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PEIRCE'S CALCULI FOR CLASSICAL PROPOSITIONAL LOGIC

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Abstract. This article investigates Charles Peirce's development of logical calculi for classical propositional logic in 1880–1896. Peirce's 1880 work on the algebra of logic resulted in a successful calculus for Boolean algebra. This calculus, denoted by **PC**, is here presented as a sequent calculus and not as a natural deduction system. It is shown that Peirce's aim was to present **PC** as a sequent calculus. The law of distributivity, which Peirce states in 1880, is proved using Peirce's Rule, which is a residuation, in **PC**. The transitional systems of the algebra of the copula that Peirce develops since 1880 paved the way to the 1896 graphical system of the alpha graphs. It is shown how the rules of the alpha system reinterpret Boolean algebras, answering Peirce's statement that logical graphs supply a new system of fundamental assumptions to logical algebra. A proof-theoretic analysis is given for the connection between **PC** and the alpha system.

§1. Introduction. Charles Peirce produced several versions of algebraic and logical calculi when he was working on the improvements on Boole's work [5, 6] ever since the late 1860s [21, 22, 24–28, 31]. These calculi evolved with varying sets of fundamental rules. They show a clever design, which in the secondary literature has been asserted to exhibit an important transition from algebraically considered systems to proper logical languages [2, 7, 9, 12, 18, 57, 63].

What has received less attention is that Peirce's stepwise development of various calculi was destined to lead not only to modern languages of logic but precisely to the graphical method of logic, first to the theory of *entitative* and soon after the theory of *existential graphs* [7, 9, 31, 59, 66, 67]. That transition can be observed to take place both in the kinds of rules Peirce proposes for his calculi over the years as well as in the notational changes in the way he represents the consequence relation, negation, and other notions of his logical theories. What is also important to notice is that, after having invented the graphical method in 1896, Peirce did not give up the algebraic approach, even though he exploits it less than he previously had done.

Our focus in the present article is in studying the systems of rules of his various propositional calculi. We limit our investigation to the propositional systems and to the alpha graphs. Work on the notational and other significant changes that led to the method of

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representing logical reasoning in systems of graphs, and to trace the factors that led to the development of his quantificational theory in the beta graphs and the graphical treatment of quantifiers in terms of lines of identity, has been carried out elsewhere [3, 4, 54].

We begin our investigation of Peirce's calculi with his 1880 article on the algebra of logic [22]. (There are many earlier articles, but 1880 is a definite source.) We then traverse the related developments through the 1885 sequel on the philosophy of notation [24], the 1891 modification on the algebra of the copula [25, 26], the 1893 remodelling of the notation and the rules [27, 28], and finally, the emergence of the system of alpha graphs and its calculus he invented in 1896 [31]. All these systems provide complete logical systems and they correspond to Boolean algebras, albeit in importantly different ways.

It is widely believed that Peirce's calculus can be understood or reinterpreted in the style of the modern system of natural deduction ([7, 58, 62]). However, through a careful reading of Peirce's 1880 article, we show that Peirce's calculus is indeed a calculus of *sequents*. In the modern terminology, a sequent calculus is a theory about *consequence relation*, and unlike natural deduction does not yield to assumptions that need to be discharged. The present article established that Peirce was exactly developing such a sequent calculus for Boolean algebras. This does not come out as too surprising, however, given the central role consequence relation (the sign of illation) has in all of his logical investigations.

The next section explains Peirce's sequent calculus for Boolean algebras, shows the centrality of its leading principle (Peirce's Rule), proofs the famous distributivity laws from its rules, and concludes that Peirce's calculus was indeed that of sequents. §3 studies Peirce's follow-up explorations in the algebra of logic, which can be considered as transitional systems on the path that led to the graphical method of logic. These systems include his five icons system of 1885, the qualitative logic of 1886 and its later versions, the revisions of the algebra of the copula in 1891, and the new sign of consequence in the reworking of those system for his 1893–4 magnum opus of the *Grand Logic*. §4 takes up his graphical system of alpha graphs, first proposed in 1896, and presents an algebraic remodelling of its rules. This, we propose, answers Peirce's comment that logical graphs supply "an entirely new system of fundamental assumptions to logical algebra" (Peirce to Ladd-Franklin, November 9, 1900 [34]). A further proof-theoretic analysis establishes the connection between Peirce's sequent calculus for Boolean algebras and the alpha system. Conclusions put Peirce's rules for his calculi in a wider philosophical perspective.

§2. Peirce's sequent calculus for boolean algebras. Peirce's famous 1880 article on the algebra of logic [22] presents a calculus for Boolean algebras. Using modern Lindenbaum–Tarski construction, one can prove that his calculus is sound and complete with respect to the class of all Boolean algebras. It is in this way, by improving upon Boole's work, that he came to develop various calculi for classical propositional logic. In this section, we give a comprehensive analysis of the main ideas behind Peirce's conception of his calculus.

2.1. The leading principle. In §2 of his 1880 article, Peirce began with the treatment of illation (deduction). He described the general form of inference as follows:

> The general type of inference is
>
> $$P$$
> $$\therefore\ C,$$
>
> where $\therefore$ is the sign of illation. ([22, p. 17])

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P is the premiss (or a set of premises), and C is the conclusion obtained by using rules of inference. A general rule of inference is also called a "habit" by Peirce. He then introduced the vital *leading principle*:

> A habit of inference may be formulated in a proposition which shall state that every proposition c, related in a given general way to any true proposition p, is true. Such a proposition is called the *leading principle* of the class of inferences whose validity it implies.[1] ([22, p. 17])

Peirce then introduced a sign $\prec$ of the copula to express this leading principle. The form $P \therefore C$ expresses an argument, and $P_i \prec C_i$ expresses *the truth of its leading principle*. Peirce presents the meaning of the copula in a modern, model-theoretic fashion:

> The symbol $\prec$ is the copula, and signifies primarily that every state of things in which a proposition of the class P_i is true is a state of things in which the corresponding propositions of the class C_i is true. ([22, p. 18])

In 1904, this semantic character of the logical consequence is accentuated in terms of any state of things that connects the consequent and the antecedent:

> *Hypothesis No 2.* In case any two propositions, x and y, should be such that we are entitled to write
>
> $$x \prec\!\!\!\!\sim\ y$$
>
> (and we do not here assume that this is the case), this implies that the possible state of things represented by y is in a certain relation to whatever possible state of things there may be to which the state of things represented by x is in that relation. (Peirce to Huntington, February 14, 1904 [45])

(The sign $\prec\!\!\!\!\sim$ is Peirce's later, cursive notation for $\prec$.)

Form Peirce's interpretation it follows that the copula $\prec$ is the sign that stands for logical consequence. The calculus that he develops is indeed about the copula and its properties.

Peirce emphasizes the significance of the leading principle or the copula. He identifies the copula of the form $A \prec B$ with a compound proposition built from A (the premiss) and B (the conclusion) by the sentential operation of material implication. In §3 of the 1880 article on forms of propositions, he stated the following:

> The forms $A \prec B$, or A implies B, and $A \not\prec B$, or A does not imply B, *embrace* both hypothetical and categorial propositions. . . . To say, 'if A, then B' is obviously the same as to say that from A, B follows, logically or extralogically. By thus identifying the relation expressed by the copula with that of illation, we identify the proposition with the inference, and the term with the proposition. This identification, by means of which all that is found true of term, proposition, or inference is at once known to

[1] Peirce presumed that proof transformations that result in illative or valid necessary inferences may ultimately have an evolutionary reason: those are associated with habits of inference that are "stable and endure", while the unsound inferences are performed by habits that are "unstable and be destroyed" (R 485 [50], 1897–8 [33]). Incidentally, one might sense some connection to contemporary evolutionary game theory in these remarks.

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> be true of all three, is a most important engine of reasoning, which we have gained by beginning with a consideration of the genesis of logic. ([22, p. 21–22], added emphasis)

It has been believed, perhaps ever since [60], that Peirce (and also Schröder) somehow confused the metalogical consequence relation with the material implication. However, we shall show that the upshot of Peirce's important identification of the two actually marks a vital discovery in the history of logic, and that it is this identification that justifies Peirce's calculus as a calculus for Boolean algebras.

2.2. The algebra of the copula. For the sake of clarity, we separate the two meanings of the copula $\prec$ (or $\succ$) by using two signs: (1) the consequence relation (the sign of illation) $\Rightarrow$; and (2) the material implication $\rightarrow$. Peirce introduced the algebra of the copula in §4 of his 1880 article. His algebra of the copula is a calculus of the consequence relation. An expression of the form $x \Rightarrow y$ is called a *sequent*, according to the proof-theoretic terminology.

