Ergod. Th. & Dynam. Sys. (2004), **24**, 1753–1786
DOI: 10.1017/S0143385704000458
© *2004 Cambridge University Press*
Printed in the United Kingdom

KAM theorem for Gevrey Hamiltonians

G. POPOV

Université de Nantes, Département de Mathématiques, UMR 6629 du CNRS,
2, rue de la Houssinière, BP 92208, 44072 Nantes Cedex 03, France
(e-mail: popov@math.univ-nantes.fr)

(*Received* 2 *April* 2003 *and accepted* 29 *March* 2004)

Abstract. We consider Gevrey perturbations H of a completely integrable non-degenerate Gevrey Hamiltonian H^0. Given a Cantor set Ω_κ defined by a Diophantine condition, we find a family of Kolmogorov–Arnold–Moser (KAM) invariant tori of H with frequencies $\omega \in \Omega_\kappa$ which is Gevrey smooth in a Whitney sense. Moreover, we obtain a symplectic Gevrey normal form of the Hamiltonian in a neighborhood of the union Λ of the invariant tori. This leads to effective stability of the quasi-periodic motion near Λ.

1. *KAM theorem for Gevrey Hamiltonians*

Let D^0 be a bounded domain in $\mathbb{R}^n$, and $\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$, $n \geq 2$. We consider a class of real-valued Gevrey Hamiltonians H in $\mathbb{T}^n \times D^0$ which are small perturbations of a real-valued non-degenerate Gevrey Hamiltonian $H^0(I)$ depending only on the action variables $I \in D^0$. Our aim is to obtain a family of Kolmogorov–Arnold–Moser (KAM) invariant tori Λ_ω of H with frequencies ω in a suitable Cantor set Ω_κ defined by a Diophantine condition and to prove Gevrey regularity for it. A KAM invariant torus Λ_ω of H of frequency ω is a smooth embedded submanifold of the phase space which is invariant under the corresponding Hamiltonian flow $\exp(tX_H)$ and such that the restriction of $\exp(tX_H)$ to Λ_ω is smoothly conjugated to a rotation with $t\omega$ at $\mathbb{T}^n$. It turns out that, for each $\omega \in \Omega_\kappa$, Λ_ω is Gevrey smooth having the same Gevrey regularity as the Hamiltonian H. Moreover, we shall prove that the family Λ_ω, $\omega \in \Omega_\kappa$, is Gevrey smooth with respect to ω in a Whitney sense, with a Gevrey index depending on the Gevrey class of H and on the exponent in the Diophantine condition. In other words, we shall obtain integrability over a Cantor set of tori in Gevrey classes. This naturally involves anisotropic Gevrey classes. Integrability over a Cantor set of tori for C^∞ Hamiltonians is obtained by Lazutkin (see [**7**] for references), Pöschel [**12**], and Chierchia and Gallavotti [**2**].

We are going to work with anisotropic Gevrey classes which are defined as follows. Let $\rho_1, \rho_2 \geq 1$ and L_1, L_2 be positive constants. Given a domain $D \subset \mathbb{R}^n$, we denote

https://doi.org/10.1017/S0143385704000458 Published online by Cambridge University Press

by $\mathcal{G}^{\rho_1,\rho_2}_{L_1,L_2}(\mathbb{T}^n \times D)$ the set of all C^∞ real-valued Hamiltonians H in $\mathbb{T}^n \times D$ such that

$$\|H\|_{L_1,L_2} := \sup_{\alpha,\beta\in\mathbb{N}^n} \sup_{(\theta,I)\in\mathbb{T}^n\times D} (|\partial^\alpha_\theta \partial^\beta_I H(\theta, I)| L_1^{-|\alpha|} L_2^{-|\beta|} \alpha!^{-\rho_1} \beta!^{-\rho_2}) < \infty, \quad (1.1)$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\alpha! = \alpha_1! \cdots \alpha_n!$ for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$. In the same way we define $\mathcal{G}^{\rho_1,\rho_2}_{L_1,L_2}(\mathbb{T}^n \times \overline{D})$, where $\overline{D}$ is the closure of D. If $\rho_1 = \rho_2 = \rho$ we also write $\mathcal{G}^{\rho}_{L_1,L_2}(\mathbb{T}^n \times D)$, and sometimes we do not indicate the Gevrey constants L_1, L_2.

1.1. *KAM theorem.* Let $\mathbb{T}^n \times D^0 \ni (\theta, I) \to H^0(I) \in \mathbb{R}$ be a completely integrable real-valued Gevrey smooth Hamiltonian. We suppose that H^0 is non-degenerate, which means that the map $\nabla H^0 : D^0 \to \Omega^0$, $\Omega^0 := \nabla H^0(D^0)$, is a diffeomorphism. Denote by $g^0 \in C^\infty(\Omega^0)$ the Legendre transform of H^0 (then $\nabla g^0 : \Omega^0 \to D^0$ is the inverse map to ∇H^0). We also suppose that there are positive constants $\rho > 1$, $A_0 > 1$, and $L_0 \geq 1$ such that $H^0 \in \mathcal{G}^{\rho}_{L_0}(D^0)$, $g^0 \in \mathcal{G}^{\rho}_{L_0}(\Omega^0)$, and

$$\|H^0\|_{L_0}, \|g^0\|_{L_0} \leq A_0 \quad (1.2)$$

in the corresponding norms, defined as in (1.1). In particular, Ω^0 is a bounded domain. Given a subdomain D of D^0 we set $\Omega := \nabla H^0(D) \subset \Omega^0$. Fix $\tau > n - 1$ and $\kappa > 0$. We denote by Ω_κ the set of all frequencies $\omega \in \Omega$ having distance greater than or equal to κ to the boundary of Ω and also satisfying the Diophantine condition

$$|\langle \omega, k\rangle| \geq \frac{\kappa}{|k|^\tau} \quad \text{for all } 0 \neq k \in \mathbb{Z}^n, \quad (1.3)$$

where $|k| = |k_1| + \cdots + |k_n|$.

We are going to find a Gevrey family of KAM invariant tori with frequencies in Ω_κ for small perturbations of H^0 in $\mathcal{G}^{\rho}_{L_1,L_2}$ with $L_2 \geq L_1 \geq 1$ and $L_2 \geq L_0$. In what follows we fix the constants A_0 and L_0, and allow the constants $L_2 \geq L_1$ to be arbitrarily large. This occurs in the case of the elliptic equilibrium for example ($L_2 \gg 1$). Moreover, we are going to obtain Gevrey-type estimates for the corresponding canonical transformation and for the transformed Hamiltonian with Gevrey constants given explicitly in terms of L_1, L_2, and κ. For any $\omega \in \Omega$ we denote by $\mathcal{L}_\omega = \langle \omega, \partial_\varphi \rangle = \sum_{j=1}^n \omega_j \partial/\partial\varphi_j$ the corresponding vector field on $\mathbb{T}^n$. Fix $0 < \varsigma \leq 1$.

THEOREM 1.1. *Let H^0 be a real-valued non-degenerate $\mathcal{G}^\rho$-smooth Hamiltonian, $\rho > 1$, depending only on $I \in D^0$ and satisfying (1.2). Let D be a subdomain of D^0 with $\overline{D} \subset D^0$, and $\Omega = \nabla H^0(D)$. Fix $L_2 \geq L_1 \geq 1$ and $\kappa \leq L_2^{-1-\varsigma}$ such that $L_2 \geq L_0$ and $\Omega_\kappa \neq \emptyset$. Then there exist $N = N(n, \rho, \tau) > 0$ and $\epsilon > 0$ independent of κ, L_1, L_2, and of the domain $D \subset D^0$, such that for any $H \in \mathcal{G}^{\rho}_{L_1,L_2}(\mathbb{T}^n \times \overline{D})$ with norm*

$$\epsilon_H := \kappa^{-2} \|H - H^0\|_{L_1,L_2} \leq \epsilon L_1^{-N},$$

there exists a map $\overline{\Phi} := (\overline{U}, \overline{V}) : \mathbb{T}^n \times \Omega \to \mathbb{T}^n \times D$ of an anisotropic Gevrey class $\mathcal{G}^{\rho,\rho'}$, $\rho' = \rho(\tau + 1) + 1$, such that:

https://doi.org/10.1017/S0143385704000458 Published online by Cambridge University Press

(i) *for each* $\omega \in \Omega_\kappa$, $\Lambda_\omega := \{(\overline{\Phi}(\theta, \omega)) : \theta \in \mathbb{T}^n\}$ *is an embedded Lagrangian invariant torus of* H *and* $X_H \circ \overline{\Phi}(\cdot, \omega) = D\overline{\Phi}(\cdot, \omega) \cdot \mathcal{L}_\omega$;

(ii) *there are constants* $A > 0$ *and* $C > 0$, *independent of* κ, L_1, L_2, *and of* $D \subset D^0$, *such that*

$$|\partial_\theta^\alpha \partial_\omega^\beta (\overline{U}(\theta; \omega) - \theta)| + \kappa^{-1} |\partial_\theta^\alpha \partial_\omega^\beta (\overline{V}(\theta; \omega) - \nabla g^0(\omega))| \\ \le A C_1^{|\alpha|} (C_2/\kappa)^{|\beta|} \alpha!^\rho \beta!^{\rho(\tau+1)+1} L_1^{N/2} \sqrt{\epsilon_H},$$

uniformly in $(\theta, \omega) \in \mathbb{T}^n \times \Omega$ *and for any* $\alpha, \beta \in \mathbb{N}^n$, *where* $C_1 = CL_1$ *and* $C_2 = CL_1^{\tau+1}$.

Note that $\overline{\Phi}$ belongs to $\mathcal{G}^{\rho,\rho'}_{\bar{L}_1,\bar{L}_2}(\mathbb{T}^n \times \overline{D})$ with Gevrey constants $\bar{L}_1 = CL_1$ and $\bar{L}_2 = C_2\kappa^{-1} \ge CL_1^{\tau+1} L_2^{1+\varsigma}$. As a consequence, we obtain a symplectic normal form of H near the union of the invariant tori.

1.2. *Symplectic normal form and effective stability.* We say that a real-valued function $\Phi \in C^\infty(\mathbb{R}^n \times D)$ is a generating function of an exact symplectic map $\chi : \mathbb{T}^n \times D \to \mathbb{T}^n \times D$ if $\Phi(x, I) - \langle x, I\rangle$ is 2π-periodic with respect to x, $|\mathrm{Id} - \Phi_I| < 1$, and

$$\{(\varphi, I; \chi(\varphi, I)) : (\varphi, I) \in \mathbb{T}^n \times D\} \\ = \{(p(\Phi_I(x, I)), I; p(x), \Phi_x(x, I)) : (x, I) \in \mathbb{R}^n \times D\},$$

where $p : \mathbb{R}^n \to \mathbb{T}^n$ is the natural projection. Fix $\widetilde{\kappa} = \widetilde{\kappa}(D)$ so that the Lebesgue measure of $\Omega_{\widetilde{\kappa}}$ is positive. Define $\widetilde{\Omega}_\kappa$, $0 < \kappa \le \widetilde{\kappa}(D)$, to be the set of points of a positive Lebesgue density in Ω_κ. In other words, $\omega \in \widetilde{\Omega}_\kappa$ if, for any neighborhood U of ω in Ω, the Lebesgue measure of $U \cap \Omega_\kappa$ is positive. Obviously, $\widetilde{\Omega}_\kappa$ and Ω_κ have the same Lebesgue measure.

COROLLARY 1.2. *Suppose that the hypothesis of Theorem 1.1 holds and* $0 < \kappa \le \widetilde{\kappa}$. *Then there exist* $N = N(n, \rho, \tau) > 0$ *and* $\epsilon > 0$ *independent of* κ, L_1, L_2, *and* D, *such that for any* $H \in \mathcal{G}^\rho_{L_1,L_2}(\mathbb{T}^n \times \overline{D})$ *with* $\epsilon_H \le \epsilon L_1^{-N}$ *there is a* $\mathcal{G}^{\rho'}$*-diffeomorphism* $\omega : D \to \Omega$, *and an exact symplectic transformation* $\chi \in \mathcal{G}^{\rho,\rho'}(\mathbb{T}^n \times D, \mathbb{T}^n \times D)$ *defined by a generating function* $\Phi(x, I) = \langle x, I\rangle + \phi(x, I)$, $\phi \in \mathcal{G}^{\rho,\rho'}(\mathbb{T}^n \times D)$, *such that the transformed Hamiltonian* $\tilde{H}(\varphi, I) := H(\chi(\varphi, I))$ *belongs to* $\mathcal{G}^{\rho,\rho'}(\mathbb{T}^n \times D)$ *and* $\mathbb{T}^n \times \{I\}$ *is an invariant torus of* $\tilde{H}$ *for each* $I \in \omega^{-1}(\widetilde{\Omega}_\kappa)$. *The functions* $K(I) := \tilde{H}(0, I)$ *and* $R(\varphi, I) := \tilde{H}(\varphi, I) - K(I)$ *satisfy*

$$\text{for all } \alpha \in \mathbb{N}^n, \text{ for all } (\varphi, I) \in \mathbb{T}^n \times \omega^{-1}(\widetilde{\Omega}_\kappa), \quad \partial^\alpha \nabla K(I) = \partial^\alpha \omega(I), \quad \partial_I^\alpha R(\varphi, I) = 0.$$

Moreover, there exist $A, C > 0$ *independent of* κ, L_1, L_2, *and of the domain* $D \in D^0$, *such that*

$$|\partial_\varphi^\alpha \partial_I^\beta \phi(\varphi, I)| + |\partial_I^\beta (\omega(I) - \nabla H^0(I))| + |\partial_\varphi^\alpha \partial_I^\beta (\tilde{H}(\varphi, I) - H^0(I))| \\ \le A\kappa C_1^{|\alpha|} (C_2/\kappa)^{|\beta|} \alpha!^\rho \beta!^{\rho'} L_1^{N/2} \sqrt{\epsilon_H},$$

uniformly with respect to $(\varphi, I) \in \mathbb{T}^n \times D$ *and for any* $\alpha, \beta \in \mathbb{N}^n$, *where* $C_1 = CL_1$ *and* $C_2 = CL_1^{\tau+1}$.

https://doi.org/10.1017/S0143385704000458 Published online by Cambridge University Press

Denote by $\Omega \ni \omega \mapsto I(\omega) \in D$ the inverse map to the diffeomorphism $I \mapsto \omega(I)$ and set $E_\kappa = \omega^{-1}(\widetilde{\Omega}_\kappa)$. Then, for each $I \in E_\kappa$, the restriction of the Hamiltonian flow of $\tilde{H}$ to the invariant torus $\mathbb{T}^n \times \{I\}$ is given by $(t, \varphi, I) \mapsto (\varphi + t\nabla K(I), I)$.

Expanding $\partial_\varphi^\alpha \partial_I^\beta R(\varphi, I)$ in a Taylor series at some $I_0 \in E_\kappa$ such that $|I_0 - I| = |E_\kappa - I| = \inf_{I' \in E_\kappa} |I' - I|$, we obtain for any $\alpha, \beta \in \mathbb{N}^n$ and $m \in \mathbb{N}$

$$|\partial_\varphi^\alpha \partial_I^\beta R(\varphi, I)| \le \kappa A C_1^{|\alpha|} (C_2/\kappa)^{|\beta|+m} \alpha!^\rho \beta!^{\rho'} m!^{\rho'-1} |E_\kappa - I|^m, \quad (\varphi, I) \in \mathbb{T}^n \times D,$$

where the positive constants A, C_1, and C_2 are as above. Using Stirling's formula, we minimize the right-hand side with respect to $m \in \mathbb{N}$ for $I \notin E_\kappa$. An optimal choice for m will be

$$m \sim (C_2 |E_\kappa - I|/\kappa)^{-1/[\rho(\tau+1)]},$$

which leads to

$$|\partial_\varphi^\alpha \partial_I^\beta R(\varphi, I)| \le \kappa A C_1^{|\alpha|} (C_2/\kappa)^{|\beta|} \alpha!^\rho \beta!^{\rho'} \exp(-(C_2 |E_\kappa - I|/\kappa)^{-1/[\rho(\tau+1)]}) \qquad (1.4)$$

for any $\alpha, \beta \in \mathbb{N}^n$ uniformly with respect to $(\varphi, I) \in \mathbb{T}^n \times D,\ I \notin E_\kappa$, where $C_1 = CL_1$, $C_2 = CL_1^{\tau+1}$, and the positive constants A and C are independent of κ, L_1, L_2, and of the domain $D \in D^0$. This estimate yields effective stability of the quasi-periodic motion near the invariant tori as in [**14**].

COROLLARY 1.3. *For any $\lambda > 1$ and for any integral curve*

$$\exp(tX_{\widetilde{H}})(\varphi_0, I_0) = (\varphi(t), I(t))$$

of the Hamiltonian vector field $\widetilde{H}$ with initial data $(\varphi(0), I(0)) = (\varphi_0, I_0) \in \mathbb{T}^n \times D$, $I_0 \notin E_\kappa$, we have

$$|I(t) - E_\kappa| \le \lambda |I_0 - E_\kappa|$$

for

$$|t| \le (\lambda - 1)(\kappa A C_1)^{-1} |I_0 - E_\kappa| \exp((\lambda C_2 |I_0 - E_\kappa|/\kappa)^{-1/\rho(\tau+1)}).$$

Moreover, for any $0 < \varepsilon \le 1$ we have

$$\begin{aligned} &|I(t) - I_0| + (C_1\kappa/C_2)|\varphi(t) - \varphi_0 - t\nabla K(I_0)| \\ &\quad \le (\lambda - 1)|I_0 - E_\kappa| \exp(-(1-\varepsilon)(\lambda C_2 |I_0 - E_\kappa|/\kappa)^{-1/\rho(\tau+1)}), \end{aligned}$$

provided that

$$|t| \le (\lambda - 1)(\kappa A C_1)^{-1} \exp(\varepsilon(\lambda C_2/\kappa)^{-1/\rho(\tau+1)}),$$

where A, C_1, and C_2 are as above.

Effective stability of the action along all the trajectories for Gevrey smooth Hamiltonians has been obtained recently by Marco and Sauzun [**10**]. The importance of the Gevrey category for those kinds of problems is indicated by Lochak [**9**]. Giorgilli and Morbidelli [**3, 11**] proved in the case of quasi-convex analytic Hamiltonians that the action variable is almost invariant on any integral curve of H with initial data ε close to the KAM tori in a time interval which is super-exponentially long with respect to $1/\varepsilon$. Here, we have stability not only of the action but of the quasi-periodic motion as well in an exponentially long time interval. Moreover, we do not impose any steepness condition.

https://doi.org/10.1017/S0143385704000458 Published online by Cambridge University Press

Theorem 1.1 and Corollaries 1.2 and 1.3 hold in the case of a non-degenerate elliptic equilibrium for Gevrey Hamiltonians as in [**14**]. Indeed, let us consider the Birkhoff normal form of the Hamiltonian, namely, $H(\theta, I) = H^0(I) + H^1(\theta, I)$, where $H^0(I) = \langle \alpha^0, I\rangle + \langle QI, I\rangle$, with $\det Q \neq 0$, and $H^1(\theta, I) = O(|I|^{5/2})$, (θ, I) being suitable polar symplectic coordinates. Here, H^1 is Gevrey smooth in $\mathbb{T}^n \times D_a$, $D_a = \{c_0 a \leq I_j \leq c_0^{-1} a : j = 1, \ldots, n\}$, and $0 < a \leq a_0$, where $0 < c_0 < 1$ is fixed. More precisely, $H^1 \in \mathcal{G}^{\rho}_{L_1,L_2}(\mathbb{T}^n \times D_a)$ with norm $\|H^1\|_{L_1,L_2} = O(a^{5/2})$, where $L_2 = C_0 a^{-1}$, and the positive constants L_1 and C_0 are fixed. Then Theorem 1.1 holds choosing $\kappa = \delta a^{1+\varsigma}$, $0 < \varsigma < 1/4$, $0 < \delta \leq 1$, and $0 < a \leq a_0 \ll 1$.

As in [**14**] the symplectic normal form in Corollary 1.2 can be used to obtain a Gevrey quantum normal form over the corresponding family of invariant tori and to construct quasi-modes with exponentially small discrepancy in the semi-classical limit for Schrödinger-type operators with Gevrey coefficients. Similar results could also be obtained for more general classes of non-quasi-analytic Hamiltonians.

The idea of the proof of Theorem 1.1 is close to that of [**14**, Theorem 1] (see also [**6**]). It follows from a KAM theorem for a family of Hamiltonians $P(\theta, I; \omega)$, where the frequencies ω are taken as independent parameters. Here we follow closely the exposition of Pöschel [**13**]. First we prove an approximation lemma for Gevrey Hamiltonians P with real-analytic Hamiltonians P_j in suitable complex domains in §3.1. To obtain P_j we first construct a suitable almost analytic extension of P and then we use Green's formula. In §3.2 we recall from Pöschel [**13**] the KAM step and in §3.3 we set the parameters and make the iterations. Finally, using a Whitney extension theorem due to Bruna [**1**], we complete the proof of the theorem. In §3.6 we consider the case of real-analytic Hamiltonians and we improve certain results in [**14**]. We prove in Appendix A an anisotropic version of the implicit function theorem of Komatsu [**5**] in Gevrey classes.

