Aesop's moral on success

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1. *Introduction*

Aesop's Fables is an enduring collection of short stories with morals that is credited to Aesop, a slave who lived in early Ancient Greece about 2600 years ago. Undoubtedly many later ancient Greek philosophers such as Pythagoras, Socrates, Aristotle and Archimedes were told Aesop's fables in their youth. In a race described in 'The Tortoise and the Hare', one of the most famous of Aesop's fables, a tortoise, running in a steady constant manner, beats a hare that is racing irregularly. The lesson of the fable is often interpreted as 'slow but steady wins the race' or 'consistent, effective effort leads to success' (see [1]) and is applicable to many human activities. The fable illustrates the general problem of working toward an objective when the rate of work is either constant or varies randomly.

It is proved in the present investigation that if the race between the tortoise and the hare is run many times with the hare's speed randomly varying, then the hare's average time to finish the race is greater than or equal to the tortoise's time to finish. This result validates the moral of the fable. To give a fair comparison of the finish times of the two contestants, it is assumed that the hare's average speed is equal to the tortoise's speed. Let the race length be equal to D, the tortoise's speed be equal to the constant v_T, and the hare's time for the race be equal to t_H. Then, the tortoise's time is equal to $T_T = D/v_T$ and the hare's average (expected) time is equal to $T_H = \mathbb{E}(t_H)$. It is shown that the hare's average time is greater than or equal to the tortoise's time, i.e. $T_H \geqslant T_T$.

First, as a preliminary illustrative example, a variation on the problem is briefly considered where it is assumed that the hare's speed, v_H, takes a fixed positive value for each race and $\mathbb{E}(v_H) = v_T$. That is, the hare's speed is constant for each race but may differ from one race to another with an average value equal to the tortoise's speed. The time for the hare to finish a race is $t_H = D/v_H$. By Jensen's inequality (see [2]),

$$\mathbb{E}(t_H) \geqslant \frac{1}{\mathbb{E}(1/t_H)} = \frac{1}{\mathbb{E}(v_H/D)} = \left(\frac{v_T}{D}\right)^{-1} = T_T.$$

That is, the hare's average time is greater than or equal to the tortoise's time. As a numerical example, suppose $D = 1$, $v_T = 1$, and v_H takes fixed values 1/2 and 4/3 with respective probabilities 2/5 and 3/5. Then $\mathbb{E}(v_H) = v_T = 1$, $T_T = 1$ and

$$\mathbb{E}(t_H) = \frac{2}{5} \times 2 + \frac{3}{5} \times \frac{3}{4} = \frac{5}{4} \geqslant T_T.$$

Although this interpretation of the race differs from that of the fable in which the hare's speed varies during the race, the main result is the same, i.e. the hare's average time is greater than or equal to the tortoise's time. In the remainder of the paper, the hare's speed varies randomly during the race.

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2. *Results*

Let the race distance be D with the hare's speed equal to the randomly varying (stochastic) process $v_H(t)$ where t is time during the race. The hare finishes at the first time t_H when $\int_0^{t_H} v_H(t)\,dt = D$. The tortoise runs at the constant speed v_T in time $T_T = D/v_T$. It is assumed that the average speed of the hare is equal to the tortoise's speed v_T. Let $T_H = \mathbb{E}(t_H)$ be the hare's average time, $q(t)$ be the probability density of times, and $Q(t)$ be the proportion of races not completed by time t. Then, $Q(t) = 1 - \int_0^t q(s)\,ds$ with $Q(0) = 1$, $Q(\infty) = 0$ and (see, e.g. [3])

$$q(t) = -\frac{dQ(t)}{dt} \text{ and } T_H = \int_0^\infty tq(t)\,dt = \int_0^\infty Q(t)\,dt.$$

Let $p_A(v, t)$ be the probability density function of the hare's speed for races that have not yet finished by time t and $\int_{-\infty}^{\infty} p_A(v, t)\,dv = 1$ for every $t \geqslant 0$. Let $A(v)\,dv$ be the probability that $v_H(t)$ is between v and $v + dv$ before the race finishes. Then $A(v)$ satisfies

$$A(v) = \int_0^\infty \frac{p_A(v, t)\,Q(t)}{T_H}\,dt \text{ and } \int_{-\infty}^{\infty} A(v)\,dv = 1. \tag{1}$$

