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SQUARE-FULL VALUES OF QUADRATIC POLYNOMIALS

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Abstract

A *square-full* number is a positive integer for which all prime divisors divide the number at least twice. For a relatively prime pair $(a, b) \in \mathbb{N} \times \mathbb{N} \cup \{0\}$ and an affine polynomial $f(x) = ax + b$, the number of $n \leqslant N$ such that $f(n)$ is square-full is of order $N^{1/2}$. For $\varepsilon > 0$ and an admissible quadratic polynomial $f(x) \in \mathbb{Z}[x]$, we show that the number of $n \leqslant N$ such that $f(n)$ is square-full is at most $O_{\varepsilon,f}(N^{\varpi+\varepsilon})$ for some absolute constant $\varpi < 1/2$. Under the *abc* conjecture, we expect the upper bound to be $O_{\varepsilon,f}(N^{\varepsilon})$.

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1. Introduction

A positive integer n is called *square-full* if for every prime p dividing n, we have $p^2 \mid n$. We consider 1 to be a square-full number. Throughout, $f(x)$ is a polynomial with integer coefficients. Let $S_f^{\blacksquare}(N)$ denote the number of positive integers $n \leqslant N$ such that $f(n)$ is square-full. For the identity polynomial $\mathrm{Id}(n) = n$, Bateman and Grosswald [1], improving the pioneering result [8] of Erdős and Szekeres, showed that

$$S_{\mathrm{Id}}^{\blacksquare}(N) = \frac{\zeta(3/2)}{\zeta(3)} N^{1/2} + \frac{\zeta(2/3)}{\zeta(2)} N^{1/3} + o(N^{1/6}).$$

They also indicate in the paper that the exponent $1/6$ in the error term cannot be improved without proving some kind of quasi-Riemann hypothesis. For $f(n) = an + b$, $(a, b) \neq (1, 0)$, the counting function is still of order $N^{1/2}$ up to a constant depending only on a, b. Chan [6] proved the asymptotic formula for such a polynomial with $\gcd(a, b) = 1$, with an improved error term for large a.

In 1931, Estermann [9] studied the counting function for quadratic polynomials assuming square-free values (that is, for every prime p dividing n, we have $p \,\|\, n$).

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Let $S_f^{\square}(N)$ denote the number of positive integers $n \leqslant N$ such that $f(n)$ is square-free. For $f(x) = x^2 + h$ with a nonzero integer h, Estermannn proved that

$$S_f^{\square}(N) = \prod_p \left(1 - \frac{\rho_h(p^2)}{p^2}\right) N + O(N^{2/3} \log N),$$

where $\rho_h(m) := \#\{n \in \mathbb{Z}/m\mathbb{Z} : n^2 + h \equiv 0 \bmod m\}$ for all $m \geqslant 2$. In 2012, Heath-Brown [13] improved the error term for $h = 1$ to $O_\varepsilon(N^{7/12+\varepsilon})$ for arbitrarily small $\varepsilon > 0$.

We apply Heath-Brown's method to determine the number of square-full numbers of quadratic forms.

THEOREM 1.1. *Let $\varepsilon > 0$ be given and N be a large number depending on ε. Let $\alpha \in \mathbb{N}$ and $f(x) = x^2 + \alpha^2$. The number of $n \leqslant N$ for which $f(n)$ is square-full is, uniformly for $\alpha \leqslant N^{\varpi}$, at most $O_\varepsilon(N^{\varpi+\varepsilon})$, where $\varpi = 29/100$.*

We also consider the general case of a quadratic polynomial $f(x) = ax^2 + bx + c \in \mathbb{Z}[x]$. An important property of any square-full integer is that it can be written uniquely as e^2d^3 for an integer e and a square-free d. Thus, $S_f^{\blacksquare}(N) \leqslant S_g^{\blacksquare}(2aN)$ with $g(x) = px^2 + q$, $p = a$ and $q = a(b^2 - 4ac)$ since

$$ax^2 + bx + c = \frac{p(2ax + b)^2 + q}{4a^2} \tag{1.1}$$

and the left-hand side is of the form e^2d^3 if $py^2 + q = (2ae)^2d^3 =: e'^2d^3$, which gives

$$S_f^{\blacksquare}(N) \leqslant \#\{n \leqslant 2aN : e'^2d^3 = pn^2 + q, d \text{ square-free}\} = S_g^{\blacksquare}(2aN).$$

We call this g a *majorant of* f. Hence, it suffices to consider $f(x) = px^2 + q$. We are particularly interested in polynomials for which the majorant $g(x) = px^2 + q$ satisfies $q \neq 0$, that is, when its discriminant is nonzero. We call such a polynomial $f(x)$ *admissible*.

To formulate the general result, we define

$$\varpi_0 = \frac{2\beta}{3 \log 3} \approx 0.1688, \tag{1.2}$$

where $\beta := ((2 + \sqrt{3})/2\sqrt{3}) \log((2 + \sqrt{3})/2\sqrt{3}) - ((2 - \sqrt{3})/2\sqrt{3}) \log((2 - \sqrt{3})/2\sqrt{3})$. We now state our second main theorem. The main ingredients in the proof are the improved bound of Ellenberg and Venkatesh [7] on counting 3-torsion of the class group $\mathbb{Q}(\sqrt{-D})$ with the Helfgott–Venkatesh mechanism [14]. These provide the bound on counting integral points on Mordell curves.

THEOREM 1.2. *For $\varepsilon > 0$ and an admissible quadratic polynomial $f(x)$, the number of $n \leqslant N$ for which $f(n)$ is square-full is at most $O_{\varepsilon,f}(N^{\varpi+\varepsilon})$, where $\varpi = 2(1 + 4\varpi_0)/(5 + 12\varpi_0) \approx 0.4769$. Moreover, if the Mordell curve conjecture (Conjecture 3.1 below) is true, then we can replace ϖ by $\frac{2}{5}$.*

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The problem of counting square-full values of $f(x) = x^2 + h$, for a nonzero integer h, is related to counting pairs of quasi-consecutive square-full numbers, since n^2 itself is square-full. In fact, if we fix a polynomial $f(x) = x(x + 1)$, then counting pairs of consecutive square-full integers that do not exceed N is the same as determining $S_f^{\blacksquare}(N)$. Reuss [15] studied this problem, as well as some related problems on k-free numbers. Our result for special $f(x)$ is the same as his result, which also shows that the number of such pairs of integers up to N is $O_\varepsilon(N^{29/100+\varepsilon})$.

