The New String Paradox

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I shall begin by discussing Benardete's string paradox. Then, a new puzzle involving supertasks and causality (the new string paradox) is proposed and discussed. Finally, the new string paradox facilitates the discovery of a surprising example of irreversible mechanical process.

1. Benardete's String Paradox. Benardete (1964) introduced the string paradox, a supertask formulated as follows:

> Let us take a piece of string. In the first 1/2 minute we shall form an equilateral triangle with the string; in the next 1/4 minute we shall employ the string to form a square; in the next 1/8 minute we shall form a regular pentagon; &c. ad infinitum. At the end of the minute what figure or shape will our piece of string be found to have assumed? Surely it can only be a circle. And yet how intelligible is that process? Each and every one of the polygons in our infinite series contains only a finite number of sides. There is thus a serious conceptual gap separating the circle, as the limiting case, from each and every polygon in the infinite series. How are we to negotiate the conceptual leap required to pass over from the *whole* infinite series to the circle as the terminus ad quem? (1964, 21)

Much later on in the book, Benardete makes a statement that points in the direction of what I believe is a correct solution to the paradox: "If we are to guarantee the circle here . . . we must presuppose some kind of postulate of continuity" (278). Certainly a string is a material object, and the kind of pos-

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tulate of continuity relevant here is one stating that the world line of any point belonging to a material body is continuous. Let us call it the World Line Continuity Postulate (WLCP). I shall simplify things by considering the string to have null thickness, that is, that it has length but not thickness. This move will also simplify my own version of the paradox, which I present in this article. Furthermore, the idea of string without thickness (a one-dimensional object) is an idealization no more objectionable in principle than the very useful notion of point mass (a zero-dimensional object).

Given a continuous material body, there are two types of intrinsic global changes that I am interested in here, which in principle may be carried out on it: one may change its form or its size (or both at the same time). Likewise, an arbitrary external force (or a system of external forces) exerted on the body may in general affect its form and its size. Applying all this to the case of a string with no thickness (which is the only one I shall consider in what follows), its size is clearly reduced to its length, and thus, being the one-dimensional object it is, its size varies when that length varies. Let S be any string. A force acting on any point P of S in the direction of (the tangent in P to) S is called force of traction. A force of traction may in principle change the length of S without changing its form, and, evidently, other forces may change the form of S without changing its length. I shall say that a string is inflexible when no force (or system of forces) to which it is subjected actually alters its form. An inflexible string is thus essentially like a rigid rod, although not necessarily rectilinear. Contrariwise, a string is perfectly flexible when it makes absolutely no resistance to its deformation (put more rigorously, when it presents no rigidity against flexion) and remains in its deformed state (whatever the deformation to which it has been subjected) with no tendency to recover its original form. In formulating my paradox, I consider a perfectly flexible string.[1] Just as the concept of a material point is a major component in the classical mechanics of discrete systems, it so happens that the idea of a perfectly flexible string covers an entire important chapter of the classical mechanics of continuous systems. Both idealizations have traditionally been studied in the sphere of what is known as "rational mechanics," and authors as relevant as Galileo, Euler, Lagrange, Lorentz, and Appell have made scientific contributions to the mechanics of the perfectly flexible string, giving it its present form. This explains the justified interest it has for philosophical analysis. There are many detailed expositions of the mechanics of the string, and a sim-

1. If a string is not perfectly flexible (i.e., if it presents a certain rigidity to flexion), increasingly large forces are required to deform it even further. Indeed, if the curvature of the string at a point tends to infinity, the force required also tends to infinity. The notion of perfect flexibility avoids these complications, which accounts for its theoretical usefulness.

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ple but rigorous introduction can be found, for example, in Starjinski (1984). My contribution in this article is framed in this context.