DEFINITION 2.1. *The calculus of the copula consists of the following axiom and rules*:

(1) *Identity*: (Id) $x \Rightarrow x$.

(2) *Peirce's Rule*:

$$\frac{x \wedge y \Rightarrow z}{x \Rightarrow y \rightarrow z}\,(PR).$$

(3) *Rule of Transitivity*:

$$\frac{x \Rightarrow y \quad y \Rightarrow z}{x \Rightarrow z}\,(\mathrm{Tr}).$$

The double line in (*PR*) *means that the lower sequent can be derived from the upper sequent and vice versa.*

The axiom of identity is easy to understand. Every proposition follows from itself. Peirce explained it in terms of the memory or monotonicity of belief: what we have hitherto believed we continue to believe, in the absence of any reason to the contrary. We name the second rule *Peirce's Rule*, because it is probably the first formulation of the *law of residuation*: that the material implication is a right residual of conjunction. We will remark on the nature and significance of Peirce's Rule, as well as its historical discovery, after we have presented his 1880 calculus.

The meaning of the rule (Tr) is the transitivity of the consequence relation. If y follows from x and z follows from y, then z must follow from x. Peirce mentions that the transitivity of the copula derives from De Morgan's work [20]. He also states that, "the same principle may be algebraically conceived as a rule for the elimination of y from the two propositions $x \prec y$ and $y \prec z$" [22, p. 25]. After Gentzen's 1934 work, the rule (Tr) became called a *cut rule* in proof theory. It concerns the elimination of the middle, or the cut term.

2.3. Peirce's calculus for boolean algebras. After the introduction of the algebra of the copula, Peirce continued his 1880 exposition to introduce the logic of nonrelative terms. The nonrelative terms are constructed from propositions using logical multiplication $\times$ and addition $+$. Here we change the notation into $\wedge$ for conjunction and $\vee$ for disjunction.

First of all, Peirce commented on the rule (PR) when the negation sign is introduced. For any term x, let $\overline{x}$ be the negation of x. Then the proposition $x \rightarrow y$ is equivalent with

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$\overline{x} \vee y$. Hence by (PR) we can derive:

$$\frac{x \wedge y \Rightarrow z}{x \Rightarrow \overline{y} \vee z}.$$

Moreover, Peirce stated the following derived variant of the rule (PR):

$$\frac{x \wedge \overline{y} \Rightarrow z}{x \Rightarrow y \vee z}.$$

Two important further variants of (PR) can be stated as follows:

$$\frac{x \Rightarrow y}{(\text{The possible}) \Rightarrow \overline{x} \vee y} \qquad \frac{x \Rightarrow y}{x \wedge \overline{y} \Rightarrow (\text{The impossible})}.$$

Peirce then continued to introduce two notations: 0 for the impossible, and ∞ for the possibility. We replace 0 and ∞ with $\bot$ and $\top$, respectively. Then the following axioms are given by Peirce:

$$(\top)\ x \Rightarrow \top \quad (\bot)\ \bot \Rightarrow x.$$

Moreover, from the axiom (Id), and using the two variants of (PR), one can easily derive the law of excluded middle and the law of contradiction:

$$\top \Rightarrow x \vee \overline{x} \quad \text{and} \quad \overline{x} \wedge x \Rightarrow \bot.$$

The negation sign can be defined in terms of $\rightarrow$ and $\bot$ as follows:

$$\overline{x} := x \rightarrow \bot.$$

Then $\overline{x} \wedge x \Rightarrow \bot$ is obtained from $x \rightarrow \bot \Rightarrow x \rightarrow \bot$ by (PR).

Peirce proceeds to introduce the rules for conjunction (multiplication) and disjunction (addition). The definition of his calculus is now complete. He proved all the axioms of lattices and stated the distributive laws. For the convenience of discussion, we summarize his 1880 calculus for classical propositional logic as follows:

DEFINITION 2.2. *Peirce's calculus* **PC** *consists of the following axioms and rules*:

(1) *Axioms*:

$$(\text{Id})\ x \Rightarrow x \quad (\top)\ x \Rightarrow \top \quad (\bot)\ \bot \Rightarrow x \quad (\text{Em})\ \top \Rightarrow x \vee \overline{x}.$$

(2) *Rules*:

$$\frac{x \wedge y \Rightarrow z}{x \Rightarrow y \rightarrow z}\ (\text{PR}) \qquad \frac{x \Rightarrow y \quad y \Rightarrow z}{x \Rightarrow z}\ (\text{Tr})$$

$$\frac{x_1 \Rightarrow z \quad x_2 \Rightarrow z}{x_1 \vee x_2 \Rightarrow z}\ (\vee\text{I}) \qquad \frac{z \Rightarrow x_1 \quad z \Rightarrow x_2}{z \Rightarrow x_1 \wedge x_2}\ (\wedge\text{I})$$

$$\frac{x_1 \vee x_2 \Rightarrow z}{x_i \Rightarrow z}\ (\vee\text{E}) \qquad \frac{z \Rightarrow x_1 \wedge x_2}{z \Rightarrow x_i}\ (\wedge\text{E}).$$

In ($\vee$E) *and* ($\wedge$E), $i \in \{1, 2\}$. *A* derivation *of a sequent* $x \Rightarrow y$ *in* **PC** *is a proof tree with the root* $x \Rightarrow y$ *such that each node is either an axiom or derived by a rule of inference. A sequent* $x \Rightarrow y$ *is* derivable *in* **PC** (*notation* $\vdash_{\mathbf{PC}} x \Rightarrow y$) *if there is a derivation of* $x \Rightarrow y$ *in* **PC**.

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Peirce also introduced the equality sign (=): $x = y$ is a shorthand for "$x \Rightarrow y$ and $y \Rightarrow x$". One can now easily derive the following lattice-theoretic equalities:

(Idempotency)	$x = x \vee x$	$x \wedge x = x$
(Commutativity)	$x \vee y = y \vee x$	$x \wedge y = y \wedge x$
(Associativity)	$x \vee (y \vee z) = (x \vee y) \vee z$	$x \wedge (y \wedge z) = (x \wedge y) \wedge z$
(Absorption)	$x \vee (y \wedge z) = x$	$x \wedge (y \vee z) = x$.

In the 1880 article, Peirce stated the following distributive laws:

$$\text{(D1) } (x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z); \quad \text{(D2) } (x \wedge y) \vee z = (x \vee z) \wedge (y \vee z).$$

He mentions casually that "they are easily proved. . ., but the proof is too tedious to give" [22, p. 33]. This passage provoked a challenge by Schröder, who took the laws of distributivity to be independent from the lattice axioms. A rejoinder and a lively discussion ensued and Peirce's lost and subsequently recovered proof was finally added to Huntington's 1904 article [13]. Houser [10, 11] provides a rich account of the context of Peirce's note and the debates that followed.

We can give a proof of (D1) and (D2) in Peirce's calculus. The following lemmas, which Peirce stated, can be derived in the calculus **PC**.

LEMMA 2.3. *The sequents $x_i \Rightarrow x_1 \vee x_2$ and $x_1 \wedge x_2 \Rightarrow x_i$ for $i = 1, 2$ are derivable in* **PC**.

Proof. By replacing z in ($\vee$E) and ($\wedge$E) with $x_1 \vee x_2$ and $x_1 \wedge x_2$, respectively. □

LEMMA 2.4. *The following rules are derivable in* **PC**:

$$\text{(R1)}\ \frac{x \Rightarrow y}{z \to x \Rightarrow z \to y} \qquad \text{(R2)}\ \frac{x \Rightarrow z \quad y \Rightarrow u}{x \wedge y \Rightarrow z \wedge u}$$

$$\text{(R3)}\ \frac{x \Rightarrow y}{y \to z \Rightarrow x \to z} \qquad \text{(R4)}\ \frac{x \Rightarrow z \quad y \Rightarrow u}{x \vee y \Rightarrow z \vee u}.$$

Proof. (R1) is derived as follows:

$$\dfrac{\dfrac{\dfrac{z \to x \Rightarrow z \to x}{(z \to x) \wedge z \Rightarrow x}\,\text{(PR)} \qquad x \Rightarrow y}{(z \to x) \wedge z \Rightarrow y}\,\text{(Tr)}}{z \to x \Rightarrow z \to y}\,\text{(PR)}.$$

(R2) is derived as follows:

$$\dfrac{\dfrac{x \wedge y \Rightarrow x \quad x \Rightarrow z}{x \wedge y \Rightarrow z}\,\text{(Tr)} \qquad \dfrac{x \wedge y \Rightarrow y \quad y \Rightarrow u}{x \wedge y \Rightarrow u}\,\text{(Tr)}}{x \wedge y \Rightarrow z \wedge u}\,(\wedge\text{I}).$$

For (R3), assume $x \Rightarrow y$. By (R_2), we get $x \wedge (y \to z) \Rightarrow y \wedge (y \to z)$. One easily derives that $y \wedge (y \to z) \Rightarrow z$ by (PR) and commutativity. Then by (Tr), we have $x \wedge (y \to z) \Rightarrow z$. Finally, by commutativity and (PR), we get $y \to z \Rightarrow x \to z$. The rule (R4) is likewise easily shown. □

2.4. Derivation of distributivity. Before deriving the distributive laws, we make the following observation on (PR). Since $\wedge$ is commutative, we also have the following version of Peirce's Rule:

$$\dfrac{x \wedge y \Rightarrow z}{y \Rightarrow x \to z}\,\text{(PR)}.$$

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We can thus apply (PR) without considering the first or second coordinate of the conjunction.