There are two challenges in adapting the determinant method to our problem. First, we need to deal with more than one nonlinear term during the extraction from the sum of two squares to exploit the method efficiently. We are helped by the fact that a square-full integer can be written as e^2d^3 for a square-free integer d uniquely, and the exponent 2 allows us to revert to a single linear term via Taylor's theorem. Second, we need to modify the counting argument for a certain subcase where we cannot count solutions using Thue's equation straightforwardly. Instead, we present a new vanishing equation, leveraging the effective result of Bombieri and Schmidt [3].

Recently, Browning and Shparlinski [5] proved that almost all polynomials of arbitrary degree $k \geqslant 2$, in the sense of naive height, take infinitely many square-free values. Almost all quadratic polynomials, in this sense, produce a small number of square-full values (see (2.1) for the expected asymptotic formula for $S_f^{\blacksquare}(N)$).

2. Preliminaries

Although it seems difficult to find a positive integer n so that $n^2 + 4$ is square-full, there are, in fact, infinitely many such values of n. Let $n = 2d$ be an even number and rewrite $n^2 + 4 = 4(d^2 + 1)$. There are infinitely many solutions $(d, k) \in \mathbb{Z}^2$ such that

$$4(d^2 + 1) = 8k^2 \iff d^2 - 2k^2 = -1,$$

since it is a negative Pell's equation with $(1, 1)$ as a base solution. Since $8k^2$ is always square-full, we produce infinitely many square-full numbers of the form $n^2 + 4$. A polynomial $f(x)$ of degree at least three with no repeated roots is unlikely to take square-full values infinitely often. This is thanks to the result of Granville as a consequence of the *abc* conjecture (see [10, Theorem 7] and the discussion around it). Our example $f(x) = x^2 + 4$ shows that the number of $n \leqslant N$ for which $f(n)$ is square-full is $\gg \log N$. We may ask, as mentioned by Granville, whether

$$S_f^{\blacksquare}(N) \sim c_f \log N \tag{2.1}$$

for some constant c_f depending on the coefficients of $f(x)$. In particular, this might hold for any admissible quadratic polynomial $f(x)$ with no fixed prime p such that $p \,\|\, f(n)$ for all sufficiently large n. In Proposition 3.7, we shall see that assuming the *abc* conjecture, the left-hand side of (2.1) is bounded by $C_{\varepsilon,f}N^\varepsilon$ for some constant $C_{\varepsilon,f} > 0$. By the Erdős–Szekeres theorem, the probability of n in each dyadic range $[N/2, N]$ to be square-full is of order $1/\sqrt{N/2}$. Thus, we expect that the probability of both n and $n + 1$ in the same range to be square-full is of order $1/(N/2)$. Thus, it is

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reasonable to expect the number of consecutive square-full integers up to N to be of order

$$\frac{N}{2}\cdot\frac{1}{N/2}+\frac{N}{4}\cdot\frac{1}{N/4}+\cdots\ll\log N,$$

where the sum ranges through all dyadic intervals not exceeding N. This gives a simple heuristic argument that supports (2.1).

For a fixed polynomial $f(x)$, it is convenient to study the expression

$$S_f^{\blacksquare}(2N)-S_f^{\blacksquare}(N)=\sum_{\substack{N<n\leqslant 2N\\ e^2d^3=f(n)}}\mu^2(d); \tag{2.2}$$

an upper bound gives a bound for $S_f^{\blacksquare}(N)$ itself by a dyadic sum, at the cost of a logarithmic factor. By defining

$$\mathcal{M}(E,D):=\#\{(e,d,n)\in\mathbb{N}^3 : N<n\leqslant 2N, E<e\leqslant 2E, D<d\leqslant 2D, e^2d^3=f(n)\},$$

the right-hand side of (2.2) is at most

$$\sum_{\ell\ll\log N}\sum_{\substack{N/2^{\ell}<E\leqslant N/2^{\ell-1}\\ E^2D^3\asymp_f N^2}}\mathcal{M}(E,D)\ll(\log N)\sup_{\substack{E\leqslant N\\ E^2D^3\asymp_f N^2}}\mathcal{M}(E,D),$$

where the sums are dyadic sums over E. From this, the problem is reduced to obtaining the bound for $\mathcal{M}(E,D)$ in the entire range. Estermann [9] showed that the contribution to $\mathcal{M}(E,D)$ for fixed $d\in[D,2D]\subset[1,N]$ is $O(\log N)$, so we have $\mathcal{M}(E,D)\ll D\log N$.

It seems difficult to extend Heath-Brown's determinant method to a polynomial $f(x)=x^2+h$ with nonsquare h. For general quadratic polynomials, we use a different approach in Section 3, employing the bound on counting integral points on the Mordell curve $y^2=x^3+D$ for a nonzero integer D.

In Section 4, we prove our first main theorem on $f(x)=x^2+\alpha^2$. We proceed in parallel to Heath-Brown's work [13] on square-free values of n^2+1, by refining the bound of $\mathcal{M}(E,D)$ to $N^{\eta}\sqrt{M}\min(\sqrt{E},\sqrt{D})$, with a new parameter M. This leads to the construction of the vanishing equation and addresses our challenges mentioned earlier. Finally, in Section 5, we provide some comments on the generalisation to higher-degree polynomials and the reduction of ϖ in both main theorems.

3. Integral points on Mordell curves and the proof of Theorem 1.2

3.1. Proof of Theorem 1.2 and counting integral points on Mordell curves. First, we state the conjecture on counting integral solutions on Mordell curves.

CONJECTURE 3.1 (Mordell curve conjecture). For a fixed integer $D\neq 0$ and an arbitrarily small $\varepsilon>0$, we have $\#\{(x,y)\in\mathbb{Z}^2 : y^2=x^3+D\}\ll_{\varepsilon}|D|^{\varepsilon}$.

Using the Helfgott–Venkatesh mechanism [14] with the Ellenberg–Venkatesh result [7], we prove the following proposition.