2. The New String Paradox. Like Benardete's string paradox, the new string paradox involves a supertask and makes use of a perfectly flexible string S with no thickness and a finite length L. But now the supertask will be type ω^* (the order type of the negative integers). At $t = 0$ let S be at rest on the X-axis (in which instant it therefore has the form of a straight segment), the coordinate of its left end being $x = 0$ and of its right end $x = L$. Let us also assume that at $t = 0$ it is not subject to any force whatsoever. For every subsequent instant, we shall maintain null traction on the string and its left end fixed on the X-axis in $x = 0$.

- At $t = 1/1 = 1$, we shall give S the form (without changing its length with regard to $t = 1/2$) of a saw tooth ($\wedge$) consisting of two straight segments of equal length forming an angle of 60° at the apex (they are thus two sides of an equilateral triangle). At this instant, let $x = H_1$ be the coordinate of the right end of the string (this end, like the other, left-hand end, being placed on the X-axis). Then the coordinate of the apex is $x = H_1/2$, and the apex will be at a height of $\sqrt{3}(H_1/2)$ above the X-axis. Therefore, at $t = 1$ all points of the string have a coordinate x such that $0 \leq x \leq H_1$ and are at a distance above the X-axis smaller than or equal to $\sqrt{3}(H_1/2)$.
- At $t = 1/n$, I give S the form (without changing its length with regard to $t = 1/(n + 1)$) of a sequence of $2^{(n-1)}$ contiguous identical saw teeth ($\wedge\wedge \ldots \wedge\wedge$) comprising 2^n rectilinear segments of equal length that form $2^{(n-1)}$ apexes, each with an angle of 60° (they are thus sides of $2^{(n-1)}$ equilateral triangles).[2] At that instant, let $x = H_n$ be the coordinate of the right end of the string (the right end, like the left end, being situated on the X-axis). Then the coordinates of the apexes are respectively $x = H_n/2^n$, $x = 3H_n/2^n$, $x = 5H_n/2^n$, . . . , $x = (2^n - 1)H_n/2^n$,

2. Let there be a saw tooth ($\wedge$) formed with the string or a fragment of it, and let us call the three characteristic points that define it A, B, and C (B is the apex). Throughout this article (with the exception mentioned in n. 4), I assume that the fragment of the string between A and B (and also between C and B) has a length l_{AB} equal to the spatial distance d_{AB} between A and B (or between C and B, which is the same: $d_{CB} = d_{AB}$); i.e., I assume that the fragment in question remains straight. However, this is not essential. We could perfectly well suppose, without making substantial changes to the arguments that follow, that, each time I form a saw tooth ABC, l_{AB} (or l_{CB}) is greater than d_{AB} ($= d_{CB}$). In that case, I say we have a saw tooth with play. Bearing in mind the description of the tasks that have to be performed at the instants $t = 1/n$ ($n \geq 1$), it is clear that the amount of play may be variable. For example, I might decide that, each time I form a saw tooth, $l_{AB} = l_{CB} = 3d_{AB}/2 = 3d_{CB}/2$, with 3/2 then being what might be called the "play factor." The absence of play therefore corresponds to a play factor equal to the unit.

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and all the apexes will be at a height $\sqrt{3}(H_n/2^n)$ above the X-axis. At $t = 1/n$, therefore, all points of the string have an x coordinate such that $0 \leq x \leq H_n$ and are at a distance above the X-axis smaller than or equal to $\sqrt{3}(H_n/2^n)$.