THEOREM 2.5. *The distributive laws* (*D1*) *and* (*D2*) *are derivable in* **PC**.

Proof. (D1) First, we derive $(x \wedge y) \vee (x \wedge z) \Rightarrow x \wedge (y \vee z)$ as follows:

$$\dfrac{x \wedge y \Rightarrow x \qquad \dfrac{x \wedge y \Rightarrow y \quad y \Rightarrow y \vee z}{x \wedge y \Rightarrow y \vee z}\,(\mathrm{Tr})}{x \wedge y \Rightarrow x \wedge (y \vee z)}\,(\wedge\mathrm{I}).$$

Similarly we get $x \wedge z \Rightarrow x \wedge (y \vee z)$. By ($\vee$I), we get $(x \wedge y) \vee (x \wedge z) \Rightarrow x \wedge (y \vee z)$. Second, we derive $x \wedge (y \vee z) \Rightarrow (x \wedge y) \vee (x \wedge z)$ as follows:

(1) We have the following derivation:

$$\dfrac{\dfrac{x \wedge y \Rightarrow x \wedge y}{y \Rightarrow x \to (x \wedge y)}\,(\mathrm{PR}) \qquad \dfrac{x \wedge z \Rightarrow x \wedge z}{z \Rightarrow x \to (x \wedge z)}\,(\mathrm{PR})}{y \vee z \Rightarrow (x \to (x \wedge y)) \vee (x \to (x \wedge z))}\,(\mathrm{R4}).$$

(2) We have the following derivation:

$$\dfrac{x \wedge y \Rightarrow (x \wedge y) \vee (x \wedge z)}{x \to (x \wedge y) \Rightarrow x \to (x \wedge y) \vee (x \wedge z)}\,(\mathrm{R1}).$$

Similarly, we have $x \to (x \wedge z) \Rightarrow x \to (x \wedge y) \vee (x \wedge z)$. Then by ($\vee$I), we get $(x \to (x \wedge y)) \vee (x \to (x \wedge z)) \Rightarrow x \to (x \wedge y) \vee (x \wedge z)$.

By (1) and (2), and using (Tr), we get $y \vee z \Rightarrow x \to (x \wedge y) \vee (x \wedge z)$. By (PR), we get $x \wedge (y \vee z) \Rightarrow (x \wedge y) \vee (x \wedge z)$.

(D2) First, we easily derive $x \wedge y \Rightarrow x \vee z$ and $x \wedge y \Rightarrow y \vee z$. By ($\wedge$I), we get $x \wedge y \Rightarrow (x \vee z) \wedge (y \vee z)$. From $z \Rightarrow x \vee z$ and $z \Rightarrow y \vee z$, we get $z \Rightarrow (x \vee z) \wedge (\vee z)$. By ($\vee$I), we get $(x \wedge y) \vee z \Rightarrow (x \vee z) \wedge (y \vee z)$. Second, we have the following:

(3) By (D1), we get $(x \vee z) \wedge (y \vee z) \Rightarrow (x \wedge (y \vee z)) \vee (z \wedge (y \vee z))$. Note that $z \wedge (y \vee z) \Rightarrow z$ by absorption. Then by (R4) we get $(x \wedge (y \vee z)) \vee (z \wedge (y \vee z)) \Rightarrow (x \wedge (y \vee z)) \vee z$. By (Tr), we get $(x \vee z) \wedge (y \vee z) \Rightarrow (x \wedge (y \vee z)) \vee z$.

(4) By commutativity and (D1), we get $x \wedge (y \vee z) \Rightarrow (x \wedge y) \vee (x \wedge z)$. By (R4), we get $(x \wedge (y \vee z)) \vee z \Rightarrow ((x \wedge y) \vee (x \wedge z)) \vee z$. By associativity and absorption, we get $((x \wedge y) \vee (x \wedge z)) \vee z \Rightarrow (x \wedge y) \vee z$. Then by (Tr) we get $(x \wedge (y \vee z)) \vee z \Rightarrow (x \wedge y) \vee z$.

From (3) and (4), by (Tr), we get $(x \vee z) \wedge (y \vee z) \Rightarrow (x \wedge y) \vee z$. □

One of Peirce's own proofs is found in R 417 written in 1893. At the end of that proof, he remarked: "This is what I had in mind in a statement Am. Math. J.III.33. which has been severely criticized". That 1880 statement concerned the cases of distributive principle that are "easily proved... but the proof is too tedious to give". The 1893 proof is correct but it is not equivalent to the proof that Peirce apparently recovered from his earlier work on the 1880 article and which he sent to E. V. Huntington on the Christmas Eve of 1903. The reason is that the 1893 proof uses the principle of iteration. (See §4 below on this rule and on the 1893 proof.)

Huntington published in 1904 the proof Peirce had sent to him, including Peirce's footnote on it [13]. In the published proof, the axiom—Huntington's "postulate"—number 9 is crucial. Without it, the axioms 1–8 only define uniquely complemented lattices which need not be distributive. The axiom 9 says that, if it is not the case that $a \leq \overline{b}$, then there is a nonzero element x such that $x \leq a$ and $x \leq b$. This is equivalent to the condition:

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(i) If $\forall x(x \leq a \,\&\, x \leq b \Rightarrow x = 0)$, then $a \leq \overline{b}$.

The condition (i) is also equivalent to the condition:

(ii) If $\forall x(x \leq a \wedge b \Rightarrow x = 0)$, then $a \leq \overline{b}$.

Moreover, (ii) is equivalent to the condition:

(iii) $a \wedge b \leq 0$ implies $a \leq \overline{b}$.

This is a special case of Peirce's leading principle (PR). Conversely, from the axiom 9 one can derive (PR):

> Assume $a \wedge b \leq c$ but not $a \leq \overline{b \wedge \overline{c}}$. Then there is $x \neq 0$ such that $x \leq a$ and $x \leq b \wedge \overline{c}$. Then $x \leq a \wedge b \wedge \overline{c} \leq c \wedge \overline{c} = 0$, which is a contradiction.

Hence the axiom 9 and Peirce's Rule are equivalent. This equivalence justifies our proof of distributivity in the fashion presented here.

2.5. Negation and contraposition. Peirce's Rule (PR) is also closely related to the rules for negation and implication which we will prove in what follows. Recall that Peirce defined negation as $\overline{x} := x \to \bot$.[2]

PROPOSITION 2.6. *The following sequents and rules are derivable in* **PC**:

(1) *Double negation laws*: (DB1) $x \Rightarrow \overline{\overline{x}}$; (DB2) $\overline{\overline{x}} \Rightarrow x$.

(2) *Rules of Contraposition*:

$$\frac{x \Rightarrow y}{\overline{y} \Rightarrow \overline{x}}\text{(CP)} \quad \frac{\overline{y} \Rightarrow \overline{x}}{x \Rightarrow y}\text{(ICP)}.$$

Proof. For (DB1), we first have $x \wedge (x \to \bot) \Rightarrow \bot$. By (PR), we get $x \Rightarrow \overline{\overline{x}}$. For (DB2), we start from (Em) $\top \Rightarrow x \vee \overline{x}$. Then we have $\overline{\overline{x}} \wedge \top \Rightarrow \overline{\overline{x}} \wedge (x \vee \overline{x})$. Clearly $\overline{\overline{x}} \Rightarrow \overline{\overline{x}} \wedge \top$. Then $\overline{\overline{x}} \Rightarrow \overline{\overline{x}} \wedge (x \vee \overline{x})$. By (D1), we have $\overline{\overline{x}} \wedge (x \vee \overline{x}) \Rightarrow (\overline{\overline{x}} \wedge x) \vee (\overline{\overline{x}} \wedge \overline{x})$. Clearly $\overline{\overline{x}} \wedge \overline{x} \Rightarrow \bot$. Then we have $\overline{\overline{x}} \wedge (x \vee \overline{x}) \Rightarrow (\overline{\overline{x}} \wedge x) \vee \bot$. Clearly $(\overline{\overline{x}} \wedge x) \vee \bot \Rightarrow \overline{\overline{x}} \wedge x$ and $\overline{\overline{x}} \wedge x \Rightarrow x$. By (Tr), we have $\overline{\overline{x}} \wedge (x \vee \overline{x}) \Rightarrow x$. Finally by (Tr), we have $\overline{\overline{x}} \Rightarrow x$.