To define the average speed of the hare, the corresponding races are considered where it is assumed that the races continue unrestricted with no finish distance. For such races, it is assumed that $\mathbb{E}(v_H(t)) = v_T$ for any $t \geqslant 0$. That is, the average speed of the hare is equal to the tortoise's speed at any time t for the unrestricted races. Let $p_B(v, t)$ be the probability density at time t of the hare's speed $v_H(t)$ for the unrestricted races. Then, $\int_{-\infty}^{\infty} p_B(v, t)\,dv = 1$ and $\int_{-\infty}^{\infty} vp_B(v, t)\,dv = v_T$ for any time $t \geqslant 0$. Let $B(v)\,dv$ be the probability that $v_H(t)$ is between v and $v + dv$ over the distribution of finish times. The density $B(v)$ satisfies

$$B(v) = \int_0^\infty \frac{p_B(v,t)Q(t)}{T_H}dt, \int_{-\infty}^{\infty} B(v)dv = 1 \text{ and } \int_{-\infty}^{\infty} vB(v)dv = v_T. \tag{2}$$

It is shown in Proposition 1 that $T_H \geqslant T_T = D/v_T$. To show this result, the two probability densities $A(v)$ and $B(v)$ are compared. The probability densities $A(v)$ and $B(v)$ are generally not known explicitly but two useful deductions can be made. First, although $A(v)$ is similar to $B(v)$, $A(v)$ is shifted to the left of $B(v)$. To see this, recall that a finish occurs at the first time t such that $\int_0^t v_H(z)\,dz = D > 0$ and so a finish can only occur at a time t when the speed, $v_H(t)$, is positive. For example, if the hare speed $v_H(t)$ is negative at t, the race cannot be finished. Finishing the race changes the proportion of time that the hare speed is spent in each interval $[v, v + dv]$. The proportion of time that the hare speed is spent in

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$[v, v + dv]$, in comparison with the proportion for unrestricted races, is larger if v is small and smaller if v is large, e.g. $A(v)\,dv \leqslant B(v)\,dv$ for v sufficiently large. As a result, the probability density $A(v)$ appears shifted to the left of density $B(v)$ and $\int_{-\infty}^{\infty} vA(v)\,dv \leqslant \int_{-\infty}^{\infty} vB(v)\,dv$. Furthermore, if $B(v)$ is symmetric about v_T, it follows similarly that $A(v)$ is skewed to the left and $A(v_T + z) \leqslant A(v_T - z)$ for $z \geqslant 0$. Second, $D/T_H = \int_{-\infty}^{\infty} vA(v)dv$. To see this, consider estimating $A(v)$ from N sample races by tabulating the total time, t_j, for the N races that the hare speed is in the jth interval $[v_j, v_{j+1}]$ for $-M \leqslant j \leqslant M$. Here $v_j = j\Delta v$, Δv is small, and $M\Delta v$ is sufficiently large that no time is spent outside $[-M\Delta v, M\Delta v]$. Then the probability $A(v_j)\,\Delta v$ that the speed is in the jth interval $[v_j, v_{j+1}]$ satisfies $A(v_j)\,\Delta v \approx t_j/T_{\text{tot}}$, where $T_{\text{tot}} = \sum_{j=-M}^{M} t_j$ is the total time before finishing for all races. Furthermore,

$$\int_{-\infty}^{\infty} A(v)dv \approx \sum_{j=-M}^{M} \frac{t_j}{T_H} = 1 \text{ and } \int_{-\infty}^{\infty} vA(v)dv \approx \sum_{j=-M}^{M} \frac{v_j t_j}{T_{\text{tot}}} \approx \frac{ND}{T_{\text{tot}}} \approx \frac{D}{T_{\text{tot}}}.$$

The next proposition follows from these deductions.

Proposition 1: Assume that $\mathbb{E}(v_H(t)) = v_T$ for any time t in an unrestricted race with no finish distance. Then, for a race of length D, the average time of the hare is greater than or equal to the tortoise's time, i.e. $T_H \geqslant D/v_T = T_T$.

