Euro. Jnl of Applied Mathematics (2002), *vol.* 13, *pp.* 449–477. © 2002 Cambridge University Press
DOI: 10.1017/S0956792501004697 Printed in the United Kingdom

Solvable structures and their application to a class of Cauchy problem

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(*Received 30 November 1999; revised 12 june 2001*)

We examine how symmetry and computer algebra can assist in solving the Cauchy problem for Pfaffian systems. We use recent results on integrating Frobenius integrable distributions via *solvable symmetry structures* to develop two techniques that when used in conjunction with symmetry determination software DIMSYM, allow us to solve the Cauchy problem for the special situation when there exists a one-dimensional Cauchy characteristic space. We also illustrate how our work can assist in extracting local solutions of a certain class of first and second order non-linear partial differential equations.

1 Introduction

It is generally well known that local solutions of certain types Partial Differential Equations (PDEs) can be found by solving an equivalent Cauchy problem based on Pfaffian systems. In this paper, we study how symmetry and computer algebra may be used to solve the Cauchy problem for Pfaffian systems. In the traditional approach, a system of ordinary differential equations is generated and then solved [18]. Our work examines how results of Lie [13] and Cartan [5, 6] for integrating Frobenius integrable vector field distributions using *solvable symmetry structures* (defined below) can solve the problem, without having to introduce any such ordinary differential equations. Our work is motivated by efforts from several authors [3, 7, 10, 14], who have in recent times provided valuable reviews and extensions of the work by Lie and Cartan.

To generate solvable symmetry structures, we use the symmetry determination software package DIMSYM [15] operating as a REDUCE overlay. This program can be used to find symmetries of differential equations as well as more general symmetries of arbitrary vector field distributions. For generating such symmetry structures, it is this second feature that we will utilise. While there are many other symbolic programs capable of such symmetry analysis that include SYMMGRP.MAX [12], LIE [11], CRACK [20], LIEPDE [19] and Liesymm for Maple, we choose to work with DIMSYM, since for a given vector field distribution we can easily insert the Lie bracket symmetry conditions, following which it then generates and attempts to solve the resulting determining equations.

Using solvable symmetry structures, we develop in this paper two techniques for solving the Cauchy problem for Pfaffian systems. The first deals with a Pfaffian system generated by a single one-form of rank one, while the second attempts to generalise the first to a

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system of one-forms, each of constant but perhaps different rank. Although both cases give results when the Cauchy characteristic space is of a dimension greater than or equal to one, we will see that our work only solves the Cauchy problem when the Pfaffian system has a Cauchy characteristic space of dimension no greater than one. We also provide a PDE example for each of the two techniques presented.

2 Background

Throughout this paper we assume U to be some open, convex neighbourhood of $\mathbb{R}^n$ with local coordinates $x^1,\ldots,x^n$. The set of exterior differential m-forms $\Lambda^m(U)$ is a section of the bundle of all homogeneous differential forms $\Lambda(U)$. We define $\mathfrak{X}(U)$ to be the module of all smooth vector fields over $C^\infty(U)$. The *kernel* of a differential form Ω is a submodule of vector fields defined by $\ker(\Omega) := \{Z \in \mathfrak{X}(U) : Z \lrcorner\, \Omega = 0\}$.

Following Bryant *et al.* [4], the *Cauchy characteristic* space of a differential ideal I generated by some finite collection of differential forms is denoted $A(I)$, and contains all vector fields $X \in \mathfrak{X}(U)$ such that $X \lrcorner I \subset I$. The *Cartan system* of I is defined by $C(I) := A(I)^\perp$, the annihilating co-distribution of $A(I)$.

A vector field $X \in \mathfrak{X}(U)$ is said to be a *symmetry* (isovector) of I if it satisfies a condition involving the Lie derivative that $\mathscr{L}_X I \subset I$. A vector field $X \in \mathfrak{X}(U)$ is a *symmetry* of a vector field distribution $D \subset \mathfrak{X}(U)$ if $\mathscr{L}_X D \subset D$. We say that a vector field is a *non-trivial* symmetry if, in terms of a differential ideal, it is not Cauchy characteristic, or in terms of a vector field distribution, it is not in the distribution.

For any vector field distribution D, p linearly independent vector fields $X_1,\ldots,X_p \in \mathfrak{X}(U)$ form a *solvable symmetry structure* for D if

$$\begin{gathered}
\mathscr{L}_{X_1}\left(sp\{X_2,\ldots,X_p\} \oplus D\right) \subset sp\{X_2,\ldots,X_p\} \oplus D,\\
\vdots\\
\mathscr{L}_{X_{p-1}}\left(sp\{X_p\} \oplus D\right) \subset sp\{X_p\} \oplus D,\\
\mathscr{L}_{X_p} D \subset D.
\end{gathered}$$

Generating a solvable symmetry structure with `DIMSYM` is done in stages by first finding a symmetry X_p of D, then finding a symmetry X_{p-1} of $D \oplus sp\{X_p\}$, and so on until X_1 is found that is a symmetry of $D \oplus sp\{X_2,\ldots,X_p\}$. For each stage, we input the necessary Lie bracket relations, and let `DIMSYM` solve the resulting linear determining equations. See the Appendix for further details. Note that $X_1,\ldots,X_p$ are quite general symmetries, and moreover, need not be of Lie point type. While X_p is a genuine symmetry of D, X_1 is a much weaker symmetry: X_1 is a symmetry of D, modulo $X_2,\ldots,X_p$.

Given a differential ideal I that is generated by a finite collection of linearly independent differential one-forms in $\Lambda^1(U)$, the *Cauchy problem* looks to extend a q-dimensional integral manifold of I to a $(q+1)$-dimensional integral manifold of I. To achieve this, we are given some *Cauchy data*, which is a one-to-one smooth map of maximal rank $\Phi : M^q \longrightarrow U$ that satisfies $\Phi^* I = 0$, where $M^q \subset \mathbb{R}^q$ is some open, convex neighbourhood. We are also given a Cauchy characteristic vector field $Y \in \mathfrak{X}(U)$ of I that is *transverse* to Φ. By transverse we mean that on U it is nowhere tangent to the image

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of Φ. In dealing with PDE problems, we demand that Y be transverse to Φ with respect to an independence condition, which means that the projections of the Cauchy data and Y onto the base manifold of the independent variables are nowhere tangent.

The usual and most basic treatment of the Cauchy problem for Pfaffian systems focuses on the following well known theorem [8, 9, 17]:

Theorem 1 *Let* $I := \langle \alpha^1, \ldots, \alpha^p, d\alpha^1, \ldots, d\alpha^p \rangle$ *for some linearly independent* $\alpha^1, \ldots, \alpha^p \in \Lambda^1(U)$ *with Cauchy data given by the one-to-one immersion*

$$\Phi : M^q \longrightarrow U,$$

that satisfies $\Phi^* I = 0$, *and let* $X \in \mathfrak{X}(U)$. *An immersion* $\Theta : M^q \times M^1 \longrightarrow U$ *defined by*

$$\Theta(u, t) := \exp_{\Phi(u)}(tX), \tag{2.1}$$

is a solution to the Cauchy problem if and only if

(1) X *is a Cauchy characteristic vector field of* I,

(2) X *is transverse to the Cauchy data,*

(3) $\Theta(M^q, 0) = \Phi(M^q)$.

The standard notation given in (2.1) denotes the point in U at t on the integral curve given by X which passes through $\Phi(u)$ when $t = 0$. The map given by (2.1) is typically found by solving a system of first order ODEs. In the remaining sections we hope to use symmetry to replace the need for solving such ODEs.

Finally, throughout this paper we assume on their domains of definition all vector field distributions are of constant dimension with linearly independent generators, and all mappings and differential one-forms are of constant rank.

3 Solvable structures

This section contains several results that are basically a consequence of Proposition 4.7 in Sherring & Prince [14]. For convenience, we reproduce this result here:

Theorem 2 *Let* $\Omega \in \Lambda^m(U)$ *be decomposable (i.e.* Ω *can be written as a exterior product of* m *one-forms) with* $d\Omega \equiv 0$ *mod* Ω. *Suppose there exists a solvable structure of linearly independent symmetries* $X_1, \ldots, X_m \in \mathfrak{X}(U)$ *such that* X_m *is a non-trivial symmetry of* $A(\langle \Omega \rangle)$, *and that for all integers* $i < m$, X_i *is a non-trivial symmetry of* $A(\langle \Omega \rangle) \oplus sp\{X_{i+1}, \ldots, X_m\}$. *For* $1 \leqslant i \leqslant m$, *define* ω^i *by*

$$\omega^i := \frac{X_1 \lrcorner \ldots \lrcorner X_{i-1} \lrcorner X_{i+1} \lrcorner \ldots \lrcorner X_m \lrcorner \Omega}{X_i \lrcorner X_1 \lrcorner \ldots \lrcorner X_{i-1} \lrcorner X_{i+1} \lrcorner \ldots \lrcorner X_m \lrcorner \Omega}.$$

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Then $\{\omega^1,\ldots,\omega^m\}$ *is dual to* $\{X_1,\ldots,X_m\}$ *and for all* ω^i *up to* $i=m$,

$$\begin{aligned}
\omega^1 &= d\gamma^1,\\
\omega^2 &= d\gamma^2 - X_1(\gamma^2)d\gamma^1,\\
\omega^3 &= d\gamma^3 - X_2(\gamma^3)(d\gamma^2 - X_1(\gamma^2)d\gamma^1) - X_1(\gamma^3)d\gamma^1,\\
&\vdots\\
\omega^m &\equiv d\gamma^m \mod d\gamma^1,\ldots,d\gamma^{m-1},
\end{aligned} \tag{3.1}$$

for some functionally independent $\gamma^1,\ldots,\gamma^m \in C^\infty(U)$. *Finally, define* $\gamma^0 := \Omega(X_1,\ldots,X_m)$. *Then* $\Omega = \gamma^0 d\gamma^1 \wedge \cdots \wedge d\gamma^m$.

It has been shown [1] that using the decomposability of Ω, the solvable symmetry structure condition in Theorem 2 is equivalent to

$$\begin{gathered}
\mathscr{L}_{X_1}(X_2 \lrcorner \ldots \lrcorner X_m \lrcorner \Omega) = \lambda_1(X_2 \lrcorner \ldots \lrcorner X_m \lrcorner \Omega),\\
\vdots\\
\mathscr{L}_{X_{m-1}}(X_m \lrcorner \Omega) = \lambda_{m-1}(X_m \lrcorner \Omega),\\
\mathscr{L}_{X_m}\Omega = \lambda_m \Omega,
\end{gathered}$$

for some $\lambda_1,\ldots\lambda_m \in C^\infty(U)$.

Theorem 2 is a simple extension of the famous theorem by Lie for using solvable symmetry structures to integrate by quadrature Frobenius integrable vector field distributions (see Barco & Prince [1] or Barco [2] for further details). Since our work makes extensive use of Theorem 2, given any solvable symmetry structure $X_1,\ldots,X_m$ for some vector field distribution D, we assume throughout that all vector fields in the symmetry structure are linearly independent, with X_m being a non-trivial symmetry of D and so on down to X_1 being a non-trivial symmetry of $D \oplus sp\{X_2,\ldots,X_m\}$. Furthermore, we assume $\gamma^1,\ldots,\gamma^m$ are those found in Theorem 2, and denote $\mathscr{F}(\gamma^a)$ as the ring of smooth functions of $\gamma^1,\ldots,\gamma^m$. For example, $\sin(\gamma^1-\gamma^m) \in \mathscr{F}(\gamma^a)$.

Lemma 1 *Let* $\Omega \in \Lambda^m(U)$ *be decomposable with* $d\Omega \equiv 0$ *mod* Ω. *Suppose there exists a solvable symmetry structure* $X_1,\ldots,X_m \in \mathfrak{X}(U)$ *for* $A(\langle\Omega\rangle)$ *and let* $Y \in A(\langle\Omega\rangle)$. *Then*

(1) $X_j(\gamma^i) = 0$ *for all* $1 \leqslant i < j \leqslant m$,
(2) $X_i(\gamma^i) = 1$ *(no sum) for all* $1 \leqslant i \leqslant m$,
(3) $Y(\gamma^i) = 0$ *for all* $1 \leqslant i \leqslant m$,
(4) $[Y, X_j] \lrcorner \omega^i = 0$ *for all* $1 \leqslant i \leqslant j \leqslant m$.

Proof From Theorem 2 $\{\omega^1,\ldots,\omega^m\}$ is dual to $\{X_1,\ldots,X_m\}$, and hence the expression for ω^1 in (3.1) implies $X_j(\gamma^1) = 0$ for all $j > 1$. Now this result and the expression for ω^2 in (3.1) yields $X_j(\gamma^2) = 0$ for all $j > 2$. It is clear that we can continue in this way up to ω^{m-1} to ultimately derive conclusion 1. Moreover, conclusion 1 and the expression for ω^i in (3.1) imply conclusion 2.

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Now for any $Y \in A(\langle\Omega\rangle)$, we have by the definition of ω^i that $Y \lrcorner \omega^i = 0$ for each $1 \leqslant i \leqslant m$. Therefore from ω^1 in (3.1) we obtain $Y(\gamma^1) = 0$. Using this result, the fact that $Y \lrcorner \omega^2 = 0$, and ω^2 in (3.1) then yields $Y(\gamma^2) = 0$. Continuing in this fashion for all i up to m, we obtain conclusion 3.

