A group of Pythagorean triples using the inradius

HOWARD SPORN

1. *The group*

Pythagorean triples are triples of integers (a, b, c) satisfying the equation $a^2 + b^2 = c^2$. For the purpose of this paper, we will take a, b and c to be positive, unless otherwise stated. Then, of course, it follows that a triple represents the lengths of sides of a right triangle. Also, for the purpose of this paper, we will consider the triples (a, b, c) and (b, a, c) to be distinct, even though they represent the same right triangle. A *primitive* Pythagorean triple is one for which a, b and c are relatively prime.

It is known that it is possible to define a multiplication operation such that the set of primitive Pythagorean triples forms a group. There are several ways of doing this, though they involve triples for which a and b could be 0 or negative integers. One way [1, 2] is by associating the Pythagorean triple (a, b, c) with the point $(x, y) = \left(\frac{a}{c}, \frac{b}{c}\right)$ on the unit circle $x^2 + y^2 = 1$, and then defining the group multiplication by

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1).$$

Another way [3, 4] is by associating the Pythagorean triple (a, b, c) with the point $(x, y) = \left(\frac{c}{a}, \frac{b}{a}\right)$ on the unit hyperbola $x^2 - y^2 = 1$, and then defining the group multiplication by

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1x_2 + y_1y_2, x_1y_2 + x_2y_1).$$

The purpose of this Article is to show that one can generate another such group, and to investigate its properties.

To define this group, we will make use of the inradius of triangles. The incircle of a triangle is the unique circle inscribed in the triangle. Its radius is called the inradius, (see Figure 1).

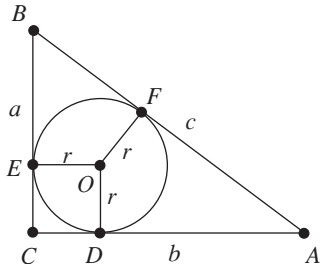


FIGURE 1: A Pythagorean triangle and its incircle.

We will derive a formula for the inradius of a right triangle [5]. In Figure 1, triangle ABC is a right triangle with a right angle at C, and sides of lengths a, b and c. We inscribe a circle of radius r and centre O, and it intersects the triangle at points D, E and F as shown.

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Now $CEOD$ is a square, and so $CD = CE = r$. It follows that $BE = a - r$ and $AD = b - r$. But $BE = BF$ and $AD = AF$ since they are tangents to the circle. So

$$c = AB = BF + AF = BE + AD = (a - r) + (b - r).$$

Therefore

$$c = a + b - 2r. \tag{1}$$

We then solve for r, and find that the inradius of the right triangle with sides (a, b, c) is

$$r = \frac{a + b - c}{2}, \tag{2}$$

For any right triangle (a, b, c) with integer sides, it is obvious from the Pythagorean Theorem that either exactly one of a, b and c is an even number, or all three are even. Therefore, the inradius r of a Pythagorean triangle must be an integer.

We now define

$$i = a - 2r \text{ and } j = b - 2r. \tag{3}$$

From (1) and (3), a formula for c follows directly:

$$c = 2r + i + j. \tag{4}$$

Then it follows [6, 7] that, from (3),

$$\begin{aligned} ij &= (a - 2)(b - 2r) \\ &= ab - 2r(a + b) + 4r^2 \\ &= \tfrac{1}{2}(a+b)^2 - \tfrac{1}{2}c^2 - 2r(a+b) + 4r^2, \text{ using the Pythagorean Theorem} \\ &= (a+b)^2 - \tfrac{1}{2}(a+b)^2 - \tfrac{1}{2}c^2 - 2r(a+b) + 4r^2 \\ &= (a+b)^2 - 2r^2 - c(a+b) - 2r(a+b) + 4r^2, \text{ from the square of (2)} \\ &= 2r^2 + 0, \text{ from (1).} \end{aligned}$$

And so we have

$$ij = 2r^2. \tag{5}$$

Now let us define

$$x = \frac{i}{r} \text{ and } y = \frac{j}{r}. \tag{6}$$

Using (5), it follows that, for any Pythagorean triple,

$$xy = 2. \tag{7}$$

These quantities and relations have been discovered and rediscovered several times.

The quantities x and y can be expressed in terms of a, b and c by using (2), (3) and (6). The result is:

$$x = \frac{2(c - b)}{a + b - c} \quad \text{and} \quad y = \frac{2(c - a)}{a + b - c}. \tag{8}$$

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Conversely, we can express a, b and c in terms of x and y by using (3), (4) and (6):

$$(a, b, c) = \big(r(x+2), r(y+2), r(x+y+2)\big) = r\big(x+2, y+2, x+y+2\big). \quad (9)$$

Note from (8) that any scalar multiple of (a, b, c) generates the same x and y, so (9) could start with (ka, kb, kc), where k is an integer constant.