2. *KAM theorem for Gevrey Hamiltonians with parameters*

Consider the Hamiltonian $H(\theta, z) = H^0(z) + H^1(\theta, z)$ in $\mathbb{T}^n \times D$. Denote by Ω' a neighborhood of Ω_κ in $\Omega = \nabla H^0(D)$ such that the distance between Ω_κ and the boundary $\partial\Omega'$ of Ω' is $\geq \kappa/2$ and the distance $|\Omega' - \partial\Omega| \geq \kappa/2$, and set $D' = \nabla g^0(\Omega')$, where g^0 is the Legendre transform of H^0. Expanding $H^0(z)$, $z \in D$, near a given $z_0 \in D'$, we write

$$H^0(z) = H^0(z_0) + \langle \nabla_z H^0(z_0), I\rangle + \int_0^1 (1-t)\langle \nabla_z^2 H^0(z_t) I, I\rangle\, dt,$$

where $z_t = z_0 + tI$, $z_1 = z$, I varies in a small ball $B_R = \{|I| < R\}$ in $\mathbb{R}^n$, and $\nabla_z^2 H^0$ stands for the Hessian matrix of H^0. In view of (1.2), we can take any positive $R \leq \kappa(4A_0 L_0^2)^{-1}$. We put $\omega = \nabla H^0(z_0)$. Then $z_0 = \nabla g^0(\omega)$, and we write

$$H^0(z) = e(\omega) + \langle \omega, I\rangle + \langle P^0(I; \omega) I, I\rangle,$$
$$H^1(\theta, z) = H^1(\theta, \nabla g(\omega) + I) = P^1(\theta, I; \omega),$$

where $e(\omega) = H^0(\nabla g(\omega))$, while P^0 stands for the quadratic term in I in the Taylor expansion of H^0. To simplify the notation, we are going to replace the constants $C\epsilon_H$, CA_0, CL_1, and CL_2 by ϵ_H, A_0, L_1, and L_2, respectively, whenever $C \geq 1$ depends

https://doi.org/10.1017/S0143385704000458 Published online by Cambridge University Press

only on L_0, ρ, τ, and n. Then using (1.1), (1.2), and Proposition A.3 we obtain $P^0 \in \mathcal{G}^{\rho}_{L_0,L_0}(B_R \times \overline{\Omega}') \subset \mathcal{G}^{\rho}_{L_2,L_2}(B_R \times \overline{\Omega}')$ and $P^1 \in \mathcal{G}^{\rho}_{L_1,L_2,L_2}(\mathbb{T}^n \times B_R \times \overline{\Omega}')$ with norms

$$\|P^0\|_{L_2,L_2} \le \|P^0\|_{L_0,L_0}, \quad \|P^1\|_{L_1,L_2,L_2} \le \kappa^2 \epsilon_H. \tag{2.1}$$

We consider the family of Hamiltonians

$$H(\theta, I; \omega) := e(\omega) + \langle \omega, I\rangle + P(\theta, I; \omega), \quad P(\theta, I; \omega) = \langle P^0(I; \omega)I, I\rangle + P^1(\theta, I; \omega) \tag{2.2}$$

in $\mathbb{T}^n \times B_R$ depending on the frequency $\omega \in \Omega'$. For any $0 < r < R$ and P as above we set

$$\langle P\rangle_r := r^2\|P^0\|_{L_2,L_2} + \|P^1\|_{L_1,L_2,L_2},$$

where the norms are taken in $\mathbb{T}^n \times B_r$ and $\mathbb{T}^n \times B_r \times \overline{\Omega}'$, respectively. Fix $0 < \varsigma \le 1$. We can now formulate our main result in this section.

THEOREM 2.1. *Suppose that H is given by (2.2) in $\mathbb{T}^n \times B_R \times \overline{\Omega}$, and let $\kappa < L_2^{-1-\varsigma}$. Then there exist $N = N(n, \rho, \tau) > 0$ and $\epsilon > 0$ independent of κ, L_1, L_2, R, and of the domain Ω in Ω_0, such that for any $0 < r < R$ and P satisfying*

$$\langle P\rangle_r = r^2\|P^0\| + \|P^1\| \le \epsilon\kappa r L_1^{-N} \tag{2.3}$$

there exist maps $\phi \in \mathcal{G}^{\rho'}(\Omega, \Omega)$ and $\Phi = (U, V) \in \mathcal{G}^{\rho,\rho'}(\mathbb{T}^n \times \Omega, \mathbb{T}^n \times B_R)$, $\rho' = \rho(\tau + 1) + 1$, such that:

(i) *for each $\omega \in \Omega_\kappa$, the map $\Phi_\omega := \Phi(\cdot, \omega) : \mathbb{T}^n \to \mathbb{T}^n \times B_R$ is a $\mathcal{G}^{\rho}$-embedding and $\Lambda_\omega := \Phi_\omega(\mathbb{T}^n)$ is an embedded Lagrangian torus invariant with respect to the Hamiltonian flow of $H_{\phi(\omega)}(\varphi, I) := H(\varphi, I; \phi(\omega))$; moreover, $X_{H_{\phi(\omega)}} \circ \Phi_\omega = D\Phi_\omega \cdot \mathcal{L}_\omega$ on $\mathbb{T}^n$;*

(ii) *there exist $A, C > 0$, independent of κ, L_1, L_2, r, and of $\Omega \subset \Omega_0$, such that*

$$|\partial_\theta^\alpha \partial_\omega^\beta (U(\theta; \omega) - \theta)| + r^{-1}|\partial_\theta^\alpha \partial_\omega^\beta V(\theta; \omega)| + \kappa^{-1}|\partial_\omega^\beta(\phi(\omega) - \omega)|$$
$$\le AC_1^{|\alpha|}(C_2/\kappa)^{|\beta|}\alpha!^{\rho}\beta!^{\rho'}\frac{\langle P\rangle_r}{\kappa r}L_1^N$$

uniformly in $(\theta, \omega) \in \mathbb{T}^n \times \Omega$ and for any α and β, where $C_1 = CL_1$ and $C_2 = CL_1^{\tau+1}$.

Remark. Note that the constant $e(\omega)$ in (2.2) plays no role in Theorem 2.1 since the Hamiltonian vector field X_H does not depend on it. From now on we take $e(\omega) \equiv 0$.

Theorem 2.1 will be proved in the next section. Theorem 1.1 and Corollary 1.2 follow from Theorem 2.1 and they will be proved in §4.

3. *Proof of Theorem 2.1*

We divide the proof of Theorem 2.1 into several steps. First, using Theorem 3.7, we extend P^j, $j = 0, 1$, to a Gevrey function $\widetilde{P}^j$ of the class $\mathcal{G}^{\rho}_{\widetilde{L}_1,\widetilde{L}_2}(\mathbb{T}^n \times \mathbb{R}^{2n})$ such that $\|\widetilde{P}^j\| \le AL_1^{n+1}\|P^j\|$, $\widetilde{L}_1 = CL_1$, and $\widetilde{L}_2 = CL_2$, where $A = A(n, \rho)$ and $C = C(n, \rho)$ are positive constants. To simplify the notation we drop the $\sim$. Multiplying P^j with a suitable cut-off function, we assume that the support of P^j with respect to (I, ω) is contained in $B_1 \times B_{\bar{R}}$, where $\bar{R} \gg 1$ is such that $\Omega_0 \subset B_{\bar{R}-1}$.

https://doi.org/10.1017/S0143385704000458 Published online by Cambridge University Press

3.1. *Approximation Lemma for Gevrey functions.* We fix $0 < \varsigma \le 1$ and choose three strictly decreasing sequences of positive numbers $\{u_j\}_{j=0}^{\infty}$, $\{v_j\}_{j=0}^{\infty}$ and $\{w_j\}_{j=0}^{\infty}$ tending to zero and such that

$$\text{for all } j \in \mathbb{N}: \quad v_j L_2, w_j L_2 \le u_j L_1 \le 1, \quad v_0, w_0 \le L_2^{-1-\varsigma}. \tag{3.1}$$

Consider the complex sets $\mathcal{U}_j^m$, $m = 1, 2$, in $\mathbb{C}^n/2\pi\mathbb{Z}^n \times \mathbb{C}^n \times \mathbb{C}^n$ consisting of all (θ, I, ω) such that

$$\begin{gathered} \operatorname{Re}\theta \in \mathbb{T}^n, \quad \operatorname{Re} I \in B_2, \quad \operatorname{Re}\omega \in B_{\bar{R}+1}, \\ |\operatorname{Im}\theta_k| \le m u_j, \quad |\operatorname{Im} I_k| \le m v_j, \quad |\operatorname{Im}\omega_k| \le m w_j, \quad 1 \le k \le n. \end{gathered} \tag{3.2}$$

Set $\mathcal{U}_j = \mathcal{U}_j^1$ and denote by $A(\mathcal{U}_j)$ the set of all real-analytic bounded functions in $\mathcal{U}_j$ equipped with the sup-norm $|\cdot|_{\mathcal{U}_j}$.

PROPOSITION 3.1. (Approximation Lemma) *Let $P \in \mathcal{G}^{\rho}_{L_1,L_2}(\mathbb{T}^n \times \mathbb{R}^{2n})$. Suppose that the support of P with respect to (I, ω) is in $B_1 \times B_{\bar{R}}$, and assume (3.1). Then there is a sequence $P_j \in A(\mathcal{U}_j)$, $j \ge 0$, such that*

$$\begin{aligned} |P_{j+1} - P_j|_{\mathcal{U}_{j+1}} &\le C_0 L_1^n \exp(-\tfrac{3}{4}(\rho-1)(2L_1u_j)^{-1/(\rho-1)})\|P\|, \\ |P_0|_{\mathcal{U}_0} &\le C_0(1 + L_1^n \exp(-\tfrac{3}{4}(\rho-1)(2L_1u_0)^{-1/(\rho-1)}))\|P\|, \end{aligned}$$

where $C_0 = \widetilde{C}_0(n, \rho, \varsigma)(\bar{R}^n + 1)$. Moreover,

$$\sup |\partial_\theta^\alpha \partial_I^\beta \partial_\omega^\gamma (P - P_j)(\theta, I, \omega)| \le C_0 L_1^n L_2 \exp(-\tfrac{3}{4}(\rho-1)(2L_1u_j)^{-1/(\rho-1)})\|P\|,$$

in $\mathbb{T}^n \times B_1 \times B_{\bar{R}}$ for $|\alpha| + |\beta| + |\gamma| \le 1$.

Remark. Instead of $3/4$ in the above we can take any positive number less than 1 in order to absorb certain polynomials of $(L_1u_j)^{-1}$. Similar estimates can be obtained without the inequalities $v_0, w_0 \le L_2^{-1-\varsigma}$. In this case $\widetilde{C}_0 = \widetilde{C}_0(n, \rho)$, but the right-hand side of the estimates above should be multiplied by L_2^{2n}. Using the 'standard' proof of the Approximation Lemma [**16**] one obtains for any $\delta > 0$ an approximation modulo $C(\rho, \delta)\exp(-c(\rho)(L_1u_j)^{-(1/\rho)+\delta})\|P\|$, $C, c > 0$.

Proof. We divide the proof into two parts.

(1) *Almost analytic extension of P.* There is a constant $C(\rho) \ge 1$, depending only on ρ, such that

$$t \in (0, 2], m \in \mathbb{N}, \quad 1 \le m \le t^{-1/(\rho-1)} + 1 \tag{3.3}$$

implies

$$t^m m!^{\rho-1} \le C(\rho) m^{(\rho-1)/2} e^{-(\rho-1)m}. \tag{3.4}$$

Indeed, by Stirling's formula, we get

$$\begin{aligned} t^m m!^{\rho-1} &\le C_1(\rho) m^{(\rho-1)/2} e^{-(\rho-1)m} \exp\{m[(\rho-1)\ln m + \ln t]\} \\ &= C_1(\rho) m^{(\rho-1)/2} e^{-(\rho-1)m} \exp\{(\rho-1)m \ln[m t^{1/(\rho-1)}]\}. \end{aligned}$$

Moreover,

$$m \ln[m t^{1/(\rho-1)}] \le m \ln[1 + t^{1/(\rho-1)}] \le m t^{1/(\rho-1)} \le 1 + 2^{1/(\rho-1)},$$

which proves (3.4).

https://doi.org/10.1017/S0143385704000458 Published online by Cambridge University Press

We define an almost analytic extension F_j of P in $\mathcal{U}_j^2$ as follows:

$$F_j(\theta + i\widetilde{\theta}, I + i\widetilde{I}, \omega + i\widetilde{\omega}) = \sum_{(\alpha,\beta,\gamma)\in\mathcal{M}_j} \partial_\theta^\alpha \partial_I^\beta \partial_\omega^\gamma P(\theta, I, \omega) \frac{(i\widetilde{\theta})^\alpha (i\widetilde{I})^\beta (i\widetilde{\omega})^\gamma}{\alpha!\beta!\gamma!}. \quad (3.5)$$

The index set $\mathcal{M}_j$ consists of all multi-indices $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n)$, and $\gamma = (\gamma_1, \dots, \gamma_n)$ such that $\alpha_k \le N_1$, $\beta_k \le N_2$ and $\gamma_k \le N_3$, $k = 1, \dots, n$, where

$$\begin{gathered} N_1 = [(2L_1u_j)^{-1/(\rho-1)}] + 1, \quad N_2 = [(2L_2v_j)^{-1/(\rho-1)}] + 1, \\ N_3 = [(2L_2w_j)^{-1/(\rho-1)}] + 1, \end{gathered} \quad (3.6)$$

and $[t]$ stands for the integer part of t. We have

$$|F_j|_{\mathcal{U}_j^2} \le \|P\| \sum_{(\alpha,\beta,\gamma)\in\mathcal{M}_j} (2L_1u_j)^{|\alpha|}(2L_2v_j)^{|\beta|}(2L_2w_j)^{|\gamma|}(\alpha!\beta!\gamma!)^{\rho-1}.$$

For $\alpha_k, \beta_k, \gamma_k \ne 0$ we estimate each term

$$(2L_1u_j)^{\alpha_k}\alpha_k!^{\rho-1}, \quad (2L_2v_j)^{\beta_k}\beta_k!^{\rho-1}, \quad (2L_2w_j)^{\gamma_k}\gamma_k!^{\rho-1}, \quad k = 1, \dots, n,$$

by $C(\rho)m^{(\rho-1)/2}e^{-(\rho-1)m}$, where $m \ge 1$ stands for α_k, β_k, and γ_k, respectively. To this end, we put $t = 2L_1u_j,\ 2L_2v_j,\ 2L_2w_j$, respectively, and we get $t \in (0, 2]$ in view of (3.1). Now (3.3) holds because of (3.6). Then using (3.4) we obtain

$$|F_j|_{\mathcal{U}_j^2} \le \|P\|\left(1 + C(\rho)\sum_{m=1}^{\infty} m^{(1-\rho)/2}e^{-(\rho-1)m}\right)^{3n} = C_1\|P\|.$$

Set $z_k = \theta_k + i\widetilde{\theta}_k$. Then applying $\bar{\partial}_{z_k} = (\partial/\partial\theta_k + i\partial/\partial\widetilde{\theta}_k)/2$ to F_j we obtain

$$2\bar{\partial}_{z_k}F_j(\theta + i\widetilde{\theta}, I + i\widetilde{I}, \omega + i\widetilde{\omega}) = \sum_{\substack{(\alpha,\beta,\gamma)\in\mathcal{M}_j \\ \alpha_k = N_1}} \partial_\theta^\alpha \partial_I^\beta \partial_\omega^\gamma \partial_{\theta_k} P(\theta, I, \omega) \frac{(i\widetilde{\theta})^\alpha (i\widetilde{I})^\beta (i\widetilde{\omega})^\gamma}{\alpha!\beta!\gamma!}. \quad (3.7)$$

We estimate each term in the sum by

$$L_1(2L_1u_j)^{|\alpha|}(2L_2v_j)^{|\beta|}(2L_2w_j)^{|\gamma|}(\alpha!\beta!\gamma!)^{\rho-1}(\alpha_k + 1)^\rho\|P\|$$

in $\mathcal{U}_j^2$, where $\alpha_k = N_1$. Since

$$(2L_1u_j)^{-1/(\rho-1)} \le \alpha_k = N_1 \le (2L_1u_j)^{-1/(\rho-1)} + 1,$$

we obtain from (3.4) (with $t = 2L_1u_j$ and $m = N_1$)

$$(2L_1u_j)^{\alpha_k}\alpha_k!^{\rho-1}(\alpha_k + 1)^\rho \le C'(L_1u_j)^{-\rho/(\rho-1)-1/2}\exp(-(\rho-1)(2L_1u_j)^{-1/(\rho-1)}).$$

This implies as above

$$\begin{gathered} |\bar{\partial}_{z_k}F_j|_{\mathcal{U}_j^2} \le C''L_1(L_1u_j)^{-\rho/(\rho-1)-1/2}\exp(-(\rho-1)(2L_1u_j)^{-1/(\rho-1)})\|P\| \\ |\bar{\partial}_{z_k}F_j|_{\mathcal{U}_j^2} \le CL_1\exp(-\tfrac{3}{4}(\rho-1)(2L_1u_j)^{-1/(\rho-1)})\|P\|, \end{gathered}$$

https://doi.org/10.1017/S0143385704000458 Published online by Cambridge University Press

where $C = C(\rho, n) > 0$. In the same way, differentiating (3.7), we get with $\beta + \gamma = 1$

$$|\partial_{\theta_p}^{\beta}\partial_{\widetilde{\theta}_p}^{\gamma}\bar{\partial}_{z_k}F_j|_{\mathcal{U}_j^2},\ |\partial_{I_p}^{\beta}\partial_{\widetilde{I}_p}^{\gamma}\bar{\partial}_{z_k}F_j|_{\mathcal{U}_j^2},\ |\partial_{\omega_p}^{\beta}\partial_{\widetilde{\omega}_p}^{\gamma}\bar{\partial}_{z_k}F_j|_{\mathcal{U}_j^2} \le CL_1L_2\exp(-\tfrac{3}{4}(\rho-1)(2L_1u_j)^{-1/(\rho-1)})\|P\|,$$

where $C = C(\rho, n) > 0$ (we recall that $L_2 \ge L_1 \ge 1$). Using (3.1), we obtain the same estimates for $\bar{\partial}_{I_k}F_j$, $\bar{\partial}_{\omega_k}F_j$, and for their derivatives of order one in $\mathcal{U}_j^2$. Indeed, putting $z_k = I_k + i\widetilde{I}_k$ we obtain

$$\begin{aligned}|\bar{\partial}_{z_k}F_j|_{\mathcal{U}_j^2} &\le C'L_2(L_2v_j)^{-\rho/(\rho-1)-1/2}\exp(-(\rho-1)(2L_2v_j)^{-1/(\rho-1)})\|P\|\\ &\le C\exp(-\tfrac{3}{4}(\rho-1)(2L_2v_j)^{-1/(\rho-1)})\|P\|\\ &\le C\exp(-\tfrac{3}{4}(\rho-1)(2L_1u_j)^{-1/(\rho-1)})\|P\|,\end{aligned}$$

where $C = C(\rho, n, \varsigma) > 0$, since $L_2 \le (L_2v_j)^{-1/\varsigma}$ by (3.1). We generalize these estimates as follows. Set $z = (\theta, I, \omega) \in \mathbb{C}^n/2\pi\mathbb{Z}^n \times \mathbb{C}^{2n}$; denote by x_k and y_k, respectively, the real and the imaginary part of z_k, $1 \le k \le 3n$, and put $\bar{\partial}_{z_k} = (\partial_{x_k} + i\partial_{y_k})/2$. Then using (3.1), (3.4), and (3.6), we obtain, for any $\delta = (\delta_1, \dots, \delta_{3n}) \in \mathbb{N}^{3n}$ with $0 \le \delta_k \le 1$ and $|\delta| \ge 1$, the estimate

$$|\bar{\partial}_z^{\delta}F_j|_{\mathcal{U}_j^2} \le CL_1^n\exp(-\tfrac{3}{4}(\rho-1)(2L_1u_j)^{-1/(\rho-1)})\|P\|,$$

where $C = C(n, \rho, \varsigma) > 0$. To this end, differentiating (3.5), we obtain an expression similar to (3.7), where for each k such that $\delta_k = 1$ we have $\alpha_k = N_1$ if $1 \le k \le n$, $\alpha_k = N_2$ if $n+1 \le k \le 2n$, and $\alpha_k = N_3$ if $2n+1 \le k \le 3n$, and then we proceed as above. More generally, for any $\delta = (\delta_1, \dots, \delta_{3n}) \in \mathbb{N}^{3n}$ with $0 \le \delta_k \le 1$ and $|\delta| \ge 1$, and any $\beta, \gamma \in \mathbb{N}^{3n}$ with $0 \le |\beta| + |\gamma| \le 1$, we obtain as above the estimate

$$|\partial_x^{\beta}\partial_y^{\gamma}\bar{\partial}_z^{\delta}F_j|_{\mathcal{U}_j^2} \le CL_1^nL_2^{|\beta|+|\gamma|}\exp(-\tfrac{3}{4}(\rho-1)(2L_1u_j)^{-1/(\rho-1)})\|P\|, \tag{3.8}$$

where $C = C(n, \rho, \varsigma) > 0$. Obviously, the same estimate holds for $\bar{\partial}_z^{\beta}\partial_z^{\gamma}\bar{\partial}_z^{\delta}F_j$ if $0 \le |\beta| + |\gamma| \le 1$. By construction we have $F_j(z) = \overline{F_j(\overline{z})}$.