Finally, it is not hard to see that the results in conclusions 1, 2 and 3 give $[Y, X_j] \lrcorner d\gamma^i = 0$ for all $1 \leqslant i \leqslant j \leqslant m$. Hence using each ω^i in (3.1) we get conclusion 4. □

Using Lemma 1 we obtain the following:

Theorem 3 *Let $\Omega \in \Lambda^m(U)$ be such that Ω is decomposable and $d\Omega \equiv 0$ mod Ω. Suppose there exists a solvable symmetry structure $X_1, \ldots, X_m \in \mathfrak{X}(U)$ for $A(\langle\Omega\rangle)$. Then for any $Y \in A(\langle\Omega\rangle)$, we have, modulo $A(\langle\Omega\rangle)$,*

$$
\begin{aligned}
[Y, X_{m-1}] &\equiv Y\,(X_{m-1}(\gamma^m))\,X_m,\\
[Y, X_{m-2}] &\equiv Y\,(X_{m-2}(\gamma^m))\,X_m + Y\left(X_{m-2}(\gamma^{m-1})\right)\{X_{m-1} - X_{m-1}(\gamma^m)X_m\},\\
[Y, X_{m-3}] &\equiv Y\,(X_{m-3}(\gamma^m))\,X_m + Y\left(X_{m-3}(\gamma^{m-1})\right)\{X_{m-1} - X_{m-1}(\gamma^m)X_m\}\\
&\quad + Y\left(X_{m-3}(\gamma^{m-2})\right)\{X_{m-2} - X_{m-2}(\gamma^{m-1})[X_{m-1} - X_{m-1}(\gamma^m)X_m]\\
&\quad\quad - X_{m-2}(\gamma^m)X_m\},\\
&\vdots\\
[Y, X_1] &\equiv Y\left(X_1(\gamma^2)\right)X_2 \mod X_3, \ldots, X_m.
\end{aligned}
$$

(see the remark below for further details on this arrangement).

Proof We begin with $[Y, X_{m-1}]$. Since $\{\omega^1, \ldots, \omega^m\}$ is dual to $\{X_1, \ldots, X_m\}$ and $A(\langle\Omega\rangle) \oplus sp\{X_1, \ldots, X_m\}$ spans the tangent space of U, we may write (with sum)

$$[Y, X_{m-1}] \equiv \left([Y, X_{m-1}] \lrcorner \omega^i\right) X_i \mod A(\langle\Omega\rangle).$$

Conclusion 4 in Lemma 1 then gives the simplification

$$[Y, X_{m-1}] \equiv ([Y, X_{m-1}] \lrcorner \omega^m)\,X_m \mod A(\langle\Omega\rangle).$$

We note from (3.1) that $\omega^m \equiv d\gamma^m \mod \omega^1, \ldots, \omega^{m-1}$. This fact and conclusion 3 in Lemma 1 then yields

$$
\begin{aligned}
[Y, X_{m-1}] &\equiv ([Y, X_{m-1}] \lrcorner d\gamma^m)\,X_m \mod A(\langle\Omega\rangle),\\
&\equiv Y\,(X_{m-1}(\gamma^m))\,X_m \mod A(\langle\Omega\rangle),
\end{aligned}
$$

which proves the first result.

Following the same argument as before, but for $[Y, X_{m-2}]$, we obtain

$$
\begin{aligned}
[Y, X_{m-2}] &\equiv \left([Y, X_{m-2}] \lrcorner \omega^{m-1}\right) X_{m-1} + ([Y, X_{m-2}] \lrcorner \omega^m)\,X_m \mod A(\langle\Omega\rangle),\\
&\equiv \left([Y, X_{m-2}] \lrcorner d\gamma^{m-1}\right) X_{m-1}\\
&\quad + \left([Y, X_{m-2}] \lrcorner \left(d\gamma^m - X_{m-1}(\gamma^m)d\gamma^{m-1}\right)\right) X_m \mod A(\langle\Omega\rangle),\\
&\equiv Y\left(X_{m-2}(\gamma^{m-1})\right) X_{m-1} + Y\,(X_{m-2}(\gamma^m))\,X_m\\
&\quad - Y\left(X_{m-2}(\gamma^{m-1})\right) X_{m-1}(\gamma^m)X_m \mod A(\langle\Omega\rangle),
\end{aligned}
$$

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initially using the expressions for ω^{m-1} and ω^m in (3.1), and then several times each of conclusions 4 and 3 in Lemma 1.

In a similar way to above, we can repeat the above process to eventually prove the remaining $m-3$ results in the theorem. □

Remark The pattern in the expressions for $[Y, X_i]$ in the conclusion of the theorem becomes obvious if we express, for example $[Y, X_{m-4}]$, modulo $A(\langle\Omega\rangle)$, in the following form:

$$[Y, X_{m-4}] \equiv Y\,(X_{m-4}(\gamma^m))\,\overline{X}_m + Y\,\left(X_{m-4}(\gamma^{m-1})\right)\overline{X}_{m-1} + Y\,\left(X_{m-4}(\gamma^{m-2})\right)\overline{X}_{m-2} + Y\,\left(X_{m-4}(\gamma^{m-3})\right)\overline{X}_{m-3},$$

where

$$\begin{aligned}
\overline{X}_m &:= X_m,\\
\overline{X}_{m-1} &:= X_{m-1} - X_{m-1}(\gamma^m)\overline{X}_m,\\
\overline{X}_{m-2} &:= X_{m-2} - X_{m-2}(\gamma^{m-1})\overline{X}_{m-1} - X_{m-2}(\gamma^m)\overline{X}_m,\\
\overline{X}_{m-3} &:= X_{m-3} - X_{m-3}(\gamma^{m-2})\overline{X}_{m-2} - X_{m-3}(\gamma^{m-1})\overline{X}_{m-1} - X_{m-3}(\gamma^m)\overline{X}_m.
\end{aligned}$$

Lemma 2 *Let $\Omega \in \Lambda^m(U)$ such that Ω is decomposable and $d\Omega \equiv 0 \bmod \Omega$, and let $X_1, \ldots, X_m \in \mathfrak{X}(U)$ be a solvable symmetry structure for $A(\langle\Omega\rangle)$. If we define the vector fields $\overline{X}_1, \ldots, \overline{X}_m$ by*

$$\begin{aligned}
\overline{X}_m &:= X_m,\\
\overline{X}_{m-1} &:= X_{m-1} - X_{m-1}(\gamma^m)\overline{X}_m,\\
\overline{X}_{m-2} &:= X_{m-2} - X_{m-2}(\gamma^{m-1})\overline{X}_{m-1} - X_{m-2}(\gamma^m)\overline{X}_m,\\
&\vdots\\
\overline{X}_1 &\equiv X_1 \mod \overline{X}_2, \ldots, \overline{X}_m,
\end{aligned} \tag{3.2}$$

then $\{d\gamma^1, \ldots, d\gamma^m\}$ is dual to $\{\overline{X}_1, \ldots, \overline{X}_m\}$.

Proof We work by induction. From the definition of $\overline{X}_m$ in (3.2) and conclusions 1 and 2 in Lemma 1, it is obvious that $\overline{X}_m(\gamma^m) = 1$ and $\overline{X}_m(\gamma^j) = 0$ for all $1 \leqslant j < m$.

Now let k be any integer such that $2 \leqslant k \leqslant m$, and assume for each $k \leqslant l \leqslant m$ and $1 \leqslant p \leqslant m$ with $p \neq l$ that

$$\overline{X}_l(\gamma^l) = 1, \tag{3.3}$$

$$\overline{X}_l(\gamma^p) = 0. \tag{3.4}$$

We wish to show (3.3) and (3.4) hold for $l = k-1$. First suppose q is any integer such that $k-1 < q \leqslant m$. Consider $\overline{X}_{k-1}$ in (3.2) operated on γ^q, i.e.

$$\overline{X}_{k-1}(\gamma^q) = X_{k-1}(\gamma^q) - X_{k-1}(\gamma^k)\overline{X}_k(\gamma^q) - \cdots - X_{k-1}(\gamma^q)\overline{X}_q(\gamma^q) - \ldots - X_{k-1}(\gamma^m)\overline{X}_m(\gamma^q). \tag{3.5}$$

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Equation (3.3) then implies

$$X_{k-1}(\gamma^q) - X_{k-1}(\gamma^q)\overline{X}_q(\gamma^q) = 0,$$

and from (3.4) all the other terms in (3.5) become zero. Thus $\overline{X}_{k-1}(\gamma^q) = 0$.

Now suppose $q = k-1$. The expression

$$\overline{X}_{k-1}(\gamma^q) = X_{k-1}(\gamma^q) - X_{k-1}(\gamma^k)\overline{X}_k(\gamma^q) - \cdots - X_{k-1}(\gamma^m)\overline{X}_m(\gamma^q), \tag{3.6}$$

simplifies to give

$$\overline{X}_{k-1}(\gamma^{k-1}) = X_{k-1}(\gamma^{k-1}),$$

since (3.4) implies all the remaining terms in (3.6) vanish. Hence by using conclusion 2 in Lemma 1 we obtain $\overline{X}_{k-1}(\gamma^{k-1}) = 1$.

Finally, suppose q is any integer such that $1 \leqslant q < k-1$. Using (3.6) for such q, and then inserting (3.4) leaves

$$\overline{X}_{k-1}(\gamma^q) = X_{k-1}(\gamma^q).$$

Then from conclusion 1 in Lemma 1 we find that $\overline{X}_{k-1}(\gamma^q) = 0$. This completes the induction. □

From the definitions of $\overline{X}_1, \ldots, \overline{X}_m$ in (3.2), we have the following corollary to Theorem 3:

Corollary 1 *Let $\Omega \in \Lambda^m(U)$ be decomposable with $d\Omega \equiv 0$ mod Ω. Given any solvable symmetry structure $X_1, \ldots, X_m \in \mathfrak{X}(U)$ for $A(\langle\Omega\rangle)$, let $Y \in A(\langle\Omega\rangle)$. Then with $\overline{X}_1, \ldots, \overline{X}_m$ defined as in Lemma 2, we have, modulo $A(\langle\Omega\rangle)$,*

$$\begin{aligned}
[Y, X_{m-1}] &\equiv Y\,(X_{m-1}(\gamma^m))\,\overline{X}_m,\\
[Y, X_{m-2}] &\equiv Y\,(X_{m-2}(\gamma^m))\,\overline{X}_m + Y\,\left(X_{m-2}(\gamma^{m-1})\right)\overline{X}_{m-1},\\
[Y, X_{m-3}] &\equiv Y\,(X_{m-3}(\gamma^m))\,\overline{X}_m + Y\,\left(X_{m-3}(\gamma^{m-1})\right)\overline{X}_{m-1}\\
&\qquad + Y\,\left(X_{m-3}(\gamma^{m-2})\right)\overline{X}_{m-2},\\
&\vdots\\
[Y, X_1] &\equiv Y\,\left(X_1(\gamma^2)\right)\overline{X}_2 \mod \overline{X}_3, \ldots, \overline{X}_m.
\end{aligned}$$

Using Corollary 1, we have the following major result:

Theorem 4 *Let $\Omega \in \Lambda^m(U)$ be decomposable with $d\Omega \equiv 0$ mod Ω. Given any solvable symmetry structure $X_1, \ldots, X_m \in \mathfrak{X}(U)$ for $A(\langle\Omega\rangle)$, let $Y \in A(\langle\Omega\rangle)$. Then with $\overline{X}_1, \ldots, \overline{X}_m$ defined as in Lemma 2, we have for each $1 \leqslant i \leqslant m$ $[Y, \overline{X}_i] \equiv 0$ mod $A(\langle\Omega\rangle)$.*

Proof Instead of proving the result by a tedious induction, we simply examine the first three $[Y, \overline{X}_i]$ in descending order. Since $\overline{X}_m = X_m$ and X_m is a symmetry of $A(\langle\Omega\rangle)$, it is obvious that

$$[Y, \overline{X}_m] \equiv 0 \mod A(\langle\Omega\rangle). \tag{3.7}$$

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Next consider $[Y, \overline{X}_{m-1}]$. By the definition of $\overline{X}_{m-1}$,

$$\begin{aligned}[Y, \overline{X}_{m-1}] &= [Y, X_{m-1} - X_{m-1}(\gamma^m)\overline{X}_m], \\ &= [Y, X_{m-1}] - Y\,(X_{m-1}(\gamma^m))\,\overline{X}_m - X_{m-1}(\gamma^m)[Y, \overline{X}_m].\end{aligned}$$

Now using the expression for $[Y, X_{m-1}]$ in Corollary 1, we have that

$$[Y, \overline{X}_{m-1}] = -X_{m-1}(\gamma^m)[Y, \overline{X}_m].$$

Then inserting (3.7) gives our second result, namely

$$[Y, \overline{X}_{m-1}] \equiv 0 \mod A(\langle \Omega \rangle). \tag{3.8}$$

Finally, we consider $[Y, \overline{X}_{m-2}]$. Once again by definition, and then inserting the expression for $[Y, X_{m-2}]$ in Corollary 1, we get

$$\begin{aligned}[Y, \overline{X}_{m-2}] &= [Y, X_{m-2} - X_{m-2}(\gamma^{m-1})\overline{X}_{m-1} - X_{m-2}(\gamma^m)\overline{X}_m], \\ &= [Y, X_{m-2}] - Y\left(X_{m-2}(\gamma^{m-1})\right)\overline{X}_{m-1} - Y\,(X_{m-2}(\gamma^m))\,\overline{X}_m \\ &\quad - X_{m-2}(\gamma^{m-1})[Y, \overline{X}_{m-1}] - X_{m-2}(\gamma^m)[Y, \overline{X}_m], \\ &= -X_{m-2}(\gamma^{m-1})[Y, \overline{X}_{m-1}] - X_{m-2}(\gamma^m)[Y, \overline{X}_m].\end{aligned}$$

Then using (3.7) and (3.8), we obtain

$$[Y, \overline{X}_{m-2}] \equiv 0 \mod A(\langle \Omega \rangle).$$

At this point, it is easy to see how induction may be used to formally prove the theorem. □

4 The Cauchy problem for a one-form of rank one

In this section we examine a symmetry technique for solving the Cauchy problem for a differential ideal generated by a single one-form of constant rank one on U. It is well known (and illustrated later in an example) that such an ideal can be generated from a single first order PDE of one dependent variable and two independent variables by pulling back the first order contact form onto the regular submanifold of the first jet bundle described by the PDE. In this space, the Cauchy characteristic space is one-dimensional [17].

Before we present the main result, Theorem 6, we will need several preliminary results, most of which also hold for differential one-forms of constant rank higher than one.