We now have enough to define a group of Pythagorean triples. This group will be associated with the hyperbola $xy = 2$ (specifically, the branch in the first quadrant), just as the groups mentioned above are associated with the circle $x^2 + y^2 = 1$ and the hyperbola $x^2 - y^2 = 1$ (see Figure 2).

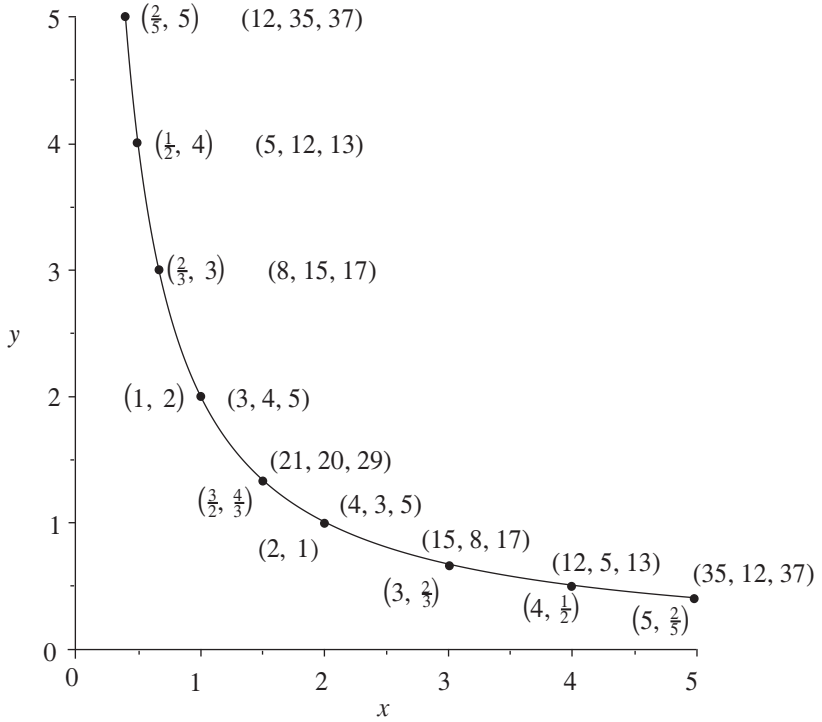


FIGURE 2: The hyperbola $xy = 2$ in the first quadrant, showing some Pythagorean triples and their associated coordinates

First we define a commonly-used equivalence relation, such that any non-primitive Pythagorean triple is defined to be equivalent to the primitive triple obtained by dividing by a common factor. That is, if k is a positive integer, then $(ka, kb, kc) \sim (a, b, c)$, where ~ means equivalence.

Next, using equations (2) to (7), we represent a Pythagorean triple by $(x, y) = \left(x, \dfrac{2}{x}\right)$. From (9), we see that any ordered pair (x, y) of positive rational numbers satisfying (7) is associated with exactly one primitive

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Pythagorean triple (a, b, c). Conversely, from (8) it follows that any primitive Pythagorean triple (a, b, c) is associated with exactly one pair (x, y). Furthermore, a scalar multiple of a triple gives the same pair (a, b, c).

Now we define the group multiplication by ordinary multiplication of the x values. Thus

$$(x_1, y_1) \cdot (x_2, y_2) = \left(x_1, \frac{2}{x_1}\right) \cdot \left(x_2, \frac{2}{x_2}\right) = \left(x_1x_2, \frac{2}{x_1x_2}\right) = \left(x_1x_2, \frac{y_1y_2}{2}\right). \quad (10)$$

The y-values are obtained from (7). From (9) and (7), it is clear that the x's may take on the value of any positive rational number. Thus the multiplication is closed. Since it is ordinary multiplication of rational numbers, it is associative and commutative, so our group is abelian. Our group is isomorphic to Q^+, the positive rational numbers under ordinary multiplication.

Let us do an example of this multiplication. We will compute the product of the triples (4, 3, 5) and (15, 8, 17). For (4, 3, 5), we use (8) and find that $(x, y) = (2, 1)$.