(2) *Construction of* P_j. We are going to approximate F_j by functions which are real analytic in $\mathcal{U}_j^2$ using Green's formula,

$$\frac{1}{2\pi i}\int_{\partial D}\frac{f(\eta)}{\eta-\zeta}\,d\eta + \frac{1}{2\pi i}\iint_D\frac{\bar{\partial}_\eta f(\eta)}{\eta-\zeta}\,d\eta\wedge d\bar{\eta} = \begin{cases} f(\zeta) & \text{if } \zeta \in D,\\ 0 & \text{if } \zeta \notin \bar{D},\end{cases} \tag{3.9}$$

where $D \subset \mathbb{C}$ is a bounded domain symmetric with respect to the real axis and with a piecewise smooth boundary ∂D which is positively oriented with respect to D, $\bar{D} = D \cup \partial D$, and $f \in C^1(\bar{D})$. Notice that

$$F(\zeta) = \frac{1}{2\pi i}\int_{\partial D}\frac{f(\eta)}{\eta-\zeta}\,d\eta$$

is real analytic in D provided that $\overline{f(\overline{\eta})} = f(\eta)$ on ∂D.

We denote by $D_k \subset \mathbb{C}$ the open rectangle $\{|x_k| < a_k,\ |y_k| < b_k\}$, where $a_k = \pi$ and $b_k = 2u_j$ for $1 \le k \le n$, $a_k = 2$ and $b_k = 2v_j$ for $n+1 \le k \le 2n$, and $a_k = \bar{R} + 1$

https://doi.org/10.1017/S0143385704000458 Published online by Cambridge University Press

and $b_k = 2w_j$ for $2n+1 \le k \le 3n$. We also denote by ∂D_k the boundary of D_k which is positively oriented with respect to D_k and by Γ the union of the oriented segments $[-\pi - 2iu_j, \pi - 2iu_j] \cup [\pi + 2iu_j, -\pi + 2iu_j]$. Note that D_k and Γ also depend on j but we omit it here. Given $\eta \in \mathbb{C}$, we consider the 2π-periodic meromorphic function

$$\zeta \mapsto K(\eta, \zeta) = \frac{1}{\eta - \zeta} + K_1(\eta, \zeta),$$
$$K_1(\eta, \zeta) = \lim_{N \to +\infty} \sum_{k=1}^{N} \left(\frac{1}{\eta - \zeta + 2\pi k} + \frac{1}{\eta - \zeta - 2\pi k} \right).$$

Consider the function

$$F_{j,1}(z) := \frac{1}{2\pi i} \int_\Gamma F_j(\eta_1, z_2, \dots, z_{3n}) K(\eta_1, z_1)\, d\eta_1, \quad z \in \mathcal{U}_j^2.$$

It is analytic and 2π-periodic with respect to z_1 in the strip $\{|\operatorname{Im} z_1| < 2u_j\}$, and $F_{j,1}(z) = \overline{F_{j,1}(\overline{z})}$. Moreover, for $z_1 \in D_1$, we have

$$F_{j,1}(z) = \frac{1}{2\pi i} \int_{\partial D_1} F_j(\eta_1, z_2, \dots, z_{3n}) K(\eta_1, z_1)\, d\eta_1,$$

since the function under the integral is 2π-periodic with respect to η_1, and using (3.9) we obtain

$$F_{j,1}(z) = F_j(z) - \frac{1}{2\pi i} \int_{D_1} \bar\partial_{\eta_1} F_j(\eta_1, z_2, \dots, z_{3n}) K(\eta_1, z_1)\, d\eta_1 \wedge d\bar\eta_1.$$

By continuity, the last formula remains true for $\operatorname{Re} z_1 = \pm\pi$. Set $F_{j,0}(z) := F_j(z)$ and $\mathcal{U}_{j,1}^2 := \mathcal{U}_j^2 \cap \{|\operatorname{Im} z_1| \le u_j\}$. We claim that for any multi-index $\alpha = (0, \alpha_2, \dots, \alpha_{3n}) \in \mathbb{N}^{3n}$ with $0 \le \alpha_m \le 1$, $1 \le m \le 3n$, any index k, and $\beta, \gamma \in \mathbb{N}$ such that $0 \le \beta + \gamma \le 1$, we have

$$|\partial_{z_k}^\beta \bar\partial_{z_k}^\gamma \bar\partial_z^\alpha (F_{j,1} - F_{j,0})|_{\mathcal{U}_{j,1}^2} \le C L_1^n L_2^{\beta+\gamma} \exp(-\tfrac{3}{4}(\rho-1)(2L_1 u_j)^{-1/(\rho-1)}) \|P\|, \quad (3.10)$$

where $C = C(n, \rho, \varsigma) > 0$. For $k \ne 1$ the claim follows directly from (3.8), differentiating under the integral. To prove it for $k = 1$, we use the same argument for

$$\frac{1}{2\pi i} \int_{D_1} \bar\partial_{\eta_1} F_j(\eta_1, z_2, \dots, z_{3n}) K_1(\eta_1, z_1)\, d\eta_1 \wedge d\bar\eta_1.$$

On the other hand,

$$\frac{1}{2\pi i} \int_{D_1} \frac{\bar\partial_{\eta_1} F_j(\eta_1, z_2, \dots, z_{3n})}{\eta_1 - z_1}\, d\eta_1 \wedge d\bar\eta_1$$
$$= -\bar z_1 \bar\partial_{z_1} F_j(z) + \frac{1}{2\pi i} \int_{D_1} \frac{\bar\partial_{\eta_1} F_j(\eta_1, z_2, \dots, z_{3n}) - \bar\partial_{z_1} F_j(z)}{\eta_1 - z_1}\, d\eta_1 \wedge d\bar\eta_1$$

for $z_1 \in D_1$, which follows from (3.9) applied to $f(z_1) = \bar z_1$. Differentiating the last equality and using (3.8) we get the estimate. Moreover, if $|\alpha| \ge 1$, then (3.8) and (3.10) imply

$$|\partial_{z_k}^\beta \bar\partial_{z_k}^\gamma \bar\partial_z^\alpha F_{j,1}|_{\mathcal{U}_{j,1}^2} \le C L_1^n L_2^{\beta+\gamma} \exp(-\tfrac{3}{4}(\rho-1)(2L_1 u_j)^{-1/(\rho-1)}) \|P\|. \quad (3.11)$$

https://doi.org/10.1017/S0143385704000458 Published online by Cambridge University Press

We define $F_{j,m}(z)$, $2 \le m \le n$, by recurrence and we prove that it satisfies (3.10) in $\mathcal{U}^2_{j,m} := \mathcal{U}^2_{j,m-1} \cap \{|\operatorname{Im} z_m| \le u_j\}$ for $\alpha = (0, \dots, 0, \alpha_{m+1}, \dots, \alpha_{3n})$. Moreover, $F_{j,m}(z)$ satisfies (3.11) for $|\alpha| \ge 1$.

For $n < m \le 3n$ we define

$$F_{j,m}(z) = \frac{1}{2\pi i} \int_{\partial D_m} \frac{F_{j,m-1}(z_1, \dots, z_{m-1}, \eta_m, z_{m+1}, \dots, z_{3n})}{\eta_m - z_m} \, d\eta_m, \quad z \in \mathcal{U}^2_j,$$

and set $\mathcal{U}^2_{j,m} := \mathcal{U}^2_{j,m-1} \cap \{|\operatorname{Im} z_m| \le p_m\}$, where $p_m = v_j$ for $n+1 \le m \le 2n$ and $p_m = w_j$ for $2n+1 \le m \le 3n$. By recurrence with respect to m, we obtain (3.10) for $F_{j,m}$ in $\mathcal{U}^2_{j,m}$ for any $n < m \le 3n$, $\alpha = (0, \dots, 0, \alpha_{m+1}, \dots, \alpha_{3n})$, $0 \le \alpha_q \le 1$ ($\alpha = 0$ if $m = 3n$), for any index k and β, γ such that $0 \le \beta + \gamma \le 1$. Moreover, $F_{j,m}(z)$ satisfies (3.11) for $|\alpha| \ge 1$ and $m < 3n$. For $2n < m \le 3n$ the constant C should be replaced by $C'(1+\bar{R})^{m-2n}$, where $C' = C'(\rho, n, \varsigma)$. The factor $1+\bar{R}$ comes from the measure of D_m, $2n < m \le 3n$.

Set $P_j = F_{j,3n}$. By construction P_j is real analytic in $\mathcal{U}_j$ and for any index k and $\ell = 0, 1$, we have

$$|\partial^\ell_{x_k}(P_j - F_j)|_{\mathcal{U}_j} \le C\, L_1^n L_2^\ell \exp(-\tfrac{3}{4}(\rho-1)(2L_1 u_j)^{-1/(\rho-1)})\|P\|.$$

In particular,

$$\begin{aligned} |P_{j+1} - P_j|_{\mathcal{U}_{j+1}} &\le |P_{j+1} - F_{j+1}| + |P_j - F_j| + |F_{j+1} - F_j| \\ &\le CL_1^n \exp(-\tfrac{3}{4}(\rho-1)(2L_1 u_j)^{-1/(\rho-1)})\|P\|. \end{aligned}$$

Moreover, for any index k and $\ell = 0, 1$,

$$|\partial^\ell_{x_k}(P_j(x) - P(x))| \le CL_1^n L_2^\ell \exp(-\tfrac{3}{4}(\rho-1)(2L_1 u_j)^{-1/(\rho-1)})\|P\|$$

in $\mathcal{U}_j \cap \{\operatorname{Im} z = 0\}$, since $F_j(x) = P(x)$ for x real. Finally,

$$|P_0|_{\mathcal{U}_0} \le |F_0| + |P_0 - F_0| \le C(1 + L_1^n \exp(-\tfrac{3}{4}(\rho-1)(2L_1 u_0)^{-1/(\rho-1)}))\|P\|.$$

This completes the proof of the proposition. □

3.2. *The KAM step.* Introduce the complex domains

$$D_{s,r} = \{\theta \in \mathbb{C}^n/2\pi\mathbb{Z}^n : |\operatorname{Im}\theta| < s\} \times \{I \in \mathbb{C}^n : |I| < r\},$$
$$O_h = \{\omega \in \mathbb{C}^n : |\omega - \Omega_\kappa| < h\}.$$

The sup-norm of functions in $\mathcal{V} := D_{s,r} \times O_h$ will be denoted by $|\cdot|_{s,r,h}$. Fix $0 < \upsilon < 1/6$ and set $\widetilde{\upsilon} = 1/2 - 3\upsilon$ (later we shall choose $\upsilon = 1/54$ and $\widetilde{\upsilon} = 4/9$). Fix $0 < s, r < 1$, $0 < \eta < 1/8$, $0 < \sigma < s/5$, and $K \ge 1$. Consider the real-valued Hamiltonian $H(\theta, I; \omega) = N(I; \omega) + H_1(\theta, I; \omega)$, $N(I; \omega) = e(\omega) + \langle \omega, I \rangle$. We shall denote by 'Const.' a positive constant depending only on n and τ, and denote this constant by 'const.' if it is less than or equal to 1. We recall the following from Pöschel [**13**].

https://doi.org/10.1017/S0143385704000458 Published online by Cambridge University Press

PROPOSITION 3.2. *Let H be real analytic in $\mathcal{V}$. Suppose that $|H - N|_{s,r,h} \le \varepsilon$ with:*
(a) $\varepsilon \le \text{const.}\ \kappa\eta r\sigma^{\tau+1}$;
(b) $\varepsilon \le \text{const.}\ \upsilon hr$;
(c) $h \le \kappa/2K^{\tau+1}$.
Then there exists a real-analytic transformation

$$\mathcal{F} = (\Phi, \phi), \quad \Phi : D_{s-5\sigma,\eta r} \times O_h \longrightarrow D_{s,r} \times O_h, \quad \phi : O_{\widetilde{\upsilon}h} \longrightarrow O_h,$$

of the form $\Phi(\theta, I; \omega) = (U(\theta; \omega), V(\theta, I; \omega))$, with V affine linear with respect to I, where the transformation $\Phi(\cdot; \omega)$ is canonical for each ω, and such that $H \circ \mathcal{F} = N_+ + P_+$ with $N_+(I; \omega) = e_+(\omega) + \langle \omega, I \rangle$, and

$$|P_+|_{s-5\sigma,\eta r,\widetilde{\upsilon}h} \le \text{Const.} \left(\frac{\varepsilon^2}{\kappa r \sigma^{\tau+1}} + (\eta^2 + K^n e^{-K\sigma})\varepsilon \right). \tag{3.12}$$

Moreover,

$$|W(\Phi - \text{id})|,\ |W(D\Phi - \text{Id})W^{-1}| \le \text{Const.}\ \frac{\varepsilon}{\kappa r \sigma^{\tau+1}},$$
$$|\phi - \text{id}|,\ \upsilon h|D\phi - \text{Id}| \le \text{Const.}\ \frac{\varepsilon}{r},$$

uniformly on $D_{s-5\sigma,\eta r} \times O_h$ and $O_{\widetilde{\upsilon}h}$, respectively, where $W = \text{diag}(\sigma^{-1}\,\text{Id}, r^{-1}\,\text{Id})$.

Remark 3.3. Set $\overline{W} = \text{diag}(\sigma^{-1}\,\text{Id}, r^{-1}\,\text{Id}, h^{-1}\,\text{Id})$ and suppose that $h \le \kappa\sigma^{\tau+1}$. Since $1 - \widetilde{\upsilon} \ge 1/3$, using the Cauchy estimate with respect to ω, we obtain

$$|\overline{W}(\mathcal{F} - \text{id})|, |\overline{W}(D\mathcal{F} - \text{Id})\overline{W}^{-1}| \le \frac{C\varepsilon}{rh}, \quad C = C(n, \tau) > 0,$$

uniformly on $D_{s-5\sigma,\eta r} \times O_{\widetilde{\upsilon}h}$, where $D\mathcal{F}$ stands for the Jacobian of $\mathcal{F}$.

The proof of Proposition 3.2 is given in [**13**]. The only difference between the statement of Proposition 1.3 and that of the KAM step in [**13**] appears in the transformation of the frequencies ($\widetilde{\upsilon} = 1/4$ in [**13**]). To prove the proposition with $\widetilde{\upsilon}$ as above we use the following analog of [**13**, Lemma A.3].

LEMMA 3.4. *Suppose $f : O_h \to \mathbb{C}^n$ is real analytic with bounded $|f|_h$. Let $0 < \upsilon < 1/6$ and $\widetilde{\upsilon} = 1/2 - 3\upsilon$. If $|f - \text{id}|_h \le \upsilon h$, then f has a real-analytic inverse $\phi : O_{\widetilde{\upsilon}h} \to O_{2(\upsilon+\widetilde{\upsilon})h}$ and*

$$|\phi - \text{id}|_{\widetilde{\upsilon}h},\ 3\upsilon h|D\phi - \text{id}|_{\widetilde{\upsilon}h} \le |f - \text{id}|_h.$$

A sketch of proof of the Lemma is given in Appendix A. We are going to prepare the next iteration. We choose a 'weighted error' $0 < E < 1$, fix $0 < \widehat{\varepsilon} \le 1$, and set

$$\eta = E^{1/2}, \quad \varepsilon = \widehat{\varepsilon}\kappa E r \sigma^{\tau+1}, \quad 0 < E < 1/64.$$

We define K and h by

$$K^n e^{-K\sigma} = E, \quad h = \frac{\kappa}{2K^{\tau+1}}.$$

Setting $x = K\sigma$ we get the equation $x^n e^{-x} = E\sigma^n$, which has a unique solution with respect to $x \in [1, +\infty)$, since $0 < E < 1/64 < 1/e$. Then $K = x\sigma^{-1} > 1$. We set

$$r_+ = \eta r, \quad s_+ = s - 5\sigma, \quad \sigma_+ = \delta\sigma,$$

https://doi.org/10.1017/S0143385704000458 Published online by Cambridge University Press

where $0 < \delta < 1$. Later we shall choose $\delta = \delta(\rho)$ as a function of ρ only. Now the KAM step gives the estimate

$$\begin{aligned} |P_+|_{s_+,r_+,\widetilde{\upsilon}h} &< \text{Const.}\,\widehat{\varepsilon}\kappa r\sigma^{\tau+1}(E^2 + (\eta^2 + K^n e^{-K\sigma})E) \\ &= \text{Const.}\,\widehat{\varepsilon}\kappa r\sigma^{\tau+1}E^2 = \text{Const.}\,\delta(\rho)^{-\tau-1}\widehat{\varepsilon}\kappa r_+\sigma_+^{\tau+1}E^{3/2}. \end{aligned}$$

Hence there is a constant $c_1 > 1$ depending only on n, ρ, and τ such that

$$|P_+|_{s_+,r_+,\widetilde{\upsilon}h} \le \tfrac{1}{2}\widehat{\varepsilon}c_1^{1/2}\kappa r_+\sigma_+^{\tau+1}E^{3/2}.$$

We fix the weighted error for the iteration by $E_+ = c_1^{1/2}E^{3/2}$, set $\varepsilon_+ = \widehat{\varepsilon}\kappa r_+\sigma_+^{\tau+1}E_+$, and then define $\eta_+,\ x_+,\ K_+$, and h_+ as above. Note that $c_1E_+ = (c_1E)^{3/2}$. We also require $c_1E < 1$, which leads to an exponentially converging scheme. Suppose that

$$h_+ \le \widetilde{\upsilon}h. \tag{3.13}$$

Then we obtain

$$|P_+|_{s_+,r_+,h_+} \le \tfrac{1}{2}\varepsilon_+. \tag{3.14}$$

3.3. *Setting the parameters and iteration.* As in [**13**] we are going to iterate the KAM step infinitely many times choosing appropriately the parameters $0 < s, r, \sigma, h, \eta < 1$ and so on. Our goal is to get a convergent scheme in the Gevrey spaces $\mathcal{G}^{\rho,\rho(\tau+1)+1}$. We are going to define suitable strictly decreasing sequences of positive numbers $\{s_j\}_{j=0}^{\infty}$, $\{r_j\}_{j=0}^{\infty}$ and $\{h_j\}_{j=0}^{\infty}$, tending to zero, and denote

$$D_j = D_{s_j,r_j}, \quad O_j = O_{h_j}, \quad \mathcal{V}_j = D_j \times O_j.$$

Fix $\delta \in (0, 1)$ (δ will depend only on ρ) and set

$$s_j = s_0\delta^j, \quad \sigma_j = \sigma_0\delta^j, \quad s_0(1-\delta) = 5\sigma_0.$$

Obviously, $s_{j+1} = s_j - 5\sigma_j$ and $\sigma_j = 5^{-1}(1-\delta)s_j$ for $j \ge 0$. We set

$$u_j = 4s_j = 4s_0\delta^j, \quad v_j = 4r_0\delta^j, \quad w_j = 4h_0\delta^j,$$

We assume for the moment that these sequences verify (3.1) and denote by $\mathcal{U}_j^1$ the corresponding complex sets defined by (3.2). Set

$$\mathcal{U}_j := \{(\theta, I, \omega) \in \mathcal{U}_j^1 : |I| < r\}.$$

Then applying Proposition 3.1 to P^0 and P^1, we obtain functions P_j^0 and P_j^1 which are real analytic in $\mathcal{U}_j^1$ such that $P_j(I;\omega) = \langle P_j^0(I;\omega)I, I\rangle + P_j^1(\varphi, I;\omega)$ satisfy

$$\begin{aligned} |P_0|_{\mathcal{U}_0} &\le C_0L_1^{2n+1}\langle P\rangle_r \\ |P_j - P_{j-1}|_{\mathcal{U}_j} &\le C_0L_1^{2n+1}\langle P\rangle_r e^{-\widetilde{B}_0 s_j^{-1/(\rho-1)}} \\ &= C_0L_1^{2n+1}\langle P\rangle_r e^{-B_0\sigma_j^{-1/(\rho-1)}}, \ j \ge 1. \end{aligned} \tag{3.15}$$

https://doi.org/10.1017/S0143385704000458 Published online by Cambridge University Press

We recall that the norm $\langle P\rangle_r = r^2\|P^0\|_{L_2,L_2} + \|P^1\|_{L_1,L_2,L_2}$ is taken in $\mathbb{T}^n \times B_r \times \overline{\Omega}$ and the positive constants $\widetilde{B}_0$ and B_0 depend only on ρ and δ. In view of (2.3) for any $0 < a \le 1$ we have

$$\widehat{\varepsilon} := \langle P\rangle_r L_1^{N-n-2}(a\kappa r)^{-1} \le \langle P\rangle_r L_1^N(\kappa r)^{-1} \le 1. \tag{3.16}$$

Introduce

$$\widetilde{\varepsilon}_j = \widehat{\varepsilon}\kappa r_0 \sigma_0^{\tau+1} \exp(-B_0\sigma_j^{-1/(\rho-1)}).$$