Lemma 3 *Let $\alpha \in \Lambda^1(U)$. Suppose α is of constant rank $r \geqslant 1$ on U and define $\Omega := (d\alpha)^r \wedge \alpha$. Then $d\Omega = 0 \mod \Omega$. Furthermore, Ω is decomposable.*

Proof See Barco & Prince [1] or Barco [2]. □

Lemma 4 *Let $\Omega \in \Lambda^p(U)$ for some integer p, and $I := \langle \Omega, d\Omega \rangle$. If $d\Omega \equiv 0$ mod Ω, then* $\ker(\Omega) = A(I)$.

Proof See Barco & Prince [1] or Barco [2]. □

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The remaining preparatory results given below incorporate symmetry, however first we must examine an existence issue:

Theorem 5 *Let $\alpha \in \Lambda^1(U)$ such that $Y_1, \ldots, Y_q \in \mathfrak{X}(U)$ generate the Cauchy characteristic space of $\langle \alpha, d\alpha \rangle$. Then there exists some non-zero $h \in C^\infty(U)$ such that for all $1 \leqslant i \leqslant q$, $Y_i \lrcorner\, d(h\alpha) = 0$.*

Proof Let the Cauchy characteristic space $A(\langle \alpha, d\alpha \rangle)$ be q-dimensional and generated by $Y_1, \ldots, Y_q \in \mathfrak{X}(U)$. Thus for each $1 \leqslant i \leqslant q$ we have

$$Y_i \lrcorner\, d\alpha = \mu_i \alpha,$$

for some $\mu_i \in C^\infty(U)$. Since for all $h \in C^\infty(U)$ and i,

$$\begin{aligned} Y_i \lrcorner\, d(h\alpha) &= Y_i \lrcorner\, (dh \wedge \alpha + h d\alpha), \\ &= Y_i(h)\alpha + h\,(Y_i \lrcorner\, d\alpha), \\ &= (Y_i(h) + h\mu_i)\,\alpha, \end{aligned}$$

we wish to show there exists some non-zero h such that for each i,

$$Y_i(h) + h\mu_i = 0. \tag{4.1}$$

Since $A(\langle \alpha, d\alpha \rangle)$ is Frobenius integrable, we can therefore write for each $1 \leqslant j, k \leqslant q$,

$$[Y_j, Y_k] = \rho^1_{jk} Y_1 + \cdots + \rho^q_{jk} Y_q, \tag{4.2}$$

for some $\rho^1_{jk}, \ldots, \rho^q_{jk} \in C^\infty(U)$ with $\rho^l_{jk} = -\rho^l_{kj}$ for all $1 \leqslant l \leqslant q$. For there to exist a local solution h of the system of PDEs in (4.1), the integrability conditions that must be satisfied are

$$Y_j(\mu_k) - Y_k(\mu_j) = \rho^1_{jk}\mu_1 + \cdots + \rho^q_{jk}\mu_q. \tag{4.3}$$

To show (4.3) holds, we have that

$$\begin{aligned} \mathscr{L}_{[Y_j, Y_k]}\alpha &= \mathscr{L}_{Y_j}\left(\mathscr{L}_{Y_k}\alpha\right) - \mathscr{L}_{Y_k}\left(\mathscr{L}_{Y_j}\alpha\right), \\ &= \mathscr{L}_{Y_j}\left(Y_k \lrcorner\, d\alpha\right) - \mathscr{L}_{Y_k}\left(Y_j \lrcorner\, d\alpha\right), \\ &= \mathscr{L}_{Y_j}\left(\mu_k \alpha\right) - \mathscr{L}_{Y_k}\left(\mu_j \alpha\right), \\ &= \left(Y_j(\mu_k) - Y_k(\mu_j)\right)\alpha. \end{aligned}$$

But from using (4.2),

$$\begin{aligned} \mathscr{L}_{[Y_j, Y_k]}\alpha &= \mathscr{L}_{\rho^1_{jk} Y_1 + \cdots + \rho^q_{jk} Y_q}\alpha, \\ &= \left(\rho^1_{jk} Y_1 + \cdots + \rho^q_{jk} Y_q\right) \lrcorner\, d\alpha, \\ &= \rho^1_{jk}\mu_1\alpha + \cdots + \rho^q_{jk}\mu_q\alpha. \end{aligned}$$

Hence the conditions in (4.3) are satisfied. □

Remark The result in Theorem 5 is expected due to the Frobenius integrability condition given in (4.2).

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At this point we introduce symmetry. We begin with a simple result which we state without proof:

Lemma 5 *Let $\alpha \in \Lambda^1(U)$. Then $Z \in \mathfrak{X}(U)$ is a non-trivial symmetry of $\langle \alpha, d\alpha \rangle$ if and only if Z is a non-trivial symmetry of α.*

From this we obtain the following:

Lemma 6 *Let $\alpha \in \Lambda^1(U)$ and let $Y_1, \ldots, Y_q \in \mathfrak{X}(U)$ generate the Cauchy characteristic space of $\langle \alpha, d\alpha \rangle$. If $Z \in \mathfrak{X}(U)$ is a non-trivial symmetry of $\langle \alpha, d\alpha \rangle$, then for all $1 \leqslant i \leqslant q$,*

$$Y_i \lrcorner\, d\left(\frac{\alpha}{Z \lrcorner\, \alpha}\right) = 0.$$

Proof Lemma 5 implies $Z \in \mathfrak{X}(U)$ is a non-trivial symmetry of α, so $Z \lrcorner\, \alpha \neq 0$. Now for each $1 \leqslant i \leqslant q$,

$$\begin{aligned} Y_i \lrcorner\, d\left(\frac{\alpha}{Z \lrcorner\, \alpha}\right) &= Y_i \lrcorner\, \left(\frac{(Z \lrcorner\, \alpha) d\alpha - d(Z \lrcorner\, \alpha) \wedge \alpha}{(Z \lrcorner\, \alpha)^2}\right), \\ &= \frac{Y_i}{(Z \lrcorner\, \alpha)^2} \lrcorner\, \left((Z \lrcorner\, \alpha) d\alpha + (Z \lrcorner\, d\alpha) \wedge \alpha\right), \\ &= \frac{1}{(Z \lrcorner\, \alpha)^2} Y_i \lrcorner\, Z \lrcorner\, (d\alpha \wedge \alpha). \end{aligned}$$

In the second line we have used the symmetry condition that $Z \lrcorner\, d\alpha + d(Z \lrcorner\, \alpha) = \mathscr{L}_Z \alpha = \rho\alpha$ for some $\rho \in C^\infty(U)$. We have

$$Y_i \lrcorner\, (d\alpha \wedge \alpha) = (Y_i \lrcorner\, d\alpha) \wedge \alpha + (Y_i \lrcorner\, \alpha) d\alpha = 0,$$

since Y_i is a Cauchy characteristic vector field of $\langle \alpha, d\alpha \rangle$. Thus

$$\frac{1}{(Z \lrcorner\, \alpha)^2} Y_i \lrcorner\, Z \lrcorner\, (d\alpha \wedge \alpha) = 0.$$

Hence the result. □

To find a non-trivial symmetry of $\langle \alpha, d\alpha \rangle$ for Lemma 6, we simply use Lemma 5 and look for a non-trivial symmetry of α, or equivalently, the vector field dual space of α using `DIMSYM`. The latter is true because a vector field is a symmetry of a decomposable differential form if and only if it is a symmetry of its kernel [14].

We illustrate Lemma 6 with an example which we will refer to later:

Example 1 Suppose $\alpha \in \Lambda^1(U^4)$ is defined by

$$\alpha := dx^3 - x^4 dx^1 + x^3 x^4 dx^2.$$

The Cauchy characteristic space of the differential ideal $\langle \alpha, d\alpha \rangle$ is of dimension one and generated by

$$Y := \frac{\partial}{\partial x^2} + x^3 \frac{\partial}{\partial x^1} - (x^4)^2 \frac{\partial}{\partial x^4}.$$

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Here, $Y \lrcorner d\alpha = -x^4\alpha$. It is obvious that $Z := \frac{\partial}{\partial x^1}$ is a non-trivial symmetry of α with $Z \lrcorner \alpha = -x^4$. So

$$Y \lrcorner d\left(\frac{\alpha}{x^4}\right) = 0,$$

on some suitably chosen subset of U^4.

The final result required is contained in the following simple lemma which we give without proof (see Barco [2] for a proof):

Lemma 7 *Let $\alpha \in \Lambda^1(U)$ be of constant rank one on U. Then $A(\langle \alpha, d\alpha\rangle) = A(\langle d\alpha \wedge \alpha\rangle)$, and is $(n-3)$-dimensional.*

At this point we are now in a position to present the main result of this section. In what follows, we assume $\alpha \in \Lambda^1(U)$ is of constant rank one on U with the $(n-3)$-dimensional Cauchy characteristic space of $\langle \alpha, d\alpha\rangle$ generated by some $Y_1, \ldots, Y_{n-3} \in \mathfrak{X}(U)$. Further, it is assumed that we have applied Lemma 6, so that $Y_j \lrcorner d\alpha = 0$ for each $1 \leqslant j \leqslant n-3$. In Theorem 6 below, we also introduce for the first time the notation $\mathfrak{X}^m(U)$, where $m \leqslant n$, to denote an m-dimensional submodule of $\mathfrak{X}(U)$ over the smooth ring of functions $\mathscr{F}(\gamma^a)$. This simply means that for any $W_1, \ldots W_m \in \mathfrak{X}(U)$ generating $\mathfrak{X}^m(U)$, if $Z \in \mathfrak{X}^m(U)$, then

$$Z = \mu^1 W_1 + \cdots + \mu^m W_m,$$

for some $\mu^1, \ldots, \mu^m \in \mathscr{F}(\gamma^a)$. In the theorem, $W_1, \ldots W_m$ are by construction linear combinations of the symmetries making up the solvable symmetry strucure for $A(\langle d\alpha \wedge \alpha\rangle)$.

Theorem 6, when $n = 4$, gives a symmetry approach to the Cauchy problem with one-dimensional Cauchy data, thus generating an integral manifold of dimension two.

Theorem 6 *Define $\Omega := d\alpha \wedge \alpha$ and let $X_1, X_2, X_3 \in \mathfrak{X}(U)$ be a solvable symmetry structure for $A(\langle \Omega\rangle)$. There exists a two-dimensional submodule $\mathfrak{X}^2(U)$ of $\mathfrak{X}(U)$ over $\mathscr{F}(\gamma^a)$ such that for any non-zero $Z \in \mathfrak{X}^2(U)$, we have that $[Y_j, Z] \equiv 0$ mod $A(\langle \alpha, d\alpha\rangle)$ for each $1 \leqslant j \leqslant n-3$. Moreover, on U, $sp\{Y_1, \ldots, Y_{n-3}, Z\}$ is Frobenius integrable and generates an $(n-2)$-dimensional integral manifold of $\langle \alpha, d\alpha\rangle$.*

Proof Lemma 3 tells us that Ω is decomposable and $d\Omega \equiv 0$ mod Ω. Then applying Theorem 2 with $\omega^1, \omega^2, \omega^3$ defined as in the theorem, we have that $\{\omega^1, \omega^2, \omega^3\}$ is dual to $\{X_1, X_2, X_3\}$ with

$$\begin{aligned}
\omega^1 &= d\gamma^1,\\
\omega^2 &= d\gamma^2 - X_1(\gamma^2)d\gamma^1,\\
\omega^3 &= d\gamma^3 - X_2(\gamma^3)(d\gamma^2 - X_1(\gamma^2)d\gamma^1) - X_1(\gamma^3)d\gamma^1,
\end{aligned}$$

for some functionally independent $\gamma^1, \gamma^2, \gamma^3 \in C^\infty(U)$. Now from Lemma 7, it follows that $A(\langle \alpha, d\alpha\rangle) = A(\langle \Omega\rangle) = sp\{Y_1, \ldots, Y_{n-3}\}$, so

$$\begin{aligned}
\alpha &= (X_1 \lrcorner \alpha)\,\omega^1 + (X_2 \lrcorner \alpha)\,\omega^2 + (X_3 \lrcorner \alpha)\,\omega^3,\\
&= \lambda_1 d\gamma^1 + \lambda_2 d\gamma^2 + \lambda_3 d\gamma^3,
\end{aligned}$$

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for some $\lambda_1, \lambda_2, \lambda_3 \in C^\infty(U)$. Since for each $1 \leqslant j \leqslant n-3$, $Y_j \lrcorner\, \alpha = 0 = Y_j \lrcorner\, d\alpha$, we therefore have

$$\mathscr{L}_{Y_j} \alpha = 0. \tag{4.4}$$

We know from conclusion 3 in Lemma 1 that $Y_j \lrcorner\, d\gamma^i = 0$ for each $1 \leqslant i \leqslant 3$, so (4.4) implies

$$Y_j(\lambda_1) d\gamma^1 + Y_j(\lambda_2) d\gamma^2 + Y_j(\lambda_3) d\gamma^3 = 0.$$

Therefore, as $d\gamma^1, d\gamma^2, d\gamma^3$ are linearly independent, it follows that

$$Y_j(\lambda_i) = 0, \tag{4.5}$$

for all i and j. Since $sp\{Y_1, \ldots, Y_{n-3}\}$ is Frobenius integrable with $\gamma^1, \gamma^2, \gamma^3$ as first integrals, we can then conclude from (4.5) that each λ_i is some function of $\gamma^1, \gamma^2, \gamma^3$. Hence α only depends on $\gamma^1, \gamma^2, \gamma^3$ and their exterior derivatives.

Suppose we now define the vector fields $\overline{X}_1, \overline{X}_2, \overline{X}_3$ as in (3.2), i.e.

$$\begin{aligned} \overline{X}_3 &= X_3, \\ \overline{X}_2 &= X_2 - X_2(\gamma^3)\overline{X}_3, \\ \overline{X}_1 &= X_1 - X_1(\gamma^2)\overline{X}_2 - X_1(\gamma^3)\overline{X}_3. \end{aligned}$$

Then from Lemma 2 we have that $\{\overline{X}_1, \overline{X}_2, \overline{X}_3\}$ is dual to $\{d\gamma^1, d\gamma^2, d\gamma^3\}$. If we choose an $1 \leqslant r \leqslant 3$ such that $\lambda_r \neq 0$, and define for each two remaining $1 \leqslant k \leqslant 3$ with $k \neq r$,

$$W_k := \lambda_k \overline{X}_r - \lambda_r \overline{X}_k,$$

then $\ker(\alpha) = sp\{Y_1, \ldots, Y_{n-3}, W_{k_1}, W_{k_2}\}$. Here k_1 and k_2 are our two choices of k. From Theorem 4 and (4.5), it then follows that for each j and $l \in \{1, 2\}$, $[Y_j, W_{k_l}] \equiv 0 \bmod Y_1, \ldots, Y_{n-3}$.