The triple (4, 3, 5) is represented by $(x, y) = (2, 1)$. (11)

By an analogous computation, for (15, 8, 17), we find $(x, y) = \left(3, \frac{2}{3}\right)$. We multiply the x values, and obtain $(2, 1) \cdot \left(3, \frac{2}{3}\right) = \left(6, \frac{1}{3}\right)$ keeping in mind that $xy = 2$.

We now need to find the Pythagorean triple associated with $(x, y) = \left(6, \frac{1}{3}\right)$. Using (9), we find that $a = 8r$, $b = \frac{7}{3}r$ and $c = \frac{25}{3}r$. We use the equivalence relation, and multiply a, b and c by $\frac{3}{r}$. Then we get the triple (24, 7, 25). Therefore,

$$(4, 3, 5)\ (15, 8, 17) = (24, 7, 25).$$

Now we need to find the identity element of the group. Clearly, it is $(x, y) = (1, 2)$, since 1 is the identity for multiplication, and x and y satisfy (7). What Pythagorean triple does it represent? Using (9), we have $a = 3r$, $b = 4r$ and $c = 5r$. So, by the equivalence relation, the identity element of the group is the Pythagorean triple (3, 4, 5).

The triple (3, 4, 5) is represented by $(x, y) = (1, 2)$. (12)

Note that if (x, y) represents the triple (a, b, c), then (y, x) represents the triple (b, a, c). For instance, we had above, in (11) and (12), that for $(x, y) = (2, 1)$ we have the triple (4, 3, 5), and for $(x, y) = (1, 2)$ we have the triple (3, 4, 5).

Finally, we need to find inverse elements in the group. Since our group multiplication corresponds to ordinary multiplication of the x values, we take the ordinary reciprocal of x to get the x-value of its inverse. By (9), the Pythagorean triple associated with the pair (x, y) is

$$\mathbf{u} = r \cdot (x + 2, y + 2, x + y + 2).$$

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By (7), its inverse is associated with the pair $\left(\frac{1}{x},\ 2x\right)$ which, again by (7), is equal to $\left(\frac{y}{2},\ 2x\right)$. By (9), the inverse triple is of the form $\mathbf{v} = r' \cdot \left(\frac{y}{2} + 2,\ 2x + 2,\ \frac{y}{2} + 2x + 2\right)$.

We now claim that there is a matrix M satisfying $M\mathbf{u}^T \sim \mathbf{v}^T$. Computation shows that

$$M = \begin{bmatrix} \frac{1}{2} & 1 & -\frac{1}{2} \\ 1 & -1 & 1 \\ \frac{1}{2} & -1 & \frac{3}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 2 & -1 \\ 2 & -2 & 2 \\ 1 & -2 & 3 \end{bmatrix}. \tag{13}$$

As an example, let us find the inverse of (5, 12, 13).

$$M\begin{pmatrix} 5 \\ 12 \\ 13 \end{pmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 2 & -1 \\ 2 & -2 & 2 \\ 1 & -2 & 3 \end{bmatrix}\begin{pmatrix} 5 \\ 12 \\ 13 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 16 \\ 12 \\ 20 \end{pmatrix} = \begin{pmatrix} 8 \\ 6 \\ 10 \end{pmatrix},$$

And so, by the equivalence relation, we get the Pythagorean triple (4, 3, 5).

Note that the identity element (3, 4, 5) and all inverse elements are Pythagorean triples consisting of positive integers a, b and c. This is not true for the groups given in [1, 2, 3, 4]. Those groups involve Pythagorean triples with elements that can be 0 or negative integers. Therefore the group introduced in this paper is not isomorphic to the others and is different.

The matrix given in (13) was used by Jingcheng Tong in [8]. Tong defines the *conjugate* of a Pythagorean triple, and uses the transformation given by our matrix M. Tong's goal is not to set up a group, but to define Pythagorean triples using two parameters, such that exchanging the two parameters also gives a Pythagorean triple. Tong's conjugate of a triple is precisely the inverse in our group.

Note, too, that M^2 is the 3×3 identity matrix, i.e. $M^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and so the inverse of an inverse is the original triple, as desired.

It is interesting to consider the eigenvalues and eigenvectors of M. There are two eigenvalues, $\lambda = -1$ and $\lambda = 1$. For $\lambda = -1$ the eigenvector is $(-1,\ 2,\ 1)$. For $\lambda = 1$ there are two eigenvectors, (2, 1, 0) and $(-1,\ 0,\ 1)$. Note that (3, 4, 5) is a linear combination of these latter two eigenvectors, and that $M(3,\ 4,\ 5)^T = (3,\ 4,\ 5)^T$ as desired, since the inverse of the identity element is itself.