The rule (CP) is an instance of (R3). The inverse of contraposition (ICP) follows immediately from (CP) and double negation laws. □

LEMMA 2.7. *The following rules are derivable in* **PC**:

$$\frac{x \wedge y \Rightarrow z}{y \Rightarrow \overline{x} \vee z}\text{(R5)} \quad \frac{y \Rightarrow \overline{x} \vee z}{x \wedge y \Rightarrow z}\text{(R6)}.$$

Proof. For (R5), assume $x \wedge y \Rightarrow z$. Then $\overline{x} \vee (x \wedge y) \Rightarrow \overline{x} \vee z$. By distributivity and (Tr), we have $(\overline{x} \vee x) \wedge (\overline{x} \vee y) \Rightarrow \overline{x} \vee z$. Clearly $\overline{x} \vee x = \top$. Then we have $\overline{x} \vee y \Rightarrow \overline{x} \vee z$. Since $y \Rightarrow \overline{x} \vee y$, we have $y \Rightarrow \overline{x} \vee z$. (R6) is shown similarly. □

PROPOSITION 2.8. *The sequent* $x \to y = \overline{x} \vee y$ *is derivable in* **PC**.

[2] This definition of negation, that "from x anything you please necessarily follows" was, from the "*formal* point of view", perfectly acceptable to Peirce. But he also thought that it does not "really define denial in terms of consequence" (Peirce to Huntington, February 14, 1904 [45]).

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Proof. First, we derive $\overline{x} \vee y \Rightarrow x \rightarrow y$ as follows. It is easy to show $x \wedge (\overline{x} \vee y) \Rightarrow x \wedge y$. Clearly $x \wedge y \Rightarrow y$. By (Tr), $x \wedge (\overline{x} \vee y) \Rightarrow y$. By (PR), we get $\overline{x} \vee y \Rightarrow x \rightarrow y$. Second, we have $x \wedge (x \rightarrow y) \Rightarrow y$. By (R5), we get $x \rightarrow y \Rightarrow \overline{x} \vee y$. □

Peirce also admitted the importance of the following De Morgan's laws and proved them.

PROPOSITION 2.9. *The De Morgan laws* (DM1) $\overline{x \wedge y} = \overline{x} \vee \overline{y}$ *and* (DM2) $\overline{x \vee y} = \overline{x} \wedge \overline{y}$ *are derivable in* **PC**.

Proof. For (DM1), first we have $\overline{x} \Rightarrow \overline{x} \vee \overline{y}$. By (CP), we get $\overline{\overline{x} \vee \overline{y}} \Rightarrow \overline{\overline{x}}$. By (DB2) and (Tr), we have $\overline{\overline{x} \vee \overline{y}} \Rightarrow x$. Similarly we get $\overline{\overline{x} \vee \overline{y}} \Rightarrow y$. Then by ($\wedge$I), we get $\overline{\overline{x} \vee \overline{y}} \Rightarrow x \wedge y$. By (CP), (DB2) and (Tr), we ge $\overline{x \wedge y} = \overline{x} \vee \overline{y}$. The law (DM2) is shown similarly. □

REMARK 2.10. Peirce's *fifth icon*, namely $((x \rightarrow y) \rightarrow x) \rightarrow x$, which he proposed in his 1885 sequel "On the Algebra of Logic: A Contribution to the Philosophy of Notation" (see §3), and which is well-known as Peirce's Law, is also derivable in **PC**. To prove $\top \Rightarrow ((x \rightarrow y) \rightarrow x) \rightarrow x$, it suffices to show that $(x \rightarrow y) \rightarrow x \Rightarrow x$. Since $(x \rightarrow y) \rightarrow = \overline{x \rightarrow y} \vee x$, it suffices to show that $\overline{x \rightarrow y} \vee x \Rightarrow x$. By ($\vee$I) and (Id), it suffices to show that $\overline{x \rightarrow y} \Rightarrow x$. Since $x \rightarrow y = \overline{x} \vee y$, by contraposition, De Morgan and double negation rules, we have $\overline{x \rightarrow y} = x \wedge \overline{y}$. Then $x \wedge \overline{y} \Rightarrow x$. By (Tr), we get $\overline{x \rightarrow y} \vee x \Rightarrow x$.

2.6. Completeness. We now establish the soundness and completeness of the calculus **PC** with respect to the class of all Boolean algebras. The soundness is quite easy and is obtained by induction on the derivation of sequents in **PC**. For the completeness, the Lindenbaum–Tarski construction is applied.

The set of all terms T is constructed from a set of propositional variables Prop and $\bot$ using $\wedge, \vee, \rightarrow$. As usual, the negation sign is defined in terms of implication and the impossible. For any Boolean algebra B, a term x induces a polynomial function x^B. We say that a sequent $x \Rightarrow y$ is valid in B if $x^B \leq y^B$ whatever the values of variables occurred in the sequent are given. The point is that the consequence relation $\Rightarrow$ is interpreted as the lattice order in a Boolean algebra.

In order to construct the Lindenbaum–Tarski algebra, one needs to define a congruence relation on T. The sign of equality $=$ suggested by Peirce can be applied for this purpose. The equality sign can be viewed as a binary relation on the set of all terms. Obviously one can show that $=$ has the following properties with respect to the derivability in **PC**:

(1) (Reflexivity) $x = x$.
(2) (Transitivity) If $x = y$ and $y = z$, then $x = z$.
(3) (Symmetry) If $x = y$, then $y = x$.

Hence $=$ is an equivalence relation. It is in fact a congruence relation. Let $|x| = \{y \mid x = y$ is derivable in **PC**$\}$ be the equivalence class of x, and $T/_=$ the set of all equivalence classes. Let $B = (T/_=, \wedge, \vee, \neg, 0)$ be the Lindenbaum–Tarski algebra on $T/_=$. It is a Boolean algebra. Then one can easily obtain the following completeness theorem:

THEOREM 2.11 (Completeness). *A sequent $x \Rightarrow y$ is derivable in* **PC** *if and only if $x \Rightarrow y$ is valid in all Boolean algebras.*

Peirce's calculus is therefore fully adequate as a calculus for Boolean algebras. It follows that the calculus **PC**, as Peirce had presented it, is a successful calculus that agrees with

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classical propositional logic. Peirce achieved his sequent calculus for his logic immediately in the 1880 article, without further ado. It dates back to Arthur Prior's work [56, 57] who showed, by different means, that Peirce's 1880 article provides a complete basis for propositional logic.

2.7. Peirce's rule in perspective. The rule (PR) is a formulation of Peirce's leading principle of inference. Leading principles that have a "maximum abstractness" ([51] NEM 4, p. 175, 1898) are logical principles. The maximal abstractness means that such principles add nothing to the premises of the inference which they govern.

The distributivity laws are derivable from (PR), together with the lattice rules for conjunction and disjunction, the identity and transitivity (Tr). But they are not derivable without (PR). Now Schröder thought that distributive laws are independent of the theory of lattices [61, Chapter XII]. Peirce stated the point clearly in a footnote to his 1885 article:

> It is interesting to observe that [the] reasoning [in Peirce's example in the 1885 paper] is dilemmatic. In fact, the dilemma involves the fifth icon [see Remark 2.10]. The dilemma was only introduced into logic from rhetoric by the humanists of the *renaissance*; and at that time logic was studies with so little accuracy that the peculiar nature of this mode of reasoning escaped notice. I was thus led to suppose that the whole non-relative logic was derivable from the principles of the ancient syllogistic, and this error[3] is involved in Chapter II of my paper in the third volume of this Journal [the 1880 paper]. My friend, Professor Schröder, detected the mistake and showed that the distributive formulæ
>
> $$(x + y)z \prec xz + yz$$
> $$(x + z)(y + z) \prec xy + z$$
>
> could not be deduced from syllogistic principles. I had myself independently discovered and virtually stated the same thing. (*Studies in Logic*, p. 189.)[4] There is some disagreement as to the definition of the dilemma (see Keynes's excellent *Formal Logic*, [14, p. 214]); but the most useful definition would be a syllogism depending on the above distribution formulæ.[5] The distribution formulæ

[3] Peirce later added a note in the margin of his copy of the article, which reads: "But it was not an error!!! See my original demonstration in marginal note". This marginal note has not been recovered (but see [17]).