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PROPOSITION 3.2. *Let $\varepsilon > 0$ be given and let $\varpi_0 = 2\beta/3\log 3 \approx 0.1688$ be as in (1.2). For a fixed nonzero integer D, the number of integer solutions on the Mordell curve $y^2 = x^3 + D$ is at most $O_\varepsilon(|D|^{\varpi_0+\varepsilon})$.*

PROOF. Substitute the bound for the 3-torsion of the class group $h_3(\mathbb{Q}(\sqrt{-D})) \ll_\varepsilon D^{1/3+\varepsilon}$ [7, Proposition 2] into the mechanism as in the proof of [14, Theorem 4.5]. □

REMARK 3.3. Proposition 3.2 gives a better bound than the significantly improved result on the 2-torsion of the class group of Bhargava *et al.* [2]. This is because [2, Theorem 1.2(c)] gives the bound on general elliptic curves rather than Mordell curves. It would be interesting to know if a bound on the 2-torsion can be exploited to improve upon Conjecture 3.1.

REMARK 3.4. It is possible to prove a weak version of Conjecture 3.1 for special D. The weak version means that for each $\varepsilon > 0$ and a fixed integer $D \neq 0$,

$$\#\{(x, y) \in \mathbb{Z}^2 : |x|, |y| \leqslant B, y^2 = x^3 + D\} \ll_\varepsilon (B|D|)^\varepsilon. \tag{3.1}$$

For any D, this weak version follows from Heath-Brown's result [11, Theorem 2]. We take $G(X, Y, Z) = ZY^2 - X^3 - DZ^3$ in Heath-Brown's theorem. This cubic form covers all solutions on our Mordell curve and leads to the estimate in (3.1). Heath-Brown assumed the Birch–Swinnerton-Dyer conjecture and the Riemann hypothesis for L-functions of elliptic curves to obtain his result. Under these hypotheses, this version suffices to prove the second assertion in Theorem 1.2. This is because we may take $B = N$ in the weak version above and use the bound as in the proof below.

PROOF OF THEOREM 1.2. As in (2.2), we consider a majorant $g(x)$ of $f(x)$, say $g(x) = ax^2 + b$ with $b \neq 0$. Then, we find an upper bound for $\mathcal{M}(E, D)$ by fixing $e \in [E, 2E]$ and count the number of solutions (d, n) to $an^2 = e^2d^3 - b$. By Proposition 3.2 with $(x, y) = (e^2ad, e^2a^2n)$ and $D = -e^4ba^3$, we obtain $\mathcal{M}(E, D) \ll_{\varepsilon,f} E \cdot |E|^{4\varpi_0+\varepsilon} = E^{1+4\varpi_0+\varepsilon}$. Thus, we obtain the total bound

$$\ll_{\varepsilon,f} N^\varepsilon\Big((\log N) \sup_{\substack{E\leqslant N \\ E^2D^3\asymp N^2}} \min(D, E^{1+4\varpi_0})\Big) \ll N^{\varpi+\varepsilon},$$

where $\varpi = 2(1 + 4\varpi_0)/(5 + 12\varpi_0)$ for the optimal choice $E = N^{2/(5+12\varpi_0)}$. The second assertion can be proved similarly by replacing ϖ_0 by 0. The theorem follows. □

REMARK 3.5. It seems difficult to tackle this problem with our naive method using the mechanisms of Heath-Brown [11] or Salberger (as mentioned in [4]). We estimate the number of points (d, n) that satisfy $q(d) - p(n) - b := e^2d^3 - an^2 - b = 0$, where $b \neq 0$, $p(x)$ is a polynomial of degree 2 and $q(x)$ is a polynomial of degree 3. Then, we proceed as in Reuss's work [15] and obtain via [11, Theorem 15] that the points (d, n) lie on at most $N^\eta\sqrt{D}$ auxiliary curves. Thus, using Bézout's theorem, the number of points considered is $\ll_\eta N^\eta\sqrt{D}$. Upon invoking Estermann's bound in the first sum,

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we obtain the upper bound

$$\ll_\eta N^\eta\Big((\log N)\sup_{\substack{E\leqslant N\\ E^2D^3\asymp N^2}}\min(D,E\sqrt{D})\Big)\ll N^{1/2+\eta},$$

where the supremum occurs at $E=N^{1/4}$. This is, however, weaker than what we expected in Theorem 1.2.

3.2. Conditional result based on the *abc* conjecture. In this subsection, we show that the number ϖ in Theorems 1.1 and 1.2 can be taken arbitrarily close to 0 under the *abc* conjecture (Conjecture 3.6 below).

For a positive integer n, we write $\mathrm{rad}(n)=\prod_{p\mid n}p$.

CONJECTURE 3.6 (*abc* conjecture). For any fixed $\varepsilon>0$, if $a+b=c$ for pairwise coprime positive integers a,b,c, then $c<K_\varepsilon\cdot\mathrm{rad}(abc)^{1+\varepsilon}$ for some constant K_ε.

PROPOSITION 3.7. *Let $\varepsilon>0$ and $f(x)$ be an admissible quadratic polynomial. Under the abc conjecture, the number of $n\leqslant N$ for which $f(n)$ is square-full is at most $O_{\varepsilon,f}(N^\varepsilon)$.*

PROOF. Assume that the *abc* conjecture holds. We focus on counting the solutions (n,e,d) in $ce^2d^3=n^2+b$ for fixed constants c,b in (1.1). Here, we remove the factor a from both p and q in (1.1), and the remaining denominator $4a$ is moved and relabelled as c. The sizes of c and b are dependent on f. Then, we determine the condition on d to satisfy $ce^2d^3=n^2+b$. For $b>0$ and $\gcd(b,n)=\ell$, we transform the equation into

$$\frac{ce^2d^3}{\ell\ell_1}=\ell'n'^2+b'',$$

where $n=\ell n'$ and $b=\ell b'$ with $\gcd(n',b')=1$. We also let $\ell_1=\gcd(\ell,b')$ with $b'=\ell_1b'',\ell=\ell_1\ell'$. Thus, there exists an absolute constant C_ε such that

$$\begin{aligned}\frac{ce^2d^3}{\ell\ell_1}&\leqslant C_\varepsilon\mathrm{rad}\Big(\frac{ce^2d^3}{\ell\ell_1}\cdot\ell'n'^2\cdot b''\Big)^{1+\varepsilon}\\&\leqslant C_\varepsilon\mathrm{rad}\Big(\frac{ce^2d^3}{\ell_1^2}n'b''\Big)^{1+\varepsilon}\leqslant C_\varepsilon\mathrm{rad}(ce^2d^3n'b'')^{1+\varepsilon}\ll_{\varepsilon,f}e^{1+\varepsilon}d^{1+\varepsilon}n'^{1+\varepsilon}.\end{aligned}$$

Hence, $e^{1-\varepsilon}d^{2-\varepsilon}\ll_{\varepsilon,f}n^{1+\varepsilon}$, and since $e^2d^3\asymp_f n^2$, we obtain $d\ll_{\varepsilon,f}N^\varepsilon$ for arbitrarily small $\varepsilon>0$. Estermann's bound and the number of possibilities of d imply the claim. For $b<0$, the argument is similar. □

REMARK 3.8. Since certain $f(x)$ produce infinitely many square-full numbers, the bound in Proposition 3.7 is best possible as a power of N.