Later we will choose $N = N(n,\tau,\rho)$ and $0 < a \le 1$, $a = a(n,\rho,\tau,\varsigma,\overline{R})$ independent of κ, L_1, L_2, and r, so that $|P_0|_{\mathcal{U}_0} \le \widetilde{\varepsilon}_0$ and $|P_j - P_{j-1}|_{\mathcal{U}_j} \le \widetilde{\varepsilon}_j$ for $j \ge 1$. Now we put

$$E_j := c_1^{-1}\exp(-B\sigma_j^{-1/(\rho-1)}) \quad \text{with } B := \frac{B_0}{2}(\delta^{-1/(\rho-1)} - 1),$$

where $c_1 > 1$ is the constant in the KAM step. We find $\delta \in (0,1)$ from the equalities

$$\text{for all } j \in \mathbb{N}, \quad E_{j+1} = c_1^{1/2}E_j^{3/2}.$$

This is equivalent to $\sigma_{j+1} = (2/3)^{\rho-1}\sigma_j$, and we get $\delta = (\frac{2}{3})^{\rho-1}$, which implies $B = B_0/4 = A_0 {L_1}^{-1/(\rho-1)}$, where $A_0 = A_0(\rho) > 0$ depends only on ρ. Now we set $\eta_j = E_j^{1/2}$, $r_{j+1} = \eta_j r_j$, and put

$$\varepsilon_j = \widehat{\varepsilon}\kappa r_j \sigma_j^{\tau+1} E_j.$$

The choice of the 'weighted error' E_j above is motivated by the inequality $\widetilde{\varepsilon}_j \le \varepsilon_{j+1}/2$, $j \ge 0$, which will be proved in (3.21). This inequality will allow us to put $P_j - P_{j-1}$ in the error term of the iteration of order j. Next we determine K_j from the equation $K_j^n e^{-K_j\sigma_j} = E_j$. Setting $x_j = K_j\sigma_j$, we obtain

$$x_j^n e^{-x_j} = E_j\sigma_j^n = c_1^{-1}\sigma_j^n \exp(-B\sigma_j^{-1/(\rho-1)}).$$

Consider the equation

$$x_j - n\ln x_j = B\sigma_j^{-1/(\rho-1)} - n\ln(\sigma_j) + \ln c_1. \tag{3.17}$$

We set

$$\sigma_0 = \sigma L_1^{-1}(\ln(L_1+e))^{-(\rho-1)}, \quad 0 < \sigma \le \widetilde{\sigma}(n,\rho) \ll 1. \tag{3.18}$$

Obviously, $\sigma_0 L_1 < \sigma \le \widetilde{\sigma}(n,\rho) \ll 1$, and for any $j \in \mathbb{N}$ we obtain

$$B\sigma_j^{-1/(\rho-1)} - n\ln(\sigma_j) + \ln c_1 \ge B\sigma_0^{-1/(\rho-1)} = A_0(L_1\sigma_0)^{-1/(\rho-1)} > A_0\sigma^{-1/(\rho-1)} \gg 1.$$

Hence, choosing $0 < \sigma \le \widetilde{\sigma}(n,\rho) \ll 1$, we obtain for each $j \in \mathbb{N}$ a unique solution $x_j = x_j(\sigma)$ of (3.17) such that

$$x_j \ge x_j - n\ln x_j \ge B\sigma_j^{-1/(\rho-1)} \ge A_0\sigma^{-1/(\rho-1)} \gg 1.$$

Then $x_j - n\ln x_j = x_j(1+o(1))$ as $\sigma \searrow 0$. On the other hand, again using (3.18), we get

$$\begin{aligned} x_j - n\ln x_j &\le B\sigma_j^{-1/(\rho-1)}[1 - nA_0^{-1}(L_1\sigma_j)^{1/(\rho-1)}\ln(L_1\sigma_j) \\ &\quad + nA_0^{-1}(L_1\sigma_0)^{1/(\rho-1)}(\ln L_1 + \ln(c_1))] \\ &= B\sigma_j^{-1/(\rho-1)}(1+o(1)), \end{aligned}$$

https://doi.org/10.1017/S0143385704000458 Published online by Cambridge University Press

uniformly with respect to $j \in \mathbb{N}$. Hence,

$$B\sigma_j^{-1/(\rho-1)} \le x_j \le B\sigma_j^{-1/(\rho-1)}(1+o(1)), \quad \sigma \searrow 0, \tag{3.19}$$

uniformly with respect to $j \in \mathbb{N}$. We set $h_j = \kappa 2^{-1} K_j^{-\tau-1}$ and fix $\upsilon = 1/54$.

We are going to check the hypothesis (a) and (b) in Proposition 3.2 for any $j \ge 0$ ((c) is fulfilled by definition). To prove (a) we use (3.16) and the fact that $\eta_j^2 = E_j = o(1)$ as $\sigma \searrow 0$. Using (3.18) and (3.19) we obtain

$$\begin{aligned}\frac{\varepsilon_j}{r_j h_j} &= 2\widehat{\varepsilon} E_j x_j{}^{\tau+1} \le 2c_1^{-1}\exp(-B\sigma_j^{-1/(\rho-1)})(B\sigma_j^{-1/(\rho-1)})^{\tau+1}(1+o(1)) \\ &\le c(\rho,\tau)\exp\left(-\frac{A_0}{2}(L_1\sigma_j)^{-1/(\rho-1)}\right) \le c(\rho,\tau)\exp\left(-\frac{A_0}{2}(\sigma\delta^j)^{-1/(\rho-1)}\right).\end{aligned}$$

This implies that $\varepsilon_j (r_j h_j)^{-1} \ll$ const. υ for $0 < \sigma \le \widetilde{\sigma}(n,\rho,\tau) \ll 1$ which proves (b). In the same way, we obtain

$$\prod_{j=0}^{\infty}\left(1+\frac{C\varepsilon_j}{r_j h_j}\right) \le \exp\left(\sum_{j=0}^{\infty}\frac{C\varepsilon_j}{r_j h_j}\right) \le 2 \tag{3.20}$$

for $0 < \sigma \le \widetilde{\sigma}(n,\rho,\tau) \ll 1$, where $C = C(n,\tau) > 0$ is the constant in Remark 3.3. We are going to check (3.13) with $\upsilon = 1/54$. Using (3.19) we obtain $x_j/x_{j+1} = (\sigma_{j+1}/\sigma_j)^{1/(\rho-1)}(1+o(1))$. This implies

$$\frac{h_{j+1}}{h_j} = \left(\frac{x_j}{x_{j+1}}\right)^{\tau+1}\left(\frac{\sigma_{j+1}}{\sigma_j}\right)^{\tau+1} = \delta^{(\tau+1)\rho/(\rho-1)}(1+o(1)) = \left(\frac{2}{3}\right)^{\rho(\tau+1)}(1+o(1))$$

for $\sigma \searrow 0$, uniformly with respect to $j \in \mathbb{N}$. Since $\rho > 1$ and $\tau + 1 > n \ge 2$, we obtain

$$\frac{h_{j+1}}{h_j} < \left(\frac{4}{9}\right)^{\rho} < \frac{4}{9} = \widetilde{\upsilon},$$

for any $0 < \sigma \le \widetilde{\sigma}(n,\rho,\tau) \ll 1$, which proves (3.13).

Using the special choice of E_j, we are going to prove by induction that

$$\text{for all } j \in \mathbb{N}, \quad \widetilde{\varepsilon}_j \le \tfrac{1}{2}\varepsilon_{j+1}. \tag{3.21}$$

To obtain the estimate for $j = 0$ we write $\varepsilon_1 = \widehat{\varepsilon}\kappa r_1 \sigma_1^{\tau+1} E_1 = \widehat{\varepsilon}\kappa r_0 \sigma_0^{\tau+1}\delta^{\tau+1} c_1^{1/2} E_0^2$, and we obtain

$$\widetilde{\varepsilon}_0/\varepsilon_1 = c_1^{-1/2}(\tfrac{3}{2})^{(\tau+1)(\rho-1)}\exp(-2B\sigma_0^{-1/(\rho-1)}) \le 1/2$$

for $0 < \sigma \le \widetilde{\sigma}(n,\rho,\tau) \ll 1$, since $B_0 = 4B$. To prove it for $j+1 \ge 1$ we write

$$\varepsilon_{j+2} = \widehat{\varepsilon}\kappa r_{j+2}\sigma_{j+2}^{\tau+1} E_{j+2} = \widehat{\varepsilon}\kappa r_{j+1}\sigma_{j+1}^{\tau+1}\delta^{\tau+1} c_1^{1/2} E_{j+1}^2 = \varepsilon_{j+1} c_1^{1/2} E_{j+1}\delta^{\tau+1}.$$

Then for $j \ge 0$ we obtain

$$\frac{\widetilde{\varepsilon}_{j+1}}{\varepsilon_{j+2}}\left(\frac{\widetilde{\varepsilon}_j}{\varepsilon_{j+1}}\right)^{-1} = c_1^{-1/2}\left(\frac{3}{2}\right)^{(\tau+1)(\rho-1)}\exp\left(-\frac{1}{2}B\sigma_j^{-1/(\rho-1)}\right) \le 1$$

for $0 < \sigma \le \widetilde{\sigma}(n,\rho,\tau) \ll 1$ which implies (3.21) by recurrence. From now on we fix $\sigma = \sigma(n,\rho,\tau) \ll 1$, so that all the estimates above hold, and define σ_0 by (3.18).

https://doi.org/10.1017/S0143385704000458 Published online by Cambridge University Press

Then we set $s_0 = 5\sigma_0(1-\delta)^{-1}$. We are going to prove that the sequences $u_j = 4s_0\delta^j$, $v_j = 4r_0\delta^j$, and $w_j = 4h_0\delta^j$ verify (3.1) choosing $r_0 = cr$ and $c = c(n, \rho, \tau, \varsigma) \ll 1$. We have $4s_0L_1 \le 1$ in view of (3.18). Moreover, $h_0 < \kappa\sigma_0^{\tau+1} < \kappa s_0 \le L_2^{-1-\varsigma}s_0$, and we obtain $w_jL_2 \le u_jL_1$, and $w_j < L_2^{-1-\varsigma}$. Finally, $r_0 = cr \le cL_2^{-1-\varsigma} < L_2^{-1-\varsigma}$, and $r_0L_2 < cL_1^{-\varsigma} \le s_0L_1$, choosing appropriately $c = c(n, \rho, \tau, \varsigma) \ll 1$.

It remains to show that

$$|P_0|_{\mathcal{U}_0} \le \widetilde{\varepsilon}_0, \quad |P_j - P_{j-1}|_{\mathcal{U}_j} \le \widetilde{\varepsilon}_j, \quad j \ge 1, \tag{3.22}$$

for $a \ll 1$. In view of (3.15) we have

$$|P_0|_{\mathcal{U}_0} \le C_0\langle P\rangle_r L_1^{2n+1} = \widehat{\varepsilon}\kappa r_0 C_0 \frac{r}{r_0} L_1^{-N+3n+3}a.$$

On the other hand, using (3.18) we get

$$\begin{aligned}\sigma_0^{\tau+1}E_0 &= c_1^{-1}\sigma_0^{\tau+1}\exp(-A_0(L_1\sigma_0)^{-1/(\rho-1)}) \\ &= C'(n,\rho,\tau)L_1^{-\tau-1}(\ln(L_1+e))^{-(\rho-1)(\tau+1)}(L_1+e)^{-M} \\ &\ge C(n,\rho,\tau)L_1^{-M-\tau-2},\end{aligned}$$

where $M = A_0(\rho)\sigma(n,\rho,\tau)^{-1/(\rho-1)}$. Now we fix $\widetilde{N} = M + \tau + 3n + 5$ and chose

$$a = C(n,\rho,\tau)C_0^{-1}\frac{r_0}{r} = C(n,\rho,\tau)c(n,\rho,\tau,\varsigma)C_0^{-1}.$$

Then $|P_0|_{\mathcal{U}_0} \le \widetilde{\varepsilon}_0$. Recall that $C_0 = C_0(n,\rho)$ comes from the Approximation Lemma and the extension in the beginning of §3; hence, $a = a(n,\tau,\rho,\varsigma,\overline{R})$ is independent of κ, L_1, L_2, and of r. Using (3.15), for each $j \ge 1$ we obtain

$$|P_j - P_{j-1}|_{\mathcal{U}_j} \le \widetilde{\varepsilon}_0 \exp(-B_0{\sigma_j}^{-1/(\rho-1)}) \le \widetilde{\varepsilon}_j.$$

We are ready to make the iterations. We consider the Hamiltonian $H_j(\varphi, I; \omega) = N_0(I;\omega) + P_j(\varphi, I;\omega)$ which is real analytic in $\mathcal{U}_j$, where $N_0(I;\omega) := \langle\omega, I\rangle$. For any $j \in \mathbb{N}$, we denote by $\mathcal{D}_j$ the class of real-analytic diffeomorphisms $\mathcal{F}_j : D_{j+1}\times O_{j+1} \to D_j \times O_j$ of the form

$$\mathcal{F}_j(\theta, I;\omega) = (\Phi_j(\theta, I;\omega), \phi_j(\omega)), \quad \Phi_j(\theta, I;\omega) = (U_j(\theta;\omega), V_j(\theta, I;\omega)), \tag{3.23}$$

where $\Phi_j(\theta, I;\omega)$ is affine linear with respect to I, and the transformation $\Phi_j(\cdot,\cdot;\omega)$ is canonical for any fixed ω. To simplify the notation we denote the sup-norm in $D_j \times O_j$ by $|\cdot|_j$ instead of $|\cdot|_{s_j,r_j,h_j}$. Obviously, $D_j \times O_j \subset \mathcal{U}_j$.

PROPOSITION 3.5. *Suppose P_j, $j \ge 0$, is real analytic on $\mathcal{U}_j$ with*

$$|P_0|_{\mathcal{U}_0} \le \widetilde{\varepsilon}_0, \quad |P_j - P_{j-1}|_{\mathcal{U}_j} \le \widetilde{\varepsilon}_j, \quad j \ge 1.$$

Then for each $j \ge 0$ there exists a real-analytic normal form $N_j(I;\omega) = e_j(\omega) + \langle\omega, I\rangle$ and a real-analytic transformation $\mathcal{F}^j$, where $\mathcal{F}^0 = \mathrm{Id}$ and

$$\mathcal{F}^{j+1} = \mathcal{F}_0 \circ \cdots \circ \mathcal{F}_j : D_{j+1}\times O_{j+1} \longrightarrow (D_0 \times O_0) \cap \mathcal{U}_j, \quad j \ge 0,$$

https://doi.org/10.1017/S0143385704000458 Published online by Cambridge University Press

with $\mathcal{F}_j \in \mathcal{D}_j$ *such that* $H_j \circ \mathcal{F}^{j+1} = N_{j+1} + R_{j+1}$ *and* $|R_{j+1}|_{j+1} \le \varepsilon_{j+1}$. *Moreover,*

$$|\overline{W}_j(\mathcal{F}_j - \mathrm{id})|_{j+1}, |\overline{W}_j(D\mathcal{F}_j - \mathrm{Id})\overline{W}_j^{-1}|_{j+1} < \frac{C\varepsilon_j}{r_j h_j}, \tag{3.24}$$

$$|\overline{W}_0(\mathcal{F}^{j+1} - \mathcal{F}^j)|_{j+1} < \frac{c\varepsilon_j}{r_j h_j}, \tag{3.25}$$

where $C = C(n, \rho) > 0$ *is the constant in Remark 3.3,* $c = c(n, \rho) > 0$, $D\mathcal{F}^j$ *stands for the Jacobian of* $\mathcal{F}^j$ *with respect to* (θ, I, ω), *and* $\overline{W}_j = \mathrm{diag}(\sigma_j{}^{-1}\,\mathrm{Id}, r_j{}^{-1}\,\mathrm{Id}, h_j{}^{-1}\,\mathrm{Id})$.

Proof. The proof is similar to that of the Iterative Lemma from [**13**]. First, applying the KAM step we find $\mathcal{F}^1 = \mathcal{F}_0$ such that $H_0 \circ \mathcal{F}_0 = N_1 + R_1$, where R_1 is real analytic in $D_1 \times O_1$ and $|R_1|_1 \le \varepsilon_1$. By recurrence we define, for any $j \ge 1$, the transformation $\mathcal{F}^{j+1} = \mathcal{F}^j \circ \mathcal{F}_j$, where $\mathcal{F}_j$ belongs to $\mathcal{D}_j$. By the inductive assumption we have $H_{j-1} \circ \mathcal{F}^j = N_j + R_j$, where $N_j(I; \omega) = e_j(\omega) + \langle \omega, I \rangle$ is a real-analytic normal form, R_j is real analytic in $D_j \times O_j$, and $|R_j|_j \le \varepsilon_j$. Then we write

$$\begin{aligned} H_j \circ \mathcal{F}^{j+1} &= (N_0 + P_{j-1}) \circ \mathcal{F}^{j+1} + (P_j - P_{j-1}) \circ \mathcal{F}^{j+1} \\ &= [H_{j-1} \circ \mathcal{F}^j] \circ \mathcal{F}_j + (P_j - P_{j-1}) \circ \mathcal{F}^{j+1} \\ &= (N_j + R_j) \circ \mathcal{F}_j + (P_j - P_{j-1}) \circ \mathcal{F}^{j+1}. \end{aligned}$$

We apply Proposition 3.2 to the Hamiltonian $N_j + R_j$ which is real analytic in $D_j \times O_j$. In this way, using (3.14), we find a real-analytic map $\mathcal{F}_j : D_{j+1} \times O_{j+1} \to D_j \times O_j$ which belongs to the class $\mathcal{D}_j$ and such that $(N_j + R_j) \circ \mathcal{F}_j = N_{j+1} + R_{j+1,1}$, where

$$|R_{j+1,1}|_{j+1} \le \tfrac{1}{2}\widehat{\varepsilon}\kappa r_{j+1}\sigma_{j+1}^{\tau+1} c_1^{1/2} E_j^{3/2} = \tfrac{1}{2}\varepsilon_{j+1}.$$

Moreover, $\mathcal{F}_j$ satisfies (3.24) in view of Remark 3.3 and as in [**13**] we obtain (3.25). We are going to show that

$$\mathcal{F}^{j+1} : D_{j+1} \times O_{j+1} \longrightarrow \mathcal{U}_j. \tag{3.26}$$

This inequality combined with (3.21) and (3.22) implies

$$|(P_j - P_{j-1}) \circ \mathcal{F}^{j+1}|_{j+1} \le |P_j - P_{j-1}|_{\mathcal{U}_j} \le \widetilde{\varepsilon}_j \le \tfrac{1}{2}\varepsilon_{j+1},$$

and we obtain $H_j \circ \mathcal{F}^{j+1} = N_{j+1} + R_{j+1}$, where $|R_{j+1}|_{j+1} \le \varepsilon_{j+1}$.

To prove (3.26) we note that

$$|\overline{W}_k \overline{W}_{k+1}^{-1}| = \sup\{s_{k+1}/s_k, r_{k+1}/r_k, h_{k+1}/h_k\} = s_{k+1}/s_k = \delta,$$

since $r_{k+1}/r_k = E_k^{1/2} \ll \delta$, and $h_{k+1}/h_k \sim \delta^{(\tau+1)\rho(\rho-1)^{-1}} < \delta$ for any $k \in \mathbb{N}$. Then, using (3.20) and (3.24), we estimate the Jacobian of $\mathcal{F}^{j+1}$ in $D_{j+1} \times O_{j+1}$ as follows (see [**13**]):

$$\begin{aligned} |\overline{W}_0 D\mathcal{F}^{j+1}\overline{W}_j^{-1}|_{j+1} &= |\overline{W}_0 D(\mathcal{F}_0 \circ \cdots \circ \mathcal{F}_j)\overline{W}_j^{-1}|_{j+1} \\ &\le \prod_{k=0}^{j-1} (|\overline{W}_k D\mathcal{F}_k \overline{W}_k^{-1}|_{k+1} |\overline{W}_k \overline{W}_{k+1}^{-1}|) |\overline{W}_j D\mathcal{F}_j \overline{W}_j^{-1}|_{j+1} \\ &\le \delta^j \prod_{j=0}^{\infty} \left(1 + \frac{C\varepsilon_j}{r_j h_j}\right) \le 2\delta^j. \end{aligned}$$

https://doi.org/10.1017/S0143385704000458 Published online by Cambridge University Press

Set $z = (\theta, I, \omega) = x + iy \in D_{j+1} \times O_{j+1}$, where x and y are the real and the imaginary parts, respectively, of z. Then

$$\mathcal{F}^{j+1}(x + iy) = \mathcal{F}^{j+1}(x) + \overline{W}_0^{-1} T_{j+1}(x, y)\overline{W}_j y,$$
$$T_{j+1}(x, y) = i \int_0^1 \overline{W}_0 D\mathcal{F}^{j+1}(x + ity)\overline{W}_j^{-1}\, dt.$$

Moreover, $|T_{j+1}(x, y)| \le 2\,\delta^j$ and using the fact that $|\overline{W}_j y| \le |\overline{W}_{j+1} y| \le \sqrt{3}$ we get

$$|T_{j+1}(x, y)\overline{W}_j y| \le 4\delta^j, \quad x + iy \in D_{j+1} \times O_{j+1}.$$

This implies $\mathcal{F}^{j+1}(x + iy) \in \mathcal{U}_j$, since $\mathcal{F}^{j+1}(x)$ is real, and we complete the proof of Proposition 3.5. □

We are going to prove suitable Gevrey estimates for $\mathcal{F}^j$. We set

$$\widetilde{D}_j = \{(\theta, I) \in D_j : |\mathrm{Im}\,\theta| < s_j/2\}, \quad \widetilde{O}_j = \{\omega \in \mathbb{C}^n : |\omega - \Omega_\kappa| < h_j/2\},$$

and we denote $\mathcal{S}^j = \mathcal{F}^{j+1} - \mathcal{F}^j$. For any multi-indices α and β and $m \in \mathbb{N}$ with $|\beta| \le m$, we denote

$$R^m_\omega(\partial^\alpha_\theta \partial^\beta_\omega \mathcal{S}^j)(\theta, I, \omega') := \partial^\alpha_\theta \partial^\beta_\omega \mathcal{S}^j(\theta, I, \omega') - \sum_{|\beta+\gamma|\le m} (\omega' - \omega)^\gamma \partial^\alpha_\theta \partial^{\beta+\gamma}_\omega \mathcal{S}^j(\theta, I, \omega)/\gamma!.$$

Recall that $\rho' = \rho(\tau + 1) + 1$.