It is obvious that all we do on going from $t = 1/(n + 1)$ to $t = 1/n$ is reduce the number of saw teeth by half; making two contiguous teeth give rise to one new one, similar to any one of the others; double the width at the base; and double the height (which leaves the length of the piece of string involved unaltered). This means that the coordinate of the right end of the string does not vary; that is, $H_1 = H_2 = H_3 = \ldots = H_n = \ldots = H$. Consequently, at $t = 1/n$ all points of the string have an x coordinate such that $0 \leq x \leq H$ and are at a distance above the X-axis smaller than or equal to $\sqrt{3}\ (H/2^n)$. Taking the limit $n \to \infty$ and taking account of WLCP, we deduce that at $t = 0$ all points of the string must have an x coordinate such that $0 \leq x \leq H$ and be situated on the X-axis. In other words, at $t = 0$ we should have a straight string from $x = 0$ to $x = H$. As we know that at $t = 0$ the string is rectilinear in form with ends at $x = 0$ and $x = L$, we deduce that $H = L$. This has a surprising consequence. We have already seen that for $t > 0$ the length of the string does not vary. Therefore, its length in any instant $t > 0$ is simply its length in, let us say, $t = 1$. This last is the sum of the lengths of two sides of an equilateral triangle with sides $H = L$. To put it another way, the string of length L at $t = 0$ suddenly has a length $2L$ for $t > 0$. This discontinuous change in the length of the string not only is compatible with WLCP but is in fact required (as we saw above) by WLCP.[3] Furthermore, it cannot have been produced by a force of traction, as throughout the entire process the string was not subjected to any traction whatsoever. This has some noteworthy consequences:

3. Although the family of curves that describe the form of the string for $t > 0$ converges pointwise to the curve that describes the form of the string for $t = 0$, it is not the case that the lengths of the curves of the family that describe the form of the string for $t > 0$ converge to the length of the curve that describes the form of the string for $t = 0$. Therefore, the paradox is not to be found in the supposed fact that the length of the string at $t = 0$ is L and is also $2L$ (which is false: the length of the string at $t = 0$ is L and is not $2L$). The paradox lies in the instantaneous change in the length of the string, from L at $t = 0$ to $2L$ for $t > 0$. Therefore, and most important, this is a paradox involving physics and philosophy; it is not a mathematical paradox. And the genuine philosophical problem is how to explain causally this instantaneous change of length. Should the ω*-type supertask be performed by constructing the saw teeth with play in the sense specified in n. 2, the instantaneous change in the length of the string may be greater. If the play factor is k, the length of the string will go from L to $2kL$. With $k = 3/2$, we achieve a supertask that instantaneously triples the string's original length.

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- Let us imagine a rectilinear string whose natural length will not be altered however great the traction force it is subjected to. We would tend to think that such a string is inextensible, but this would be a mistake because we could increase its length by the basically kinematic procedure described in the new string paradox.
- As the string was only moved after $t = 0$ (i.e., when its length had already doubled), it is clear that none of the particular movements reducing the number of apexes could have been the cause of this doubling. In some way, the cause was rather the infinite sequence of such movements.
- WLCP implies that, in an evident sense, the form of a string must vary continuously over time (Benardete's string paradox). But, surprisingly, it does not also imply that its size should vary continuously over time (new string paradox).

To corroborate the coherence of the process described above, let us see what the maximum average velocity is for a point of the string in the temporal interval $[0, 1/n]$. As we have seen, the maximum height it can attain above the X-axis is $\sqrt{3}(L/2^n)$, and therefore the maximum average velocity is $V_M = \sqrt{3}(L/2^n)/(1/n) = \sqrt{3}n(L/2^n)$. As $\lim_{n\to\infty} V_M = 0$, it follows that all the points of the string had null instantaneous velocity at $t = 0$, which is in accordance with the initial assumption that the string was at rest at $t = 0$. Analogously, it can be demonstrated that all points of the string had null instantaneous acceleration at $t = 0$, which means that the force on it was also null at that instant.