4. The determinant method and the proof of Theorem 1.1

In this section, we prove Theorem 1.1, following Heath-Brown [13] and Reuss [15]. Fixing $\eta>0$, we determine the solutions (e,d,n) to $e^2d^3=n^2+\alpha^2$ with d square-free.

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Recall that we have parameters E, D, N, where $E < e \leqslant 2E, D < d \leqslant 2D$ and $N < n \leqslant 2N$. Thus, $E^2D^3 \asymp N^2$ since $\alpha \leqslant N^{\varpi}$ and $\varpi = 29/100$. We have $e = x_1^2 + x_2^2$ and $d = y_1^2 + y_2^2$ by the sum of two squares argument, and d is square-free. By the unique factorisation property of $\mathbb{Z}[i]$, $(x_1 + ix_2)^2(y_1 + iy_2)^3 = n + \alpha i$ and, after taking the imaginary part,

$$(x_1^2 - x_2^2)(3y_1^2y_2 - y_2^3) + 2x_1x_2(y_1^3 - 3y_1y_2^2) = \alpha. \tag{4.1}$$

We shall assume that $|x_1| \leqslant |x_2|$ and x_1, x_2 have the same signs. If $|x_1| \geqslant |x_2|$, we may change the roles of x_1 and x_2, and change the sign of y_2. If x_1, x_2 have different signs, we can change the signs of y_1 and x_2. Now, we claim that

$$\max(|3y_1^2y_2 - y_2^3|, |y_1^3 - 3y_1y_2^2|) \gg D^{3/2}.$$

The proof will assume $|y_1| \leqslant |y_2|$; by symmetry, the other case can be handled similarly. For the case $|y_2| \cdot |3y_1^2 - y_2^2| \geqslant |y_1| \cdot |y_1^2 - 3y_2^2|$, we first suppose that $y_2^2 < 3y_1^2$. Then, $|y_2|(3y_1^2 - y_2^2) \geqslant |y_1||y_1^2 - 3y_2^2| \geqslant |y_1|(3y_2^2 - y_1^2)$. Since d is square-free, we see that $|y_1| \neq |y_2|$, for otherwise, $d = 2y_1^2$ is not square-free and $d \gg D$ is large for $N \gg_{\eta} 1$, so that $3t^2 - 1 \geqslant t(3 - t^2)$ for $1 \geqslant t := |y_1/y_2|$. Thus, $0 \leqslant t^3 + 3t^2 - 3t - 1 \leqslant 3t^2 - 3t = 3t(t - 1)$ and $t = 1$ so $|y_1| = |y_2|$, which is not possible. Therefore, $y_2^2 \geqslant 3y_1^2$ and, with the same setup, we obtain $0 \leqslant t^3 - 3t^2 - 3t + 1 = (t - 1)^3 - 6t + 2 \leqslant 2 - 6t$ and $|y_1| \leqslant \frac{1}{3}|y_2|$. The maximum in this case is $\gg |y_2|(y_2^2 - 3y_1^2) \gg D^{3/2}$. Similarly, for the case $|y_2||3y_1^2 - y_2^2| < |y_1||y_1^2 - 3y_2^2|$, with the same notation, we have either

$$t(3 - t^2) > 3t^2 - 1 \quad \text{or} \quad t(3 - t^2) > 1 - 3t^2,$$

and the second inequality is impossible, as it is equivalent to $(t + 1)(t^2 - 4t + 1) < 0$. The first corresponds to $(t - 1)(t^2 + 4t + 1) < 0$, which is always true, and corresponds to $t^2 < 3$, which implies $y_2^2 < 3y_1^2$. Again, the maximum of the two terms is $\gg D^{3/2}$.

On taking $q_1(y_1, y_2)$ to be the larger of the two, which has the property that its absolute value is $\gg D^{3/2}$, and taking z_1, z_2 appropriately to be polynomials in x_1, x_2, we obtain $q_1(y_1, y_2)z_1 + q_2(y_1, y_2)z_2 = \alpha$.

We may assume that $D \geqslant N^{\varpi}$. Estermann [9, (16)] provides a bound on

$$\#\{(n, e) \in \mathbb{Z}^2 : |n|, |e| \leqslant N, n^2 = d^3e^2 - \alpha^2\} \ll \log(N + |d\alpha|).$$

For $D < N^{\varpi}$, the contribution is

$$\sum_{\substack{N<n\leqslant 2N \\ e^2d^3=n^2+\alpha^2 \\ D<d\leqslant 2D}} \mu^2(d) \leqslant \sum_{d\leqslant 2N^{\varpi}} \sum_{\substack{n^2=d^3e^2-\alpha^2 \\ |n|,|e|\leqslant 2N}} 1 \ll N^{\varpi} \log N,$$

which is acceptable. By the triangle inequality,

$$D^{3/2}|z_1| \ll |q_1(y_1, y_2)||z_1| \leqslant \alpha + |q_2(y_1, y_2)||z_2| \ll D^{3/2}|z_2|,$$

since $\alpha \leqslant N^{\varpi} \leqslant D$ and here, we assume that $|z_1z_2| \neq 0$. For if $|z_1z_2| = 0$, then $x_1 = x_2$, and (4.1) becomes $2x_1^2y_1(y_1^2 - 3y_2^2) = \alpha$. The number of (x_1, x_2, y_1, y_2) satisfying such

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an equation is $\ll \tau(\alpha)^{10} \ll N^{\varepsilon}$. Thus, $|z_1| \ll |z_2|$ and $|z_2| \gg \max(|z_1|, |z_2|) \gg E$. Hence,

$$t = -\frac{q_2(s,1)}{q_1(s,1)} + O(E^{-1}D^{-3/2}) = -\frac{q_2(s,1)}{q_1(s,1)} + O(N^{-1}) =: \phi(s) + O(N^{-1}), \tag{4.2}$$

where $s = y_1/y_2$ and $t = z_1/z_2$ if $|y_1| \leqslant |y_2|$ and $s = y_2/y_1$ in the other case, which can be handled in a similar way. So we may assume $s, t \ll 1$. Therefore, the problem reduces to counting points (s,t) that lie close to the curve $t = \phi(s)$. We write $w = x_1/x_2$ which is positive since x_1, x_2 have the same signs and $w \ll 1$.