LEMMA 3.6. *Under the assumptions of Proposition 3.5 we have*

$$|\overline{W}_0 \partial^\alpha_\theta \partial^\beta_\omega \mathcal{S}^j(\theta, 0, \omega)| \le \widehat{\varepsilon} A C^{|\alpha+\beta|} L_1^{|\alpha|+|\beta|(\tau+1)+1} \kappa^{-|\beta|} \alpha!^\rho \beta!^{\rho'} E_j^{1/2},$$
$$(\theta, 0, \omega) \in \widetilde{D}_{j+1} \times \widetilde{O}_{j+1},$$
$$|\overline{W}_0 R^m_\omega(\partial^\alpha_\theta \partial^\beta_\omega \mathcal{S}^j(\theta, 0, \omega'))| \le \widehat{\varepsilon} A C^{m+|\alpha|+1} L_1^{|\alpha|+(m+1)(\tau+1)+1} \kappa^{-m-1}$$
$$\times \frac{|\omega - \omega'|^{m-|\beta|+1}}{(m - |\beta| + 1)!} \alpha!^\rho (m + 1)!^{\rho'} E_j^{1/2},$$
$$\theta \in \mathbb{T}^n, \omega, \omega' \in \Omega_\kappa,$$

for any $m \in \mathbb{N}$, $\alpha, \beta \in \mathbb{N}^n$, $|\beta| \le m$, where the constants A and C depend only on τ, ρ, n, and ς.

Proof. Using (3.25) and the Cauchy estimate, we evaluate $\partial^\alpha_\theta \partial^\beta_\omega \mathcal{S}^j(\theta, 0, \omega)$ for any $j \ge 0$ and $|\alpha + \beta| \ge 1$ in $\widetilde{D}_{j+1} \times \widetilde{O}_{j+1}$. We have

$$M_{j,\alpha,\beta}(\theta, 0, \omega) := |W_0 \partial^\alpha_\theta \partial^\beta_\omega \mathcal{S}^j(\theta, 0, \omega)| \le c \frac{2^{|\alpha+\beta|}\alpha!\beta!\varepsilon_j}{r_j h_j s_{j+1}^{|\alpha|} h_{j+1}^{|\beta|}} = c\kappa\widehat{\varepsilon} \frac{2^{|\alpha+\beta|}\alpha!\beta! E_j \sigma_j^\tau}{h_j s_{j+1}^{|\alpha|} h_{j+1}^{|\beta|}}.$$

Recall that $s_j = 5(1 - \delta)^{-1}\sigma_j = 5(1 - \delta)^{-1} A_0^{\rho-1} L_1^{-1} (B\sigma_j^{-1/(\rho-1)})^{-(\rho-1)}$, and

$$h_j = \frac{\kappa}{2} K_j^{-\tau-1} = \frac{\kappa}{2} \sigma_j^{\tau+1} x_j^{-\tau-1}.$$

Then by (3.19) we get

$$h_{j+1}^{-1} \le \kappa^{-1} \widetilde{C}_0 L_1^{\tau+1} (B\sigma_j^{-1/(\rho-1)})^{\rho(\tau+1)},$$

https://doi.org/10.1017/S0143385704000458 Published online by Cambridge University Press

where $\widetilde{C}_0$ depends only on τ and ρ. This implies

$$M_{j,\alpha,\beta} \le \widehat{\varepsilon} A_1 C_1^{|\alpha+\beta|} L_1^{|\alpha|+|\beta|(\tau+1)+1} \kappa^{-|\beta|} \alpha!\beta! \\ \times (B\sigma_j^{-1/(\rho-1)})^{(\rho-1)(|\alpha|-\tau)+\rho(\tau+1)(|\beta|+1)} \exp(-B\sigma_j^{-1/(\rho-1)}),$$

where A_1 and C_1 depend only on τ and ρ. Then we obtain

$$M_{j,\alpha,\beta} \le \widehat{\varepsilon} A C^{|\alpha+\beta|} L_1^{|\alpha|+|\beta|(\tau+1)+1} \kappa^{-|\beta|} \alpha!^{\rho} \beta!^{\rho(\tau+1)+1} E_j^{1/2}, \tag{3.27}$$

where A and C depend only on τ and ρ.

We are going to prove the second estimate for $\omega, \omega' \in \Omega_\kappa$. First, we suppose that $|\omega' - \omega| \le h_{j+1}/8$. Expanding the function $\omega \to \partial_\omega^\alpha \mathcal{S}^j(\theta, 0, \omega)$, $\theta \in \mathbb{T}^n$, which is analytic in O_{j+1} in Taylor series with respect to ω at ω', and using as above the Cauchy estimate for $M_{j,\alpha,\gamma}$, we evaluate

$$L_{j,\alpha,\beta}^m := |W_0(R_\omega^m \partial_\theta^\alpha \partial_\omega^\beta \mathcal{S}^j)(\theta, 0; \omega')|, \quad \theta \in \mathbb{T}^n, \ \omega, \omega' \in \Omega_\kappa.$$

For $|\beta| \le m+1$ we have

$$\frac{(\beta+\gamma)!}{\gamma!} \le 2^{|\beta+\gamma|}\beta! \le 2^{|\beta+\gamma|} \frac{(m+1)!}{(m-|\beta|+1)!}.$$

Then as above we obtain

$$\begin{aligned} L_{j,\alpha,\beta}^m &\le \sum_{|\gamma|\ge m-|\beta|+1} |\omega'-\omega|^{|\gamma|} M_{j,\alpha,\beta+\gamma}(\theta,0,\omega')/\gamma! \\ &\le c\alpha!(m+1)! \frac{|\omega'-\omega|^{m-|\beta|+1}}{(m-|\beta|+1)!} \frac{4^{|\alpha|+m+1}\varepsilon_j}{r_j h_j s_{j+1}^{|\alpha|} h_{j+1}^{m+1}} \\ &\quad \times \sum_{|\beta+\gamma|\ge m+1} (4|\omega'-\omega| h_{j+1}^{-1})^{|\beta+\gamma|-m-1}, \end{aligned}$$

and we get

$$\begin{aligned} L_{j,\alpha,\beta}^m &\le 2c\alpha!(m+1)! \frac{|\omega'-\omega|^{m-|\beta|+1}}{(m-|\beta|+1)!} \frac{4^{|\alpha|+m+1}\varepsilon_j}{r_j h_j s_{j+1}^{|\alpha|} h_{j+1}^{m+1}} \\ &\le \widehat{\varepsilon} A C^{|\alpha|+m+1} L_1^{|\alpha|+(m+1)(\tau+1)+1} \kappa^{-m-1} \\ &\quad \times \frac{|\omega'-\omega|^{m-|\beta|+1}}{(m-|\beta|+1)!} \alpha!^{\rho} (m+1)!^{\rho(\tau+1)+1} E_j^{1/2}, \end{aligned}$$

where A and C depend only on τ, ρ, and n. For $|\omega' - \omega| \ge h_{j+1}/8$ we obtain the same inequality by estimating $L_{j,\alpha,\beta}^m$ term by term as in (3.27). This proves the lemma. □

According to Proposition 3.5 and Lemma 3.6, the limit

$$\partial_\theta^\alpha \mathcal{H}^\beta(\theta,\omega) := \lim_{j\to\infty} \partial_\theta^\alpha \partial_\omega^\beta [\mathcal{F}^j(\theta,0,\omega) - (\theta,0,\omega)], \quad (\theta,\omega) \in \mathbb{T}^n \times \Omega_\kappa,$$

exists for each $\alpha, \beta \in \mathbb{N}$ and it is uniform since $\sum E_j^{1/2} < \infty$. Moreover, the partial derivatives of $\partial_\theta^\alpha(\mathcal{H}^\beta) = \partial_\theta^\alpha \mathcal{H}^\beta$ exist and they are continuous on $\mathbb{T}^n \times \Omega_\kappa$. Consider the

https://doi.org/10.1017/S0143385704000458 Published online by Cambridge University Press

jet $\mathcal{H} = (\partial_\theta^\alpha \mathcal{H}^\beta)$, $\alpha, \beta \in \mathbb{N}^n$, of continuous functions $\partial_\theta^\alpha \mathcal{H}^\beta : \mathbb{T}^n \times \Omega_\kappa \to \mathbb{T}^n \times D \times \Omega$, and set

$$(R_\omega^m \partial_\theta^\alpha \mathcal{H})_\beta(\theta, \omega') := \partial_\theta^\alpha \mathcal{H}^\beta(\theta, \omega') - \sum_{|\beta+\gamma|\le m} (\omega' - \omega)^\gamma \partial_\theta^\alpha \mathcal{H}^{\beta+\gamma}(\theta, \omega)/\gamma!.$$

In view of Lemma 3.6, we have

$$\begin{aligned} |\overline{W}_0 \partial_\theta^\alpha \mathcal{H}^\beta(\theta, \omega)| &\le \widehat{\varepsilon} A L_1 (CL_1)^{|\alpha|} (CL_1^{\tau+1}/\kappa)^{|\beta|} \alpha!^\rho \beta!^{\rho'} \\ |\overline{W}_0 (R_\omega^m \partial_\theta^\alpha \mathcal{H})_\beta(\theta, \omega')| &\le \widehat{\varepsilon} A L_1 (CL_1)^{|\alpha|} (CL_1^{\tau+1}/\kappa)^{m+1} \frac{|\omega - \omega'|^{m-|\beta|+1}}{(m - |\beta| + 1)!} \alpha!^\rho (m+1)!^{\rho'} \end{aligned} \tag{3.28}$$

for each α, and β satisfying $0 \le |\beta| \le m$, and $\theta \in \mathbb{T}^n$, $\omega, \omega' \in \Omega_\kappa$, where A and C depend only on τ, ρ, ρ', and n. We are going to extend $\mathcal{H}$ to a Gevrey function on $\mathbb{T}^n \times \Omega$.

3.4. *Whitney extension in Gevrey classes.* Let K be a compact set in $\mathbb{R}^n$ and $\rho \ge 1$, $\rho' > 1$. We consider a jet (f^β), $\beta \in \mathbb{N}^n$, of functions $f^\beta : \mathbb{T}^n \times K \to \mathbb{R}$, such that for each $\alpha \in \mathbb{N}^n$ the partial derivative $\partial_\theta^\alpha f^\beta$ exists, it is continuous on $\mathbb{T}^n \times K$, and there are positive constants A, C_1, and C_2 such that

$$\begin{aligned} |\partial_\theta^\alpha f^\beta(\theta, \omega)| &\le A C_1^{|\alpha|} C_2^{|\beta|} \alpha!^\rho \beta!^{\rho'}, \\ |(R_\omega^m \partial_\theta^\alpha f)_\beta(\theta, \omega')| &\le A C_1^{|\alpha|} C_2^{m+1} \frac{|\omega - \omega'|^{m-|\beta|+1}}{(m - |\beta| + 1)!} \alpha!^\rho (m+1)!^{\rho'} \end{aligned} \tag{3.29}$$

for each $\alpha, \beta \in \mathbb{N}^n$, $\theta \in \mathbb{T}^n$, and $\omega, \omega' \in K$.

THEOREM 3.7. *There exist positive constants $A_0 = A_0(n, \rho, \rho')$ and $C_0 = C_0(n, \rho, \rho')$, and for any compact set K and any jet $f = (f^\beta)$, $\beta \in \mathbb{N}^n$, satisfying (3.29) there exists $\widetilde{f} \in \mathcal{G}^{\rho,\rho'}(\mathbb{T}^n \times \mathbb{R}^n)$ such that $\partial_\theta^\alpha \partial_\omega^\beta \widetilde{f} = \partial_\theta^\alpha f^\beta$ on $\mathbb{T}^n \times K$ for each α, β, and*

$$|\partial_\theta^\alpha \partial_\omega^\beta \widetilde{f}(\theta, \omega)| \le A_0 A \max(C_1, 1)(C_0 C_1)^{|\alpha|+n} (C_0 C_2)^{|\beta|} \alpha!^\rho \beta!^{\rho'}, \quad (\theta, \omega) \in \mathbb{T}^n \times \mathbb{R}^n.$$

Remark. We point out that the positive constants $A_0 = A_0(n, \rho, \rho')$ and $C_0 = C_0(n, \rho, \rho')$ are independent of the jet f, the compact set K, and of the constants A, C_1, and C_2.

Proof. Consider the Fourier coefficients

$$f_k^\beta(\omega) = (2\pi)^{-n} \int_{\mathbb{T}^n} e^{-i\langle k, \varphi\rangle} f^\beta(\varphi, \omega)\, d\varphi, \quad k \in \mathbb{Z}^n,$$

and denote by $\mathcal{A}$ the set of all Whitney jets $g_k = (g_k^\beta)$, $\beta \in \mathbb{N}^n$, where $g_k^\beta = e^{r|k|^{1/\rho}} f_k^\beta$ and $|k| = |k_1| + \cdots + |k_n|$. Choosing $r = c_0 C_1^{-1/\rho}$ with $0 < c_0 = c_0(n, \rho) \ll 1$, we are going to show that $(g_k^\beta) \in \mathcal{A}$ satisfy

$$\begin{aligned} |g_k^\beta(\omega)| &\le A_2 C_2^{|\beta|} \beta!^\rho, \quad \omega \in K \\ |(R_\omega^m g_k)_\beta(\omega')| &\le A_2 C_2^{m+1} \frac{|\omega - \omega'|^{m-|\beta|+1}}{(m - |\beta| + 1)!} (m+1)!^{\rho'}, \quad \omega, \omega' \in K \end{aligned} \tag{3.30}$$

https://doi.org/10.1017/S0143385704000458 Published online by Cambridge University Press

where $A_2 = 2A\max(C_1, 1)$. We decompose $j = \rho\ell_j + q_j$, $j, \ell_j \in \mathbb{N}$, $0 \le q_j < \rho$. For any $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$ we have $|k| \le n \max_{1\le i\le n} |k_i| = n|k_p|$ for some p. Then integrating by parts and using (3.29), we estimate

$$\frac{r^j |k|^{j/\rho}}{j!} |f_k^\beta(\omega)| \le A \frac{r^j}{j!} n^{\ell_j+1} C_1^{\ell_j+1} C_2^{|\beta|} (\ell_j+1)!^\rho \beta!^{\rho'} \le 2^{-j} A \max(C_1, 1) C_2^{|\beta|} \beta!^{\rho'},$$

choosing $r = c_0 C_1^{-1/\rho}$ with $0 < c_0 = c_0(n, \rho) \ll 1$. This proves the first part of (3.30). To prove the second part, we notice that $(R_\omega^m f_k)_\beta(\omega')$ is just the Fourier coefficient of $(R_\omega^m f)_\beta(\theta, \omega')$ corresponding to k. Now we use a variant of the Whitney extension theorem due to Bruna (see [**1**, Theorem 3.1]).

THEOREM 3.8. *There exist positive constants $A_0 = A_0(n, \rho')$ and $C_0 = C_0(n, \rho')$ and for any compact set K and a jet $g = (g^\beta)$, $\beta \in \mathbb{N}^n$, satisfying (3.30) on K with some A_2, $C_2 > 0$, there exists $\widetilde{g} \in \mathcal{G}^{\rho'}(\mathbb{R}^n)$ such that $\partial_\omega^\beta \widetilde{g} = g^\beta$ on K for any β, and*

$$|\partial_\omega^\beta \widetilde{g}(\omega)| \le A_0 A_2 (C_0 C_2)^{|\beta|} \beta!^{\rho'}, \quad \omega \in \mathbb{R}^n.$$

Proof. The proof of Theorem 3.8 is given in [**1**]. Here, for the sake of completeness, we only recall the main lines in the proof, verifying that the positive constants $A_0 = A_0(n, \rho')$ and $C_0 = C_0(n, \rho')$ do not depend on the compact set K on the jet g or on the constants A_2 and C_2.

Step 1. Setting

$$f^\beta(\omega) = A_2^{-1} C_2^{-|\beta|} g^\beta(C_2^{-1}\omega), \quad \omega \in \widetilde{K} := \{C_2\omega : \omega \in K\},$$

we obtain $|f^\beta(\omega)| \le \beta!^{\rho'}$ and

$$|(R_{\omega'}^m f)_\beta(\omega)| = A_2^{-1} C_2^{-|\beta|} |(R_{C_2^{-1}\omega'}^m g)_\beta(C_2^{-1}\omega)| \le \frac{|\omega - \omega'|^{m-|\beta|+1}}{(m-|\beta|+1)!} (m+1)!^{\rho'}$$

for $\omega, \omega' \in \widetilde{K}$. As usual, we denote $M_k = k!^{\rho'}$, $k \in \mathbb{N}$. Recall from Bruna [**1**] that the jet $h = (h^\beta)_{\beta\in\mathbb{N}^n}$ belongs to the Whitney space $C_M^\infty(\widetilde{K})$ if it satisfies

$$\begin{gathered} |h^\beta(\omega)| \le A\epsilon^{|\beta|} M_{|\beta|}, \quad \beta \in \mathbb{N}^n,\ \omega \in \widetilde{K} \\ |(R_\omega^m h)_\beta(\omega')| \le A\epsilon^{m+1} \frac{|\omega - \omega'|^{m-|\beta|+1}}{(m-|\beta|+1)!} M_{m+1}, \quad \beta \in \mathbb{N}^n,\ |\beta| \le m,\ \omega, \omega' \in \widetilde{K} \end{gathered} \tag{3.31}$$

for some $A > 0$ and $\epsilon > 0$. Hence, $f \in C_M^\infty(\widetilde{K})$ and it satisfies the estimates above with $A = \epsilon = 1$. In particular, for any $\omega \in \widetilde{K}$ the sequence $(f^\beta(\omega))_{\beta\in\mathbb{N}^n}$ belongs to

$$\mathcal{B} := \{(a_\beta)_{\beta\in\mathbb{N}^n} : a_\beta \in \mathbb{C}, |a_\beta| \le M_{|\beta|}\},$$

which is a bounded set in the space c_M^n of all sequences $(a_\beta)_{\beta\in\mathbb{N}^n}$, $a_\beta \in \mathbb{C}$, such that $|a_\beta| \le A\epsilon^{|\beta|} M_{|\beta|}$ for some positive constants A and ϵ. On the other hand, c_M^n and $\mathcal{D}_M^\infty(\mathbb{R}^n)$ (the space of C^∞ functions h with compact supports satisfying for any $\beta \in \mathbb{N}^n$ and $\omega \in \mathbb{R}^n$ the inequality $|\partial^\beta h(\omega)| \le A\epsilon^{|\beta|} M_{|\beta|}$, where A and ϵ are positive constants) are Silva spaces, that is, inductive limits of Banach spaces such that the canonical mappings are compact [**8**]. Now we use the following argument of Bruna [**1**].

https://doi.org/10.1017/S0143385704000458 Published online by Cambridge University Press

LEMMA 3.9. *Suppose that E and F are Silva spaces and $u : E \to F$ is a linear mapping, continuous and onto. Then, for every bounded set B in F, there exists a bounded set A in E such that $u(A) = B$.*

We apply the lemma for the bounded set $\mathcal{B}$ above and the mapping $u : \mathcal{D}_M^\infty(\mathbb{R}^n) \to c_M^n$ defined by $u(h)_\beta = \partial^\beta h(0)$, $\beta \in \mathbb{N}^n$. It is proved in [**1**, §2], that the map u is onto as far as the sequence $M = (M_k)$ satisfies the so-called strong non-quasi-analytic condition, a particular case of which is $M_k = k!^{\rho'}$ with $\rho' > 1$. In this way we find a bounded set $\mathcal{C} \subset \mathcal{D}_M^\infty(\mathbb{R}^n)$, such that $u(\mathcal{C}) = \mathcal{B}$. According to [**8**], there exist $R_0 > 0$, $A > 0$, and $\epsilon > 0$ such that the support of any $\psi \in \mathcal{C}$ is contained in the ball $B_{R_0} \subset \mathbb{R}^n$ and

$$|\partial^\beta \psi(\omega)| \le A\epsilon^{|\beta|} M_{|\beta|}, \quad \beta \in \mathbb{N}^n,\ \omega \in \mathbb{R}^n.$$

In this way we assign to any $n \in \mathbb{N}$, $n \ge 1$, and $\rho' > 1$ positive constants $R_0 = R_0(n, \rho')$, $A = A(n, \rho')$, and $\epsilon = \epsilon(n, \rho')$ such that for any $\psi \in \mathcal{C}$ the estimates above hold and $\operatorname{supp}\psi \subset B_{R_0}$. In particular, for each $\omega \in \widetilde{K}$ there exists $g_\omega \in \mathcal{C}$ such that $\operatorname{supp} g_\omega \subset B_{R_0}$, $\partial^\beta g_\omega(0) = f^\beta(\omega)$ for any $\beta \in \mathbb{N}^n$, and it satisfies the above estimates. We set $f_\omega(z) = g_\omega(z - \omega)$, $\omega \in \widetilde{K}$, $z \in \mathbb{R}^n$. Then we have

$$\begin{aligned} \partial_z^\beta f_\omega(\omega) &= f^\beta(\omega), \quad \omega \in \widetilde{K},\ \beta \in \mathbb{N}^n, \\ |\partial_z^\beta f_\omega(z)| &\le A\epsilon^{|\beta|} M_{|\beta|}, \quad z \in \mathbb{R}^n,\ \omega \in \widetilde{K},\ \beta \in \mathbb{N}^n. \end{aligned} \tag{3.32}$$

Step 2. Consider the function h defined for $x > 0$ by

$$h(x) = \sup_k \frac{k!}{x^k M_k} = \sup_k \frac{1}{x^k k!^{\rho'-1}} = \exp\left[\sup_k \left(k \ln \frac{1}{x} - (\rho' - 1)\ln k!\right)\right].$$

We recall that $h(x) \ge 1$ is a decreasing function, $h(x) = 1$ for $x \ge 1$, and by Stirling's formula $h(x) \sim (\sqrt{2\pi})^{1-\rho'}\sqrt{x}\exp((\rho' - 1)x^{-1/(\rho'-1)})$ as $x \to +0$. It is easy to see that, for any $q \in \mathbb{N}$, there exist positive constants $C = C(q, \rho')$ and $c = c(q, \rho')$ such that $h(x)^q \le Ch(cx)$, $x > 0$.