Proof: The result follows from

$$\frac{D}{T_H} = \int_{-\infty}^{\infty} vA(v)\,dv \leqslant \int_{-\infty}^{\infty} vB(v)\,dv = v_T.$$

So, $D/v_T \leqslant T_H$ and the hare's average time is greater than or equal to the tortoise's time of $T_T = D/v_T$.

Example 1: *Hare speed discontinuous*

In this example, the hare either runs with speed $v_1 > 0$ or with speed 0. This is similar to Aesop's original race in which the hare naps. Specifically, it is assumed that during each time interval of length h, the hare is either running with speed v_1 or napping. The probability of running is p_1 and the probability of napping is $(1 - p_1)$ during each time interval. For the corresponding unrestricted problem, $\mathbb{E}(v_H(t)) = (1 - p_1)0 + p_1v_1 = v_T$. It is assumed in this example that $D/(hv_1) = N$ where N is a positive integer and so, the hare finishes each race after a number of intervals of napping and running but, importantly, always as soon as the total number of intervals of running reaches N. Applying Proposition 1 gives $T_H \geqslant D/v_T$. However, for this example, the average time of the hare can be determined exactly.

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Consider $k \geqslant N$ time steps each of length h. Let

$$P(N-1, k-1, p_1) = \binom{k-1}{N-1} p_1^{N-1} (1-p_1)^{k-N}$$

be the binomial probability that during exactly $N-1$ of the $k-1$ time steps the hare is running at speed v_1. Considering the kth time step as the Nth running step, then the probability of finishing at the kth time step is given by $P_f(k) = p_1 P(N-1, k-1, p_1)$. For this example, the following proposition shows that the average time of the hare is equal to the tortoise's time.

Proposition 2: Assume for each time interval h that the hare runs with speed v_1 with probability p_1 or rests with probability $(1-p_1)$. Also assume that $D/(hv_1) = N$ where N is a positive integer and $\mathbb{E}(v_h(t)) = p_1 v_1 = v_T$. Then the average time of the hare is identical to the tortoise's time, i.e. $T_H = D/v_T$.

Proof: The average time of the hare is

$$\begin{aligned}
\mathbb{E}(t_H) &= \sum_{k=N}^{\infty} khP_f(k) = \sum_{k=N}^{\infty} khp_1 \binom{k-1}{N-1} p_1^{N-1} (1-p_1)^{k-N} \\
&= Nhp_1^N \sum_{k=N}^{\infty} \binom{k}{N} (1-p_1)^{k-N} \\
&= Nhp_1^N (1-(1-p_1))^{-N-1} = \frac{Nh}{p_1} = \frac{D}{p_1 v_1} = T_T.
\end{aligned}$$

Examples 2-4: *Applications with rate continuous*

In the remaining examples, the randomly varying rate $v_H(t)$ is continuous in time t. The rate $v_H(t)$ is hare speed in Example 2, population growth rate in Example 3, and dividend rate in Example 4. In the three examples, the following procedure is used to define the rate as a continuous randomly varying function of time t. Letting h be a small time step, the discrete times $\{t_i\}_{i=0}^{\infty}$ are defined as $t_i = ih$ for $i = 0, 1, 2, \ldots$. The rate $v_H(t)$ is described at these discrete times in a simple iterative way using random numbers from a standard normal distribution as

$$v_H(t_i) = v_H(t_{i-1}) + \gamma(v_T - v_H(t_{i-1}))h + \beta\sqrt{h}\,\eta_i, \text{with } v_H(t_0) = v_H(0) = v_T, \quad (3)$$

for $i = 1, 2, \ldots$ where $\gamma > 0$ and β are constants and η_i are independent normally distributed random numbers with zero mean and unit variance. Given the value of the rate $v_H(t_{i-1})$ at time t_{i-1}, the first two terms on the right-hand side cause the rate $v_H(t_i)$ at time t_i to move closer to v_T while the third term causes $v_H(t_i)$ to move randomly away from v_T. As a result, the rate $v_H(t_i)$, for $i = 1, 2, \ldots$, randomly varies about the average value v_T. To make $v_H(t)$ continuous for $t \geqslant 0$, $v_H(t)$ is defined on each subinterval

