal{A}) &\le P(\text{at least } h+1 \text{ edges open prior to } x u_N) \\ &\le \frac{(K x u_N)^{h+1}}{h!} = \frac{K^{h+1}((\frac{x^h}{N})^{1/h})^{h+1}}{h!} \\ &= K_3 \left(\frac{1}{N}\right)^{1+(1/h)}, \end{aligned} \tag{6}$$

where $K_3 = (xK)^{h+1}/h!$.

When $|j_1 - j_2| > h$,

$$\begin{aligned} P(\mathcal{A}) &\le P(\text{at least } 2h \text{ edges open prior to } x u_N) \\ &\le \frac{(K x u_N)^{2h}}{(2h-1)!} = \frac{K^{2h}\left(\left(\frac{x^h}{N}\right)^{1/h}\right)^{2h}}{(2h-1)!} \\ &= K_4 \frac{1}{N^2}, \end{aligned} \tag{7}$$

where $K_4 = (xK)^{2h}/(2h-1)!$.

Therefore,

$$\begin{aligned} &\limsup_N N \sup_{i \in I} \sup_{j_1 \in J} \sum_{j_2 \in J \setminus \{j_1\}} P(\tau_{N,(i-1)N'+j_1} \le x u_N, \tau_{N,(i-1)N'+j_2} \le x u_N) \\ &\le \lim_{N \to \infty} N \left[N' \left(\frac{1}{N^2} + o\left(\frac{x^{2h}}{N^2}\right)\right) + 2h \left(\frac{1}{N}\right)^{1+1/h} K_3 + \frac{N'}{N^2} K_4 \right] \\ &= \lim_{N \to \infty} \frac{x^{2h}}{k} + \frac{N^2}{k} o\left(\frac{1}{N^2}\right) + 2h \left(\frac{1}{N}\right)^{1/h} K_3 + \frac{1}{k} K_4 \\ &= \frac{\max(x^{2h}, 1)}{k}(1 + K_4) \to 0 \quad \text{as } k \to \infty, \end{aligned}$$

which completes the proof. ∎

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We can now complete the proof.

Proof of the Theorem 2.1: Fix an integer k. For each block $B_{N,i}$,

$$\{\tau_{B_{N,i}} \le xu_N\} = \bigcup_{j=(i-1)N'+1}^{iN'} \{\tau_{N,j} \le xu_N\}.$$

Hence,

$$\begin{aligned} 1 - &\sum_{j=(i-1)N'+1}^{iN'} P(\tau_{N,j} \le xu_N) \\ &+ \sum_{j_1 \in J} \sum_{j_2 \in J \setminus \{j_1\}} P(\tau_{N,(i-1)N'+j_1} \le xu_N, \tau_{N,(i-1)N'+j_2} \le xu_N) \\ &\ge P(\tau_{B_{N,i}} \le xu_N) \ge 1 - \sum_{j=(i-1)N'+1}^{iN'} P(\tau_{N,j} \le xu_N) \end{aligned}$$

From Lemma 3.2, $P(\tau_{N,j} \le xu_N) = x^h/N + o(1/N)$. The first sum has $[N/k]$ terms. Therefore,

$$\sum_{j=(i-1)N'+1}^{iN'} P(\tau_{N,j} \le xu_N) \le \frac{x^h}{k} + o\left(\frac{1}{N}\right)$$

which converges to x^h/k as $N \to \infty$. By Lemma 3.6,

$$\begin{aligned} &\sum_{j_1 \in J} \sum_{j_2 \in J \setminus \{j_1\}} P(\tau_{N,(i-1)N'+j_1} \le xu_N, \tau_{N,(i-1)N'+j_2} \le xu_N) \\ &\quad \le \limsup_N \frac{N}{k} \sup_{i \in I} \sup_{j_1 \in J} \sum_{j_2 \in J \setminus \{j_1\}} P(\tau_{N,(i-1)N'+j_1} \le xu_N, \tau_{N,(i-1)N'+j_2} \le xu_N) \\ &\quad = o\left(\frac{1}{k}\right). \end{aligned}$$

Hence,

$$1 - \frac{x^h}{k} + o\left(\frac{1}{k}\right) \ge \limsup_N P(\tau_{B_{N,i}} \le xu_N) \ge \liminf_N P(\tau_{B_{N,i}} \le xu_N) \ge 1 - \frac{x^h}{k}$$

By Lemmas 3.4 and 3.5,

$$\left(1 - \frac{x^h}{k} + o\left(\frac{1}{k}\right)\right)^k \ge \limsup_N P(\tau_N \le xu_N) \ge \liminf_N P(\tau_N \le xu_N) \ge \left(1 - \frac{x^h}{k}\right)^k.$$

Letting $k \to \infty$, we obtain $P(\tau_N \ge xu_N) \to e^{-x^h}$, which completes the proof of the theorem. ■

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Proof of Corollary 2.2: We repeat the proofs of Lemmas 3.1–3.6 where in system $\mathbb{S}_N$ the rate for the exponential random variables X_e's is $u_N = (1/N)^{1/h}$. We obtain

$$P(\tau_{N,j} \le x) = \frac{C(Kxu_N)^h}{h!} e^{-Ku_N} + o(u_N^h)$$
$$= \left(\left(\frac{x^h}{N}\right)^{1/h}\right)^h + o\left(\left(\left(\frac{1}{N}\right)^{1/h}\right)^h\right) = \frac{x^h}{N} + o\left(\frac{1}{N}\right).$$

Note: When the rate of X_e is 1, the number of jumps made in a block, having K edges, prior to xu_N is Poisson with rate Kxu_N. Similarly, when the rate of X_e is u_N, the number of jumps made prior to 1 is Poisson with rate Ku_N.

Theorem 2.1 now gives

$$\lim_{N\to\infty} P(\tau_N > x) = e^{-x^h}.$$

■

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