As in [**1**, §3], we obtain

$$|\partial_z^\beta (f_\omega(z) - f_{\omega'}(z))| \le A\frac{\epsilon^{|\beta|} M_{|\beta|}}{h(\epsilon(|z - \omega| + |z - \omega'|))}, \quad z \in \mathbb{R}^n,\ \omega,\ \omega' \in \widetilde{K},\ \beta \in \mathbb{N}^n, \tag{3.33}$$

where $A = A(n, \rho') > 0$ and $\epsilon = \epsilon(n, \rho') > 0$ are independent of the compact set $\widetilde{K}$ and of the jet f.

Step 3. We recall the following two lemmas from Bruna [**1**, §3].

LEMMA 3.10. *There exists a collection of closed cubes with sides parallel to the axes, $Q_1, Q_2, \ldots$, such that:*

(a) $\mathbb{R}^n \setminus \widetilde{K} = \bigcup_i Q_i$;

(b) *the interiors of Q_i are disjoint;*

(c) *if $\delta_i = \operatorname{diam}(Q_i)$ and d_i denotes the distance from Q_i to $\widetilde{K}$, then $\delta_i \le d_i \le 4\delta_i$;*

(d) *if $0 < \lambda < 1/4$, and Q_i^* denotes the cube which has the same center as Q_i, expanded by the factor $1 + \lambda$, then there is a constant $c = c(n, \lambda) > 1$ such that $c^{-1}\delta_i \le d(z, \widetilde{K}) \le c\delta_i$ for $z \in Q_i^*$;*

(e) *if $D = (12)^{2n}$, then each Q_i^* intersects at most D cubes Q_j^*;*

https://doi.org/10.1017/S0143385704000458 Published online by Cambridge University Press

(f) *there is a constant* $c = c(n, \lambda) > 1$ *such that* $c^{-1}\delta_j \le \delta_i \le c\delta_j$ *if* $Q_i^* \cap Q_j^* \neq \emptyset$.

We fix $\lambda = 1/8$.

LEMMA 3.11. *For each* $\eta > 0$ *there exists a family of functions* $\phi_i \in \mathcal{D}_M^\infty(\mathbb{R}^n)$ *such that:*
(a) $\phi_i \ge 0$;
(b) $\operatorname{supp}\phi_i \subset Q_i^*$;
(c) $\sum_i \phi_i(z) = 1,\ z \in \mathbb{R}^n \setminus \widetilde{K}$;
(d) *there exist positive constants* $B = B(n, \rho')$ *and* $C = C(n, \rho')$ *such that*

$$|\partial^\beta \phi_i(z)| \le Ch(B\eta d(z, \widetilde{K}))\eta^{|\beta|}M_{|\beta|}, \quad z \in Q_i^*.$$

Now for each i we choose $\omega_i \in \widetilde{K}$ such that $d(\omega_i, Q_i) = d(\widetilde{K}, Q_i)$ and define

$$\widetilde{f}(z) = \begin{cases} f^0(z) & \text{if } z \in \widetilde{K}, \\ \displaystyle\sum_i \phi_i(z) f_{\omega_i}(z) & \text{if } z \notin \widetilde{K}. \end{cases}$$

Step 4. For any $\omega \in \widetilde{K}$ fixed, we compare, as in [**1**], $\widetilde{f}(z)$ with $f_\omega(z)$ for $z \notin \widetilde{K}$. We have

$$\partial_z^\beta \widetilde{f}(z) - \partial_z^\beta f_\omega(z) = \sum_{\gamma \le \beta} \binom{\beta}{\gamma} \sum_i \partial^\gamma \phi_i(z) \partial^{\beta-\gamma}\{f_{\omega_i}(z) - f_\omega(z)\}.$$

Evaluating the terms for which $\gamma = 0$ and using (3.33) we get

$$\left| \sum_i \phi_i(z) \partial^\beta \{f_{\omega_i}(z) - f_\omega(z)\} \right| \le A \frac{\epsilon^{|\beta|} M_{|\beta|}}{h(\epsilon|z - \omega|)}, \quad z \in \mathbb{R}^n,\ \omega \in \widetilde{K},\ \beta \in \mathbb{N}^n,$$

where $A = A(n, \rho) > 0$ and $\epsilon = \epsilon(n, \rho) > 0$ are independent of the compact set $\widetilde{K}$ and of the jet f. On the other hand, for $\gamma \neq 0$, we have $\sum_i \partial^\gamma \phi_i(z) = 0$ on $\mathbb{R}^n \setminus \widetilde{K}$, and we can write

$$\sum_i \partial^\gamma \phi_i(z) \partial^{\beta-\gamma}\{f_{\omega_i}(z) - f_\omega(z)\} = \sum_i \partial^\gamma \phi_i(z) \partial^{\beta-\gamma}\{f_{\omega_i}(z) - f_{\omega'}(z)\}|_{\omega'=\bar{z}},$$

where $\bar{z} \in \widetilde{K}$ is such that $|z - \bar{z}| = d(z, \widetilde{K})$. Then (3.33) and Lemma 3.11(d) imply

$$\left| \sum_i \partial^\gamma \phi_i(z) \partial^{\beta-\gamma}\{f_{\omega_i}(z) - f_\omega(z)\} \right| \le A\epsilon^{|\beta|-|\gamma|}\eta^{|\gamma|}M_{|\beta|}h(\epsilon d(z, \widetilde{K}))^{-1}h(B\eta d(z, \widetilde{K})).$$

As was mentioned in Step 3, there exist constants $c = c(\rho') > 0$ and $C_0 = C_0(\rho') > 0$ such that $h(cx)^{-1}h(x) \le C_0 h(x)^{-1}$ for $x \in (0, +\infty)$, and choosing $\eta = \eta(n, \rho') = \epsilon/cB$ we get, as in [**1**],

$$\left| \sum_i \partial^\gamma \phi_i(z) \partial^{\beta-\gamma}\{f_{\omega_i}(z) - f_\omega(z)\} \right| \le A\epsilon^{|\beta|-|\gamma|}\eta^{|\gamma|}M_{|\beta|}h(\widetilde{\epsilon}|z - \omega|)^{-1}, \quad \widetilde{\epsilon} = \frac{\epsilon}{c}.$$

The last two inequalities yield

$$\left| \sum_{\gamma \le \beta} \binom{\beta}{\gamma} \sum_i \partial^\gamma \phi_i(z) \partial^{\beta-\gamma}\{f_{\omega_i}(z) - f_\omega(z)\} \right| \le A\,(\eta + \epsilon)^{|\beta|}\, M_{|\beta|}h(\widetilde{\epsilon}|z - \omega|)^{-1}$$

https://doi.org/10.1017/S0143385704000458 Published online by Cambridge University Press

for any $z \in \mathbb{R}^n$, $\omega \in \widetilde{K}$, and $\beta \in \mathbb{N}^n$. Arguing as in [**1**], we obtain that $\widetilde{f} \in \mathcal{D}_M^\infty(\mathbb{R}^n)$, $\partial^\beta \widetilde{f} = f^\beta$ on $\widetilde{K}$ and that

$$|\partial^\beta \widetilde{f}(z)| \le A\epsilon^{|\beta|} M_{|\beta|}, \quad z \in \mathbb{R}^n,$$

where $A = A(n, \rho')$ and $\epsilon = \epsilon(n, \rho')$ are positive constants. Scaling back by C_2 we obtain the desired estimates with constants $A_0 = A_0(n, \rho') > 0$ and $C_0 = C_0(n, \rho') > 0$. □

Applying Theorem 3.8 to the family of jets $\mathcal{A}$, we obtain a family of functions $\widetilde{g}_k \in \mathcal{G}^{\rho'}$ such that $\partial_\omega^\beta \widetilde{g}_k = g_k^\beta$ on K for each β, and

$$|\partial_\omega^\beta \widetilde{g}_k(\omega)| \le A_0 A \max(C_1, 1)(C_0 C_2)^{|\beta|} \beta!^{\rho'} \quad \text{for all } \omega \in \mathbb{R}^n,\ k \in \mathbb{Z}^n,\ \beta \in \mathbb{N}^n.$$

Now it is easy to see that the function

$$\widetilde{f}(\theta, \omega) = \sum_{k \in \mathbb{Z}^n} e^{i\langle k, \theta\rangle - r|k|^{1/\rho}} \widetilde{g}_k(\omega)$$

satisfies the requirements of Theorem 3.7, since $\sum_{k\in\mathbb{Z}^n} e^{-\frac{1}{2}r|k|^{1/\rho}} \le C(n, \rho) C_1^n$. □

3.5. *Proof of Theorem 2.1.* Using (3.28) and Theorem 3.7 we extend the jet $\mathcal{H}$ to a Gevrey function $\mathcal{H} = (\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3) : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^n$. We have

$$|\overline{W}_0 \partial_\theta^\alpha \partial_\omega^\beta \mathcal{H}(\theta, \omega)| \le \widehat{\varepsilon} A L_1^{n+2} (CL_1)^{|\alpha|} (CL_1^{\tau+1}/\kappa)^{|\beta|} \alpha!^\rho \beta!^{\rho(\tau'+1)+1},$$

where $\widehat{\varepsilon} = \langle P\rangle_r L_1^{N-n-2} (a\kappa r)^{-1}$, and $A = A(n, \tau, \rho)$ and $C = C(n, \tau, \rho)$ are positive constants (independent of L_1, L_2, κ, r, and $\Omega \subset B_{\bar{R}}$). We set $\mathcal{F} = (\Phi, \phi)$, $\Phi = (U, V)$, where $U(\theta, \omega) = \mathcal{H}_1(\theta, \omega) + \theta$, $V(\theta, \omega) = \mathcal{H}_2(\theta, \omega)$, and $\phi(\omega) = \mathcal{H}_3(\omega) + \omega$. Recall that $r_0 = cr$, where $c = c(n, \tau, \rho, \varsigma) > 0$ is fixed in §3.3. On the other hand, $h_0 \le \kappa \sigma_0^{\tau+1} < \kappa$, and we obtain

$$\begin{aligned} &|\partial_\theta^\alpha \partial_\omega^\beta (U(\theta; \omega) - \theta)| + r^{-1} |\partial_\theta^\alpha \partial_\omega^\beta V(\theta; \omega)| + \kappa^{-1} |\partial_\omega^\beta (\phi(\omega) - \omega)| \\ &\qquad \le A \frac{\langle P\rangle_r L_1^N}{\kappa r} (CL_1)^{|\alpha|} (CL_1^{\tau+1}/\kappa)^{|\beta|} \alpha!^\rho \beta!^{\rho'} \end{aligned}$$

for some positive constants A and C as above. Choosing $\epsilon < 1/A$ in (2.3) we obtain $|V(\theta, \omega)| \le A\langle P\rangle_r L_1^N \kappa^{-1} \le A\epsilon r < r \le R$. In the same way, we get $\phi(\omega) \in \Omega$ for $\omega \in \Omega_\kappa$. This proves the estimates in Theorem 2.1. As in [**13**, §5.d], we obtain that $|X_{H_j} \circ \mathcal{F}^j - D\Phi^j \cdot X_N| \le c\varepsilon_j / r_j h_j$ on $\mathbb{T}^n \times \{0\} \times \Omega_\kappa$ for all $j \ge 0$, where X_{H_j} and X_{N_ω} stand for the Hamiltonian vector fields of $H_j(\theta, I; \omega)$ and $N_\omega = \langle \omega, I\rangle$, respectively. On the other hand, ∇H_j converges uniformly to ∇H as $j \to \infty$ in view of Proposition 3.1, hence

$$X_{H(\cdot;\phi(\omega))} \circ \Phi = D\Phi \cdot X_{N_\omega}$$

on $\mathbb{T}^n \times \{0\} \times \Omega_\kappa$. Then $\{\Phi(\theta; \omega) : \theta \in \mathbb{T}^n\}$ is an embedded invariant torus of the Hamiltonian $H(\theta, I; \phi(\omega))$ with frequency $\omega \in \Omega_\kappa$. It is Lagrangian by construction (see also [**4**, §I.3.2]). This completes the proof of Theorem 2.1. □

https://doi.org/10.1017/S0143385704000458 Published online by Cambridge University Press

3.6. *Real-analytic Hamiltonians.* Consider a real-analytic Hamiltonian

$$P \in \mathcal{G}^1_{L_1,L_2,L_2}(\mathbb{T}^n \times B_R \times \Omega)$$

with norm

$$\|P\|_{L_1,L_2} = \sup(|\partial_\theta^\alpha \partial_I^\beta \partial_\omega^\gamma P(\theta, I; \omega)| L_1^{-|\alpha|} L_2^{-|\beta|-|\gamma|} (\alpha!\beta!\gamma!)^{-1}) < +\infty.$$

Then P can be extended as an analytic Hamiltonian in $D_{s,r} \times O_h$, where $s = (2L_1)^{-1}$ and $r = h = (2L_2)^{-1}$, and with sup-norm satisfying $\|P\|_{2L_1,2L_2} \le \|P\|_{s,r,h} \le C\|P\|_{L_1,L_2}$, where $C = C(n) > 0$. Fix $\tau' > \tau > n-1$. We can slightly improve Theorem 3.1 [**14**]. Given $s > 0$ we denote by $\mathcal{U}_s$ the set of all $\theta \in \mathbb{C}^n/2\pi\mathbb{Z}^n$ such that $|\mathrm{Im}\,\theta| < s$.

THEOREM 3.12. *Suppose that H is given by (2.2) where $P \in \mathcal{G}^1_{L_1,L_2,L_2}(\mathbb{T}^n \times B_R \times \Omega)$. Fix $\kappa > 0$ and $r > 0$ such that $\kappa, r < L_2^{-1}$ and $r \le R$. Then there exist $0 < s_0 \le s$ and $\epsilon > 0$ both independent of κ, L_2, r, R, and of $\Omega \subset \Omega_0$, such that if*

$$\|P\| \le \epsilon\kappa r$$

in $\mathcal{G}^1_{L_1,L_2,L_2}(\mathbb{T}^n \times B_r \times \Omega)$, then there exist maps

$$\phi \in \mathcal{G}^{\tau'+2}(\Omega, \Omega), \quad \Phi = (U, V) \in \mathcal{G}^{1,\tau'+2}(\mathcal{U}_{s_0/2} \times \Omega, \mathcal{U}_{s_0} \times B_R),$$

satisfying Theorem 2.1(i), with $\rho = 1$. Moreover, there exist $A, C > 0$, independent of κ, L_2, r, and Ω, such that

$$|\partial_\omega^\beta(U(\theta;\omega) - \theta)| + r^{-1}|\partial_\omega^\beta V(\theta;\omega)| + \kappa^{-1}|\partial_\omega^\beta(\phi(\omega) - \omega)| \le AC^{|\beta|}\kappa^{-|\beta|}\beta!^{\tau'+2}\frac{\|P\|}{\kappa r}$$

uniformly in $(\theta, \omega) \in \mathcal{U}_{s_0/2} \times \Omega$ and for any $\beta \in \mathbb{N}^n$.

Proof. Fix $\rho = (\tau' - \tau)(\tau+1)^{-1} + 1$ and set as above $\sigma_j = \sigma_0\delta^j$, and $s_{j+1} = s_j - 5\sigma_j$, $j \ge 0$, where $s_0(1-\delta) = 20\sigma_0$ (then $s_j \to 3s_0/4$) and $s_0 \le s = (2L_1)^{-1}$. As above we define $E_j := c_1^{-1}\exp(-\sigma_j^{-1/(\rho-1)})$ and set $\varepsilon_j = \widehat{\varepsilon}\kappa r_j \sigma_j^{\tau+1} E_j$, where $\widehat{\varepsilon} = \|P\|_{s,r,h}(a\kappa r)^{-1}$. Choose $r_0 = r$ and $h_0 = h$ and define r_j, x_j, K_j, and h_j as above. We consider $H = \langle\omega, I\rangle + P$ in $D_j \times O_j$. We fix $\sigma_0 = \sigma_0(n, \rho, \tau, s) \le s(1-\delta)/20$, so that Proposition 3.1 (a)–(c) and (3.13) hold for any $j \ge 0$. Next we choose $a = a(n, \rho, \tau, L_1) \le c_1^{-1}\sigma_0^{-\tau-1}\exp(-\sigma_0^{-1/(\rho-1)})$. Then $|P|_0 \le \varepsilon_0$, and we can apply Proposition 3.5. Moreover, as in Lemma 3.6 we obtain with $\rho' = \rho(\tau+1) + 1 = \tau' + 2$,

$$|\overline{W}_0 \partial_\omega^\beta \mathcal{S}^j(\theta, 0, \omega)| \le \widehat{\varepsilon}AC^{|\beta|}\kappa^{-|\beta|}\beta!^{\rho'}E_j^{1/2}, \quad (\theta, 0, \omega) \in \mathcal{U}_{s_0/2} \times \widetilde{O}_{j+1},$$

$$|\overline{W}_0 R^m_{\omega'}(\partial_\omega^\beta \mathcal{S}^j(\theta, 0, \omega))| \le \widehat{\varepsilon}AC^{m+1}\kappa^{-m-1} \times \frac{|\omega - \omega'|^{m-|\beta|+1}}{(m-|\beta|+1)!}(m+1)!^{\rho'}E_j^{1/2}, \quad \theta \in \mathcal{U}_{s_0/2},\ \omega, \omega' \in \Omega_\kappa,$$

for any $m \in \mathbb{N}$, $\beta \in \mathbb{N}^n$, $|\beta| \le m$, where the constants A and C depend only on τ, ρ, n, and L_1. We complete the proof of the theorem as above. □

https://doi.org/10.1017/S0143385704000458 Published online by Cambridge University Press

4. *Proof of the main results*

Proof of Theorem 1.1. Set $r = R = \kappa\sqrt{\epsilon_H}$ and suppose that $\epsilon_H \leq \epsilon^2(1 + \|P^0\|_{L_0})^{-2}L_1^{-2N}$, where N and ϵ are the constants in Theorem 2.1. Then (2.1) implies $\langle P\rangle_r \leq (\|P^0\|_{L_0} + 1)\kappa r\sqrt{\epsilon_H} \leq \epsilon r L_1^{-N}$, and we can apply Theorem 2.1. Consider the map $\overline{\Phi} : \mathbb{T}^n \times \Omega \to \mathbb{T}^n \times \mathbb{R}^n$ given by

$$\overline{\Phi}(\theta, \omega) = (U(\theta; \omega), \nabla g^0(\phi(\omega)) + V(\theta; \omega)),$$

where U and V are obtained in Theorem 2.1. Multiplying V by a suitable Gevrey cut-off function which equals 1 in a neighborhood of Ω_κ and 0 outside Ω' (recall that $|\Omega_\kappa - \partial\Omega'| \geq \kappa/2$), we can suppose that $V = 0$ outside Ω' keeping the same Gevrey estimates for V. Then $\overline{\Phi} : \mathbb{T}^n \times \Omega \to \mathbb{T}^n \times D$. We have $H(\theta, I; \phi(\omega)) = H(\theta, \nabla g^0(\phi(\omega)) + I)$, $I \in B_R$, in the notations of Section 2. Then Theorem 2.1(i) implies that $\Lambda_\omega = \{\overline{\Phi}(\theta, \omega) : \theta \in \mathbb{T}^n\}$, $\omega \in \Omega_\kappa$, is an embedded Lagrangian invariant torus of the Hamiltonian H for each $\omega \in \Omega_\kappa$ with frequency ω. The corresponding estimates for $\overline{\Phi}$ follow directly from those in Theorem 2.1. □