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$[t_{i-1}, t_i]$ as the continuous piecewise-linear function

$$v_H(t) = \frac{v_H(t_i)(t - t_{i-1})}{h} + \frac{v_H(t_{i-1})(t_i - t)}{h}, \tag{4}$$

for $i = 1, 2, \ldots$. In particular, for the unrestricted race, $v_H(t_i)$ is normally distributed with average $\mathbb{E}(v_H(t_i)) = v_T$ and variance

$$\mathrm{Var}(v_H(t_i)) \approx \frac{\beta^2(1 - (1 - \gamma h)^{2i})}{2\gamma - \gamma^2 h}$$

for each i. In addition, for h small,

$$\mathrm{Var}(v_H(t_i)) \approx \frac{\beta^2(1 - \exp(-2\gamma t_i))}{2\gamma}.$$

Randomly varying processes such as (3) and (4) are commonly used, for example, in mathematical finance and in population biology. Indeed, as the time step h approaches zero, $v_H(t)$ approaches the solution of a mean-reverting stochastic differential equation (see [4, 5, 6, 7, 8]).

Example 2: *Hare speed continuous*

In this example, the hare's speed $v_H(t)$ satisfies (3) and (4) until the first time t_H such that the hare's distance $x_H(t_H) = \int_0^{t_H} v_H(t)\,dt = D$. The hare's speed $v_H(t)$ is continuous but varies randomly about the average value v_T. Indeed, for h small, the probability density $B(v)$ is approximately

$$\int_0^\infty \frac{Q(t)}{T_H} \frac{1}{\sqrt{2\pi\sigma^2(t)}} \exp\left(\frac{-(v - v_T)^2}{2\sigma^2(t)}\right) dt, \tag{5}$$

where $\sigma^2(t) = \dfrac{\beta^2(1 - \exp(-2\gamma t))}{2\gamma}$. Furthermore, as $\mathbb{E}(v_H(t)) = v_T$ for the unrestricted race, applying Proposition 1 gives $T_H \geqslant D/v_T$.

For this example, as $A(v)$ and $B(v)$ are not explicitly known, the probability densities are computed from 10,000 sample race simulations of (3) and (4) with $h = 0.01$ for $\gamma = 2$, $\beta = 2$, $v_T = 1$ and $D = 1$. For the simulations, the calculated average time of the hare is $T_H = 1.35 > T_T = 1$. In Figure 1, the hare's speed $v_H(t)$ and position $x_H(t) = \int_0^t v_H(t)\,dt$ are shown up to the finish distance of $D = 1$ for two different sample races. The calculated probability densities $A(v)$ and $B(v)$ are illustrated in Figure 2 showing that $B(v)$ is symmetric about v_T and $A(v)$ is slightly skewed to the left.

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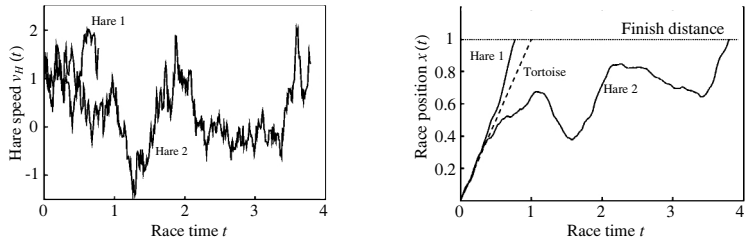


FIGURE 1: Hare speed $v_H(t)$ for two sample races up to the finish (left), Race distance $x_H(t) = \int_0^t v_H(t)\,dt$ for two sample races of the hare along with the tortoise distance $x_T(t) = v_T t$ up to the finish distance $D = 1$ (right)

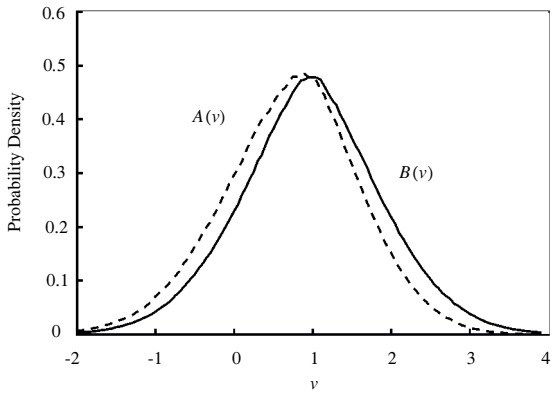


FIGURE 2: Calculated probability densities $A(v)$ (dashed) and $B(v)$ (solid) from 10,000 sample race simulations with $\gamma = 2, \beta = 2, v_T = 1, D = 1$