4.1. The determinant method. In this subsection, we construct another equation in a bi-homogeneous form of the same variables as in (4.2) for each interval that we split. The bound is plausible since we have the existing bound for counting points on $G(x_1, x_2; y_1, y_2) = 0$, for an absolutely irreducible bi-homogeneous polynomial $G(x_1, x_2; y_1, y_2) \in \mathbb{Z}[x_1, x_2, y_1, y_2]$.

Our plan is as follows. We choose a parameter $M \leqslant N$, which will be specified later in terms of E, D. Then, we divide the intervals of possible values of $s \ll 1$ into $O(M)$ intervals of the form $I = (s_0, s_0 + M^{-1}]$. By Taylor's theorem, if we write $s = s_0 + u$ with $u = O(M^{-1})$, then $\phi(s) = \phi(s_0) + P(u) + O(N^{-1})$, where $P(u)$ is a polynomial in u without a constant term and all coefficients of size $O(1)$. Thus, $|P(u)| \ll M^{-1}$. Write $\tau(w) := t = \phi(s) + O(N^{-1})$, where $0 < w = x_1/x_2 \ll 1$ from our conditions. If $\tau(w) = (w^2 - 1)/2w$, then we obtain

$$\frac{w^2 - 1}{2w} = \phi(s_0) + P(u) + O(N^{-1}).$$

Since $w > 0$ and $P(u)$ is small,

$$w = \phi(s_0) + P(u) + \sqrt{(\phi(s_0) + P(u))^2 + 1 + O(N^{-1})}.$$

By Taylor's theorem, $\sqrt{s + c} = \sqrt{c} + (2\sqrt{c})^{-1}s + O(s^2)$ for $s \in \mathbb{R}$ around zero and a constant $c \geqslant 1$. Hence,

$$\begin{aligned} w &= \phi(s_0) + P(u) + \sqrt{1 + (\phi(s_0) + P(u))^2} + \frac{O(N^{-1})}{2\sqrt{1 + (\phi(s_0) + P(u))^2}} \\ &= c_{s_0} + Q(u) + O(N^{-1}), \end{aligned}$$

using Taylor's theorem for the square root, and the equation holds for some polynomial Q in u. Similarly, if $\tau(w) = 2w/(w^2 - 1)$, we obtain

$$w = \frac{1 + \sqrt{1 + (\phi(s_0) + P(u) + O(N^{-1}))^2}}{\phi(s_0) + P(u)} \ll 1,$$

and as $P(u)$ is small, $|\phi(s_0)| \gg 1$. By Taylor's theorem applied to $\sqrt{1 + (c + s)^2}$ for $s \in \mathbb{R}$ around zero and a constant $c \geqslant 1$,

$$w = \frac{1 + \sqrt{1 + (\phi(s_0) + P(u))^2}}{\phi(s_0) + P(u)} + \frac{O(N^{-1})}{\sqrt{1 + (\phi(s_0) + P(u))^2}} = c_{s_0} + Q(u) + O(N^{-1}),$$

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since $|\phi(s_0)| \gg 1$ and $P(u)$ is small. Since $P(u) \ll M^{-1}$ and M will be chosen so that $\log M \asymp \log N$, finitely many terms are accumulated in $Q(u)$ with the residual term of order $O(N^{-1})$. Therefore, in either case, w can be represented by a polynomial $Q(u)$ such that $Q(u) \ll M^{-1}$ with error $O(N^{-1})$.

Suppose that there are J solutions to (4.2) in the interval I. Upon introducing new parameters K, L that will be dependent merely on η, we construct a $J \times H$ matrix

$$\mathfrak{M} = \begin{pmatrix} 1 & s_1 & w_1 & \cdots & s_1^k w_1^\ell & \cdots & s_1^K w_1^L \\ 1 & s_2 & w_2 & \cdots & s_2^k w_2^\ell & \cdots & s_2^K w_2^L \\ \vdots & & & \vdots & & & \vdots \\ 1 & s_J & w_J & \cdots & s_J^k w_J^\ell & \cdots & s_J^K w_J^L \end{pmatrix},$$

where $H = (K+1)(L+1)$. If $\operatorname{rank}(\mathfrak{M}) < H$, there is a nonzero vector $\mathbf{c}$ such that

$$\mathfrak{M}\mathbf{c} = \mathbf{0}.$$

With this nonzero vector $\mathbf{c}$, we can consider the polynomial $C_I(s,w) = \sum_{h=1}^{H} c_h m_h(s,w)$, where we use the label $m_h(s,w)$ to run through the monomials $s^k w^\ell$ appearing in each row of $\mathfrak{M}$ for $k \leqslant K$ and $\ell \leqslant L$ so that $C_I(s_j, w_j) = 0$ for all $j \leqslant J$. By reducing the matrix $\mathfrak{M}$ to row-reduced echelon form, we see that $\mathbf{c} \in \mathbb{Q}^H$ has rational entries and we may clear the denominator of C_I, which is $\ll N^A$ for some fixed A. Then, the coefficients are integral of size $\ll N^{\kappa(K,L)}$. The fact that $\operatorname{rank}(\mathfrak{M})$ is strictly less than H follows from

$$\Big(\prod_{j \leqslant H} y_{2,j}^K x_{2,j}^L\Big)\Delta \in \mathbb{Z}, \tag{4.3}$$

where Δ is the subdeterminant of the first $H \times H$ matrix, and we assume that $J \geqslant H$. We can choose M appropriately so that $\Delta = 0$. It suffices to prove, for an appropriate M, that $|\Delta| \leqslant C_{K,L} D^{-KH/2} E^{-LH/2}$ for a sufficiently small constant $C_{K,L}$, which is chosen so that the absolute value of the term in (4.3) is smaller than 1. This forces Δ to be 0.