[4] In that 1883 publication of the "Note B" in his *Studies in Logic* [23] Peirce stated that two relatives are "undistributed" in a relative product and in a relative sum.

[5] Keynes distinguishes five different formulations of dilemmatic arguments: those given by (i) Mansel, Whately, and Jevons, (ii) by Fowler, (iii) Keynes's own formulation, (iv) by Hamilton, and (v) by Thomson. Peirce appears to mean none of theirs as "the most useful definition". For example, he proposes the rule of dilemma to be "If $(a\overline{b} + c)(\overline{a} + c)$ then c" (see R 736, [51] NEM IV, p. 115). This is proved using the distributivity principle thus: $c + (a\overline{b}\overline{a})$ implies c, and by the law of contradiction, $c + 0$ implies c. This direction of the derivation of the dilemmatic rule depends on the second distribution principle as given here in Peirce's footnote, namely one that is not derivable from the lattice rules alone. In the other direction, the distribution principle applied is strictly syllogistic.

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$$xz + yz \prec (x + y)z$$
$$xy + z \prec (x + z)(y + z)$$

> are strictly syllogistic. De Morgan's added moods are virtually dilemmatic, depending on the principle of excluded middle.[6] [24, p. 190]

Clearly $(x \wedge z) \vee (y \wedge z) \Rightarrow (x \vee y) \wedge z$ (D1, right-to-left) and $(x \wedge y) \vee z \Rightarrow (x \vee z) \wedge (y \vee z)$ (D2, left-to-right) are derivable using only lattice rules and without Peirce's Rule. But our reformulation of Peirce's calculus explains the derivability of the distributive laws $(x \vee y) \wedge z \Rightarrow (x \wedge z) \vee (y \wedge z)$ (D1, left-to-right) and $(x \vee z) \wedge (y \vee z) \Rightarrow (x \wedge y) \vee z$ (D2, right-to-left) perfectly. The calculus **PC** is essentially not to be conceived in the style of natural deduction but as a sequent calculus. Consequently, the proof of distributive laws is not hard.

The second aspect of Peirce's Rule is concerned with implication which is intuitionistic. As mentioned, (PR) can be viewed algebraically as the law of residuation:

$$\text{(RES) } a \wedge c \leq b \text{ if and only if } c \leq a \rightarrow b.$$

This is exactly the way in which intuitionistic implication is introduced. A *Heyting algebra* is identified with an algebra $(H, \wedge, \vee, 0, 1, \rightarrow)$, where $(A, \wedge, \vee, 0, 1)$ is a bounded lattice, and $\rightarrow$ is a binary operation on H satisfying (RES). Now distributivity can be derived from the lattice rules and (PR). Equally, the algebraic distributive laws can also be derived from (RES). Moreover, if we remove the law of excluded middle (Em) from the calculus **PC**, we would obtain not a propositional calculus but a calculus for Heyting algebras, and hence intuitionistic logic. Peirce was not far from such invention, and there are further and collateral reasons as to why his overall approach to logic and its philosophy can be argued to have taken steps towards an intuitionistic approach ([53, Chapter 3]).

§3. From algebras to graphs: the transitional period.

3.1. Peirce's icon system. The first step from Peirce's calculus for Boolean algebras towards the calculus for logical graphs was taken in his truth-value semantics and in the Hilbert-style axiomatization of classical propositional logic, both of which were proposed in his 1885 article "The Algebra of Logic: A Contribution to the Philosophy of Notation" [24].

Peirce proposes a system of five icons as a contribution to such notation [63]. One important aspect of it is the *truth-value semantics* for hypothetical propositions. To do this, the two constants of the possible ∞ and the impossible 0 are interpreted as two truth values, **v** and **f**, respectively.[7] Peirce then provided the meaning for hypothetical propositions in the general sense:

> The range of possibilities is in one case taken wider, in another narrower; in the present case it is limited to the actual state of things. Here,

[6] Later in 1893 Peirce takes a dilemmatic argument to be "any argument whose validity depends upon the principle of excluded middle" ([49] CP 2.474). Dilemmatic arguments would thus not be intuitionistically valid.

[7] The explicit use of **v** and **f** to denote the truth values *true* and *false* appeared already in Peirce's *Logic Notebook* in 1880 ([52] W4, p. 216).

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> therefore, the proposition $a \prec b$ is true if a is case or if b is true, but is false if a is true while b is case. [24, p. 187]

This is exactly the truth table for material implication [1].[8] Peirce then proceeds to introduce the *five icons* that give rise to the algebra of logic:

(I1) $x \to x$
(I2) $(x \to (y \to z)) \to ((y \to (x \to z)))$
(I3) $(x \to y) \to ((y \to z) \to (x \to z))$
(I4) $\bot \to x$
(I5) $((x \to y) \to x) \to x$.

The icons (I1)–(I3) are obtained by the properties of the copula. The icon (I4) is a bit complicated. Peirce's purpose in it was to introduce negation:

> The third *icon* introduces the image of a chain of consequence. We must now again *enlarge the notation* so as to introduce negation. We have already seen that if a is true, we can write $x \prec a$, what ever x may be. Let b be such that we can write $b \prec x$ whatever x may be. Then b is false. We have here a *fourth icon*, which gives a new sense to several formulæ. ([24, p. 189], added emphasis)

The fourth icon enlarges the notation so that the negation can be defined. This is done by introducing the constant 0, written as $\bot$ (*falsum*) in the standardized notation. The law $\bot \to x$, whatever x may be, belongs to Peirce's nonrelative logic. Thus using the constant 0 one can define negation: $\overline{x} := x \to \bot$. Peirce did not explain this definition explicitly here in 1885, but it is invariably assumed and he applied this definition when he presented the new formulæ related to negation. For example, from $x \to (y \to z)$ we can infer $y \to (x \to z)$ by (I2). Here z is an arbitrary proposition. If we replace it with the falsity $\bot$, we get the inference that $x \to \overline{y}$ implies $y \to \overline{x}$.

Peirce also obtained the rule of *modus ponens* after the introduction of the second icon: (MP) From x and $x \to y$ infer y. But rather than having it as an explicit rule of inference, Peirce took (MP) to be a "consequence" of the first and second icons, and as such "the most rudimentary form of reasoning". From $(x \to y) \to (x \to y)$ we can indeed infer $x \to ((x \to y) \to y)$. Peirce considers the meaning of this formula as follows:

> This is the same as to say that if in any state of things x is true, and if the proposition "if x, then y" is true, then in that state of things y is true. This is the *modus ponens* of hypothetical inference, and is the most rudimentary form of reasoning. [24, p. 188]

The justification of (MP) is not syntactic but semantic, resorting to the propositions being true "in any state of things"—that is, in a model. Peirce was arguing for the soundness of the rule of *modus ponens* in semantic terms. Conversely, if we use (MP) as a primitive rule of inference, which is the most rudimentary form of reasoning, we can easily derive $x \to ((x \to y) \to y)$ from (I1) and (I2) by (MP).

A similar situation in Peirce's work is that, from the icon (I2), he argued for the validity of the formula $x \to (y \to x)$ from the validity of $y \to (x \to x)$. This proof is also a

[8] Such truth tables and the ensuing truth-functional analysis was presented already in a manuscript from 1884, see [52] W3, p. 112.

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semantic one. Indeed there is no way to derive $y \to (x \to x)$ from the first or the second icon in a proof-theoretic manner. Put it in another way, Peirce proposed the following rule of *weakening*: (W) from x infer $y \to x$. The rule (W) cannot be derived from (I2) using (MP). But the formula $x \to (y \to x)$ can be easily derived using (I2) and (MP).

The validity of the rules of (MP) and (W) is therefore first of all a semantical, or as Peirce used the term, a *semeiotic* question, namely a question concerning the philosophy of notation.

Now the five icons, plus the rule (MP) and (W), give a sound and complete axiomatization of classical propositional logic. We may call this *Peirce's Icon System* (cf. [63]). The completeness can be shown by its equivalence with the Tarski–Bernays–Wajsberg axiomatic system, which consists of the rule (MP), plus the icons (I3), (I4), and (I5), and the axiom $x \to (y \to x)$.