Example 3: *Population persistence*

Besides the tortoise-hare problem, another application of Proposition 1 involves the average persistence time of a population in a randomly varying environment. Let $y(t)$ be the population size and $r(t)$ be the intrinsic growth rate at time t. The size $y(t)$ satisfies the differential equation

$$\frac{dy(t)}{dt} = y'(t) = r(t)\,y(t).$$

Suppose that the intrinsic growth rate, $r(t)$, of the population is randomly varying due to changes in the environment (see [5, 9, 10]) as in (3) and (4) where η_i are independent normally distributed random numbers with zero mean and unit variance. A mathematical model for the population size in the varying environment has the form

$$\begin{cases} r(t_i) = r(t_{i-1}) + \gamma(r_\alpha - r(t_{i-1}))h + \hat{\beta}\sqrt{h}\,\eta_i, & \text{with } r(0) = r_\alpha, \\ r(t) = r(t_i)(t - t_{i-1})/h + r(t_{i-1})(t_i - t)/h, & \text{for } t_{i-1} \leqslant t \leqslant t_i, \\ y'(t) = r(t)\,y(t), & \text{with } y(0) = y_0 > 1, \end{cases} \tag{6}$$

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where it is assumed in this example that the initial population size, $y(0)$, is greater than unity and the average intrinsic growth rate, r_a, is negative. That is, $\mathbb{E}\left(r(t_i)\right) = r_a < 0$. Between discrete time points t_i for $i \geqslant 0$, $r(t)$ is continuous piecewise-linear. To study the average persistence time (average time until extinction) T_H, it is assumed that the population fails to persist if the population size reaches unity. At that point, $y(t_H) = 1$, the time, t_H, being the time for the population size, $y(t)$, to fall to unity. Changing variables to $x_H(t) = -\ln\left(y(t)/y_0\right)$, $v_H(t) = -r(t)$, $v_T = -r_a$ and $\beta = -\hat{\beta}$, (6) can be written as

$$\begin{cases} v_H(t_i) &= v_H(t_{i-1}) + \gamma(v_T - v_H(t_{i-1}))h + \beta\sqrt{h}\,\eta_i, \quad \text{with } v_H(0) = v_T, \\ v_H(t) &= v_H(t_i)(t - t_{i-1})/h + v_H(t_{i-1})(t_i - t)/h, \quad \text{for } t_{i-1} \leqslant t \leqslant t_i, \\ x_H'(t) &= v_H(t), \qquad\qquad \text{with } x_H(0) = 0. \end{cases} \tag{7}$$

For model (7), the time t_H satisfies

$$x_H(t_H) = \int_0^{t_H} v_H(s)\,ds = D = -\ln\left(\frac{1}{y_0}\right) = \ln(y_0).$$

For the unrestricted problem, $\mathbb{E}\left(v_H(t)\right) = v_T$.

The race problem given by (7) is identical to the problem in the previous example. Proposition 1 says that the average persistence time of the population, $y(t)$, is bounded below by $D/v_T = -\ln(y_0)/r_a$, i.e. $T_H \geqslant -\ln(y_0)/r_a$. In particular, if the average intrinsic growth rate r_a increases to zero, the average persistence time of the population goes to infinity, i.e. $T_H \to \infty$ as $r_a \to 0$.