Following the proof of Heath-Brown [13], for $V \asymp N$, the Taylor expansion $w = c_{s_0} + P(u) + v$ has $v \ll V^{-1}$ and $u \ll M^{-1}$. We now order the values of $M^{-k}V^{-\ell}$, $k \leqslant K$, $\ell \leqslant L$, decreasingly as $1 = M_0, M_1, \ldots$. From Heath-Brown [12, Lemma 3],

$$\Delta \ll_H \prod_{h \leqslant H} M_h.$$

Let us denote $M_H = W^{-1}$. Then, $M^{-k}V^{-\ell} \geqslant M_H$ if and only if $k \log M + \ell \log V \leqslant \log W$. The number of pairs k, ℓ satisfying this inequality is

$$\frac{\log^2 W}{2 \log M \log V} + O\Big(\frac{\log W}{\log N}\Big) + O(1),$$

which equals H. Since $\log M, \log V \ll \log N$,

$$\log W = \sqrt{2H \log M \log V} + O(\log N).$$

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Moreover,

$$\log \prod_{h\leqslant H} M_h = \sum_{\substack{k,\ell \\ k\log M+\ell\log V\leqslant \log W}} -(k\log M+\ell\log V)$$
$$= -\frac{\log^3 W}{3\log M\log V} + O\Big(\frac{\log^2 W}{\log N}\Big)$$
$$= -H^{3/2}\cdot\frac{2\sqrt{2}}{3}\sqrt{\log M\log V} + O(H\log N),$$

by using the estimate for $\log W$ above. This gives

$$\log|\Delta| \leqslant O_H(1) - H^{3/2}\cdot\frac{2\sqrt{2}}{3}\sqrt{\log M\log V} + O(H\log N).$$

Thus, $\Delta = 0$ if

$$\frac{K}{2}\log D + \frac{L}{2}\log E \leqslant (KL)^{1/2}\frac{2\sqrt{2}}{3}\sqrt{\log M\log V} + O_{K,L}(1) + O(\log N).$$

Taking $K = [L\log E/\log D]$, the condition becomes

$$L\log E \leqslant L\cdot\frac{2\sqrt{2}}{3}\sqrt{\log M\log V}\sqrt{\frac{\log E}{\log D}} + O_L(1) + O(\log N).$$

For small fixed $\delta > 0$, if

$$\frac{2\sqrt{2}}{3}\sqrt{\log M\log V}\sqrt{\frac{\log E}{\log D}} \geqslant (1+\delta)\log E$$

and $L, N \gg_\delta 1$, we will obtain the desired inequality. Since $V \gg N$, we have proved the following lemma.

LEMMA 4.1. *Let $\eta > 0$ be given and let $M \in [D, N]$ satisfy*

$$\log M \geqslant \frac{9}{8}(1+\eta)\frac{\log E\log D}{\log N}.$$

Then, for any interval $I = [s_0, s_0 + M^{-1}]$, there exists a nonzero polynomial $C_I(s,t)$ with integral coefficients such that all solutions satisfy

$$C_I(y_1/y_2, x_1/x_2) = 0,$$

with $y_1/y_2 \in I$. Moreover, $\deg(C_I) = O_\eta(1)$ and its coefficients are $O_\eta(N^\kappa)$ for some $\kappa = \kappa(\eta)$.

As in [13], we may assume without loss of generality that this polynomial C_I is absolutely irreducible. So we can apply the existing bound on counting solutions noted at the beginning of this subsection.

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4.2. Counting the solutions and finishing the proof. In the previous subsection, we constructed an auxiliary equation involving an absolutely irreducible bi-homogeneous polynomial C_I in each fixed interval $I = (s_0, s_0 + M^{-1}]$. In the interval $I = (s_0, s_0 + M^{-1}]$,

$$s_0 < s = \frac{y_1}{y_2} \leqslant s_0 + M^{-1}.$$

It follows that $|y_1 - s_0 y_2| \leqslant D^{1/2} M^{-1}$ because $|y_2| \leqslant D^{1/2}$. Let

$$\Lambda = \{(D^{-1/2} M(y_1 - s_0 y_2), D^{-1/2} y_2) : (y_1, y_2) \in \mathbb{Z}^2\}.$$

Then, Λ is a lattice of determinant $D^{-1}M$. We are interested in points $(\alpha_1, \alpha_2) \in \Lambda$ falling in the square

$$S = \{(\alpha_1, \alpha_2) : \max(|\alpha_1|, |\alpha_2|) \leqslant 1\}.$$

Let $\mathbf{g}^{(1)}$ be the shortest nonzero vector in the lattice and $\mathbf{g}^{(2)}$ be the shortest vector not parallel to $\mathbf{g}^{(1)}$. The two vectors form a basis for Λ and $\lambda_1 \mathbf{g}^{(1)} + \lambda_2 \mathbf{g}^{(2)} \in S$ only when $|\lambda_1| \ll |\mathbf{g}^{(1)}|^{-1} =: L_1$ and $|\lambda_2| \ll |\mathbf{g}^{(2)}|^{-1} =: L_2$. Thus, $L_1 \gg L_2$ and $(L_1 L_2)^{-1} = |\mathbf{g}^{(1)}||\mathbf{g}^{(2)}| \ll \det(\Lambda) = D^{-1}M$, which implies $L_1 L_2 \gg DM^{-1}$. Hence, we may replace (y_1, y_2) by (λ_1, λ_2). We may argue in exactly the same way to replace x_1, x_2 by τ_1, τ_2. Now, we have the equations

$$G_0(\lambda_1, \lambda_2; \tau_1, \tau_2) = \alpha, \tag{4.4}$$

$$G_1(\lambda_1, \lambda_2; \tau_1, \tau_2) = 0, \tag{4.5}$$

which correspond, respectively, to (4.1) and the vanishing equation from Lemma 4.1. We know that G_0 is bi-homogeneous of degree $(3; 2)$. Suppose that a bi-homogeneous polynomial G_1 has degree $(a; b)$. When $a, b \geqslant 2$, we can get a satisfactory bound from the following lemma derived from Heath-Brown [13, Lemma 2].

LEMMA 4.2. *Let $G(x_1, x_2; y_1, y_2)$ be an absolutely irreducible bi-homogeneous polynomial with integer coefficients of degree $(a; b)$ with $a, b \geqslant 1$. Let $\varepsilon > 0$ be given. Then, for $X \geqslant 1$, there exist $O_{a,b,\varepsilon}(X^{2/b+\varepsilon}\|G\|^{\varepsilon})$ points $(a_1, a_2, b_1, b_2) \in \mathbb{Z}^4$ satisfying the conditions* $\gcd(a_1, a_2) = \gcd(b_1, b_2) = 1, G(a_1, a_2; b_1, b_2) = 0$ *and* $\max\{a_1, a_2\} \leqslant X$.