Now let $\mathfrak{X}^2(U)$ be generated by W_{k_2} and W_{k_1}, and let $Z \in \mathfrak{X}^2(U)$. Then

$$Z = \mu^{k_1} W_{k_1} + \mu^{k_2} W_{k_2},$$

for some $\mu^{k_1}, \mu^{k_2} \in \mathscr{F}(\gamma^a)$. From conclusion 3 in Lemma 1 we then have that for each j, $[Y_j, Z] \equiv 0 \bmod Y_1, \ldots, Y_{n-3}$.

Finally, it is clear that $Z \lrcorner\, \alpha = 0$, and since $Y_1, \ldots, Y_{n-3}$ are linearly independent of Z, we now have an $(n-2)$-dimensional Frobenius integrable distribution spanned by $Y_1, \ldots, Y_{n-3}$ and Z, that generates an integral manifold of $\langle \alpha \rangle$, and hence an integral manifold of $\langle \alpha, d\alpha \rangle$ since the exterior derivative and the pull-back commute. □

If, in applying Theorem 6 to the Cauchy problem (i.e. $n = 4$) we are given some Cauchy data in the form of a one-dimensional curve, and we are able to establish some vector field tangent to the curve that is in $\mathfrak{X}^2(U)$, then the theorem will generate a unique two-dimensional foliation of U in which there exists a unique two-dimensional leaf containing the Cauchy data. If, however, we are only given a vector field $Z \in \mathfrak{X}^2(U)$ for the Cauchy data, then the theorem only guarantees *uniqueness up to foliation*. This means that we obtain, at most, a unique two-dimensional foliation of U, where the tangent space of each two-dimensional leaf is spanned by Z and the Cauchy characteristic vector field Y_1.

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Thus the Cauchy problem is 'solved' in terms of two arbitrary constants. We illustrate this further in the example in the next section.

5 An application

In this section we present an example showing how one may use Theorem 6 to generate local solutions of first order PDEs of one dependent variable and two independent variables of the form

$$\frac{\partial u}{\partial x^2} = F\left(x^1, x^2, u, \frac{\partial u}{\partial x^1}\right), \tag{5.1}$$

for some smooth function F. We work in the first jet bundle $J^1(U^2, V^1)$ with local coordinates $x^1, x^2, z^1, z^1_1, z^1_2$, where U^2 has the local coordinates x^1, x^2, V^1 has the local coordinate z^1, and z^1_1, z^1_2 are local coordinates for the derivatives of z^1 with respect to, respectively, the independent variables x^1, x^2. Equation (5.1) can be seen to describe a four-dimensional regular submanifold of the five-dimensional jet bundle, which we denote by $M^4 \subset J^1(U^2, V^1)$. Replacing the dependent variable u with the coordinate z^1, M^4 is therefore described by the locus of

$$z^1_2 = F(x^1, x^2, z^1, z^1_1).$$

With coordinates x^1, x^2, z^1, z^1_1 for M^4, the Pfaffian equation corresponding to (5.1) is

$$\alpha := dz^1 - z^1_1 dx^1 - F dx^2 = 0. \tag{5.2}$$

Before we examine a PDE example, we recall a well known fact (see, for example, Stormark [17]) that with α defined as in (5.2), the Cauchy characteristic space of $\langle \alpha, d\alpha \rangle$ is one-dimensional and generated by

$$Y := \frac{\partial}{\partial x^2} - \frac{\partial F}{\partial z^1_1}\frac{\partial}{\partial x^1} + \left(F - z^1_1 \frac{\partial F}{\partial z^1_1}\right)\frac{\partial}{\partial z^1} + \left(\frac{\partial F}{\partial x^1} + z^1_1 \frac{\partial F}{\partial z^1}\right)\frac{\partial}{\partial z^1_1}.$$

Example 2 Consider the Hopf equation:

$$\frac{\partial u}{\partial x^2} + u\frac{\partial u}{\partial x^1} = 0. \tag{5.3}$$

Here $M^4 \subset J^1(U^2, V^1)$ is the regular submanifold described by the equation $z^1_2 + z^1 z^1_1 = 0$. Using coordinates x^1, x^2, z^1, z^1_1 for M^4, we define $\alpha := dz^1 - z^1_1 dx^1 + z^1 z^1_1 dx^2$. By dimension, α has a maximum rank of one, so we suppose α has constant rank on M^4. The Cauchy characteristic space of $\langle \alpha, d\alpha \rangle$ is generated by

$$Y := \frac{\partial}{\partial x^2} + z^1\frac{\partial}{\partial x^1} - (z^1_1)^2\frac{\partial}{\partial z^1_1}.$$

It is not necessary at this point to ensure that $Y \lrcorner d\alpha = 0$. This can be done following the application of Theorem 2. With $\Omega := d\alpha \wedge \alpha$, we find that $X_3 := \frac{\partial}{\partial x^1}$ is a non-trivial symmetry of Ω, that $X_2 := \frac{\partial}{\partial x^2}$ is a non-trivial symmetry of $X_3 \lrcorner \Omega$, and finally that

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$X_1 := \frac{\partial}{\partial z^1}$ is a non-trivial symmetry of $X_2 \lrcorner X_3 \lrcorner \Omega$. Applying Theorem 2 gives

$$\omega^1 = \frac{X_2 \lrcorner X_3 \lrcorner \Omega}{X_1 \lrcorner X_2 \lrcorner X_3 \lrcorner \Omega} = dz^1,$$
$$\omega^2 = \frac{X_1 \lrcorner X_3 \lrcorner \Omega}{X_2 \lrcorner X_1 \lrcorner X_3 \lrcorner \Omega} = d\left(x^2 - \frac{1}{z_1^1}\right),$$
$$\omega^3 = \frac{X_1 \lrcorner X_2 \lrcorner \Omega}{X_3 \lrcorner X_1 \lrcorner X_2 \lrcorner \Omega} = d\left(x^1 - \frac{z^1}{z_1^1}\right) + \frac{1}{z_1^1} dz^1,$$

on some suitably restricted domain. Next, we scale α so that $Y \lrcorner d\alpha = 0$. From Example 1 we obtain

$$Y \lrcorner d\left(\frac{\alpha}{z_1^1}\right) = 0.$$

We can therefore replace α with

$$\overline{\alpha} := \frac{\alpha}{z_1^1} = \frac{dz^1}{z_1^1} - dx^1 + z^1 dx^2.$$

Hence

$$\begin{aligned} \overline{\alpha} &= (X_1 \lrcorner \overline{\alpha})\,\omega^1 + (X_2 \lrcorner \overline{\alpha})\,\omega^2 + (X_3 \lrcorner \overline{\alpha})\,\omega^3, \\ &= z^1 d\left(x^2 - \frac{1}{z_1^1}\right) - d\left(x^1 - \frac{z^1}{z_1^1}\right). \end{aligned} \tag{5.4}$$

Now we have from Lemma 2 that

$$\overline{X}_1 = \frac{\partial}{\partial z^1} + \frac{1}{z_1^1}\frac{\partial}{\partial x^1}, \qquad \overline{X}_2 = \frac{\partial}{\partial x^2}, \qquad \overline{X}_3 = \frac{\partial}{\partial x^1},$$

so that

$$\left\{dz^1, d\left(x^2 - \frac{1}{z_1^1}\right), d\left(x^1 - \frac{z^1}{z_1^1}\right)\right\}$$

is dual to

$$\left\{\frac{\partial}{\partial z^1} + \frac{1}{z_1^1}\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^1}\right\}.$$

Thus from the expression for $\overline{\alpha}$ in (5.4), we let our Cauchy data be generated by

$$Z := \mu^1\left(\frac{\partial}{\partial z^1} + \frac{1}{z_1^1}\frac{\partial}{\partial x^1}\right) + \mu^2\left(z^1\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2}\right),$$

where μ^1, μ^2 are any smooth functions of $z^1, x^2 - 1/z_1^1, x^1 - z^1/z_1^1$, so that $Z \lrcorner \overline{\alpha} = 0$. Note that with this construction for the Cauchy data, applying Theorem 6 will only give us uniqueness up to foliation. Therefore any solution of (5.3) will depend on two arbitrary constants.

Now from Theorem 6, $[Y, Z] \equiv 0 \mod Y$, and hence $sp\{Y, Z\}$ is a two-dimensional Frobenius integrable distribution that generates an integral manifold of the differential ideal $\langle \alpha, d\alpha \rangle$. To obtain a local solution to (5.3), we choose Z so that the transverse condition $dx^1 \wedge dx^2 \neq 0$ is satisfied on the distribution $sp\{Y, Z\}$.

We now proceed to pick two different Z's, and derive a local solution to (5.3) for each

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choice. First choose $\mu^1 = 1$ and $\mu^2 = 0$ so that

$$Z = \frac{\partial}{\partial z^1} + \frac{1}{z_1^1}\frac{\partial}{\partial x^1}.$$

On M^4, the distribution spanned by Y and Z is annihilated by the two one-forms $\overline{\alpha}$ and $(z_1^1)^2 dx^2 + dz_1^1$. Define

$$\overline{\Omega} := \overline{\alpha} \wedge \left((z_1^1)^2 dx^2 + dz_1^1\right).$$

Since $sp\{Y, Z\}$ is a Frobenius integrable distribution, we have that $d\overline{\Omega} \equiv 0 \bmod \overline{\Omega}$. Hence we can proceed to once again apply Theorem 2. We find that $\frac{\partial}{\partial x^1}$ and $\frac{\partial}{\partial x^2}$ are two linearly independent commuting non-trivial symmetries of $\overline{\Omega}$. We can use Corollary 3.4 in Sherring & Prince [14] (or Theorem 3.16 in Barco & Prince [1]), which is essentially a particular case of Theorem 2, so that we obtain the two closed forms

$$\frac{\frac{\partial}{\partial x^1} \lrcorner\, \overline{\Omega}}{\frac{\partial}{\partial x^2} \lrcorner\, \frac{\partial}{\partial x^1} \lrcorner\, \overline{\Omega}} = d\left(x^2 - \frac{1}{z_1^1}\right), \qquad \frac{\frac{\partial}{\partial x^2} \lrcorner\, \overline{\Omega}}{\frac{\partial}{\partial x^1} \lrcorner\, \frac{\partial}{\partial x^2} \lrcorner\, \overline{\Omega}} = d\left(x^1 - \frac{z^1}{z_1^1}\right).$$

In general, we will not produce the same first integrals we obtained before. Put

$$x^1 - \frac{z^1}{z_1^1} = c^1, \qquad x^2 - \frac{1}{z_1^1} = c^2,$$

for any choice of constant functions c_1 and c_2. Then equating the z_1^1 term and replacing z^1 with u, we get the following local solution (where $x^2 \neq c_2$) of the PDE in (5.3):

$$u = \frac{x^1 - c_1}{x^2 - c_2}.$$

For our second choice for Z, we will be a little more ambitious and suppose that

$$Z = \frac{\partial}{\partial z^1} + \frac{1}{z_1^1}\frac{\partial}{\partial x^1} + \frac{1}{z^1}\left(x^2 - \frac{1}{z_1^1}\right)\left(z^1\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2}\right).$$

Following the same procedure as before, and using the same notation, we define

$$\overline{\Omega} := \overline{\alpha} \wedge \left(dz_1^1 + (z_1^1)^2 dx^2 + \frac{z_1^1}{z^1}\left(1 - z_1^1 x^2\right) dz^1\right).$$

It is easy to see that $\frac{\partial}{\partial x^1}$ is a non-trivial symmetry of $\overline{\Omega}$. Using `DIMSYM`, we find that $z^1\frac{\partial}{\partial z^1}$ is a non-trivial symmetry of $\frac{\partial}{\partial x^1} \lrcorner\, \overline{\Omega}$. Applying Theorem 2 yields

$$\begin{aligned}
&\frac{\frac{\partial}{\partial x^1} \lrcorner\, \overline{\Omega}}{z^1\frac{\partial}{\partial z^1} \lrcorner\, \frac{\partial}{\partial x^1} \lrcorner\, \overline{\Omega}} = d\left(\ln\left|\frac{z^1 z_1^1}{1 - z_1^1 x^2}\right|\right),\\
&\frac{z^1\frac{\partial}{\partial z^1} \lrcorner\, \overline{\Omega}}{\frac{\partial}{\partial x^1} \lrcorner\, z^1\frac{\partial}{\partial z^1} \lrcorner\, \overline{\Omega}} \equiv d\left(x^1 + \frac{(x^2)^2 z^1 z_1^1}{2(1 - z_1^1 x^2)} - \frac{z^1}{2z_1^1(1 - z_1^1 x^2)}\right)\\
&\quad \bmod d\left(\ln\left|\frac{z^1 z_1^1}{1 - z_1^1 x^2}\right|\right).
\end{aligned}$$

We put

$$\frac{z^1 z_1^1}{1 - z_1^1 x^2} = \pm\exp(c_1), \qquad x^1 + \frac{(x^2)^2 z^1 z_1^1}{2(1 - z_1^1 x^2)} - \frac{z^1}{2z_1^1(1 - z_1^1 x^2)} = c_2,$$

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for any constant functions c_1 and c_2. Removing the z_1^1 term, and replacing z^1 with u, we then get on one branch the following local solution to (5.3):

$$u = \sqrt{\exp(c_1)}\left(\sqrt{2(x^1 - c_2) + \exp(c_1)(x^2)^2} - \sqrt{\exp(c_1)}x^2\right).$$

6 Pfaffian systems in general

In this section we give in Theorem 8 a generalisation of Theorem 6 that will ultimately allow us to use symmetry to solve the Cauchy problem for Pfaffian systems that are generated by a finite number of linearly independent one-forms, each of constant rank on the domain of definition.