Example 4: *Financial floating rate fund*

In a financial market, floating rate exchange-traded funds are composed of securities that have interest payments which change periodically. One such fund is Invesco Variable Rate Investment Grade VRIG which has at least 80% of its assets in variable or floating rate debt securities. It is assumed, for a fixed number of shares of the fund, that the randomly varying dividend payment rate (in dollars/month) can be modelled using a Vasicek interest rate model [11] in which the dividend rate, $v_H(t)$, satisfies (3) and (4) where $\gamma > 0$ and β are constants, v_T is the average dividend rate, and η_i are independent normally distributed random numbers with zero mean and unit variance. Let the total dividend payment amount by time t be denoted as $x_H(t)$. Then the dividend payment rate $v_H(t)$ and total payment amount $x_H(t)$ satisfy

$$\begin{cases} v_H(t_i) &= v_H(t_{i-1}) + \gamma(v_T - v_H(t_{i-1}))h + \beta\sqrt{h}\,\eta_i, \quad \text{with } v_H(0) = v_T, \\ v_H(t) &= v_H(t_i)(t - t_{i-1})/h + v_H(t_{i-1})(t_i - t)/h, \quad \text{for } t_{i-1} \leqslant t \leqslant t_i, \\ x_H'(t) &= v_H(t), \qquad\qquad \text{with } x_H(0) = 0, \end{cases} \tag{8}$$

for $i = 1, 2, \ldots$. The time to reach a total dividend payment amount of D is

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t_H where $x_H(t_H) = \int_0^{t_H} v_H(t)\,dt = D$. For model (8), $\mathbb{E}(v_H(t)) = v_T$ for the corresponding unrestricted problem and, by Proposition 1, the average time to reach the amount D is $\mathbb{E}(t_H) = T_H \geqslant D/v_T$. That is, the average time to reach a total dividend payment amount of D is bounded below by the time taken if the payments are made at the fixed average rate v_T. This example illustrates that a floating rate fund, which has some risk, needs to have a higher average dividend rate than a risk-free rate (such as the interest rate of a bank savings account) in order to motivate investors to buy it.

As an example, the 21 VRIG monthly dividends from April 2018 to December 2019 were 0.056, 0.058, 0.060, 0.062, 0.062, 0.062, 0.064, 0.065, 0.070, 0.069, 0.069, 0.068, 0.069, 0.070, 0.067, 0.064, 0.064, 0.063, 0.059, 0.055, 0.064. Fitting these data values to Vasicek's model (8), using a maximum likelihood estimation procedure, gives $v_T \approx 0.0647$, $\gamma \approx 0.297$ and $\beta \approx 0.00265$. Two sample simulations of $v_H(t)$ for model (8) with $h = 0.01$ and the fitted parameter values are displayed in Figure 3 along with the dividend data. Performing 10,000 simulations of (8) until $x_H(t)$ reaches $D = 21v_T$ produces the probability density of finish times shown in Figure 3. For the calculations, the finish times have average

$$T_H = 21.02 > T_T = D/v_T = 21 \text{ months}$$

and standard deviation 0.55. The results support Proposition 1 and imply that if a risk-free rate is equal to the average dividend rate v_T, then the average time for the dividend payments of the floating rate fund to reach a total amount D is greater than or equal to the time required for the risk-free investment.

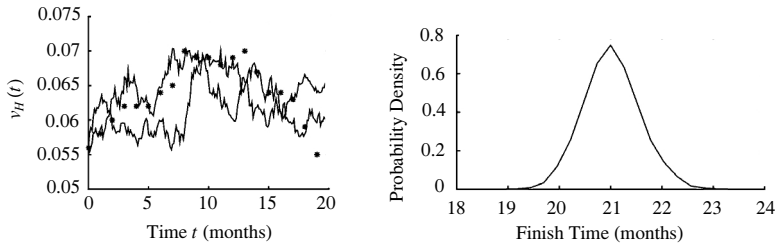


FIGURE 3: VRIG monthly dividends (stars) and two sample simulations of $v_H(t)$ from model (8) with $v_T = 0.0647$, $\gamma = 0.297$ and $\beta = 0.00265$ (left), probability density of finish times from 10,000 simulations of (8) (right)

3 *Summary*

The moral of Aesop's fable 'The Tortoise and the Hare', that steady effort leads to success, is examined. It is shown that the average time to finish a race is greater than or equal to the time taken if the race is run at the average speed. The result supports the message of Aesop's fable and is useful in applications. In general, the result indicates that the average time to complete a project performed at a randomly varying rate is greater than or

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equal to the time required when the project is performed steadily at the average rate.

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