It is possible to assume that $\gcd(\lambda_1, \lambda_2) = \gcd(\tau_1, \tau_2) = 1$ at the expense of a factor of α^{ε}. Suppose that $\gcd(\lambda_1, \lambda_2) = \lambda$. Then, $\lambda^3 \mid \alpha$ as G_0 is homogeneous in λ_1, λ_2. If this happens with $\lambda > 1$, we split this case and count the solutions from it. There are $O(\tau(\alpha)) = O(\alpha^{\varepsilon/2})$ possibilities of such equations. Similarly, there exist $O(\alpha^{\varepsilon/2})$ possible equations when we determine a pair of τ_1, τ_2.

We present here an elementary result on counting integral solutions with some conditions on certain bivariate forms.

LEMMA 4.3. *Let N be a large positive number and α be as in Theorem 1.1. Let c, d be a pair of integers with $|c|, |d| \leqslant N^{1000}$. Then, there are $O_{\varepsilon}(N^{\varepsilon})$ solutions $(y_1, y_2) \in \mathbb{Z}^2$ with $|y_1|, |y_2| \leqslant N$ to the equation*

$$P_{c,d}(y_1, y_2) := c(3y_1^2 y_2 - y_2^3) + d(y_1^3 - 3y_1 y_2^2) = \alpha.$$

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PROOF. Without loss of generality, we assume that $\gcd(c,d)=1$ since $\alpha \leqslant N^{29/100}$ and there are at most $\tau(\alpha) \ll_\varepsilon N^\varepsilon$ possible cases to determine. We first prove that $P(y_1,y_2) := P_{c,d}(y_1,y_2)$ is not a product of linear factors. Suppose that there exists a linear factor $qy_1 - py_2$ with $p,q \in \mathbb{Z}$ and $\gcd(p,q)=1$. Then,

$$dp^3 + 3cp^2q - 3dpq^2 - cq^3 = 0,$$

which implies that $p \mid c$ and $q \mid d$. Let $c = pc', d = qd'$ with $\gcd(c',d')=1$. Substituting in the equation, we obtain

$$\frac{c'+3d'}{3c'+d'} = \frac{p^2}{q^2},$$

from which, by parity checking, $\gcd(c'+3d', 3c'+d') = 1, 2$ or 4. If we restrict ourselves to the cases $\gcd(c'+3d', 3c'+d') = 1, 2$, we have either $c'+3d' = p'^2$ and $3c'+d' = q'^2$ or $c'+3d' = 2p'^2$ and $3c'+d' = 2q'^2$ as the possible systems of equations in integers. In each case, $p'^2 - 3q'^2 \equiv 0 \pmod 4$, which is impossible for $\gcd(p',q')=1$. In this case, $P(y_1,y_2)$ is irreducible over $\mathbb{Q}$ and we obtain from the main theorem of Bombieri and Schmidt [3] involving Thue's equation that the bound is $O(3^{1+\sup_{d|\alpha}\omega(d)})$, where $\omega(d)$ is the number of distinct prime factors of d. We have $3^{\sup_{d|\alpha}\omega(d)} \leqslant 3^{\omega(\alpha)} \leqslant \tau(\alpha)^{\log 3/\log 2} \ll_\varepsilon N^\varepsilon$. The total bound is therefore $O_\varepsilon(N^\varepsilon)$.

When $\gcd(c'+3d', 3c'+d') = 4$, we have

$$c = \frac{p(3q^2-p^2)}{2}, \quad d = \frac{q(3p^2-q^2)}{2}.$$

The discriminant of the quadratic polynomial

$$\frac{P(y_1,y_2)}{qy_1 - py_2} =: ay_1^2 + by_1y_2 + ey_2^2 \in \mathbb{Q}[y_1,y_2]$$

is $b^2 - 4ae = 3(p^2-q^2)^2$. This means that it is not possible to factor further into two linear factors over $\mathbb{Q}$. If there is no such linear factor, $P(y_1,y_2)$ is irreducible over $\mathbb{Q}$. Thus, $P(y_1,y_2)$ is not a product of linear factors.

This leaves the case in which $P(y_1,y_2)$ has a quadratic factor. In this case, we bound the number of solutions of $ay_1^2 + by_1y_2 + ey_2^2 = \beta$ for a certain divisor β of α. Since there are at most $\tau(\alpha) \ll_\varepsilon N^\varepsilon$ divisors, we may consider the case when β produces the maximum number of solutions. The equation can be rearranged as

$$(2ay_1 + by_2)^2 - (4ae - b^2)y_2^2 = 4a\beta. \tag{4.6}$$

The number of pairs (y_1,y_2) that satisfy (4.6) is at most the number of pairs $(z_1,z_2) \in \mathbb{Z}^2$ satisfying $z_1^2 - \gamma z_2^2 := z_1^2 - (4ae-b^2)y_2^2 = 4a\beta$ and $|z_1|, |z_2| \ll N^{1001}$. We obtain the bound $O_\varepsilon(N^\varepsilon)$ in this case, since $|\gamma| \ll N^{2000}$ and $4a\beta \ll N^{1001}$. Hence, the lemma follows. □

Now, let us count the number of solutions to (4.4) and (4.5). When $b \geqslant 2$, we set $X = L_1$ in Lemma 4.2. In (4.5), Lemma 4.1 implies $\|G_1\| \ll N^\kappa$, where $\kappa = \kappa(\eta)$.

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Then, by Lemma 4.2, the number of solutions to (4.5) is $O_\eta(L_1^{1+\eta}N^\eta)$. Each of these solutions produces at most one solution in (4.4). Therefore, the total number of solutions that satisfy both equations is $O_\eta(N^\eta L_1^{1+\eta})$. Similarly, for $a \geqslant 2$, we obtain the contribution $O_\eta(N^\eta T_1^{1+\eta})$.

For $b = 0$, (4.5) becomes $\tilde{G}_1(\lambda_1, \lambda_2) = 0$ for some homogeneous polynomial $\tilde{G}_1$. This corresponds to $O(1)$ pairs of (λ_1, λ_2), since $\gcd(\lambda_1, \lambda_2) = 1$. Each such d produces $O(\log N)$ pairs (n, e) satisfying $N/2 \leqslant n < N$ and $E/2 \leqslant e < E$ in the equation $e^2d^3 = n^2 + \alpha^2$ from the Estermann bound. The total contribution is $O(\log N)$. In the case $a = 0$, (4.5) becomes $\tilde{G}_1(\tau_1, \tau_2) = 0$ for a certain homogeneous polynomial $\tilde{G}_1$. The number of possible (τ_1, τ_2) is $O(1)$ in this case. We then substitute each (τ_1, τ_2) into (4.4) and count the (y_1, y_2) in (4.1) for fixed (x_1, x_2). By Lemma 4.3, the total contribution is $O_\eta(N^\eta)$.