Many philosophers tend to dissolve problems like the one posed by the new string paradox by rejecting the legitimacy of the idealizations employed in their formulation: the idea of a perfectly flexible string in the new string paradox. I think this attitude impoverishes philosophical activity. The new string paradox poses conceptual difficulties for theoretical mechanics (a well-established scientific discipline), and philosophical discussion of them is relevant for this theory. In his work, the philosopher studies and reflects on a scientific theory, not directly on reality, and this is precisely how he can make a significant contribution to the task of conceptual clarification. Even if, given the new string paradox, someone argues that a concept like perfect flexibility (or the one of a string of null thickness) should be excluded from the theory in favor of other less problematic concepts, he would seem to be admitting implicitly the importance of the new string paradox as an argument for justifying such an exclusion. Finally, I would like to consider the following interesting criticism. Let us suppose that the string S is inextensible as well as perfectly flexible. In other words, S can be deformed freely, but its length cannot be changed; in particular, it cannot be stretched. Then the duplication of length described in the new string paradox is impossible, as

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it contradicts inextensibility. In these conditions, we are apparently obliged to specify in more detail the process of deformation of S mentioned in the formulation of the paradox to indicate clearly how and where it will fail, as we know that it must. But no such subsequent specification would avoid having to admit what is by any reckoning counterproductive, namely, that for some mysterious reason the mechanism supposed to reduce by half the number of saw teeth of the string S on going from $t = 1/(n + 1)$ to $t = 1/n$ (a mechanism we may easily imagine in multiple forms and that is trivial) would have to fail an infinite number of times (for infinite values of n). This is the only way the conclusion of the doubling of length is avoidable. My reply to this criticism consists in rejecting its starting point, that is, in denying that string S may be inextensible. What the new string paradox shows is rather that no perfectly flexible string may be inextensible. In the scientific literature on continuous media, the idealization of a perfectly flexible, inextensible string is frequently used, but there is no incompatibility whatsoever with my previous conclusion because, when one analyzes the context, one soon discovers that in the theoretical mechanics of the string, inextensibility is always understood as inextensibility under forces of traction (this is the form in which the concept is extremely useful for modeling a multitude of situations from the real world). The new string paradox obliges us to qualify the nature of the inextensibility by underscoring the fact that inextensibility under forces of traction is not equivalent to inextensibility *simpliciter*: the former does not imply the latter.

I would like to take briefly into more detailed consideration the elastic properties of the string S. I shall assume, as is usual in a string with no thickness, that they are gathered in a constant k, the quotient between the force of traction applied to S and the lengthening it undergoes. As in the supertask of the new string paradox no force of traction acts on S at $t = 0$, there is no such force that stretches it (neither is there after $t = 0$ because S underwent no lengthening for $t > 0$), and therefore we are not authorized to speak of an increase in the elastic potential energy of the string. On the contrary, as S is not tensed at any moment, its elastic potential energy will remain constant and null.

3. Defining a Mechanism. To highlight the paradoxical nature of the instantaneous doubling in length in the new string paradox, I assumed in the previous section that the manipulations of the string's form were performed maintaining null traction on it all times, so that the change in the length experienced may in no case be attributed to what until now has been considered as the only possible cause of the doubling, that is, a force of traction. This means that, in the process of reducing the number of saw teeth by half between $t = 1/n$ and $t = 1/(n - 1)$, the net forces on each point of the string must always be normal to the string itself. It is difficult to specify in detail

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how such forces must act. But we may radically simplify the problem if we use, from the beginning, the notion of a string not only perfectly flexible but also inextensible by traction. The resulting doubling of length will not, by definition, be attributable to any force of traction exerted on the string. In these conditions it is very easy to specify a string manipulation mechanism. We should remember that all we need to do is to indicate how it has to go from the configuration of the string at $t = 1/n$ to the configuration at $t = 1/(n - 1)$, which implies reducing the number of saw teeth by half—making two contiguous teeth give rise to one new one, similar to any one of the others—double the width at the base, and double the height. This is done in three steps, *a–c*:

a) At the instant $t = 1/n$, we fix spatially the points of the string whose x coordinates are $x = L/2^n$, $x = 3L/2^n$, $x = 5L/2^n$, . . . , $x = (2^n - 1)\ L/2^n$ such that they do not move during the operation to be performed between $t = 1/n$ and $t = 1/(n - 1)$.