3.2. The 1891 algebra of the copula. A few years later, Peirce resumed his interest in the algebra of copula and began to reformulate the algebraic rules of reasoning. In spring 1891, he drafted two versions on the algebra of the copula. In Version 1 ([25, p. 212], W8, pp. 210–211), he introduced five algebraic rules. The rules 1–4 are quite thoughtful, but the fifth one is very intangible. Here we cite the first four rules:

> Rule 1. The order of antecedents is immaterial.
> Rule 2. Any antecedent may be inserted under an even number of parentheses, or none, and omitted under an odd number.
> Rule 3. The insertion of an antecedent of a proposition a second time as antecedent of the proposition or of a clause of it has no effect.
> Rule 4. The insertion of a clause having an antecedent identical with the consequent of the proposition has no effect; and a proposition having an antecedent identical with its consequent is identically true. [25, pp. 201–211]

Rule 1 is familiar: from $a \prec (b \prec c)$ we can infer $b \prec (a \prec c)$. This has the same effect as the icon (I2). Rules 2–4 are about the insertion or deletion of antecedents in a formula. Peirce's example of Rule 2 is that from $(a \prec b) \prec c$ we can infer $(d \prec (a \prec b)) \prec c$ and $b \prec c$. One can also observe that from $a \prec (a \prec b)$ we can infer $a \prec b$ because of the second occurrence of a is within the odd number of parentheses. Peirce explains Rules 3 and 4 with similar examples. In five year's time, the emerging idea of insertions and deletions would become central in the form of the transformation rules of his graphical method of logic.

The second version of the algebra of the copula is more detailed. Peirce first rewrote the three basic rules (α, β, γ) to define the copula of consequence:

> α. If b is true, then $a \prec b$ is true.
> β. Either a or $a \prec b$ is true.
> γ. If $a \prec b$ and a are true, b is true.[9] [26, p. 212]

[9] In the earlier draft, he had spoken of "scriptibility" instead of truth values: α. "If b is scriptible, then $a \prec b$ is scriptible", etc. The difference is not only a terminological one, as it is here in the latter draft that the philosophy of notation becomes increasingly pronounced. He defined *scriptibility* as that which is "capable of being written conformably to the purpose" (R 501, c.1902). This semantic characterisation suggests that Peirce was thinking about a generalisation of truth-values to other values besides truth and falsity, which he then goes on to examine

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The rules α and γ are the rules (W) and (MP) which we find in Peirce's 1885 icon system. The rules α and β are obtained by the meaning of $\prec\!\!\!-$. Peirce then introduces seven algebraic rules, which are easier to understand than the rules of Version 1. For the sake of brevity, we mention three of them.

The rule number 3 is the *principle of insertions and omissions*: "any antecedent may be inserted under an even number of parentheses (or none) or dropped under an odd number" [26, p. 213]. The crucial example can be expressed as follows:

$$\frac{b}{a \prec\!\!\!- b}\text{(R3)}.$$

Peirce is now explicit about the rules of insertion and omission.

The fourth rule is the *principle of repetition*: "If an expression occurs twice as antecedent, and one of these times under no parenthesis which does not enclose the second occurrence, then this second has no effect and may be inserted or dropped" [26, p. 213]. Formally, we can express this rule as follows:

$$\frac{a \prec\!\!\!- (a \prec\!\!\!- c)}{a \prec\!\!\!- c}\ \text{(R4)}.$$

Here we can observe a new rule being born. Principle of repetition corresponds to that of the rule of iteration ("may be inserted") and deiteration ("may be dropped") in the alpha system.

After having formulated (R3) and (R4), Peirce derives some new rules. One important rule is the fifth one (R5). His proof of (R5) is again a semantic one. By the basic rule β, we have that $a \vee (a \prec\!\!\!- b)$ is true. If $(a \prec\!\!\!- b) \prec\!\!\!-$ is true, in each case, we get that a is true.

The fifth rule is the *principle of breaking parentheses*. This rule is for any copula within an even number of parentheses [26, p. 215]:

(R5.1) Bring the beginnings of the two innermost parentheses forward to the beginning of that copula.

(R5.2) Repeat all to the left of the innermost parenthesis from the beginning of the next innermost parenthesis follows by one of the antecedents to the left of the break in the innermost parenthesis. This is to be done for each of these antecedents.

What makes this fifth rule peculiar is that it proposes applying inferential rules to parentheses, that is, to what may be considered as punctuation marks (the "Klammern"). Later Peirce explains: "The Klammern, which are commonly spoken of in the elementary algebra-books as mere subsidiary signs, are in reality the very heart of algebraic notation" (R 530, 1904 [44]).

The sixth rule is the principle of identity and the seventh the principle of double consequent. The first rule is about commutation and the second the principle of contraposition.

Peirce's reformulation of algebraic rules considers positions that antecedents have in a formula. The really interesting rules are the insertions and omissions of antecedents (R3),

in detail in R 501. As early as 1886, Peirce had used modal notions when referring to "the sheet of conceivable states of things" ([52] W5:331–2). The allied notion to scriptibility is *transformability*, which concerns graphs or formulas that are "capable of being transformed without changing anything scriptible into anything non-scriptible" (R 430 [35]).

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the repetition (R4), as well as the fifth rule of breaking the parentheses (R5). They are all germane to his next major revision of his algebra; they would soon give rise to the invention of an entirely different-looking system of logical graphs.

3.3. Qualitative logic and the sign of consequence. A further step in the journey towards these graphical systems of logic takes place in Peirce's *Qualitative Logic* ([51] NEM IV, 101–115; R 736; W5). The treatise was at first written for students of logic in 1886. They were retained and later revised for the projected 1893–4 *Grand Logic*.[10]

Qualitative logic introduces a new notation for the sign of illation: $\overline{x\,|}$ or $\overline{x\,\curlywedge}$. This sign consists of $\curlywedge$ (an aggregate, a special variant of the plus sign), with a connected "streamer" or vinculum covering the antecedent of the conditional. Again, Peirce's treatment of this new connective is well-nigh semantic: "To say that N follows from M, is to say that every possible state of things in which M is true is a state of things in which N is true" ([27] R 411, 1893). In the proofs of this system Peirce attaches all propositions with a truth value.

Peirce's "purpose for selecting the special sign" was that it does something that "would not be accomplished by any of the signs that have been proposed hitherto" (R 411).[11] This new sign of consequence can denote, without any additional signs such as parentheses, also the scope of the antecedent.

With this new sign of illation, Peirce regains confidence in his algebra of the copula. One can now express "the most complicated relations without the slightest ambiguity" ([51] NEM IV, p. 107; R 736). The new sign in fact makes all other logical constants, as well as parentheses, superfluous. Notice that falsum is represented by the right hand side of this sign being empty: $\curlyvee$, and thus the negation of x is represented as $\overline{x\,\curlywedge}$.

One of the simplified codes Peirce proposes to be the set of formal rules of translation of this language consists of the following three rules (R 737 [32], 1900?):

(1) $\overline{x\,\curlywedge}\,x$.

(2) If from x and y, follows z, then if x, $\overline{y\,\curlywedge}\,z$ and *conversely*.

(3) Hence $\overline{x\,\curlywedge}\,\overline{y\,\curlywedge}\,z = \overline{y\,\curlywedge}\,\overline{x\,\curlywedge}\,z$.

The rule (1) is the law of identity (cf. (R6) above). The clause (2) expresses Peirce's Rule (PR). Axiom (3) is the principle of commutation (Peirce's second icon I2 in 1885). It follows from the other two rules. The law of excluded middle is also immediate from the identity law. It is clear that this parenthesis-free calculus of qualitative logic is a complete system and that it agrees with Boolean algebra.

As usual, Peirce exercises with many different systems and sets of formal codes. Among them is the following, which he calls substitution:

> The general rule of substitution is that if $\overline{m\,\curlywedge}\,n$, then n may be substituted for m under an even number of streamers (or under none), while under an odd number m may be substituted for n. (R 736)

This rule is also valid.

---

[10] This work, entitled, *How to Reason*: *A Critick of Arguments*, is projected to appear as Vol. 11 of the chronological edition *Writings of Charles S. Peirce*.

[11] Among the earlier suggestions he lists Peano's reversed-C or $\supset$, Schröder's subsumption $\Subset$, MacColl's colon :, mathematicians' $\leqq$, De Morgan's), Hamilton's cuneiform line $\blacktriangleright$, as well as Peirce's own previous signs for illation, $\prec$ and $\succ$.

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The upshot of introducing the sign of consequence ⊦ and the revised rules for his system of qualitative logic is that we need not wait until his 1897 "The Logic of Relatives" was written and published to be able to argue that the real transition from the algebraic treatment of logic to the graphical period—or what somewhat ill-advisedly has been termed the transition from the symbolic approach to logical languages to the diagrammatic or iconic approach—had already taken place.[12] In fact the rules of an equivalent and complete calculus were established by 1894. It was only the notation that underwent some further modifications from the algebra of copula to the sign of consequence and then, finally, to the notation that he employed in logical graphs.