Then, on considering the cases $b = 1$ and $a = 1$ as in the work of Heath-Brown (see the discussion before [13, Lemma 3]), we obtain the following lemma.

LEMMA 4.4. *For $\eta > 0$, in an interval I, we have $\mathcal{M}(E, D) \ll_\eta N^\eta \min(L_1^{1+\eta}, T_1^{1+\eta})$.*

PROOF OF THEOREM 1.1. After fixing a parameter M as in Lemma 4.1, we count the possible intervals by fixing $L \ll N$ and consider each interval $L < L_1 \leqslant 2L$. Then, we sum the total contribution dyadically on L by applying Lemma 4.4. If (y_1, y_2) corresponds to $\mathbf{g}^{(1)}$, then $L_1(y_1 - s_0y_2) \ll M^{-1}\sqrt{D}$ and $L_1y_2 \ll \sqrt{D}$. Fix $\varepsilon > 0$. If $L \gg D^{1/2+\varepsilon}$, then $y_2 = 0$ because $L|y_2| \ll L_1|y_2| \ll \sqrt{D}$ and $y_2 \in \mathbb{Z}$. Also, $y_1 = 0$ from the first inequality, which is not possible. Hence, $L_1 \ll_\varepsilon D^{1/2+\varepsilon}$. Write $s_0 = y_3M^{-1}$ in $I = (s_0, s_0 + M^{-1}]$ so that $y_3 \ll M$ since $s_0 \ll 1$. Therefore, the number of intervals for which $L < L_1 \leqslant 2L$ is at most the number of triples $(y_1, y_2, y_3) \in \mathbb{Z}^3$ with $\gcd(y_1, y_2) = 1$ for which

$$y_2y_3 = My_1 + O(L^{-1}\sqrt{D}), \quad y_2 \ll L^{-1}\sqrt{D} \quad \text{and} \quad y_3 \ll M.$$

Recall that $L_1 \gg L_2$ and $L_1L_2 \gg DM^{-1}$, so $L \gg D^{1/2}M^{-1/2}$. For each value of y_1, there are $O_\eta(N^\eta L^{-1}\sqrt{D})$ pairs (y_2, y_3). Since L_1 is of order L, there are $O_\eta(N^\eta L^{-2}D)$ intervals since $y_1 \ll L^{-1}\sqrt{D}$. From Lemma 4.4, $\mathcal{M}(E, D) \ll_\eta N^{2\eta}L^{-2}D(L^{1+\eta}) \ll N^{3\eta}\sqrt{DM}$ since $L \gg D^{1/2}M^{-1/2}$. Analogously, we obtain $\mathcal{M}(E, D) \ll N^{3\eta}\sqrt{ME}$. Therefore, we conclude that $\mathcal{M}(E, D) \ll N^{3\eta}\sqrt{M}\min(\sqrt{E}, \sqrt{D})$.

Let M satisfy the condition in Lemma 4.1, dependent on E, D. By Lemma 4.4, the total contribution in each dyadic range is at most

$$N^\eta\Bigg(\sum_{\ell \ll \log N} \sum_{\substack{N/2^\ell < E \leqslant N/2^{\ell-1} \\ E^2D^3 \asymp N^2}} \sqrt{M}\min(\sqrt{E}, \sqrt{D})\Bigg)$$
$$\ll_\eta N^\eta\Big((\log N) \sup_{\psi \in [0,1]} N^{3\psi(1-\psi)/8 + \min(\psi/2, (1-\psi)/3)}\Big) \ll_\eta N^{\varpi+\eta},$$

where $\varpi = 29/100$ for the optimal choice $\psi = \frac{2}{5}$. This completes the proof. □

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5. Comments

We give some comments on possible unconditional extensions of this work. There are two interesting approaches that seem feasible for further investigation.

(1) *The extension to higher-degree polynomials.* A similar approach to that used in Theorem 1.1 can be used to show for all $m \geqslant 1$ and $\alpha \in \mathbb{N}$ that

$$\#\{n \leqslant N : n^{2m} + \alpha^2 \text{ is square-full}\} \ll_{\varepsilon,\alpha,m} N^{1/2-3(m+6)/4(2m+3)^2+\varepsilon}$$

for any fixed $\varepsilon > 0$. When $m > 1$, the bound is worse than in the quadratic case because we are unable to exploit the mth power condition. If we could modify the argument and exploit this condition, it might be possible to show that the upper bound approaches N^{ε} as the degree of the polynomial increases. Such a result would be significant since we tend to lose control of the error terms for higher-degree polynomials, similar to the square-free case. In fact, we expect that $f(n)$ assumes square-full values finitely many times for any fixed polynomial $f(x)$ of degree at least 3 without repeated roots. For general polynomials with no repeated roots, one might need to develop analogous results on the counting of integral points on *superelliptic curves*, defined by the equation $y^k = f(x)$, where $k \geqslant 2$ and $\deg(f) \geqslant 3$.

(2) *Reducing* ϖ. To reduce ϖ in the quadratic case, particularly in Theorem 1.1, one might focus on the contribution when $e \in [N^{2/5-\delta}, N^{2/5+\delta}]$ for some $\delta > 0$ as this region determines the general bound. As in Remark 3.5 and the discussion around (2.2), we consider counting the number of $(e, d, n) \in \mathbb{N}^3$ satisfying $e^2d^3 = an^2 + b$ for a majorant $g(x) = ax^2 + b$ with conditions $E < e \leqslant 2E, D < d \leqslant 2D$ and $N < n \leqslant 2N$. It suffices to count rationals s, t corresponding to

$$t := \frac{e}{n} = \frac{a^{1/2}}{d^{3/2}} + O_b(N^{-2}) =: \phi_a(s) + O_b(N^{-2}),$$

where $s = 1/d$ and $t = e/n$. Then, we can employ the determinant method saving on the number of possible intervals since $e \leqslant N^{2/5+\delta}$. However, we might need to modify the argument to exploit the fact that s is not just rational, but is of the form $1/d$. It would also be interesting if one could push the reduction beyond the $\frac{2}{5}$ threshold assuming the Mordell curve conjecture.

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