Note that these points are in fact the apexes of the string at $t = 1/n$; as in the previous section, we saw that their x coordinates were $x = H_n/2^n$, $x = 3H_n/2^n$, $x = 5H_n/2^n$, . . . , $x = (2^n - 1)H_n/2^n$ and that $H_1 = H_2 = H_3 = \ldots = H_n = \ldots = H = L$. To introduce step *b*, we call the portion of string located between two consecutive apexes a valley and the lowest point of a valley the antiapex. Obviously the x coordinates of the antiapexes at $t = 1/n$ are $x = 2L/2^n = 2 \times 1L/2^n$, $x = 4L/2^n = 2 \times 2L/2^n$, $x = 6L/2^n = 2 \times 3L/2^n, \ldots,$ $x = (2^n - 2)L/2^n = 2(2^{(n-1)} - 1)L/2^n$, and the number of antiapexes (an odd number) is one unit less than the number of apexes, that is, $2^{(n-1)} - 1$. We number them naturally beginning from the left such that the x coordinates of the odd antiapexes at $t = 1/n$ are $x = 2 \times 1L/2^n$, $x = 2 \times 3L/2^n$, $x = 2 \times 5L/2^n$, . . . , $x = 2(2^{(n-1)} - 1)L/2^n$.

b) Let us exert between $t = 1/n$ and $t = 1/(n - 1)$ a constant vertical force of magnitude F upward on each of the points of the string whose x coordinates are $x = 2 \times 1L/2^n$, $x = 2 \times 3L/2^n$, $x = 2 \times 5L/2^n$, . . . , $x = 2(2^{(n-1)} - 1)L/2^n$.

Note that these points are in fact the odd antiapexes of the string at $t = 1/n$. Under the effect of this force, every one of them will become one of the apexes of the string at $t = 1/(n - 1)$. Thus, every single apex of the string is obtained at $t = 1/(n - 1)$, whose x coordinates we know to be $x = L/2^{(n-1)} = 2 \times 1L/2^n$, $x = 3L/2^{(n-1)} = 2 \times 3L/2^n$, $x = 5L/2^{(n-1)} = 2 \times 5L/2^n$, . . . , $x = (2^{(n-1)} - 1)L/2^{(n-1)} = 2(2^{(n-1)} - 1)L/2^n$. To complete the discussion, we need to confirm that a constant force of magnitude F independent of n acting between $t = 1/n$ and $t = 1/(n - 1)$ (the force men-

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tioned in b) is capable of converting a point of the string that is an antiapex at $t = 1/n$ into an apex at $t = 1/(n-1)$. This is not evident a priori because perhaps the interval of time of the operation is too small for the force in question, independently of n, to be able to displace the portion of string required to convert a previous antiapex into a subsequent apex in the form described above. The point of the string that goes from being an antiapex to being an apex needs to be elevated vertically a distance of $\sqrt{3}(L/2^{(n-1)})$, which is in fact (as we saw in the previous section, given that $H_n = \ldots = H = L$) the height of the apexes above the X-axis at $t = 1/(n-1)$. Every portion of string that needs to be displaced vertically is located, at $t = 1/n$, between two consecutive apexes (which, by a, remain fixed during the operation), and, consequently, its total length, as is immediately clear, is $2 \times 2L/2^n = L/2^{(n-2)}$. This will also be its mass, assuming the string has a linear density that is constant and equal to the unit. We thus arrive at the least favorable scenario, in which we assume all that mass is concentrated at $t = 1/n$ in the form of a point particle at rest on the X-axis that must be displaced a distance of $\sqrt{3}(L/2^{(n-1)})$ under the action of a constant force of magnitude F and in the interval of time $\Delta t = 1/[n(n-1)] = [1/(n-1)] - [1/n]$. The acceleration of the movement will be $A = F/[L/2^{(n-2)}]$; that is, $A = [F2^{(n-2)}]/L$. If we want to guarantee that in $\Delta t = 1/[n(n-1)]$ the distance traveled is at least $\sqrt{3}(L/2^{(n-1)})$, then it must be that $(1/2)A(\Delta t)^2 = (1/2) \times [(F2^{(n-2)})/L][1/(n^2(n-1)^2)] \geq \sqrt{3}(L/2^{(n-1)})$, which leads to $F \geq [n^2/2^n] \times [(n-1)^2/2^{(n-1)}]8L^2\sqrt{3}$. In view of the fact that for positive integer n the maximum value of $[n^2/2^n][(n-1)^2/2^{(n-1)}]$ is 9/8, it is sufficient that $F \geq (9/8)8L^2\sqrt{3} = 9L^2\sqrt{3}$. More accurately, it is evident that, for any $n \geq 2$, with a force $F = [n^2/2^n][(n-1)^2/2^{(n-1)}]8L^2\sqrt{3}$ acting starting from $t = 1/n$ on all the points of the string that are odd antiapexes at $t = 1/n$, each of those points will become an apex of the string before $t = 1/(n-1)$.[4] As was required, $\lim_{n\to\infty}[n^2/2^n][(n-1)^2/2^{(n-1)}]8L^2\sqrt{3} = 0$.