Proof of Corollary 1.2. The proof is close to that of [**14**, Theorem 2.1], and we will be concerned mainly with the corresponding Gevrey estimates. Let $\epsilon_H \leq \epsilon L_1^{-N-2}$. Then in the notation of Theorem 1.1(ii) we get $AC_1L_1^{N/2}\sqrt{\epsilon_H} \leq AC\sqrt{\epsilon}$. Choosing ϵ small enough and using Proposition A.2 as well as Theorem 1.1(ii), we obtain a solution $\theta = \theta(\varphi, \omega)$ of the equation $\overline{U}(\theta, \omega) = \varphi$ such that $\theta(\varphi, \omega) - \varphi$ satisfies the same Gevrey estimates as $\overline{U}$. Set $F(\varphi, \omega) = \overline{V}(\theta(\varphi, \omega), \omega)$. We have $\Lambda_\omega = \{(\varphi, F(\varphi, \omega)) : \varphi \in \mathbb{T}^n\}$ for each $\omega \in \Omega_\kappa$. Moreover, by Proposition A.4 we obtain

$$|\partial_\varphi^\alpha \partial_\omega^\beta (F(\varphi, \omega) - \nabla g^0(\omega))| \leq \kappa AC_1^{|\alpha|}(C_2/\kappa)^{|\beta|}\alpha!^\rho \beta!^{\rho(\tau+1)+1}L_1^{N/2}\sqrt{\epsilon_H}$$

uniformly in $(\theta, \omega) \in \mathbb{T}^n \times \Omega$. Hereafter, A, $C_1 = CL_1$, and $C_2 = CL_1^{\tau+1}$ are positive constants as in Theorem 1.1. Denote by $p : \mathbb{R}^n \to \mathbb{T}^n$ the natural projection. As in [**14**, Lemma 2.2], we shall find $\psi \in \mathcal{G}^{\rho,\rho'}(\mathbb{R}^n \times \Omega)$ and $R \in \mathcal{G}^{\rho'}(\Omega)$ such that $Q(x, \omega) := \psi(x, \omega) - \langle x, R(\omega)\rangle$ is 2π-periodic with respect to x and

(i) for all $(x, \omega) \in \mathbb{R}^n \times \Omega_\kappa$, $\nabla_x \psi(x, \omega) = F(p(x), \omega)$,

(ii) $|\partial_x^\alpha \partial_\omega^\beta Q(x, \omega)| + |\partial_I^\beta (R(\omega) - \nabla g^0(\omega))| \leq \kappa AC_1^{|\alpha|}(C_2/\kappa)^{|\beta|}\alpha!^\rho \beta!^{\rho(\tau+1)+1}L_1^{N/2}\sqrt{\epsilon_H}$,

for $(x, \omega) \in \mathbb{R}^n \times \Omega$. To obtain ψ we consider the function

$$\widetilde{\psi}(x, \omega) = \int_{\gamma_x} \sigma = \int_0^1 \langle F(p(tx), \omega), x\rangle\, dt, \quad (x, \omega) \in \mathbb{R}^n \times \Omega,$$

where $\gamma_x = \{(tx, F(p(tx), \omega)) : 0 \leq t \leq 1\}$ and $\sigma = \xi\, dx$ is the canonical 1-form on $T^*\mathbb{R}^n$. Then $\widetilde{\psi}(x, \omega) - \langle \nabla g^0(\omega), x\rangle$ satisfies the Gevrey estimates (ii) in $[0, 4\pi]^n \times \Omega$. We set $2\pi R_j(\omega) = \widetilde{\psi}(2\pi e_j, \omega)$, $\omega \in \Omega$, $\{e_j\}$ being an unitary basis in $\mathbb{R}^n$. Then $R - \nabla g^0$ satisfies (ii) in Ω. Since Λ_ω, $\omega \in \Omega_\kappa$, is Lagrangian, we obtain as in [**14**] that for such a ω the function $\nabla_x \widetilde{\psi}(x, \omega)$ is 2π-periodic with respect to x and $\widetilde{\psi}(x + 2\pi m, \omega) - \widetilde{\psi}(x, \omega) = \langle 2\pi m, R(\omega)\rangle$ for $m \in \mathbb{Z}^n$. Consider the function $\widetilde{Q}(x, \omega) = \widetilde{\psi}(x, \omega) - \langle x, R(\omega)\rangle$. It satisfies the Gevrey estimates (ii) in $[0, 4\pi]^n \times \Omega$, and it is 2π-periodic with respect to x for $\omega \in \Omega_\kappa$. We are going to average $\widetilde{Q}$

https://doi.org/10.1017/S0143385704000458 Published online by Cambridge University Press

on $\mathbb{T}^n$. Let $f \in \mathcal{G}_C^\rho(\mathbb{R}^n)$ with $\operatorname{supp} f \subset [\pi/2, 7\pi/2]^n$, where $C > 0$ is a positive constant, and such that $\sum_{k\in\mathbb{Z}^n} f(x - 2\pi k) = 1$ for each $x \in \mathbb{R}^n$. Consider the function $Q(x,\omega) = \sum_{k\in\mathbb{Z}^n}(f\widetilde{Q})(x - 2\pi k, \omega)$. It is 2π-periodic with respect to x by construction, it belongs to $\mathcal{G}^{\rho,\rho'}(\mathbb{R}^n \times \Omega)$, and $Q(x,\omega) = \widetilde{Q}(x,\omega)$ for $(x,\omega) \in \mathbb{R}^n \times \Omega_\kappa$. Moreover, Q satisfies the Gevrey estimates (ii) in $\mathbb{R}^n \times \Omega$. We set $\psi(x,\omega) = Q(x,\omega) + \langle x, R(\omega)\rangle$. Recall that $\operatorname{dist}(\Omega_\kappa, \mathbb{R}^n \setminus \Omega) \geq \kappa$. Then multiplying Q and $R - \nabla g^0$ by a suitable cut-off function $h \in \mathcal{G}_{\widetilde{C}}^{\rho'}(\Omega)$, with $\widetilde{C} = C/\kappa$ and $C > 0$ independent of $\Omega \subset \Omega_0$, such that $h = 1$ in a neighborhood of Ω_κ and $h(\omega) = 0$ if $\operatorname{dist}(\omega, \mathbb{R}^n \setminus \Omega) \leq \kappa/2$, we can assume that $\psi(x,\omega) = \langle x, \nabla g^0(\omega)\rangle$ for any ω such that $\operatorname{dist}(\omega, \mathbb{R}^n \setminus \Omega) \leq \kappa/2$. This does not change the corresponding Gevrey estimates for ψ.

Let $\epsilon_H L_1^{N+2(\tau+2)} \leq \epsilon \ll 1$. Then $\kappa A C_1 (C_2/\kappa) L_1^{N/2}\sqrt{\epsilon_H} \leq AC^2\sqrt{\epsilon} \ll 1$, and the map $\Omega \ni \omega \to \nabla_x \psi(x,\omega) \in D$ becomes a diffeomorphism for any x fixed, which gives a $\mathcal{G}^{\rho,\rho'}$-foliation of $\mathbb{T}^n \times D$ by Lagrangian tori $\Lambda_\omega = \{(p(x), \nabla_x\psi(x,\omega)) : x \in \mathbb{R}^n\}$, $\omega \in \Omega$. The action $I = (I_1, \dots, I_n) \in D$ on each Λ_ω, $\omega \in D$, is given by

$$I_j(\omega) = (2\pi)^{-1}\int_{\gamma_j(\omega)} \sigma = (2\pi)^{-1}(\psi(2\pi e_j, \omega) - \psi(0,\omega)) = R_j(\omega),$$

where $\gamma_j(\omega) = \{(p(te_j), \nabla_x\psi(te_j,\omega)) : 0 \leq t \leq 2\pi\}$. Then $I(\omega) - \nabla g^0(\omega) = R(\omega) - \nabla g^0(\omega)$ satisfies (ii) in Ω, and choosing $\epsilon_H L_1^{N+2(\tau+2)} \leq \epsilon \ll 1$ we obtain the fact that the frequency map $\Omega \ni \omega \mapsto I(\omega) \in D$ is a diffeomorphism of Gevrey class $\mathcal{G}^{\rho'}$. Using Remark A.1 we show that the inverse map $D \ni I \mapsto \omega(I) \in \Omega$ is in $\mathcal{G}^{\rho'}$ and

$$|\partial^\alpha(\omega(I) - \nabla H^0(I))| \leq \kappa A (C_2/\kappa)^{|\alpha|}\alpha!^{\rho'} L_1^{N/2}\sqrt{\epsilon_H},$$

uniformly with respect to $(\varphi,\omega) \in \mathbb{T}^n \times \Omega$ and for any $\alpha, \beta \in \mathbb{N}^n$. Now we set $\Phi(x, I) = \psi(x, \omega(I))$. Then $\Phi(x, I) - \langle x, I\rangle$ is 2π-periodic with respect to x, and using Proposition A.3 we get the estimates

$$|\partial_x^\alpha \partial_I^\beta(\Phi(x,I) - \langle x, I\rangle)| \leq \kappa A C_1^{|\alpha|}(C_2/\kappa)^{|\beta|}\alpha!^\rho \beta!^{\rho(\tau+1)+1} L_1^{N/2}\sqrt{\epsilon_H}.$$

Solving the equation $\Phi_I(\theta, I) = \varphi$ with respect to θ by means of Proposition A.2 we obtain the symplectic transformation χ. For any $\omega \in \Omega_\kappa$ and any θ we have $(\theta, F(\theta,\omega)) = (\theta, \Phi_\theta(\theta, I(\omega))) = \chi(\Phi_I(\theta, I(\omega)), I(\omega))$, hence $\Lambda_\omega = \chi(\mathbb{T}^n \times \{I(\omega)\})$. Set $\tilde{H}(\varphi, I) = H(\chi(\varphi, I))$. Then $\tilde{H}$ is constant on $\mathbb{T}^n \times \{I(\omega)\}$. We set $K(I) = \tilde{H}(0, I)$ and $R(\varphi, I) = \tilde{H}(\varphi, I) - K(I)$. Then $R(\varphi, I) = 0$ on $\mathbb{T}^n \times E_\kappa$, hence all the derivatives of R vanish on $\mathbb{T}^n \times E_\kappa$, since each point of E_κ is of positive Lebesgue density in E_κ. Using Proposition A.4 for $\tilde{H}(\varphi, I) = H(\theta(\varphi, I), \Phi_\theta(\theta(\varphi, I), I))$ we obtain for any $\alpha, \beta \in \mathbb{N}^n$

$$|\partial_\varphi^\alpha \partial_I^\beta(\tilde{H}(\varphi, I) - H^0(I))| \leq \kappa A C_1^{|\alpha|}(C_2/\kappa)^{|\beta|}\alpha!^\rho \beta!^{\rho(\tau+1)+1} L_1^{N/2}\sqrt{\epsilon_H}$$

uniformly with respect to $(\varphi, I) \in \mathbb{T}^n \times D$, where the constants A, C_1, and C_2 are as above. □

Proof of Corollary 1.3. Consider an integral curve $(\varphi(t), I(t)) = \exp(tX_{\widetilde{H}})(\varphi_0, I_0)$, $t \in (-a, a)$, such that $I_0 \notin E_\kappa$. Fix $p \in E_\kappa$ such that $|I_0 - p| = |I_0 - E_\kappa|$. Then $d(t) := |I(t) - p| > 0$ for any $t \in (-a, a)$. Suppose that there exists $T > 0$ such that $d(t) < \lambda d(0)$

https://doi.org/10.1017/S0143385704000458 Published online by Cambridge University Press

for $t \in [0, T)$ and $d(T) = \lambda d(0)$. Consider $\dot{d}(t) = -\langle \nabla_\varphi R(\varphi(t), I(t)), I(t) - p\rangle / d(t)$. Then

$$(\lambda - 1)d(0) \le \int_0^T |\nabla_\varphi R(\varphi(t), I(t))|\, dt \le \kappa A C_1 \int_0^T \exp(-(C_2 d(t)/\kappa)^{-1/[\rho(\tau+1)]})\, dt$$
$$< T\kappa A C_1 \exp(-(\lambda C_2 d(0)/\kappa)^{-1/[\rho(\tau+1)]}),$$

which proves the first part of the statement. The second one follows from (1.4). □

A. *Appendix*

We shall obtain a variant of the implicit function theorem of Komatsu [**5**] in anisotropic Gevrey classes. Let X and Ω_0 be domains in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Fix $\rho' \ge \rho \ge 1$. Let $F = (f_1, \dots, f_n)$ be a Gevrey function of the class $\mathcal{G}^{\rho,\rho'}(X \times \Omega, \mathbb{R}^n)$. We suppose that there are constants $A_0 > 0$ and $h_1, h_2 > 0$, and a small parameter $0 < \varepsilon \le 1$, such that

$$\text{for all } (x, \omega) \in X \times \Omega_0, \quad |\partial_x^\alpha \partial_\omega^\beta (F(x, \omega) - x)| \le \varepsilon A_0 h_1^{|\alpha|} h_2^{|\beta|} \alpha!^\rho \beta!^{\rho'} \tag{A.1}$$

for any multi-indices α, β. We choose $\varepsilon > 0$ so that $\varepsilon A_0 h_1 \le 1/2$. Then $|D_x F(x, \omega) - \mathrm{Id}| \le 1/2$ for any $(x, \omega) \in X \times \Omega_0$, where $D_x F$ stands for the Jacobian of F with respect to x. Hence, the inverse matrix $(D_x F)^{-1}$ exists and $|D_x F(x, \omega)^{-1}| \le 2$ in $X \times \Omega_0$. Now consider a local solution $x = g(y, \omega)$, $(y, \omega) \in Y \times \Omega$ of $F(x, \omega) = y$, where $Y \subset \mathbb{R}^n$ and $\Omega \subset \Omega_0$ are suitable domains.

PROPOSITION A.1. *Suppose that (A.1) holds and $0 < \varepsilon \le 1$ and $\varepsilon A_0 h_1 \le 1/2$, and consider a local solution $x = g(y, \omega)$, $(y, \omega) \in Y \times \Omega$, of $F(x, \omega) = y$. Then $g \in \mathcal{G}^{\rho,\rho'}(Y \times \Omega, X)$, and there exist positive constants A and C depending only on ρ, ρ', n, and m, such that*

$$\text{for all } \alpha, \beta, \quad \sup_{Y \times \Omega} |\partial_y^\alpha \partial_\omega^\beta (g(y, \omega) - y)| \le \varepsilon A_0 A C^{|\alpha+\beta|} h_1^{|\alpha|} h_2^{|\beta|} \alpha!^\rho \beta!^{\rho'}.$$

Remark A.1. Let $f = (f_1, \dots, f_n) \in \mathcal{G}^\rho(X, \mathbb{R}^n)$ and $|\partial_x^\alpha (f(x) - x)| \le \varepsilon A_0 h^{|\alpha|} \alpha!^\rho$ in X for any α, where $0 < \varepsilon \le 1$ and $\varepsilon A_0 h \le 1/2$, and let $g : Y \to X$ be an inverse map to f. Then the inverse map g of f belongs to $\mathcal{G}^\rho(Y, X)$ and $|\partial_y^\alpha (g(y) - y)| \le \varepsilon A_0 A C^{|\alpha|} h^{|\alpha|} \alpha!^\rho$ in Y for any α.

As a corollary we obtain the following.

PROPOSITION A.2. *Suppose that $F \in \mathcal{G}^{\rho,\rho'}(\mathbb{T}^n \times \Omega, \mathbb{T}^n)$ satisfies (A.1) with $0 < \varepsilon \le 1$ and $\varepsilon A_0 h_1 \le 1/2$. Then there exists $g \in \mathcal{G}^{\rho,\rho'}(\mathbb{T}^n \times \Omega, \mathbb{T}^n)$ and there are positive constants A and C depending only on ρ, ρ', n, and m such that*

$$\text{for all } \alpha, \beta, \quad \sup_{X \times \Omega} |\partial_y^\alpha \partial_\omega^\beta (g(y, \omega) - y)| \le \varepsilon A_0 A C^{|\alpha+\beta|} h_1^{|\alpha|} h_2^{|\beta|} \alpha!^\rho \beta!^{\rho'}.$$

Proof of Proposition A.1. We rescale the variables

$$x \mapsto h_1 x = \tilde{x} \in h_1 X := \tilde{X}, \quad y \mapsto h_1 y = \tilde{y} \in h_1 Y := \tilde{Y},$$
$$\omega \mapsto h_2 \omega = \tilde{\omega} \in h_2 \Omega := \tilde{\Omega}.$$

https://doi.org/10.1017/S0143385704000458 Published online by Cambridge University Press

Set $\tilde{F}(x,\omega) = h_1 F(h_1^{-1}x, h_2^{-1}\omega)$, $(x,\omega) \in \tilde{X}\times\tilde{\Omega}$, and $\tilde{g}(y,\omega) = h_1 g(h_1^{-1}y, h_2^{-1}\omega)$, $(y,\omega) \in \tilde{Y}\times\tilde{\Omega}$. Then $x = \tilde{g}(y,\omega)$, $(y,\omega) \in \tilde{Y}\times\tilde{\Omega}$, is a solution of the equation $\tilde{F}(x,\omega) = y$. Moreover, (A.1) implies

$$\sup_{\tilde{X}\times\tilde{\Omega}} |\partial_x^\alpha \partial_\omega^\beta (\tilde{F}(x,\omega) - x)| \le \tilde{\varepsilon}\alpha!^\rho \beta!^{\rho'}, \tag{A.2}$$

where $\tilde{\varepsilon} := \varepsilon A_0 h_1 \le 1/2$. We are going to prove that there exist positive constants A and C depending only on ρ, ρ', n, and m such that

$$\text{for all } \alpha,\beta, \quad \sup_{\tilde{Y}\times\tilde{\Omega}} |\partial_y^\alpha \partial_\omega^\beta (\tilde{g}(y,\omega) - y)| \le \tilde{\varepsilon} A C^{|\alpha+\beta|}\alpha!^\rho \beta!^{\rho'}. \tag{A.3}$$

Then rescaling back the variables we obtain the estimates in Proposition A.1.

We are now going to prove (A.3). From now on we omit '$\sim$' to simplify the notation. The implicit function theorem of Komatsu [**5**] implies that $g \in \mathcal{G}^{\rho'}(Y\times\Omega; X)$. Moreover, it follows from [**5**] that

$$\sup_{X\times\Omega} |\partial_x^\alpha \partial_\omega^\beta (g_j(y,\omega) - y_j)| \le \varepsilon A h^{|\alpha|+|\beta|}(\alpha!\beta!)^{\rho'}. \tag{A.4}$$

Hereafter, A and h are positive constants, depending only on ρ, ρ', n, and m. To obtain (A.4) we consider the inverse mapping (g, id) of (F, id) in $X\times\Omega$ and we apply the corresponding estimates in [**5**] (the estimates (A.4) also follow from the estimates of $\psi^{r,\alpha}$ below).

Take any $x^0, y^0 \in X$ and $\omega^0 \in \Omega$ such that $F(x^0,\omega^0) = y^0$. Changing the variables if necessary we assume $y^0 = 0$ and $\omega^0 = 0$. Now consider the solution $\omega \mapsto g(0,\omega)$ of $F(x,\omega) = 0$ with $g(0,0) = x^0$. Then we obtain

$$F(x,\omega) = (x - g(0,\omega))D_x F(g(0,\omega),\omega) + R(x,\omega),$$

in a neighborhood of $(0,0)$, where $R(x,\omega) = O(|x - g(0,\omega)|^2)$, $R \in \mathcal{G}^{\rho,\rho'}(X\times\Omega;\mathbb{R}^n)$. Moreover, it follows from (A.2), (A.4), and from Proposition A.3 below that

$$\sup_{X\times\Omega} |\partial_x^\alpha \partial_\omega^\beta R(x,\omega)| \le \varepsilon A h^{|\alpha+\beta|}\alpha!^\rho \beta!^{\rho'},$$

where $A > 0$ and $h > 0$ depend only on ρ, ρ', n, and n. Now me make a local change of variables

$$z = (x - g(0,\omega))D_x F(g(0,\omega),\omega),$$

and we obtain the equation (with respect to z)

$$z = y + \varphi(z,\omega), \quad z \in Y_0, y \in Y, \omega \in \Omega, \tag{A.5}$$

where Y_0 and Y are neighborhoods of 0 in $\mathbb{R}^n$, $\varphi(0,\omega) \equiv 0$, and

$$\text{for all } \alpha,\beta, \quad \sup_{Y_0\times\Omega} |\partial_z^\alpha \partial_\omega^\beta \varphi(z,\omega)| \le \varepsilon A h^{|\alpha+\beta|}\alpha!^\rho \beta!^{\rho'}$$

with some positive constants A and h as above. We are going to estimate the derivatives $\partial_y^p \partial_\omega^\alpha u(0,0)$ of the solution $z = u(y,\omega)$ of the equation above as in [**5**].

https://doi.org/10.1017/S0143385704000458 Published online by Cambridge University Press

For $j \in \mathbb{N}$ we set $M_j = j!^{\rho}$ and $N_j = j!^{\rho'}$. Then $M_j \le N_j$ and there exists $H \ge 1$ such that

$$(M_q/q!)^{1/q-1} \le H(M_p/p!)^{1/p-1}, \quad (M_q/q!)^{1/q} \le H(M_p/p!)^{1/p}, \quad 2 \le q \le p. \tag{A.6}$$

The same estimates also hold for N_j.