There is a simple relationship between (x, y) for a Pythagorean triple, and the parametric form of a triple. The well-known parametric form for a primitive Pythagorean triple with a odd and b even is

$$\left(m^2 - n^2,\ 2mn,\ m^2 + n^2\right), \tag{14}$$

where $m > n > 0$ and m and n coprime.

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If a is even and b is odd, then we can multiply a, b and c by 2, and the parametric form still holds. In both cases, it is easy to show, using (8) and (14), that

$$(x, y) = \left(\frac{m - n}{n}, \frac{2n}{m - n}\right).$$

2. *Some families within the group*

To get a feel for the group, we list some Pythagorean triples and their associated coordinates. The Pythagoras family [9, 10] consists of primitive Pythagorean triples for which $c - b = 1$. They are of the form $(2n + 1,\ 2n^2 + 2n,\ 2n^2 + 2n + 1)$. Their (x, y)'s are of the form $\left(\frac{1}{n},\ 2n\right)$. Thus the y-value is an even integer. For each triple, the inradius r takes the value n. The first three triples in the Pythagoras family are $(3, 4, 5)$, $(5,\ 12,\ 13)$ and $(7, 24, 25)$, and their corresponding (x, y)'s are $(1, 2)$, $(\frac{1}{2}, 4)$, and $(\frac{1}{3}, 6)$.

The Platonic family [9, 10] consists of primitive Pythagorean triples for which $c - b = 2$. They are of the form $(4n,\ 4n^2 - 1,\ 4n^2 + 1)$. Their (x, y)'s are of the form $\left(\frac{2}{2n - 1},\ 2n - 1\right)$. Thus the y-coordinate is an odd integer. For each triple the inradius r takes the value $2n - 1$. The first three triples in the Platonic family are $(4, 3, 5)$, $(8, 15, 17)$ and $(12, 35, 37)$, and their corresponding (x, y)'s are $(2, 1)$, $(\frac{2}{3}, 3)$ and $(\frac{2}{5}, 5)$.

Of course the group of positive rational numbers under multiplication has an infinite number of subgroups. Therefore our group of Pythagorean triples also has an infinite number of subgroups. For instance, consider all triples associated with (x, y) for which x is of the form 2^k, where k is an integer. This forms a subgroup of the Pythagorean triples. This subgroup is isomorphic to the group of integers under addition. Similarly, another subgroup consists of all triples associated with (x, y) for which x is of the form 3^k, where k is an integer.

We can construct yet another subgroup by letting x be a product of powers of 2 and 3, i.e. $x = 2^k 3^l$ where k and l are integers.

The reader may construct some of the Pythagorean triples belonging to these subgroups.

3. *A related group*

As mentioned above, our group is isomorphic to Q^+, the positive rational numbers under ordinary multiplication. We can expand the group by taking R^+, the positive *real* numbers under ordinary multiplication. Then (x, y) consists of two real numbers, and our extended group consists of *all* right triangles, including those with irrational side lengths.

One example is the 45-45-90 right triangle, with sides $(1,\ 1,\ \sqrt{2})$. One can use (8) to compute the x and y values, but it is not necessary. Since a and b are equal, it follows that x and y are equal. Then by (7) we have

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$(x, y) = (\sqrt{2}, \sqrt{2})$. Another interesting example is the triangle $(1, 2, \sqrt{5})$. If we define $\varphi = \dfrac{1 + \sqrt{5}}{2}$, the golden ratio, and $\bar{\varphi} = \dfrac{1 - \sqrt{5}}{2}$, its conjugate, then we can write $(x, y) = (-\bar{\varphi}, 2\varphi)$. The inverse of the triple $(1, 2, \sqrt{5})$ is $(\sqrt{5}, 2, 3)$, and its associated (x, y) is $(\varphi, -2\bar{\varphi})$. Yet another example is the 30-60-90 right triangle, having sides $(1, \sqrt{3}, 2)$. Using (8), we find that $(x, y) = (\sqrt{3} - 1, \sqrt{3} + 1)$.

Further study

An interesting question to consider is whether other groups of the primitive Pythagorean triples are possible. Clearly, it is easy to define a group on the hyperbola $xy = 2$ in Figure 2 by using multiplication of the y-values instead of the x-values. But are other groups defined on the hyperbola possible?

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HOWARD SPORN

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Department of Mathematics and Computer Science,
Queensborough Community College, Bayside, NY 11364, USA
e-mail: *hsporn@qcc.cuny.edu*

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