To begin with, we wish to examine a generalisation of the existence result given in Theorem 5. With $\alpha^1, \ldots, \alpha^p \in \Lambda^1(U)$ defined as p linearly independent one-forms, suppose for some $q \geqslant 1$, $Y_1, \ldots, Y_q \in \mathfrak{X}(U)$ generates the Cauchy characteristic space of $\langle \alpha^1, \ldots, \alpha^p, d\alpha^1, \ldots, d\alpha^p \rangle$. Hence for each $1 \leqslant i \leqslant p$ and $1 \leqslant j \leqslant q$,

$$Y_j \lrcorner\, d\alpha^i = \mu^i_{j1}\alpha^1 + \cdots + \mu^i_{jp}\alpha^p,$$

for some $\mu^i_{j1}, \ldots, \mu^i_{jp} \in C^\infty(U)$. Since $A(\langle \alpha^1, \ldots, \alpha^p, d\alpha^1, \ldots, d\alpha^p \rangle)$ is Frobenius integrable, we may write for each $1 \leqslant k, l \leqslant q$,

$$[Y_k, Y_l] = \rho^1_{kl} Y_1 + \cdots + \rho^q_{kl} Y_q, \tag{6.1}$$

for some $\rho^1_{kl}, \ldots, \rho^q_{kl} \in C^\infty(U)$. If, for each i, there exists some non-zero $h_i \in C^\infty(U)$ such that for all j,

$$Y_j(h_i) + h_i \mu^i_{ji} = 0, \tag{6.2}$$

then it is clear that for each i and j, (no sum)

$$\begin{aligned} Y_j \lrcorner\, d\left(h_i \alpha^i\right) &= Y_j \lrcorner\, \left(dh_i \wedge \alpha^i + h_i d\alpha^i\right), \\ &= h_i\left(\mu^i_{j1}\alpha^1 + \cdots + \mu^i_{j(i-1)}\alpha^{i-1} + \mu^i_{j(i+1)}\alpha^{i+1} + \cdots + \mu^i_{jp}\alpha^p\right) \\ &\quad + \left(Y_j(h_i) + h_i\mu^i_{ji}\right)\alpha^i, \\ &\equiv 0 \mod \alpha^1, \ldots, \alpha^{i-1}, \alpha^{i+1}, \ldots, \alpha^p. \end{aligned}$$

For there to exist some non-zero $h_1, \ldots, h_p$, the integrability conditions for the pq partial differential equations in (6.2) can easily be shown to be

$$Y_k(\mu^i_{li}) - Y_l(\mu^i_{ki}) = \rho^1_{kl}\mu^i_{1i} + \cdots + \rho^q_{kl}\mu^i_{qi}. \tag{6.3}$$

Unfortunately we show below that in general, the equations in (6.3) are not satisfied for $q > 1$.

Using the same approach given in Theorem 5, we find that for all i,

$$\mathscr{L}_{[Y_k, Y_l]}\alpha^i = \mathscr{L}_{Y_k}\left(\mu^i_{l1}\alpha^1 + \cdots + \mu^i_{lp}\alpha^p\right) - \mathscr{L}_{Y_l}\left(\mu^i_{k1}\alpha^1 + \cdots + \mu^i_{kp}\alpha^p\right),$$

which expands to give

$$\begin{aligned} \mathscr{L}_{[Y_k, Y_l]}\alpha^i = &\left[Y_k(\mu^i_{l1}) - Y_l(\mu^i_{k1})\right]\alpha^1 + \cdots + \left[Y_k(\mu^i_{li}) - Y_l(\mu^i_{ki})\right]\alpha^i + \ldots \\ &+ \left[Y_k(\mu^i_{lp}) - Y_l(\mu^i_{kp})\right]\alpha^p + \zeta^i, \end{aligned}$$

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where

$$\begin{aligned}\zeta^i := &\left[(\mu^i_{l1}\mu^1_{k1} + \cdots + \mu^i_{lp}\mu^p_{k1}) - (\mu^i_{k1}\mu^1_{l1} + \cdots + \mu^i_{kp}\mu^p_{l1})\right]\alpha^1 + \ldots \\ &+ \left[(\mu^i_{l1}\mu^1_{ki} + \cdots + \mu^i_{lp}\mu^p_{ki}) - (\mu^i_{k1}\mu^1_{li} + \cdots + \mu^i_{kp}\mu^p_{li})\right]\alpha^i + \ldots \\ &+ \left[(\mu^i_{l1}\mu^1_{kp} + \cdots + \mu^i_{lp}\mu^p_{kp}) - (\mu^i_{k1}\mu^1_{lp} + \cdots + \mu^i_{kp}\mu^p_{lp})\right]\alpha^p.\end{aligned}$$

We also have using (6.1),

$$\mathscr{L}_{[Y_k,Y_l]}\alpha^i = \mathscr{L}_{\rho^1_{kl}Y_1+\cdots+\rho^q_{kl}Y_q}\alpha^i,$$

which expands to give

$$\begin{aligned}\mathscr{L}_{[Y_k,Y_l]}\alpha^i = &\left(\rho^1_{kl}\mu^i_{11} + \cdots + \rho^q_{kl}\mu^i_{q1}\right)\alpha^1 + \cdots + \left(\rho^1_{kl}\mu^i_{1i} + \cdots + \rho^q_{kl}\mu^i_{qi}\right)\alpha^i + \ldots \\ &+ \left(\rho^1_{kl}\mu^i_{1p} + \cdots + \rho^q_{kl}\mu^i_{qp}\right)\alpha^p.\end{aligned}$$

Hence the integrability conditions in (6.3) become satisfied if and only if for each i, k and l,

$$\mu^i_{ka}\mu^a_{li} = \mu^i_{la}\mu^a_{ki}, \tag{6.4}$$

with summation only on $1 \leqslant a \leqslant p$. Of course if $q = 1$, then the conditions in (6.4) are met.

Undeterred by this inconvenience for Cauchy characteristic spaces of a higher dimension than one, in what follows we assume the integrability conditions in (6.3) are satisfied. The reason for this is that using symmetries, Lemma 6 easily generalises to give the following four results below.

Lemma 8 *Let $\alpha^1,\ldots,\alpha^p \in \Lambda^1(U)$ be p linearly independent one-forms. Suppose $Y_1,\ldots,Y_q \in \mathfrak{X}(U)$ generate the Cauchy characteristic space of $\langle\alpha^1,\ldots,\alpha^p,d\alpha^1,\ldots,d\alpha^p\rangle$. If, for each $1 \leqslant i \leqslant p$, $Z_i \in \mathfrak{X}(U)$ is a non-trivial symmetry of $\langle\alpha^i,d\alpha^i\rangle$, and $Z_i \lrcorner\, \alpha^j = 0$ for all $1 \leqslant j \leqslant p$ with $j \neq i$, then for each i and $1 \leqslant k \leqslant q$,*

$$Y_k \lrcorner\, d\left(\frac{\alpha^i}{Z_i \lrcorner\, \alpha^i}\right) \equiv 0 \mod \alpha^1,\ldots,\alpha^{i-1},\alpha^{i+1},\ldots,\alpha^p.$$

Proof For each $1 \leqslant i \leqslant p$ and $1 \leqslant k \leqslant q$, we have from the proof of Lemma 6 that

$$Y_k \lrcorner\, d\left(\frac{\alpha^i}{Z_i \lrcorner\, \alpha^i}\right) = \frac{1}{(Z_i \lrcorner\, \alpha^i)^2} Y_k \lrcorner\, Z_i \lrcorner\, (d\alpha^i \wedge \alpha^i).$$

Note from Lemma 5 that Z_i is a non-trivial symmetry of α^i, so $Z_i \lrcorner\, \alpha^i \neq 0$. We have that

$$\begin{aligned}&\left[Y_k \lrcorner\, d\left(\frac{\alpha^i}{Z_i \lrcorner\, \alpha^i}\right)\right] \wedge \alpha^1 \wedge \cdots \wedge \alpha^{i-1} \wedge \alpha^{i+1} \cdots \wedge \alpha^p \\ &\quad = -\frac{1}{(Z_i \lrcorner\, \alpha^i)^2}\left[Z_i \lrcorner\, Y_k \lrcorner\, (d\alpha^i \wedge \alpha^i)\right] \wedge \alpha^1 \wedge \cdots \wedge \alpha^{i-1} \wedge \alpha^{i+1} \cdots \wedge \alpha^p, \\ &\quad = -\frac{1}{(Z_i \lrcorner\, \alpha^i)^2} Z_i \lrcorner\, \left[\left(Y_k \lrcorner\, (d\alpha^i \wedge \alpha^i)\right) \wedge \alpha^1 \wedge \cdots \wedge \alpha^{i-1} \wedge \alpha^{i+1} \cdots \wedge \alpha^p\right],\end{aligned}$$

using the assumption that $Z_i \lrcorner\, \alpha^j = 0$ for all $1 \leqslant j \leqslant p$ with $j \neq i$. Also by assumption,

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$Y_k \in A(\langle \alpha^1, \ldots, \alpha^p, d\alpha^1, \ldots, d\alpha^p \rangle)$, so it follows that

$$\begin{aligned}\left(Y_k \lrcorner (d\alpha^i \wedge \alpha^i)\right) \wedge \alpha^1 \wedge \cdots \wedge \alpha^{i-1} \wedge \alpha^{i+1} \cdots \wedge \alpha^p \\ = \left(Y_k \lrcorner d\alpha^i\right) \wedge \alpha^i \wedge \alpha^1 \wedge \cdots \wedge \alpha^{i-1} \wedge \alpha^{i+1} \cdots \wedge \alpha^p = 0.\end{aligned}$$

Hence for all i and k,

$$Y_k \lrcorner d\left(\frac{\alpha^i}{Z_i \lrcorner \alpha^i}\right) \equiv 0 \mod \alpha^1, \ldots, \alpha^{i-1}, \alpha^{i+1}, \ldots, \alpha^p.$$

□

Lemma 9 *Let $Y \in \mathfrak{X}(U)$ be a non-zero Cauchy characteristic vector field of the differential ideal $\langle \alpha^1, \ldots, \alpha^p, d\alpha^1, \ldots, d\alpha^p \rangle$ generated by some p linearly independent one-forms $\alpha^1, \ldots, \alpha^p \in \Lambda^1(U)$. If, for each $1 \leqslant i \leqslant p$,*

$$Y \lrcorner d\alpha^i \equiv 0 \mod \alpha^1, \ldots, \alpha^{i-1}, \alpha^{i+1}, \ldots, \alpha^p, \tag{6.5}$$

then $Y \lrcorner d\left(\alpha^1 \wedge \cdots \wedge \alpha^p\right) = 0$.

Proof Since

$$d\left(\alpha^1 \wedge \cdots \wedge \alpha^p\right) = \left(d\alpha^1 \wedge \alpha^2 \wedge \cdots \wedge \alpha^p\right) + \cdots + \left(d\alpha^p \wedge \alpha^1 \wedge \cdots \wedge \alpha^{p-1}\right),$$

and $Y \lrcorner \alpha^i = 0$ for all $1 \leqslant i \leqslant p$, we therefore have that

$$\begin{aligned}Y \lrcorner d\left(\alpha^1 \wedge \cdots \wedge \alpha^p\right) = \left((Y \lrcorner d\alpha^1) \wedge \alpha^2 \wedge \cdots \wedge \alpha^p\right) + \ldots \\ + \left((Y \lrcorner d\alpha^p) \wedge \alpha^1 \wedge \cdots \wedge \alpha^{p-1}\right).\end{aligned}$$

Then because of (6.5), the result is now obvious. □

Using Lemma 9, we have the following result:

Theorem 7 *Let $Y \in \mathfrak{X}(U)$ be a non-zero Cauchy characteristic vector field of the differential ideal $\langle \alpha^1, \ldots, \alpha^p, d\alpha^1, \ldots, d\alpha^p \rangle$ generated by some p linearly independent $\alpha^1, \ldots, \alpha^p \in \Lambda^1(U)$. Suppose that for each $1 \leqslant i \leqslant p$,*

$$Y \lrcorner d\alpha^i \equiv 0 \mod \alpha^1, \ldots, \alpha^{i-1}, \alpha^{i+1}, \ldots, \alpha^p.$$

Further suppose there exist $p-1$ linearly independent vector fields $Z_1, \ldots, Z_{p-1} \in \mathfrak{X}(U)$, each not in $A(\langle \alpha^1, \ldots, \alpha^p, d\alpha^1, \ldots, d\alpha^p \rangle)$, such that with $\Omega := \alpha^1 \wedge \cdots \wedge \alpha^p$, we have

$$\begin{aligned}\mathscr{L}_{Z_{p-1}} \Omega &= \lambda_{p-1} \Omega, \\ \mathscr{L}_{Z_{p-2}}(Z_{p-1} \lrcorner \Omega) &= \lambda_{p-2}(Z_{p-1} \lrcorner \Omega), \\ &\vdots \\ \mathscr{L}_{Z_1}(Z_2 \lrcorner \ldots \lrcorner Z_{p-1} \lrcorner \Omega) &= \lambda_1(Z_2 \lrcorner \ldots \lrcorner Z_{p-1} \lrcorner \Omega),\end{aligned} \tag{6.6}$$

for some $\lambda_1, \ldots, \lambda_{p-1} \in C^\infty(U)$. Then

$$Z_1 \lrcorner \ldots \lrcorner Z_{p-1} \lrcorner \Omega = \rho_1 \alpha^1 + \cdots + \rho_p \alpha^p \neq 0,$$

for some $\rho_1, \ldots, \rho_p \in C^\infty(U)$ such that $Y \lrcorner d\left(\rho_1 \alpha^1 + \cdots + \rho_p \alpha^p\right) = 0$.