Consider the Taylor expansion of φ in a formal power series at $(0,0)$,

$$\varphi(z,\omega) = \sum_{|p|\ge 2} \sum_{\alpha\in\mathbb{N}^m} \frac{\varphi^{p,\alpha}}{p!\alpha!} z^p \omega^\alpha, \quad \varphi^{p,\alpha} = \partial_z^p \partial_\omega^\alpha \varphi(0,0).$$

Then we have

$$\varphi \ll \varepsilon A \sum_{|p|\ge 2} \sum_{\alpha\in\mathbb{N}^m} \frac{M_{|p|}}{p!} \frac{N_{|\alpha|}}{\alpha!} h^{|p|+|\alpha|} z^p \omega^\alpha,$$

which means that $|\varphi^{p,\alpha}| \le \varepsilon A M_{|p|} N_{|\alpha|} h^{|p|+|\alpha|}$ for each p, α. Consider the solution $z = u(y,\omega)$ of (A.5) and its formal Taylor series at $(0,0)$,

$$u(y,\omega) = y + \sum_{|p|\ge 2} \sum_{\alpha\in\mathbb{N}^m} \frac{u^{p,\alpha}}{p!\alpha!} y^p \omega^\alpha.$$

Then (A.5) is formally equivalent to

$$\sum_{|p|\ge 2} \sum_{\alpha\in\mathbb{N}^m} \frac{u^{p,\alpha}}{p!\alpha!} y^p \omega^\alpha = \sum_{|p|\ge 2} \sum_{\alpha\in\mathbb{N}^m} \frac{\varphi^{p,\alpha}}{p!\alpha!} \left(y + \sum_{|q|\ge 2} \sum_{\beta\in\mathbb{N}^m} \frac{u^{q,\beta}}{q!\beta!} y^q \omega^\beta \right)^p \omega^\alpha.$$

Denote by $v(y,\omega)$ the power series

$$v(y,\omega) = \sum_{|p|\ge 2} \sum_{\alpha\in\mathbb{N}^m} \frac{v^{p,\alpha}}{p!\alpha!} y^p \omega^\alpha,$$

and suppose that it is a formal solution of

$$v(y,\omega) = \varepsilon A \sum_{|p|\ge 2} \sum_{\alpha\in\mathbb{N}^m} \frac{M_{|p|}}{p!} \frac{N_{|\alpha|}}{\alpha!} h^{|p|} h^{|\alpha|} (y + v(y,\omega))^p \omega^\alpha. \tag{A.7}$$

We claim that such a solution exists and it is unique, $v^{p,\alpha} > 0$, and $u \ll y + v$. Indeed, for any $|p| \ge 2$, $\alpha \in \mathbb{N}^m$, and $1 \le j \le n$, there is a polynomial $Q_{p,\alpha,j} \not\equiv 0$ of the variables $(t_{q,\beta}, s_{r,\gamma})$ and with non-negative coefficients, where $q, r \in \mathbb{N}^n$, $\beta, \gamma \in \mathbb{N}^m$, $2 \le |q| \le |p|$, $\beta \le \alpha$, $2 \le |r| \le |p| - 1$, $\gamma \le \alpha$, and such that $u_j^{p,\alpha}$ (respectively, $v_j^{p,\alpha}$) is the value of $Q_{p,\alpha,j}$ for $t_{q,\beta} = \varphi^{q,\beta}$, $s_{r,\gamma} = u_j^{r,\gamma}$ (respectively, for $t_{q,\beta} = \varepsilon A M_{|p|} N_{|\alpha|} h^{|p|+|\alpha|}$, $s_{r,\gamma} = v_j^{r,\gamma}$). For $|p| = 2$ the polynomial $Q_{p,\alpha,j}$ is independent of $s_{r,\gamma}$, $|r| \ge 2$. Since the coefficients of $Q_{p,\alpha,j}$ are non-negative, we get $|u_j^{r,\alpha}| \le v_j^{r,\alpha}$ for $|r| = 2$ and each α. By recurrence we obtain $|u_j^{r,\alpha}| \le v_j^{r,\alpha}$ for each $|r| \ge 2$ and $\alpha \in \mathbb{N}^m$.

Now we set $y = (s, \dots, s)$, $s > 0$, and consider

$$\phi(s,\omega) = \sum_{r=2}^{\infty} \sum_{\alpha\in\mathbb{N}^m} \frac{\phi^{r,\alpha}}{r!\alpha!} s^r \omega^\alpha, \quad \phi^{r,\alpha} = \sum_{|p|=r} v^{p,\alpha} \frac{r!}{p!}.$$

https://doi.org/10.1017/S0143385704000458 Published online by Cambridge University Press

Note that for such y the right-hand side of equation (A.7) is invariant under any permutation of the components $v_j(y,\omega)$, $1 \le j \le n$, and using the uniqueness of the solution we obtain $\phi_1(s,\omega) = \cdots = \phi_n(s,\omega) = \tilde{\phi}(s,\omega)$. Then $t = s + \tilde{\phi}(s,\omega)$ is a formal solution of

$$t = s + \varepsilon A \sum_{r=2}^{\infty} \sum_{\alpha\in\mathbb{N}^m} \sum_{|p|=r} \frac{r!}{p!} \frac{M_r}{r!} \frac{N_{|\alpha|}}{\alpha!} (ht)^r (h\omega)^\alpha,$$

where $\sum_{|p|=r}(r!/p!) \le n^r$. Let $t = \psi(s,\omega) := s + \sum_{r=2}^{\infty} \sum_{\alpha\in\mathbb{N}^m} (\psi^{r,\alpha}/r!\alpha!) s^r \omega^\alpha$ be the formal solution of

$$t = s + \varepsilon A \sum_{r=2}^{\infty} \sum_{\alpha\in\mathbb{N}^m} \frac{M_r}{r!} \frac{N_{|\alpha|}}{\alpha!} (hnt)^r (h\omega)^\alpha. \tag{A.8}$$

We obtain as above $s + \tilde{\phi} \ll \psi$, which implies $|u_j^{p,\alpha}| \le \psi^{|p|,\alpha}$ for $1 \le j \le n$ and for any $|p| \ge 2, \alpha \in \mathbb{N}^m$.

We are going to estimate $\psi^{r,\alpha}$. Set $C(\omega) = A \sum_{\alpha\in\mathbb{N}^m} (N_{|\alpha|}/\alpha!)(h\omega)^\alpha$. Then (A.8) becomes

$$s = k(t,\omega) := t - \varepsilon C(\omega) \sum_{r=2}^{\infty} \frac{M_r}{r!} (hnt)^r.$$

Recall that $0 < \varepsilon \le 1$. We also suppose that $A \ge 1$ and $h \ge 1$. As in [**5**], using the Lagrange expansion theorem, we obtain for $r \ge 2$

$$\begin{aligned}
\psi^{r,\alpha} &= \left[\partial_\omega^\alpha \left\{ \left(\frac{d}{dt}\right)^{r-1} \left(\frac{t}{k(t,\omega)}\right)^r \right\}\Big|_{t=0}\right]\Big|_{\omega=0} \\
&\le \varepsilon \left[\partial_\omega^\alpha \left\{ \left(\frac{d}{dt}\right)^{r-1} \left(\sum_{q=0}^{\infty} \left(C(\omega) hn \sum_{p=1}^{r-1} \frac{M_{p+1}}{(p+1)!} (hnt)^p \right)^q \right)^r \right\}\Big|_{t=0}\right]\Big|_{\omega=0},
\end{aligned}$$

and we get

$$\begin{aligned}
\psi^{r,\alpha} &\le \varepsilon M_r (4Hhn)^{r-1} \left[\partial_\omega^\alpha \left(1 + Ahn \sum_{\beta\le\alpha} \frac{N_{|\beta|}}{|\beta|!} (h\omega)^\beta\right)^{r-1}\right]\Big|_{\omega=0} \\
&\le \varepsilon M_r (8AHh^2n^2)^{r-1} \left[\partial_\omega^\alpha \left(\sum_{\beta} \left(\left(\frac{N_{|\alpha|}}{|\alpha|!}\right)^{1/|\alpha|} h\omega \right)^\beta \right)^{r-1}\right]\Big|_{\omega=0} \\
&\le \varepsilon M_r N_{|\alpha|} (8AHh^2n^2)^{r-1} h^{|\alpha|} 2^{m(r-2)+|\alpha|}.
\end{aligned}$$

We have used the inequalities (A.6) for M_j and N_j, as well as the identity

$$\left(\sum_{\beta\in\mathbb{N}^m} \omega^\beta \right)^q = \sum_{\alpha\in\mathbb{N}^m} C_\alpha \omega^\alpha,$$

where $q = r - 1 \ge 1$ and

$$C_\alpha = \binom{q+\alpha_1-1}{\alpha_1} \cdots \binom{q+\alpha_m-1}{\alpha_m} \le 2^{m(q-1)+|\alpha|}.$$

Thus we obtain

$$\text{for all } (p,\alpha,j), \quad |u_j^{p,\alpha}| \le \psi^{|p|,\alpha} \le \varepsilon A M_{|p|} N_{|\alpha|} h^{r+|\alpha|}$$

where $A, h > 0$ depend only on ρ, ρ', n, and m. Changing back the variables and rescaling back Ω, we obtain the desired estimates for g. □

https://doi.org/10.1017/S0143385704000458 Published online by Cambridge University Press

Finally we recall the Gevrey estimates for the composition of two functions in anisotropic Gevrey classes.

PROPOSITION A.3. *Let $g \in \mathcal{G}^{\mu}_{C_1}(\Omega, Y)$ and $f \in \mathcal{G}^{\rho,\mu}_{B,C_2}(X \times Y)$, where $\mu \geq \rho \geq 1$, B, C_1, and C_2 are positive constants and X, Y, and Ω are open sets in $\mathbb{R}^n$. Suppose that $\|g\|_{C_1} = A_1$ and $\|f\|_{B,C_2} = A_2$ in the corresponding Gevrey norms with $A_1, A_2 > 0$. Then the composition $(x, \omega) \mapsto F(x, \omega) := f(x, g(\omega))$ belongs to $\mathcal{G}^{\rho,\mu}_{B,C}(X \times \Omega)$, where $C = 2^{n+\mu} n^{\mu} C_1 \max(1, A_1 C_2)$ and $\|F\|_C \leq A_2$.*

Proof. Using the Faa de Bruno formula, we write

$$\partial_x^{\gamma} \partial_{\omega}^{\alpha} F(x, \omega) = \sum_{1 \leq |\beta| = p \leq |\alpha|} \frac{(\partial_x^{\gamma} \partial_y^{\beta} f)(x, g(\omega))}{\beta!} \times \sum_{\substack{\alpha^1 + \cdots + \alpha^p = \alpha \\ |\alpha^1| \geq 1, \ldots, |\alpha^p| \geq 1}} \frac{\alpha!}{\alpha^1! \cdots \alpha^p!} (\partial_{\omega}^{\alpha^1} g)(\omega) \cdots (\partial_{\omega}^{\alpha^p} g)(\omega).$$

Since $|\partial_{\omega}^{\alpha} g(\omega)| \leq A_1 C_1^{|\alpha|} \alpha!^{\mu}$, and $j! \leq 2^j (j-1)!$ for $j \geq 1$, we estimate $|\partial_x^{\gamma} \partial_{\omega}^{\alpha} F(x, \omega)|$ above by

$$A_2 B^{|\gamma|} (2^{\mu - 1} C_1)^{|\alpha|} \gamma!^{\rho} \alpha! \times \sum_{1 \leq |\beta| = p \leq |\alpha|} \sum_{\substack{\alpha^1 + \cdots + \alpha^p = \alpha \\ |\alpha^1| \geq 1, \ldots, |\alpha^p| \geq 1}} (A_1 C_2)^p ((|\alpha^1| - 1)! \cdots (|\alpha^p| - 1)! p!)^{\mu - 1}.$$

We have $(|\alpha^1| - 1)! \cdots (|\alpha^p| - 1)! p! \leq |\alpha|! \leq n^{|\alpha|} \alpha!$. Observe that, for any $r, p \in \mathbb{N}$, $r, p \geq 1$, the number of multi-indices $\gamma = (\gamma_1, \ldots, \gamma_p) \in \mathbb{N}^p$ with $|\gamma| = r$ is given by $\binom{r+p-1}{p-1}$ (see [**15**, §1.2]). Then the number of partitions $\alpha^1 + \cdots + \alpha^p = \alpha$, $\alpha^j \in \mathbb{N}^n$, of $\alpha = (\alpha_1, \ldots, \alpha_n)$ is given by

$$\binom{\alpha_1 + p - 1}{p - 1} \cdots \binom{\alpha_n + p - 1}{p - 1} \leq 2^{|\alpha|} 2^{n(p-1)},$$

which implies

$$|\partial_x^{\gamma} \partial_{\omega}^{\alpha} F(x)| \leq A_2 2^{-n} B^{|\gamma|} (2^{n+\mu} n^{\mu} C_1 \max(1, A_1 C_2))^{|\alpha|} \gamma!^{\rho} \alpha!^{\mu} \times \sum_{\beta \in \mathbb{N}^n} 2^{-n|\beta|} < A_2 B^{|\gamma|} C^{|\alpha|} \gamma!^{\rho} \alpha!^{\mu}. \qquad \square$$

In the same way we obtain the following.

PROPOSITION A.4. *Let $g \in \mathcal{G}^{\rho,\mu}_{B_1,C_1}(X \times \Omega, Y)$ and $f \in \mathcal{G}^{\rho,\mu}_{B_2,C_2}(Y \times \Omega)$, where $\mu \geq \rho \geq 1$, B_j and C_j, $j = 1, 2$, are positive constants, and X, Y, and Ω are open sets in $\mathbb{R}^n$. Suppose that $\|g\|_{B_1,C_1} = A_1$ and $\|f\|_{B_2,C_2} = A_2$ in the corresponding Gevrey norms with $A_1, A_2 > 0$. Then the composition $(x, \omega) \mapsto F(x, \omega) := f(g(x, \omega), \omega)$ belongs to $\mathcal{G}^{\rho,\mu}_{B,C}(X \times \Omega)$, where $B = 4^n (4n)^{\rho} B_1 \max(1, A_1 B_2)$, $C = C_2 + 4^n (4n)^{\rho} C_1 \max(1, A_1 B_2)$, and $\|F\|_{B,C} \leq A_2$.*

https://doi.org/10.1017/S0143385704000458 Published online by Cambridge University Press

To prove the Gevrey estimates we set $z = (x, \omega)$ and, using the Leibnitz formula and the Faa de Bruno formula, we write

$$\partial_x^\gamma \partial_\omega^\alpha F(x,\omega) = \sum_{\alpha' \le \alpha} \binom{\alpha}{\alpha'} \sum_{1 \le |\beta| = p \le |\gamma + \alpha'|} \frac{(\partial_y^\beta \partial_\eta^{\alpha-\alpha'} f)(g(z),\eta)|_{\eta=\omega}}{\beta!} \times \sum_{\substack{\delta^1 + \cdots + \delta^p = \delta = (\gamma,\alpha') \\ |\delta^1| \ge 1, \ldots, |\delta^p| \ge 1}} \frac{\delta!}{\delta^1! \cdots \delta^p!} (\partial_z^{\delta^1} g)(z) \cdots (\partial_z^{\alpha^p} g)(z),$$

where $\delta^j = (\gamma^j, \alpha^j)$. Then

$$\begin{aligned} |\partial_x^\gamma \partial_\omega^\alpha F(x,\omega)| \le{}& A_2 B_1^{|\gamma|} \gamma! \alpha! \\ &\times \sum_{\alpha' \le \alpha} C_1^{|\alpha'|} C_2^{|\alpha|-|\alpha'|} (\alpha - \alpha')!^{\mu-1} \sum_{1 \le |\beta| = p \le |\gamma+\alpha'|} (A_1 B_2)^p \beta!^{\rho-1} \\ &\times \sum_{\substack{\delta^1 + \cdots + \delta^p = \delta = (\gamma,\alpha') \\ |\delta^1| \ge 1, \ldots, |\delta^p| \ge 1}} (\gamma^1! \cdots \gamma^p!)^{\rho-1} (\alpha'^1! \cdots \alpha'^p!)^{\mu-1}. \end{aligned}$$

On the other hand,

$$\begin{aligned} &(\gamma^1! \cdots \gamma^p!)^{\rho-1} (\alpha'^1! \cdots \alpha'^p!)^{\mu-1} \beta!^{\rho-1} \\ &\quad \le 2^{(\rho-1)|\gamma+\alpha'|} ((|\gamma^1 + \alpha'^1| - 1)! \cdots (|\gamma^p + \alpha'^p| - 1)! p!)^{\rho-1} (\alpha'^1! \cdots \alpha'^p!)^{\mu-\rho} \\ &\quad \le 2^{(\rho-1)|\gamma+\alpha'|} (2n)^{(\rho-1)|\gamma+\alpha'|} \gamma^\rho \alpha'!^\mu. \end{aligned}$$

We obtain as above

$$\begin{aligned} |\partial_x^\gamma \partial_\omega^\alpha F(x,\omega)| \le{}& A_2 ((4n)^\rho B_1 \max(1, A_1 B_2))^{|\gamma|} \gamma!^\rho \alpha!^\mu \\ &\times \sum_{\alpha' \le \alpha} ((4n)^\rho C_1 \max(1, A_1 B_2))^{|\alpha'|} C_2^{|\alpha|-|\alpha'|} \sum_{1 \le |\beta| = p \le |\gamma+\alpha'|} 4^{n(p-1)} \end{aligned}$$

which implies the estimates. □

Proof of Lemma 3.4. First, using the Cauchy formula, we obtain as in [**13**, Lemma A.3], that $f : O_{2(\tilde\upsilon+\upsilon)h} \to \mathbb{C}$ is one-to-one (injective). Note that $2(\tilde\upsilon + \upsilon) = 1 - 4\upsilon$. To show that $O_{\tilde\upsilon h} \subset f(O_{(1-4\upsilon)h})$, we take $w \in O_{\tilde\upsilon h}$ and $\omega \in \Omega_\kappa$ such that $|w - \omega| < \tilde\upsilon h$. Set $f = \mathrm{id} - F$. Then $|F|_h \le \upsilon h$ and $|DF|_{(1-4\upsilon)h} \le 1/4$ by the Cauchy estimates. Put $u_0 = \omega$ and $u_{k+1} = F(u_k) + w$ for $k \ge 0$. By recurrence we prove that

$$|u_{k+1} - u_k| \le 2^{-1} 4^{-k} (1 - 4\upsilon)h, \quad |u_k - \omega| \le (2/3)(1 - 4^{-k})(1 - 4\upsilon)h.$$

Taking the limit we find $u \in B_r(\omega)$, $r = (1 - 4\upsilon)h$, such that $u = F(u) + w$. The corresponding estimates of ϕ follow from the arguments in [**13**, Lemma A.3]. □

References

[1] J. Bruna. An extension theorem of Whitney type for non quasi-analytic classes of functions. *J. London Math. Soc.* **22**(2) (1980), 495–505.

https://doi.org/10.1017/S0143385704000458 Published online by Cambridge University Press

[2] L. Chierchia and G. Gallavotti. Smooth prime integrals for quasi-integrable Hamiltonian systems. *Nuovo Cimento* B(11) **67**(2) (1982), 277–295.
[3] A. Giorgilli and A. Morbidelli. Invariant KAM tori and global stability for Hamiltonian systems. *Z. Angew. Math. Phys.* **48**(1) (1997), 102–134.
[4] M. Herman. Inégalités 'a priori' pour des tores lagrangiens invariants par des difféomorphismes symplectiques. *Publ. Math. Inst. Hautes Études Sci.* **70** (1989), 47–101.
[5] H. Komatsu. The implicit function theorem for ultradifferentiable mappings. *Proc. Japan Acad.* A **55** (1979), 69–72.
[6] S. Kuksin. *Analysis of Hamiltonian PDE's (Oxford Lecture Series in Mathematics and its Applications, 19)*. Oxford University Press, Oxford, 2000.
[7] V. Lazutkin. *KAM Theory and Semiclassical Approximations to Eigenfunctions*. Springer, Berlin, 1993.
[8] J. L. Lions and E. Magenes. *Problèmes aux limites non homogènes et applications*, Vol. 3 *(Travaux et recherches mathématiques, 20)*. Dunod, Paris, 1970.
[9] P. Lochak. Canonical perturbation theory: an approach based on joint approximations. *Usp. Mat. Nauk* **47**(6) (1992), 59–140 (Engl. transl. *Russian Math. Surveys* **47**(6) (1992), 57–133).
[10] J.-P. Marco and D. Sauzun. Stability and instability for Gevrey quasi-convex near-integrable Hamiltonian systems. *Publ. Math. Inst. Hautes Études Sci.* **96** (2002), 199–275.
[11] A. Morbidelli and A. Giorgilli. On a connection between KAM and Nekhoroshev's theorems. *Physica* D **86** (1995), 514–516.
[12] J. Pöschel. Integrability of Hamiltonian systems on Cantor tori. *Comm. Pure Appl. Math.* **35** (1982), 653–695.
[13] J. Pöschel. A lecture on the classical KAM theorem. *Proc. Symp. Pure Math.* **69** (2001), 707–732.
[14] G. Popov. Invariant tori, effective stability and quasi-modes with exponentially small error terms I. *Ann. Henri Poincaré* **1** (2000), 223–248.
G. Popov. Invariant tori, effective stability and quasi-modes with exponentially small error terms II. *Ann. Henri Poincaré* **1** (2000), 249–279.
[15] L. Rodino. *Linear Partial Differential Operators in Gevrey Spaces*. World Scientific, Singapore, 1993.
[16] E. Zehnder. Generalized implicit function theorems with applications to some small divisor problems I. *Comm. Pure Appl. Math.* **28** (1975), 91–140.

https://doi.org/10.1017/S0143385704000458 Published online by Cambridge University Press