In September 1894 Peirce gives an outline of the *Grand Logic*, now entitled as *The Art of Reasoning*. On the algebra of the copula the outline stated that "this difficult subject is here perfectly cleared up and systematized for the first time, and made to lead directly to the Boolian algebra". Moreover, Peirce claimed that he can now "show that the rules of the nonrelative Algebra may be immensely simplified by dropping all connection with arithmetic" (Peirce to F. Russell, September 6, 1894, R L 387 [30]).

What entitled such confidence in his logic? The answer is that at this stage of the development of the algebra of the copula, Peirce already had the complete system of rules including, as the last finding of the system, the rule of iteration/deiteration. Together with the rules of deletion, insertion, and the double negation, Peirce accomplishes a complete calculus, just as he does of his graphical system of 1896.

In fact, already the unpublished *Grand Logic* proposed to deal with "Graphs and graphical logic" in its Chapter 11. It would show "the value and limitations of the geometrical way of thinking" (Peirce to Russell, September 6, 1894). Apparently this planned chapter was never written, and instead of making any final effort to complete and get his massive work eventually published, by 1896 Peirce had already drafted a long article (R 482) on logical graphs which he submitted to the *Monist* for publication in August that year and again in next March. That article was ultimately never published, either, but it opened up a new field of explicitly graphical and topological approaches to logic, which Peirce termed a "diagrammatic syntax".

§4. The alpha system. In this section, we first present Peirce's alpha system for classical propositional logic. We then present the algebraic semantics for alpha and prove that the alpha algebras are exactly Boolean algebras. The algebraic completeness of the alpha system with respect to alpha algebras justifies the correctness of the system as a completely new one for classical propositional logic.

4.1. The language of graphs. Let Prop be a denumerable set of propositional variables. The set of all formulas or terms of classical propositional logic $\mathcal{L}_C$ is defined inductively by the following rule:

$$\mathcal{L}_C \ni x ::= p \mid \top \mid (x \wedge y) \mid \overline{x},$$

where $p \in \mathsf{Prop}$. Define $x \vee y = \overline{\overline{x} \wedge \overline{y}}$ and $x \to y := \overline{x \wedge \overline{y}}$. The set of all formulas of classical propositional logic is denoted by $\mathcal{L}_C$.

[12] Dipert [9, p. 293] takes the 1897 Logic of Relatives and the Entitative Graphs as the "transitional work between the Algebraic and the Diagrammatic periods". We have argued here that all the relevant aspects of the transitional work were accomplished by 1894.

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A formula in $\mathcal{L}_C$ is written in the linear order as the formulas used by Peirce in his 1880 work on algebras. The alpha graphs are constructed from primitive ones on the *sheet of assertion*,[13] or the blank, using a *cut* operation: $\bigcirc$. The meaning of the cut is the operation of negation.[14] One can double the continuous cut, and get the *double cut* $\bigcirc\!\!\bigcirc$, which means double negation.

Peirce also has a formal language of alpha graphs. A simple proposition, that is, a proposition without any occurrence of a propositional connective, can appear in a graph. Peirce did not introduce propositional letters standing for simple propositions. For the purpose of reconstruction of Peirce's alpha system, we fix a denumerable set of propositional letters Prop, the elements of which are counted as primitive graphs. Moreover, the sheet of assertion, or the blank, is also counted as a graph. Henceforth, we denote the blank by SA. A *primitive graph* P is a propositional letter or the blank SA. Every alpha graph is constructed from primitive graphs using cuts.

DEFINITION 4.1 (Alpha graphs). *The set* $\mathfrak{G}_\alpha$ *of all* alpha graphs *is defined inductively by the following rule*:

$$\mathfrak{G}_\alpha \ni G := p \mid \mathsf{SA} \mid G_1\, G_2 \mid \textcircled{G}\,,$$

where p *is a propositional letter. The graph* $G_1\, G_2$ *is obtained from* G_1 *and* G_2 *by placing them on the same* area *on the sheet of assertion. By an* area*, we mean the continuous region placed over the sheet of assertion, as defined by the interior of a cut. By continuous, we mean uninterrupted by any cuts.*[15]

When we say "graphs" we mean "alpha graphs". To improve readability and to reduce the visual Moire effect that several nested cuts give rise to, we define some abbreviations of compound graphs. Given any graphs G and H, define graphs $G \veebar H$, $G \supset H$ and $G \equiv H$ as below:

$$G \veebar H := \textcircled{\textcircled{G}\,\textcircled{H}}\,; \quad G \supset H := \textcircled{G\,\textcircled{H}}\,; \quad G \equiv H := \textcircled{G\,\textcircled{H}}\,\textcircled{H\,\textcircled{G}}\,.$$

Here $\veebar$ and $\Rightarrow$ can be viewed as compound graph operations which can be used to construct new graphs.

---

[13] In Peirce's terms: "The *sheet*, or *sheet of assertion*, or *actual sheet*, is that surface upon which any graph that may be scribed will be understood to be thereby asserted of the universe. It is so much of the surface originally appropriated to this purpose as remains unoccupied after part of it has been occupied by instances of scribed graphs. The surface originally so appropriated is called the *original sheet*, as opposed to the actual sheet" (R 1589, 1903 [39]).

[14] This is not the only meaning of this operation, as it also signifies, among others, the grouping together of the graphs that it encircles together, as is customarily done by parentheses. Peirce in fact distinguishes altogether five offices of this operation (three for alpha and two more for beta, that is, for the first-order logic of quantifiers), only one of which is the negation, and is negation only secondarily, as the primary meaning is that of grouping or linear separation (R 430, 1902 [35]; [54]).

[15] In Peirce's terms: "The term *cut* shall be applied to a linear separation of a surface, which line of separation returns into itself, so as to enclose a space within which the material of the surface in which the cut is made is replaced by a different material, either by stripping off the original material and disclosing a new surface below, or by putting on a patch fitting into the cut. The new surface of the interior shall be called the *area* of the cut, the outer surface the place of the cut, while the cut and its area, as a single whole, shall be called the hole ~~wound~~ of the cut" (R S-28, 1903 [42]).

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There is an obvious translation from $\mathfrak{G}_\alpha$ to $\mathcal{L}_C$. For any graph $G \in \mathcal{G}_\alpha$, define its translation $\pi(G)$ inductively as follows:

$$\pi(p) = p \qquad \pi(\mathsf{SA}) = \top$$
$$\pi(G_1 G_2) = \pi(G_1) \wedge \pi(G_2) \qquad \pi(\textcircled{G}) = \overline{\pi(G)}.$$

Clearly $\pi(G \otimes H) = \pi(G) \vee \pi(H)$ and $\pi(G \supset H) = \pi(G) \to \pi(H)$.

A graph is constructed from primitive graphs using cuts. Then a natural notion is a partial graph as a part of a graph. Peirce mentioned the concept of a partial graph. The 'entire graph' is everything that is scribed on the sheet of assertion. A *partial graph* is any graph scribed alone, or in the presence of other graphs (R 450, 1903 [40]). Peirce did not present any precise definition of partial graphs.[16] This can be done by defining the parsing tree of an alpha graph.

DEFINITION 4.2. *For any graph G, the* parsing tree *of G, denoted by $T(G)$, is defined inductively as follows*:

(1) *$T(p)$ is a single root node p.*
(2) *$T(\mathsf{SA})$ is a single root node* SA.
(3) *$T(G_1 G_2)$ is a root node $G_1 G_2$ with child nodes $T(G_1)$ and $T(G_2)$.*
(4) *$T(\textcircled{G})$ is a root node $\textcircled{G}$ with one child node $T(G)$.*

A partial graph *of a graph G is a node in $T(G)$.*

DEFINITION 4.3. *For any graph G, the* history *of a node J in $T(G)$, denoted by $h(J)$, is the unique path from the root to J. We say that J is a positive* (*negative*) *node of $T(G)$ if there is an even* (*odd*) *number of cuts in $h(J)$.*

The notation $G[H]$ stands for that H is a node in $T(G)$. An occurrence of a graph J is positive (negative) *in G, notation $G^+[J]$ ($G^-[J]$), if it is a positive* (*negative*) *node in $T(G)$.*

Given a graph G with $H \in T(G)$, for any graph J, let $G[H/J]$ be the graph obtained from G by substituting J for H. Note that the substitution is not uniform. It is operated on any node of $T(G)$.