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Proof We first rewrite the equations in (6.6) in the following form:

$$\begin{aligned} Z_{p-1}\lrcorner\, d\Omega + d\left(Z_{p-1}\lrcorner\, \Omega\right) &= \lambda_{p-1}\Omega, \\ Z_{p-2}\lrcorner\, d(Z_{p-1}\lrcorner\, \Omega) + d\left(Z_{p-2}\lrcorner\, Z_{p-1}\lrcorner\, \Omega\right) &= \lambda_{p-2}(Z_{p-1}\lrcorner\, \Omega), \\ &\vdots \\ Z_1\lrcorner\, d(X_2\lrcorner\, \ldots\lrcorner\, Z_{p-1}\lrcorner\, \Omega) + d\left(Z_1\lrcorner\, \ldots Z_{p-1}\lrcorner\, \Omega\right) &= \lambda_1(Z_2\lrcorner\, \ldots\lrcorner\, Z_{p-1}\lrcorner\, \Omega). \end{aligned} \tag{6.7}$$

Now Lemma 9 gives

$$Y\lrcorner\, Z_1\lrcorner\, \ldots\lrcorner\, Z_{p-1}\lrcorner\, d\Omega = 0. \tag{6.8}$$

We can then use (6.8) and the first equation in (6.7) to show that

$$Y\lrcorner\, Z_1\lrcorner\, \ldots\lrcorner\, Z_{p-1}\lrcorner\, d\Omega = Y\lrcorner\, Z_1\lrcorner\, \ldots Z_{p-2}\lrcorner\, \left[\lambda_{p-1}\Omega - d\left(Z_{p-1}\lrcorner\, \Omega\right)\right] = 0.$$

Since $Y \in A\left(\langle \alpha^1, \ldots, \alpha^p, d\alpha^1, \ldots, d\alpha^p\rangle\right)$ implies $Y\lrcorner\, \Omega = 0$, we therefore have

$$Y\lrcorner\, Z_1\lrcorner\, \ldots Z_{p-2}\lrcorner\, d\left(Z_{p-1}\lrcorner\, \Omega\right) = 0. \tag{6.9}$$

We can repeat the above procedure, but this time using (6.9) and the second equation in (6.7) to show that

$$Y\lrcorner\, Z_1\lrcorner\, \ldots Z_{p-3}\lrcorner\, d\left(Z_{p-2}\lrcorner\, Z_{p-1}\lrcorner\, \Omega\right) = 0.$$

Continuing in this fashion, we eventually obtain

$$Y\lrcorner\, d\left(Z_1\lrcorner\, \ldots Z_{p-1}\lrcorner\, \Omega\right) = 0.$$

Since $Z_1, \ldots, Z_{p-1}$ are all linearly independent, and each not in the Cauchy characteristic space of $\langle \alpha^1, \ldots, \alpha^p, d\alpha^1, \ldots, d\alpha^p\rangle$, we have that

$$Z_1\lrcorner\, \ldots Z_{p-1}\lrcorner\, \Omega \neq 0.$$

Hence

$$Z_1\lrcorner\, \ldots Z_{p-1}\lrcorner\, \Omega = \rho_1\alpha^1 + \cdots + \rho_p\alpha^p,$$

for some $\rho_1, \ldots, \rho_p \in C^\infty(U)$, with at least one of $\rho_1, \ldots, \rho_p$ being not identically zero. □

In our work, it is the following corollary to Theorem 7 that will be useful.

Corollary 2 *Let $\alpha^1, \ldots, \alpha^p \in \Lambda^1(U)$ be some p linearly independent one-forms and suppose $A(\langle \alpha^1, \ldots, \alpha^p, d\alpha^1, \ldots, d\alpha^p\rangle)$ is generated by some $Y_1, \ldots, Y_q \in \mathfrak{X}(U)$. Further suppose that for each $1 \leqslant i \leqslant p$ and $1 \leqslant j \leqslant q$,*

$$Y_j\lrcorner\, d\alpha^i \equiv 0 \mod \alpha^1, \ldots, \alpha^{i-1}, \alpha^{i+1}, \ldots, \alpha^p.$$

Define $\Omega := \alpha^1 \wedge \cdots \wedge \alpha^p$. If, for each i, there exist $p-1$ linearly independent vector fields $Z_{i1}, \ldots, Z_{i(p-1)} \in \mathfrak{X}(U)$ each not in $A(\langle \alpha^1, \ldots, \alpha^p, d\alpha^1, \ldots, d\alpha^p\rangle)$ that satisfy the solvable

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symmetry structure property

$$\mathscr{L}_{Z_{i(p-1)}}\Omega = \lambda_{i(p-1)}\Omega,$$
$$\mathscr{L}_{Z_{i(p-2)}}(Z_{i(p-1)}\lrcorner\,\Omega) = \lambda_{i(p-2)}(Z_{i(p-1)}\lrcorner\,\Omega),$$
$$\vdots$$
$$\mathscr{L}_{Z_{i1}}(Z_{i2}\lrcorner\,\ldots\lrcorner\,Z_{i(p-1)}\lrcorner\,\Omega) = \lambda_{i1}(Z_{i2}\lrcorner\,\ldots\lrcorner\,Z_{i(p-1)}\lrcorner\,\Omega),$$

for some $\lambda_{i1},\ldots,\lambda_{i(p-1)} \in C^\infty(U)$*, and if our symmetries are such that the one-forms* $\beta^1,\ldots,\beta^p$ *defined by*

$$\beta^i := Z_{i1}\lrcorner\,\ldots\lrcorner\,Z_{i(p-1)}\lrcorner\,\Omega,$$

are linearly independent on U*, then*

$$\langle\alpha^1,\ldots,\alpha^p,d\alpha^1,\ldots,d\alpha^p\rangle = \langle\beta^1,\ldots,\beta^p,d\beta^1,\ldots,d\beta^p\rangle,$$

such that for all i *and* j*,* $Y_j\lrcorner\,\beta^i = 0 = Y_j\lrcorner\,d\beta^i$*.*

We may summarise some of the above results of this section in the following way: Given $A(\langle\alpha^1,\ldots,\alpha^p,d\alpha^1,\ldots,d\alpha^p\rangle)$ generated by some $Y_1,\ldots,Y_q \in \mathfrak{X}(U)$, we look to first apply Lemma 8, so that for each $1 \leqslant i \leqslant p$ and $1 \leqslant j \leqslant q$, (no sum)

$$Y_j\lrcorner\,d(h_i\alpha^i) \equiv 0 \mod \alpha^1,\ldots,\alpha^{i-1},\alpha^{i+1},\ldots,\alpha^p,$$

for some non-zero $h_1,\ldots,h_p \in C^\infty(U)$. Then we look to use Corollary 2 to obtain some $\beta^1,\ldots,\beta^p \in \Lambda^1(U)$ such that

$$\langle\alpha^1,\ldots,\alpha^p,d\alpha^1,\ldots,d\alpha^p\rangle = \langle\beta^1,\ldots,\beta^p,d\beta^1,\ldots,d\beta^p\rangle,$$

with the property that for all i and j, $Y_j\lrcorner\,\beta^i = 0 = Y_j\lrcorner\,d\beta^i$. We will illustrate this process in the example contained in the next section.

We now present the main result of this section, Theorem 8. In what follows, we assume $\alpha^1,\ldots,\alpha^p \in \Lambda^1(U)$ are p linearly independent one-forms, and the Cauchy characteristic space of $\langle\alpha^1,\ldots,\alpha^p,d\alpha^1,\ldots,d\alpha^p\rangle$ is generated by $Y_1,\ldots,Y_q \in \mathfrak{X}(U)$, so that the Cartan system of $\langle\alpha^1,\ldots,\alpha^p,d\alpha^1,\ldots,d\alpha^p\rangle$ is generated by some $n-q$ linearly independent one-forms $\sigma^1,\ldots,\sigma^{n-q} \in \Lambda^1(U)$. Suppose also that we have been able to apply Lemma 8, and Corollary 2, so that for all $1 \leqslant i \leqslant p$ and $1 \leqslant j \leqslant q$, $Y_j\lrcorner\,d\alpha^i = 0$. Theorem 8, when $q = 1$, once again provides us with a symmetry approach to the typical Cauchy problem when the Cauchy characteristic space is one-dimensional.

Theorem 8 *Define* $\Omega := \sigma^1\wedge\cdots\wedge\sigma^{n-q}$ *and let* $X_1,\ldots,X_{n-q} \in \mathfrak{X}(U)$ *be a solvable symmetry structure for* $A(\langle\Omega\rangle)$*. There exists a* $(n-q-p)$*-dimensional submodule* $\mathfrak{X}^{n-q-p}(U)$ *of* $\mathfrak{X}(U)$ *over* $\mathscr{F}(\gamma^a)$ *such that if we are given a* v*-dimensional regular submanifold of* U *whose tangent space is spanned by some* v *linearly independent vector fields* $Z_1,\ldots,Z_v \in \mathfrak{X}^{n-q-p}(U)$*, then for each* $1 \leqslant j \leqslant q$ *and* $1 \leqslant l \leqslant v$*,* $[Y_j,Z_l] \equiv 0$ *mod* $Y_1,\ldots,Y_q$*. Moreover, the distribution* $sp\{Y_1,\ldots,Y_q,Z_1,\ldots,Z_v\}$ *is Frobenius integrable and generates a* $(q+v)$*-dimensional integral manifold of the differential ideal* $\langle\alpha^1,\ldots,\alpha^p,d\alpha^1,\ldots,d\alpha^p\rangle$*.*

Proof Since the Cauchy characteristic space of $\langle\alpha^1,\ldots,\alpha^p,d\alpha^1,\ldots,d\alpha^p\rangle$ is Frobenius inte-

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grable, we have $d\Omega \equiv 0 \bmod \Omega$. Then applying Theorem 2 with $\omega^1,\dots,\omega^{n-q}$ defined as in the theorem yields $\{\omega^1,\dots,\omega^{n-q}\}$ dual to $\{X_1,\dots,X_{n-q}\}$ such that

$$\begin{aligned}
\omega^1 &= d\gamma^1,\\
\omega^2 &= d\gamma^2 - X_1(\gamma^2)d\gamma^1,\\
\omega^3 &= d\gamma^3 - X_2(\gamma^3)(d\gamma^2 - X_1(\gamma^2)d\gamma^1) - X_1(\gamma^3)d\gamma^1,\\
&\vdots\\
\omega^{n-q} &\equiv d\gamma^{n-q} \mod d\gamma^1,\dots,d\gamma^{n-q-1},
\end{aligned}$$

for some functionally independent $\gamma^1,\dots,\gamma^{n-q} \in C^\infty(U)$. Now since the Cartan system of any differential ideal contains all the one-forms that are in the ideal, we therefore have for each $1 \leqslant s \leqslant p$,

$$\begin{aligned}
\alpha^s &= (X_1 \lrcorner\, \alpha^s)\,\omega^1 + \cdots + \left(X_{n-q} \lrcorner\, \alpha^s\right)\omega^{n-q},\\
&= \lambda^s_1 d\gamma^1 + \cdots + \lambda^s_{n-q} d\gamma^{n-q},
\end{aligned}$$

for some $\lambda^s_1,\dots,\lambda^s_{n-q} \in C^\infty(U)$. By assumption, $Y_j \lrcorner\, \alpha^s = 0 = Y_j \lrcorner\, d\alpha^s$ for each $1 \leqslant j \leqslant q$ and s. This gives

$$\mathscr{L}_{Y_j}\alpha^s = 0. \tag{6.10}$$

But we know from conclusion 3 in Lemma 1 that $Y_j \lrcorner\, d\gamma^i = 0$, for each $1 \leqslant i \leqslant n-q$ and j, so (6.10) implies

$$Y_j(\lambda^s_1)d\gamma^1 + \cdots + Y_j(\lambda^s_{n-q})d\gamma^{n-q} = 0.$$

Since $d\gamma^1,\dots,d\gamma^{n-q}$ are linearly independent, we get

$$Y_j(\lambda^s_i) = 0,$$

for all i, j and s. By the same argument used in the proof of Theorem 6, we can then conclude that $\alpha^1,\dots,\alpha^p$ only depend on $\gamma^1,\dots,\gamma^{n-q}$ and their exterior derivatives.

The rest of the proof continues to be very similar to that of Theorem 6. Define the vector fields $\overline{X}_1,\dots,\overline{X}_{n-q}$ as in (3.2). Then from Lemma 2 we have that $\{\overline{X}_1,\dots,\overline{X}_{n-q}\}$ is dual to $\{d\gamma^1,\dots,d\gamma^{n-q}\}$. Using this and the fact that each α^s only depends on $\gamma^1,\dots,\gamma^{n-q}$ and their exterior derivatives, it is then obvious that the kernel of $\alpha^1 \wedge \cdots \wedge \alpha^p$ (which equals the distribution annihilated by each α^s) is spanned by the Cauchy characteristic fields $Y_1,\dots,Y_q$, and some $n-q-p$ linearly independent vector fields $W_1,\dots,W_{n-q-p} \in \mathfrak{X}(U)$ that are linear combinations of $\overline{X}_1,\dots,\overline{X}_{n-q}$ with coefficients in $\mathscr{F}(\gamma^a)$. Since we have expressed α^s solely in terms of $\gamma^1,\dots,\gamma^{n-q}$ and their exterior derivatives, Lemma 2 now implies that calculating $W_1,\dots,W_{n-q-p}$ is a straightforward exercise. From Theorem 4 and conclusion 3 in Lemma 1, it now follows that for each j and $1 \leqslant k \leqslant n-q-p$, $[Y_j, W_k] \equiv 0 \bmod Y_1,\dots,Y_q$.