4. The work required to convert a point of the string that is an odd antiapex at $t = 1/n$ into an apex of the string before $t = 1/(n-1)$ (for which, as we know, a vertical displacement $\sqrt{3}(L/2^{(n-1)})$ is required) is bounded above by $[n^2/2^n][(n-1)^2/2^{(n-1)}]8L^2\sqrt{3}\sqrt{3}(L/2^{(n-1)})$. As the total number of odd antiapexes to be displaced at $t = 1/n$ is $2^{(n-2)}$ (equal to the number of apexes there must be at $t = 1/(n-1)$), the total work required on the string between $t = 1/n$ and $t = 1/(n-1)$ is in turn bounded above by $[n^2/2^n][(n-1)^2/2^{(n-1)}]8L^2\sqrt{3}\sqrt{3}(L/2^{(n-1)})2^{(n-2)} = [n^2/2^n][(n-1)^2/2^{(n-1)}]12L^3 = [n(n-1)/2^n]^2 24L^3$, and then the total work performed on the string is bounded by $\Sigma_{n=1}^{n=\infty}[n(n-1)/2^n]^2 24L^3$. As for positive integer n it is $[n(n-1)/2^n] < 1$, it holds that $[n(n-1)/2^n]^2 < [n(n-1)/2^n] < [n^2/2^n]$. But the series $\Sigma_{n=1}^{n=\infty}[n^2/2^n]$ is convergent, which means that $\Sigma_{n=1}^{n=\infty}[n(n-1)/2^n]^2 24L^3$ will also be convergent. Thus, the work W performed on the string in the new string paradox is finite. This work must have been converted, after the conclusion of the supertask, into the string's internal energy. But it cannot be elastic potential energy because the forces that deform the string performed

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c) At the instant $t = 1/(n - 1)$, we free the points of the string that had been spatially fixed in *a*.

By means of the reiterated application of steps *a*–*c* above (for each positive integer $n \geq 2$), it becomes clear how the supertask of the instantaneous doubling of length for a string that is perfectly flexible but inextensible by traction may be performed. In no case is it necessary to exert on some point of the string a force of greater magnitude than the value F noted above. We should also remember that in all this process we agreed from the beginning to maintain the left end of the string spatially fixed at the point $x = 0$ of the X-axis. We have verified that then the right end of the string will also be permanently maintained at the point $x = L$ of the X-axis, despite all our manipulations. It follows, and this will be relevant in the next section, that we could have proceeded by fixing from the beginning not only the left end of the string at point $x = 0$ on the X-axis but also the right end at point $x = L$ of the X-axis.