Now let $\mathfrak{X}^{n-q-p}(U)$ be generated by $W_1,\dots,W_{n-q-p}$, and suppose we are given some v-dimensional regular submanifold of U whose tangent space is spanned by v linearly independent vector fields $Z_1,\dots,Z_v \in \mathfrak{X}^{n-q-p}(U)$. Then for each $1 \leqslant l \leqslant v$,

$$Z_l = \mu^1_l W_1 + \cdots + \mu^{n-q-p}_l W_{n-q-p},$$

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for some $\mu_l^1, \ldots, \mu_l^{n-q-p} \in \mathscr{F}(\gamma^a)$. From conclusion 3 in Lemma 1 we then get that for each j and $l, [Y_j, Z_l] \equiv 0 \bmod Y_1, \ldots, Y_q$. Since for each l, $Z_l \lrcorner \alpha = 0$ and for each j, the Y_j are linearly independent of $Z_1, \ldots, Z_v$, it follows that $sp\{Y_1, \ldots, Y_q, Z_1, \ldots, Z_v\}$ is a $(q+v)$-dimensional Frobenius integrable distribution that generates a $(q+v)$-dimensional integral manifold of $\langle \alpha^1, \ldots, \alpha^p \rangle$, and hence an integral manifold of $\langle \alpha^1, \ldots, \alpha^p, d\alpha^1, \ldots, d\alpha^p \rangle$. □

As in Theorem 6 earlier, if in applying Theorem 8 to the Cauchy problem (i.e. $q = 1$) we are given the Cauchy data only in the form of some $Z_1, \ldots, Z_v \in \mathfrak{X}^{n-1-p}(U)$ (which are constructed as linear combinations of the symmetries forming the solvable symmetry structure) that are closed under the Lie bracket, then the theorem only guarantees uniqueness up to foliation. Therefore we obtain a unique $(v+1)$-dimensional foliation of U, where the $(v+1)$-dimensional leaves depend on $n-v-1$ constants.

In the next section we discuss two PDE applications of Theorem 8 and give one example.

7 Two applications

This section examines two types of PDE problems for which there exist non-zero Cauchy characteristic vector fields. The first of these considers a single first order (possibly non-linear) PDE of one dependent variable and an arbitrary number of independent variables, while the second deals with a special class of systems of two second order PDEs of one dependent variable and two independent variables.

Beginning with the first situation, suppose we are given a first order PDE of one dependent variable and n independent variables of the form

$$\frac{\partial u}{\partial x^n} = F\left(x^1, \ldots, x^n, u, \frac{\partial u}{\partial x^1}, \ldots, \frac{\partial u}{\partial x^{n-1}}\right), \tag{7.1}$$

for smooth F. We work in the first jet bundle $J^1(U^n, V^1)$, where U^n has local coordinates $x^1, \ldots, x^n$ and V^1 has the local coordinate z^1. We express the PDE in (7.1) as

$$z_n^1 = F(x^1, \ldots, x^n, z^1, z_1^1, \ldots, z_{n-1}^1), \tag{7.2}$$

replacing the dependent variable u with the coordinate z^1. The Pfaffian equation corresponding to (7.1) on the regular submanifold $M^{2n} \subset J^1(U^n, V^1)$ defined by (7.2), with coordinates $x^1, \ldots, x^n, z^1, z_1^1, \ldots, z_{n-1}^1$ is

$$\alpha := dz^1 - z_1^1 dx^1 - \cdots - z_{n-1}^1 dx^{n-1} - F dx^n = 0. \tag{7.3}$$

From Stormark [17] we have the following lemma:

Lemma 10 *With α defined as in (7.3), the Cauchy characteristic space of $\langle \alpha, d\alpha \rangle$ is one-dimensional and generated by*

$$Y := \frac{\partial}{\partial x^n} - \sum_{i=1}^{n-1} \frac{\partial F}{\partial z_i^1} \frac{\partial}{\partial x^i} + \left(F - \sum_{i=1}^{n-1} z_i^1 \frac{\partial F}{\partial z_i^1}\right) \frac{\partial}{\partial z^1} + \sum_{i=1}^{n-1} \left(\frac{\partial F}{\partial x^i} + z_i^1 \frac{\partial F}{\partial z^1}\right) \frac{\partial}{\partial z_i^1}.$$

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Using Lemma 10, we may therefore apply Theorem 8 to find local solutions of the first order PDE in (7.1). We will not provide an example of this. Instead, we will give an example of the following material that deals with second order PDEs.

Consider the following pair of second order PDEs:

$$\begin{aligned}\frac{\partial^2 u}{\partial (x^1)^2} &= F_1\left(x^1, x^2, u, \frac{\partial u}{\partial x^1}, \frac{\partial u}{\partial x^2}\frac{\partial^2 u}{\partial x^1 \partial x^2}\right),\\ \frac{\partial^2 u}{\partial (x^2)^2} &= F_2\left(x^1, x^2, u, \frac{\partial u}{\partial x^1}, \frac{\partial u}{\partial x^2}\frac{\partial^2 u}{\partial x^1 \partial x^2}\right),\end{aligned} \tag{7.4}$$

for some smooth functions F_1 and F_2. This time, we work in the second jet bundle $J^2(U^2, V^1)$, and represent the PDEs in (7.4) by

$$\begin{aligned}z^1_{11} &= F_1(x^1, x^2, z^1, z^1_1, z^1_2, z^1_{12}),\\ z^1_{22} &= F_2(x^1, x^2, z^1, z^1_1, z^1_2, z^1_{12}),\end{aligned} \tag{7.5}$$

replacing the dependent variable u with the coordinate z^1. Define $M^6 \subset J^2(U^2, V^1)$ as the six-dimensional regular submanifold defined by (7.5). On M^6 with coordinates $x^1, x^2, z^1, z^1_1, z^1_2, z^1_{12}$, the Pfaffian system corresponding to (7.4) is generated by the one-forms

$$\begin{aligned}\alpha^1 &:= dz^1 - z^1_1 dx^1 - z^1_2 dx^2,\\ \alpha^2 &:= dz^1_1 - F_1 dx^1 - z^1_{12} dx^2,\\ \alpha^3 &:= dz^1_2 - z^1_{12} dx^1 - F_2 dx^2.\end{aligned} \tag{7.6}$$

Note that $d\alpha^1 \equiv 0 \bmod \alpha^2, \alpha^3$. We will use the following result also found in Stormark [17]:

Theorem 9 *With $\alpha^1, \alpha^2, \alpha^3$ defined as in* (7.6)*, the Cauchy characteristic space of $\langle \alpha^1, \alpha^2, \alpha^3, d\alpha^2, d\alpha^3 \rangle$ is one-dimensional and generated by*

$$\begin{aligned}Y :=& \frac{\partial}{\partial x^1} - \frac{\partial F_1}{\partial z^1_{12}}\frac{\partial}{\partial x^2} + \left(z^1_1 - z^1_2\frac{\partial F_1}{\partial z^1_{12}}\right)\frac{\partial}{\partial z^1} + \left(F_1 - z^1_{12}\frac{\partial F_1}{\partial z^1_{12}}\right)\frac{\partial}{\partial z^1_1}\\ &+ \left(z^1_{12} - F_2\frac{\partial F_1}{\partial z^1_{12}}\right)\frac{\partial}{\partial z^1_2} + \left(\frac{\partial F_1}{\partial x^2} + z^1_2\frac{\partial F_1}{\partial z^1} + z^1_{12}\frac{\partial F_1}{\partial z^1_1} + F_2\frac{\partial F_1}{\partial z^1_2}\right)\frac{\partial}{\partial z^1_{12}},\end{aligned}$$

if and only if

$$\frac{\partial F_1}{\partial z^1_{12}}\frac{\partial F_2}{\partial z^1_{12}} = 1,$$

and

$$\begin{aligned}&\frac{\partial F_1}{\partial x^2} + z^1_2\frac{\partial F_1}{\partial z^1} + z^1_{12}\frac{\partial F_1}{\partial z^1_1} + F_2\frac{\partial F_1}{\partial z^1_2}\\ &+\frac{\partial F_1}{\partial z^1_{12}}\left(\frac{\partial F_2}{\partial x^1} + z^1_1\frac{\partial F_2}{\partial z^1} + F_1\frac{\partial F_2}{\partial z^1_1} + z^1_{12}\frac{\partial F_2}{\partial z^1_2}\right) = 0.\end{aligned}$$

We close this section with the following PDE example.

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Example 3 Consider the system of linear second order PDEs

$$\frac{\partial^2 u}{\partial (x^1)^2} = \frac{\partial^2 u}{\partial x^1 \partial x^2} + \frac{\partial u}{\partial x^1} - \frac{\partial u}{\partial x^2}, \qquad \frac{\partial^2 u}{\partial (x^2)^2} = \frac{\partial^2 u}{\partial x^1 \partial x^2}, \tag{7.7}$$

for which we now use Theorem 8 to find a local solution. On $M^6 \subset J^2(U^2, V^1)$ defined by $z^1_{11} = z^1_{12} + z^1_1 - z^1_2$ and $z^1_{22} = z^1_{12}$, we let the differential ideal I defined on M^6 be generated by the one-forms

$$\begin{aligned}
\alpha^1 &:= dz^1 - z^1_1 dx^1 - z^1_2 dx^2, \\
\alpha^2 &:= dz^1_1 - \left(z^1_{12} + z^1_1 - z^1_2\right) dx^1 - z^1_{12} dx^2, \\
\alpha^3 &:= dz^1_2 - z^1_{12} dx^1 - z^1_{12} dx^2,
\end{aligned}$$

and the exterior derivatives of α^2 and α^3. Theorem 9 can be applied to (7.7). Doing so, we find that the Cauchy characteristic space of $I := \langle \alpha^1, \alpha^2, \alpha^3, d\alpha^2, d\alpha^3 \rangle$ is generated by

$$Y = \frac{\partial}{\partial x^1} - \frac{\partial}{\partial x^2} + \left(z^1_1 - z^1_2\right) \frac{\partial}{\partial z^1} + \left(z^1_1 - z^1_2\right) \frac{\partial}{\partial z^1_1}. \tag{7.8}$$

Our first step is to ensure that for all $1 \leqslant i \leqslant 3$,

$$Y \lrcorner\, d\alpha^i \equiv 0 \mod \alpha^1, \ldots, \alpha^{i-1}, \alpha^{i+1}, \ldots, \alpha^3.$$

We have

$$\begin{aligned}
Y \lrcorner\, d\alpha^1 &= \alpha^2 - \alpha^3, \\
Y \lrcorner\, d\alpha^2 &= \alpha^2 - \alpha^3, \\
Y \lrcorner\, d\alpha^3 &= 0,
\end{aligned}$$

so we only need to scale α^2 by some non-zero function f. Instead of using Lemma 8, we obtain by expanding $Y \lrcorner\, d(f\alpha^2)$ that simple inspection then yields

$$Y \lrcorner\, d\left(\exp\left(x^2\right)\alpha^2\right) = -\exp\left(x^2\right)\alpha^3.$$

Therefore, we define $\overline{\alpha}^1 := \alpha^1$, $\overline{\alpha}^2 := \exp\left(x^2\right)\alpha^2$, and $\overline{\alpha}^3 := \alpha^3$. This gives

$$\begin{aligned}
Y \lrcorner\, d\overline{\alpha}^1 &= \exp\left(-x^2\right)\overline{\alpha}^2 - \overline{\alpha}^3, \\
Y \lrcorner\, d\overline{\alpha}^2 &= -\exp\left(x^2\right)\overline{\alpha}^3, \\
Y \lrcorner\, d\overline{\alpha}^3 &= 0.
\end{aligned}$$

Our second step is to look to apply Corollary 2. It is easy to see that $\frac{\partial}{\partial x^2}$ is a non-trivial symmetry of $\overline{\alpha}^1 \wedge \overline{\alpha}^2 \wedge \overline{\alpha}^3$, and that $\frac{\partial}{\partial x^1}$ is a non-trivial symmetry of $\frac{\partial}{\partial x^2} \lrcorner \left(\overline{\alpha}^1 \wedge \overline{\alpha}^2 \wedge \overline{\alpha}^2\right)$. Thus,

$$\begin{aligned}
\frac{\partial}{\partial x^1} \lrcorner \frac{\partial}{\partial x^2} \lrcorner \left(\overline{\alpha}^1 \wedge \overline{\alpha}^2 \wedge \overline{\alpha}^3\right) = &-\exp\left(x^2\right) z^1_{12} \left(z^1_1 - z^1_2\right) \overline{\alpha}^1 + z^1_{12}(z^1_1 - z^1_2)\overline{\alpha}^2 \\
&+ \exp\left(x^2\right) \left(z^1_2 z^1_{12} + z^1_1 z^1_2 - (z^1_2)^2 - z^1_1 z^1_{12}\right) \overline{\alpha}^3,
\end{aligned}$$

with the property that

$$Y \lrcorner\, d\left(\frac{\partial}{\partial x^1} \lrcorner \frac{\partial}{\partial x^2} \lrcorner \left(\overline{\alpha}^1 \wedge \overline{\alpha}^2 \wedge \overline{\alpha}^3\right)\right) = 0.$$

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We need another pair of symmetries. We have that $\frac{\partial}{\partial z^1}$ is a non-trivial symmetry of $\overline{\alpha}^1 \wedge \overline{\alpha}^2 \wedge \overline{\alpha}^3$, and that $\frac{\partial}{\partial x^2}$ is a non-trivial symmetry of $\frac{\partial}{\partial z^1} \lrcorner \left(\overline{\alpha}^1 \wedge \overline{\alpha}^2 \wedge \overline{\alpha}^2\right)$. Hence

$$\frac{\partial}{\partial x^2} \lrcorner \frac{\partial}{\partial z^1} \lrcorner \left(\overline{\alpha}^1 \wedge \overline{\alpha}^2 \wedge \overline{\alpha}^3\right) = z^1_{12}\overline{\alpha}^2 - \exp(x^2) z^1_{12}\overline{\alpha}^3,$$

with

$$Y \lrcorner d\left(\frac{\partial}{\partial x^2} \lrcorner \frac{\partial}{\partial z^1} \lrcorner \left(\overline{\alpha}^1 \wedge \overline{\alpha}^2 \wedge \overline{\alpha}^3\right)\right) = 0.$$