4. A Surprisingly Irreversible Evolution. The idea of a perfectly flexible string inextensible by traction is especially interesting because, as we shall now see, it implies a new form of irreversibility in classical mechanics that is qualitatively different from any examples of irreversibility studied to date. Let S be a perfectly flexible string inextensible by traction that, at $t = 0$, is at rest on the X-axis (it therefore has the form of a straight segment), the coordinate of its left end being $x = 0$ and of its right end $x = L$. Furthermore, we shall maintain its left end fixed on the X-axis at $x = 0$ and its right end fixed on the X-axis at $x = L$ permanently. We saw above that a supertask may be performed on S that instantaneously doubles its length so that, after $t = 0$, this is $2L$. Alternatively, let what we do be to exert on S, at a certain $t > 0$, any force G applied to any point of S. As S cannot be deformed as a consequence of the action of G by being inextensible by traction, in these circumstances it will behave like a rigid rod with its ends fixed: there will be no kind of movement. The fact that S behaves in these circumstances like a rigid rod with its ends fixed does not contradict its being perfectly flexible. Fine-tuning the definition given in section 1, a string is perfectly flexible when it makes absolutely no resistance to its deformation (when it presents no rigidity against flexion) in those situations in which such defor-

work for $t > 0$, when the string already had a constant length $2L$. Work W must therefore be converted into kinetic energy of vibration of the different parts of the string. For this to occur, the successive deformation of the string must be carried out by making saw teeth with play, in the sense described in n. 2. Thus, for the principle of energy conservation to be strictly observed, the play factor must necessarily be greater than the unit.

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mation (flexion) is physically possible. In our case, no deformation (flexion) of the string S is physically possible by the action of the force G because S is initially straight and is inextensible by traction, and its ends are fixed in space. More specifically, let us suppose that we choose a specific integer $n^* \geq 2$ and proceed thus:

- α) At the instant $t = 1/n^*$, we fix spatially the points of the string whose x coordinates are $x = L/2^{n^*}$, $x = 3L/2^{n^*}$, $x = 5L/2^{n^*}$, . . . , $x = (2^{n^*} - 1)L/2^{n^*}$.
- β) Between $t = 1/n^*$ and $t = 1/(n^* - 1)$, we exert a constant vertical force of magnitude F directed upward on each of the points of the string whose x coordinates are $x = 2 \times 1L/2^{n^*}$, $x = 2 \times 3L/2^{n^*}$, $x = 2 \times 5L/2^{n^*}$, . . . , $x = 2(2^{(n^*-1)} - 1)L/2^{n^*}$.
- γ) At the instant $t = 1/(n^* - 1)$, we free the points of the string that had been spatially fixed in α.

For the same reasons as before, there will be no kind of movement in the string.

Now let S be a perfectly flexible string inextensible by traction that, at $t = -1$, is at rest on the X-axis (it therefore has the form of a straight segment), the coordinate of its left end being $x = 0$ and of its right end $x = L$. Further, we shall maintain its left end fixed on the X-axis at $x = 0$ and its right end fixed on the X-axis at $x = L$ permanently. Let us suppose that we choose a specific integer $n^* \geq 2$ and proceed as follows:

- $-\alpha$) At the instant $t = -1/(n^* - 1)$, we fix spatially the points of the string whose x coordinates are $x = L/2^{n^*}$, $x = 3L/2^{n^*}$, $x = 5L/2^{n^*}$, . . . , $x = (2^{n^*} - 1)L/2^{n^*}$.
- $-\beta$) Between $t = -1/(n^* - 1)$ and $t = -1/n^*$, we exert a constant vertical force of magnitude F upward on each of the points of the string whose x coordinates are $x = 2 \times 1L/2^{n^*}$, $x = 2 \times 3L/2^{n^*}$, $x = 2 \times 5L/2^{n^*}$, . . . , $x = 2(2^{(n^*-1)} - 1)L/2^{n^*}$.
- $-\gamma$) At the instant $t = -1/n^*$, we free the points of the string that had been fixed spatially in $-\alpha$.