If we now make the following non-singular linear transformation,

$$\begin{aligned}
\widehat{\alpha}^1 &:= -\exp(x^2) z^1_{12}\left(z^1_1 - z^1_2\right)\overline{\alpha}^1 + z^1_{12}(z^1_1 - z^1_2)\overline{\alpha}^2 \\
&\quad + \exp(x^2)\left(z^1_2 z^1_{12} + z^1_1 z^1_2 - (z^1_2)^2 - z^1_1 z^1_{12}\right)\overline{\alpha}^3, \\
\widehat{\alpha}^2 &:= z^1_{12}\overline{\alpha}^2 - \exp(x^2) z^1_{12}\overline{\alpha}^3, \\
\widehat{\alpha}^3 &:= \overline{\alpha}^3,
\end{aligned}$$

then $Y \lrcorner \widehat{\alpha}^i = 0 = Y \lrcorner d\widehat{\alpha}^i$ for all $1 \leqslant i \leqslant 3$. Explicitly,

$$\begin{aligned}
\widehat{\alpha}^1 &= \exp(x^2)(z^1_2 - z^1_1)\left(z^1_{12} dz^1 - z^1_{12} dz^1_1 - (z^1_2 - z^1_{12}) dz^1_2\right), \\
\widehat{\alpha}^2 &= \exp(x^2) z^1_{12}\left(dz^1_1 - dz^1_2 - (z^1_1 - z^1_2) dx^1\right), \\
\widehat{\alpha}^3 &= dz^1_2 - z^1_{12} dx^1 - z^1_{12} dx^2.
\end{aligned}$$

Applying Theorem 8, it is easy to show that on M^6, the Cartan system of I is generated by $\widehat{\alpha}^1, \widehat{\alpha}^2, \widehat{\alpha}^3, dz^1_2, dz^1_{12}$. We define

$$\Omega := \widehat{\alpha}^1 \wedge \widehat{\alpha}^2 \wedge \widehat{\alpha}^3 \wedge dz^1_2 \wedge dz^1_{12}.$$

Since the Cauchy characteristic space of I is Frobenius integrable, it follows that $d\Omega \equiv 0 \mod \Omega$. Therefore, using Theorem 2 with the solvable symmetry structure $X_1, \ldots, X_5 \in \mathfrak{X}(M^6)$ for Ω found using `DIMSYM`, where

$$\begin{aligned}
X_1 &:= \frac{\partial}{\partial z^1_1} + \frac{\partial}{\partial z^1_2}, \qquad X_2 := \frac{1}{(z^1_{12})^3}\frac{\partial}{\partial z^1_{12}}, \qquad X_3 := \frac{\partial}{\partial x^1}, \\
X_4 &:= \frac{\partial}{\partial x^2}, \qquad X_5 := \frac{\partial}{\partial z^1},
\end{aligned}$$

we find

$$\begin{aligned}
\omega^1 &= \frac{X_2 \lrcorner X_3 \lrcorner X_4 \lrcorner X_5 \lrcorner \Omega}{X_1 \lrcorner X_2 \lrcorner X_3 \lrcorner X_4 \lrcorner X_5 \lrcorner \Omega} = dz^1_2, \\
\omega^2 &= \frac{X_1 \lrcorner X_3 \lrcorner X_4 \lrcorner X_5 \lrcorner \Omega}{X_2 \lrcorner X_1 \lrcorner X_3 \lrcorner X_4 \lrcorner X_5 \lrcorner \Omega} = d\left(\frac{(z^1_{12})^4}{4}\right), \\
\omega^3 &= \frac{X_1 \lrcorner X_2 \lrcorner X_4 \lrcorner X_5 \lrcorner \Omega}{X_3 \lrcorner X_1 \lrcorner X_2 \lrcorner X_4 \lrcorner X_5 \lrcorner \Omega} = d\left(x^1 - \ln|z^1_2 - z^1_1|\right), \\
\omega^4 &= \frac{X_1 \lrcorner X_2 \lrcorner X_3 \lrcorner X_5 \lrcorner \Omega}{X_4 \lrcorner X_1 \lrcorner X_2 \lrcorner X_3 \lrcorner X_5 \lrcorner \Omega} = d\left(x^2 + \ln|z^1_2 - z^1_1|\right), \\
\omega^5 &= \frac{X_1 \lrcorner X_2 \lrcorner X_3 \lrcorner X_4 \lrcorner \Omega}{X_5 \lrcorner X_1 \lrcorner X_2 \lrcorner X_3 \lrcorner X_4 \lrcorner \Omega} = d(z^1 - z^1_1 + z^1_2).
\end{aligned}$$

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Putting

$$\gamma^1 := z_2^1, \qquad \gamma^2 := \frac{(z_{12}^1)^4}{4}, \qquad \gamma^3 := x^1 - \ln|z_2^1 - z_1^1|,$$
$$\gamma^4 := x^2 + \ln|z_2^1 - z_1^1|, \qquad \gamma^5 := z^1 - z_1^1 + z_2^1,$$

and assuming we are working in a neighbourhood where $z_2^1 > z_1^1$ and $z_{12}^1 > 0$ (the other three cases – excluding $z_1^1 = z_2^1$ and $z_{12}^1 = 0$ – can be treated similarly), we obtain

$$\begin{aligned}
\widehat{\alpha}^1 &= \left(4\gamma^2\right)^{\frac{1}{4}} \exp\left(\gamma^4\right) d\gamma^5 - \gamma^1 \exp\left(\gamma^4\right) d\gamma^1,\\
\widehat{\alpha}^2 &= \left(4\gamma^2\right)^{\frac{1}{4}} \exp\left(\gamma^4\right) d\gamma^3,\\
\widehat{\alpha}^3 &= d\gamma^1 - \left(4\gamma^2\right)^{\frac{1}{4}} d\gamma^4 - \left(4\gamma^2\right)^{\frac{1}{4}} d\gamma^3.
\end{aligned} \tag{7.9}$$

Following the procedure in Theorem 8, we find that for all $1 \leqslant i \leqslant 5$, $\overline{X}_i = X_i$, because $d\omega^i = 0$. Therefore, using the annihilating space of (7.9), we require the vector field Z describing our Cauchy data to be of the form

$$Z := \lambda_1 \left(\left(4\gamma^2\right)^{\frac{1}{4}} X_1 + X_4 + \gamma^1 X_5\right) + \lambda_2 X_2,$$

for any choice of λ_1, λ_2 that are smooth functions of $\gamma^1, \ldots, \gamma^5$. We suppose that $\lambda_1 = 1$ and $\lambda_2 = 0$, so that

$$Z = \frac{\partial}{\partial x^2} + \frac{\partial}{\partial z^1} + z_{12}^1 \frac{\partial}{\partial z_1^1} + z_{12}^1 \frac{\partial}{\partial z_1^1}.$$

On M^6, the annihilating space of $sp\{Y, Z\}$ is spanned by $\alpha^1, \alpha^2, \alpha^3, dz_{12}^1$. Defining $\overline{\Omega} := \alpha^1 \wedge \alpha^2 \wedge \alpha^3 \wedge dz_{12}^1$, and with the solvable symmetry structure $\Gamma_1, \ldots, \Gamma_4 \in \mathfrak{X}(M^6)$ for $\overline{\Omega}$ (partly found using `DIMSYM`), where

$$\Gamma_1 := \frac{1}{z_{12}^1} \frac{\partial}{\partial z_{12}^1}, \qquad \Gamma_2 := \frac{\partial}{\partial x^1}, \qquad \Gamma_3 := \frac{\partial}{\partial x^2}, \qquad \Gamma_4 := \frac{\partial}{\partial z^1},$$

we obtain using Theorem 2,

$$\begin{aligned}
\overline{\omega}^1 &:= \frac{\Gamma_2 \lrcorner\, \Gamma_3 \lrcorner\, \Gamma_4 \lrcorner\, \overline{\Omega}}{\Gamma_1 \lrcorner\, \Gamma_2 \lrcorner\, \Gamma_3 \lrcorner\, \Gamma_4 \lrcorner\, \overline{\Omega}} = d\left(\frac{(z_{12}^1)^2}{2}\right),\\
\overline{\omega}^2 &:= \frac{\Gamma_1 \lrcorner\, \Gamma_3 \lrcorner\, \Gamma_4 \lrcorner\, \overline{\Omega}}{\Gamma_2 \lrcorner\, \Gamma_1 \lrcorner\, \Gamma_3 \lrcorner\, \Gamma_4 \lrcorner\, \overline{\Omega}} = d\left(x^1 - \ln(z_2^1 - z_1^1)\right),\\
\overline{\omega}^3 &:= \frac{\Gamma_1 \lrcorner\, \Gamma_2 \lrcorner\, \Gamma_4 \lrcorner\, \overline{\Omega}}{\Gamma_3 \lrcorner\, \Gamma_1 \lrcorner\, \Gamma_2 \lrcorner\, \Gamma_4 \lrcorner\, \overline{\Omega}} \equiv d\left(x^2 + \ln(z_2^1 - z_1^1) - \frac{z_2^1}{z_{12}^1}\right) \mod d\left(\frac{(z_{12}^1)^2}{2}\right),\\
\overline{\omega}^4 &:= \frac{\Gamma_1 \lrcorner\, \Gamma_2 \lrcorner\, \Gamma_3 \lrcorner\, \overline{\Omega}}{\Gamma_4 \lrcorner\, \Gamma_1 \lrcorner\, \Gamma_2 \lrcorner\, \Gamma_3 \lrcorner\, \overline{\Omega}} \equiv d\left(z^1 - z_1^1 + z_2^1 - \frac{(z_2^1)^2}{2z_{12}^1}\right) \mod d\left(\frac{(z_{12}^1)^2}{2}\right).
\end{aligned}$$

Putting

$$z_{12}^1 = c_1, \qquad x^1 - \ln(z_2^1 - z_1^1) = c_2, \qquad x^2 + \ln(z_2^1 - z_1^1) - \frac{z_2^1}{z_{12}^1} = c_3,$$
$$z^1 - z_1^1 + z_2^1 - \frac{(z_2^1)^2}{2z_{12}^1} = c_4,$$

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for any constant functions c_1, c_2, c_3, c_4 (where we have assumed that $c_1 > 0$), yields the following local solution to the system of PDEs in (7.7):

$$u = c_5 + c_6 \exp(x^1) + c_7(x^1 + x^2) + c_8(x^1 + x^2)^2, \tag{7.10}$$

for any constants c_5, c_6, c_7, c_8. It can be shown that the assumption $c_1 > 0$ means that $c_8 > 0$, but the solution in (7.10) also holds for zero and negative c_8. Finally, note that

$$u = c_5 + c_6 \exp(x^1) + f(x^1 + x^2),$$

is also a solution to (7.7), where f is any smooth function of $x^1 + x^2$. Of course, f will depend on the Cauchy data.

8 Conclusions

Our aim has been to determine the extent to which solvable symmetry structures can be applied to the Cauchy problem. We have managed to develop techniques in Theorems 6 and 8 that solve the problem when the Cauchy characteristic space is one-dimensional. An inherent deficiency in these techniques is that the Cauchy data must be specified by vector fields of the precise form in the theorems: in generating the data, one must pick coefficients of the W_i that are functions of the γ^a. For one-dimensional data, this is all that is required; however in Theorem 8, where the Z_l must form a Frobenius integrable distribution, this may be difficult (or may not even exist) if the Cauchy data is of a higher dimension than one.

Appendix A Solvable structures using DIMSYM

Here we illustrate by way of Example 3 how to use DIMSYM to find a non-trivial symmetry of an arbitrary vector field distribution. Suppose then that given Y, we have found X_5, X_4, X_3 and wish to find X_2. Then working in REDUCE with x(1), x(2), x(3), x(4), x(5), x(6) representing $x^1, x^2, z^1, z^1_1, z^1_2, z^1_{12}$, we load the DIMSYM package and enter the vector fields like so:

```
Y := @x(1) - @x(2) + (x(4)-x(5))*@x(3) + (x(4)-x(5))*@x(4);
X5 := @x(3);
X4 := @x(2);
X3 := @x(1);
```

We then define some arbitrary functions:

```
for i := 1:6 do
xi(i) := newarb(x(1),x(2),x(3),x(4),x(5),x(6));
```

and also a symmetry vector:

```
symvec := xi(1)*@x(1) + xi(2)*@x(2) + xi(3)*@x(3)
+ xi(4)*@x(4) + xi(5)*@x(5) + xi(6)*@x(6);
```

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Now we define the following arbitrary functions:

```
for i := 1:16 do
a(i) := newarb(x(1),x(2),x(3),x(4),x(5),x(6));
```

and set up the Lie bracket symmetry relations:

```
zvec1 := comm(symvec,Y)
- a(1)*Y - a(2)*X5 - a(3)*X4 - a(4)*X3;
zvec2 := comm(symvec,X5)
- a(5)*Y - a(6)*X5 - a(7)*X4 - a(8)*X3;
zvec3 := comm(symvec,X4)
- a(9)*Y - a(10)*X5 - a(11)*X4 - a(12)*X3;
zvec4 := comm(symvec,X3)
- a(13)*Y - a(14)*X5 - a(15)*X4 - a(16)*X3;
```

Next, we introduce determining equations:

```
for i := 1:6 do
begin
deteqn(i) := vecder(zvec1,x(i));
deteqn(i+6) := vecder(zvec2,x(i));
deteqn(i+12) := vecder(zvec3,x(i));
deteqn(i+18) := vecder(zvec4,x(i));
end;
```

where `vecder` is a `DIMSYM` command which simply extracts from the vector field in the first argument, the coefficient of the basis vector in the second argument. We then read the determining equations into `DIMSYM` using:

```
readdets();
```

Ask `DIMSYM` to solve the determining equations using the standard algorithm:

```
solvedets(std);
```

Any unsolved determining equations that were too difficult for `DIMSYM` to solve will be given using:

```
showdets();
```

Finally, the infinitesimal generators of all the trivial as well as nontrivial symmetries that `DIMSYM` has been able to find are given using the following:

```
mkgens();
end;
```

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To find a solvable structure of, say p, non-trivial symmetries, the above procedure must be carried out p times. Ideally, we would like no unsolved determining equations to exist when we execute `showdets()` each time, however in most cases, there will be unsolved determining equations if the `symvec` coefficients `xi(1)`,`xi(2)`,`xi(3)`,`xi(4)`,`xi(5)`,`xi(6)` are left purely arbitrary as they are above. Typically, for at least one of these coefficients, one is required to reduce the number of arguments of the corresponding `newarb` or even define the coefficient to be zero.

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https://doi.org/10.1017/S0956792501004697 Published online by Cambridge University Press