Once again, the rigid rod type of behavior guarantees that there will be no kind of movement in the string. It is important here to see that the trio $-\alpha$, $-\beta$, and $-\gamma$ describes in this case a process that is trivially a temporal inversion of the process described by the trio α, β, and γ. In particular, a point of the string is fixed (freed) at the instant t, as per α, β, and γ, if and only if a point of the string is freed (fixed) at the instant $-t$ as per $-\alpha$, $-\beta$, and $-\gamma$. Finally let it be that, for each integer $n \geq 2$, we observe the following plan:

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$-\alpha^*$) At the instant $t = -1/(n-1)$, we spatially fix the points of the string whose x coordinates are $x = L/2^n$, $x = 3L/2^n$, $x = 5L/2^n$, . . . , $x = (2^n - 1)L/2^n$.

$-\beta^*$) Between $t = -1/(n-1)$ and $t = -1/n$, we exert a constant vertical force of magnitude F upward on each of the points of the string whose x coordinates are $x = 2 \times 1L/2^n$, $x = 2 \times 3L/2^n$, $x = 2 \times 5L/2^n$, . . . , $x = 2(2^{(n-1)} - 1)L/2^n$.

$-\gamma^*$) At the instant $t = -1/n$, we free the points of the string that had been spatially fixed in $-\alpha^*$.

It is clear that nothing substantial changes: having begun at $t = -1$ with a string in the form of a straight segment of length L, we will conclude at $t = 0$ with the same string in the form of a straight segment of length L. The process described by the trio $-\alpha^*$, $-\beta^*$, and $-\gamma^*$, for $n \geq 2$ shall be called Ω^*. Given that Ω^* is a fairly trivial process in which there is no kind of movement in the string, this remains at all times in the straight shape with the constant length L. It might then seem that we are dealing with a reversible process. Let us suppose that it is. In that case, it follows that its temporal inversion is also a possible process, which I shall call Ω. Under Ω, it begins at $t = 0$ with a string in the form of a straight segment of length L, and it ends at $t = 1$ with the same string (which behaves like a rigid rod) in the form of a straight segment of length L. But the Ω process is evidently described by the trio a, b, and c that was introduced in the previous section (once again, for $n \geq 2$). It follows that, under a, b, and c $(n \geq 2)$, the string can evolve over time starting from $t = 0$ in two different ways: by doubling instantaneously its length for $t > 0$, as we saw in the section in question (indeed, the play factor could be greater than 1, but this is not relevant here), or by remaining for $t > 0$ as a straight segment of length L that behaves like a rigid rod (and therefore not becoming deformed at all, as per the Ω process). There is no way of knowing beforehand which of the two paths it will take, and, in consequence, the string's evolution is undetermined between deformation and instantaneous increase of length or no deformation (and no change of length) whatsoever. But this indeterminism is in truth incompatible with the string's perfect flexibility. Indeed, as the deformation (flexion) of the string is physically possible (it corresponds to the possible form of evolution in which the length instantaneously increases), it follows from the more accurate definition of perfect flexibility given in this section (and from the assumption that the string is perfectly flexible) that the string will not place any resistance to its deformation (no rigidity against its flexion) and then will indeed undergo the deformation that accompanies the process of instantaneous increase of its length. Therefore, this deformation of the string is not only physically possible but also, as we now see, physically necessary: it is the only possible way for the string to evolve in the given circumstances.

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The Ω process of nondeformation is not therefore an option, which indicates that our initial assumption that the Ω* process is reversible is false. Despite it being, as we have seen above, completely trivial, Ω* is an irreversible process.

I would like to end by mentioning two immediate extensions of the new string paradox. Instead of starting with a one-dimensional string and suddenly doubling its length by deforming it in one plane, we may analogously start with a two-dimensional membrane and suddenly double its surface by deforming it in space or start with a three-dimensional body and suddenly double its volume by deforming it in hyperspace.

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