A *position* is a point on the area of a graph (but not on the cut line). Given any graph G, a position in G is *positive* (*negative*) if it is enclosed by an even (odd) number of cuts. A graph can be scribed at a position.[17]

---

[16] But observe Peirce's definition of the *nest* in R 650 (1910 [48]), effectively translating alpha graphs into that of a tree structure, as well as his CONVENTION NO. 2: "Graphs on different parts of the sheet, called *partial graphs*, shall independently assert what they would severally assert, were each the entire graph" (R 492, 1903); "An *entire graph* is a graph on a sheet otherwise blank. A *partial graph* is a part (possibly the whole) of an entire graph, which part if it were alone upon the sheet would have a meaning as a diagram" (R 491, 1903 [37, 38]), and "*Partial graphs* are graphs on the sheet of assertion that are not the only ones scribed on the sheet. The *entire graph* is all that is scribed on the sheet. The *total graph* is the entire graph together with the sheet of assertion itself" (R S-27, 1903 [41]).

[17] In Peirce's terms: "Every expression of a proposition in conformity with the conventions of this system is called an *existential graph*, or for brevity, a *graph* (although there are other kinds of graphs). Since it is sometimes awkward to say that a graph is *written* and it is sometimes awkward to say it is *drawn*, I will always say it is *scribed*. A graph scribed on the sheet of assent is said to be *accepted*. We must distinguish carefully between the *graph* and its different

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4.2. The system α. Peirce's alpha is a system defined by graph rules. A graph rule is of the form

$$\frac{G_1 \dots G_n}{G_0},$$

where $G_0, \dots, G_n$ are graphs. The graph $G_1 \dots G_n$ is called the *premiss*, and G_0 is called the conclusion. Now we can present Peirce's alpha system.

On the sheet of assertion, the rules of commutativity, associativity and conjunction can be expressed as follows:

$$\frac{G[H_1 H_2]}{G[H_2 H_1]}\ (\text{CM}) \quad \frac{G_1(G_2 G_3)}{(G_1 G_2)G_3}\ (\text{AS}) \quad \frac{G \qquad H}{G\,H}\ (\text{Jux}).$$

The commutativity rule (CM) says that the positions of H_1 and H_2 in a partial graph $H_1 H_2$ of G is immaterial. The associativity is concerned with the order of drawing graphs on the sheet of assertion. The parentheses in (AS) indicate the order of forming the graphs. The rule (AS) says that the order of forming graphs is immaterial. The rule (Jux) says that, if G and H are derived, then the juxtaposition $G\,H$ can be derived. When we have G and H on the sheet of assertion, we obtain $G\,H$.

However, when representing alpha graphs in their two-dimensional form, the commutativity, associativity and conjunction fall naturally from the properties of the space and needs no independent statement in the graph system. In any proof of a graph, we can identify the graphs obtained by the three rules (CM), (AS) and (Jux).[18]

DEFINITION 4.4. *The system α consists of the following axioms and rules*:

(1) *Axiom*: SA

(2) *Deletion rule*:

$$\frac{G^+[H]}{G^+[\mathsf{SA}]}\ (\text{DR}).$$

Every positive partial graph H in a graph G can be deleted.

(3) *Insertion rule*:

$$\frac{G^-[\mathsf{SA}]}{G^-[H]}\ (\text{IR}).$$

Any graph can be inserted into a negative position in a graph G.

(4) *Double negation rule*:

$$\frac{G[H]}{G[\textcircled{\textcircled{H}}]}\ (\text{DN}).$$

replicas. It is the *graph* which is accepted; and the graph is scribed when a *replica* of it is scribed" (R 450 [40]).

[18] Peirce notes: "Previously to 1880, the associative and commutative principles were regarded as the fundamental characters of copulation, along with a distributive principle. I proved the two first from the above definition; but Rule II shows them in their true light, as mere negations which in no way touch upon the real essence of composition. It is no more necessary to refer to them than it is to state that the graphs can be written in blue ink, if desired" (R 530, 1904 [44]). His Rule II is that "detached graphs may be inserted or erased, as if no others were on the sheet, and that regardless of distance and of direction" (*ibid.*).

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The double line means that the upper graph can derive the lower graph, and vice versa. (*DN*) *means that any partial graph H of a graph G can be replaced by the double cut of H and any double cut in G can be deleted.*

(5) *Iteration/deiteration rule*:

$$\frac{K[GH[J]]}{K[GH[J/GJ]]}(\text{IT}) \quad \frac{K[GH[J/GJ]]}{K[GH[J]]}(\text{DIT}).$$

(*IT*) *means that, in any graph $K[GH]$, the partial graph G can be iterated at any position in H.* (*DIT*) *is the converse of* (*IT*).

A *proof* of a graph G in α is a finite sequence of graphs $G_0, \ldots, G_n$ such that $G_n = G$, and each G_i is either an axiom, or derived from previous graphs by a rule. A graph G is *provable* in α, notation $\vdash_\alpha G$, if it has a derivation in α. A graph rule is *provable* in α if the conclusion is provable whenever the premiss is provable in α.

REMARK 4.5. The alpha system is essentially a sequent calculus, because it is a calculus of the consequence relation which is shown in the rules. The axiom SA is not a real axiom of the form in the system **PC**: the blank is nothing but the sheet of assertion. The rules in alpha are stated in terms of the *deep structure* of a graph: the position of a partial graph scribed in an entire graph. This is closely and interestingly related to Brünnler's deep sequent calculus for classical propositional logic [8]. In deep sequent calculus, derivation starts from SA which stands for the tautology. In α, the derivability of a graph is reduced to the transformations in deep structures by introducing or removing a partial graph and the double negation.

PROPOSITION 4.6. *The following graphs and rules are provable in α*:

(1) *Blank*: $G \supset \mathsf{SA}$.
(2) *Identity*: $G \supset G$.
(3) $G \supset GG$.
(4) *The law of excluded middle*: $G \mathbin{\circledcirc} \textcircled{G}$.
(5) $G_1\, G_2 \supset G_i$ *for* $i = 1, 2$.
(6) *De Morgan rules*:

$$\frac{\textcircled{GH}}{\textcircled{G} \mathbin{\circledcirc} \textcircled{H}}(\text{DM1}) \quad \frac{\textcircled{G \mathbin{\circledcirc} H}}{\textcircled{G}\,\textcircled{H}}(\text{DM2}).$$

(7) $G_i \supset G_1 \mathbin{\circledcirc} G_2$ *for* $i = 1, 2$.
(8) *Contraposition and transitivity rules*:

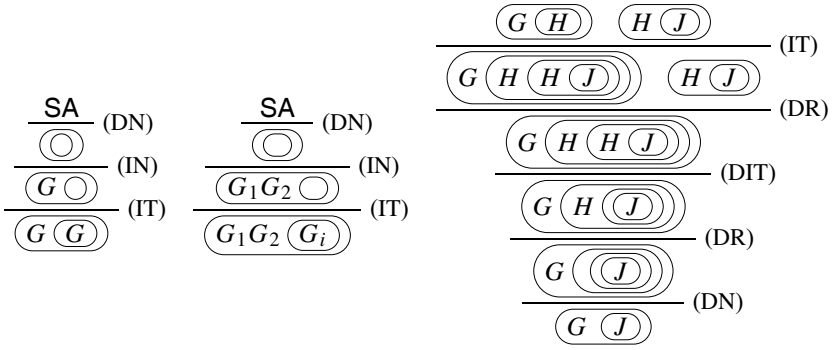
$$\frac{G \supset H}{\textcircled{H} \supset \textcircled{G}}(\text{CP}) \quad \frac{G \supset H \quad H \supset J}{G \supset J}(\text{TR}).$$

(9) *Prefixing and Modus Ponens*:

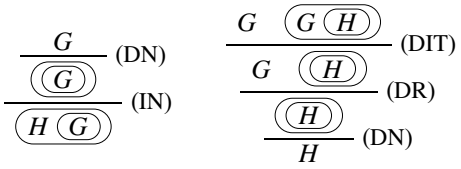
$$\frac{G}{H \supset G}(\text{PF}) \quad \frac{G \quad G \supset H}{H}(\text{MP}).$$

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Proof. (1) is obtained from SA by (DN) and (IN). (3) is obtained from (2) by (IT). (4) is obtained from (2) by (DN). De Morgan rules and (CP) are straightforward by (DN). (8) is obtained by (DN) and De Morgan rules. The proofs of (2), (5) and (Tr) are given below, respectively:



For (9), (PF) and (MP) are shown as follows:



This completes the proof. □

REMARK 4.7. In R 417 (1893 [28]) Peirce reproduces the modus ponens (MP) in the algebra of the copula using iteration and deletion in the following fashion: from t and $\bar{t} \Psi n$, compound first the premises: $t \cdot (\bar{t} \Psi n)$. Then infer (by iteration and deletion) $t \cdot \bar{t} \Psi n$, which is the same as $0 \Psi n$, and which now gives n.

PROPOSITION 4.8. *The following transformation rules are derivable in* α:

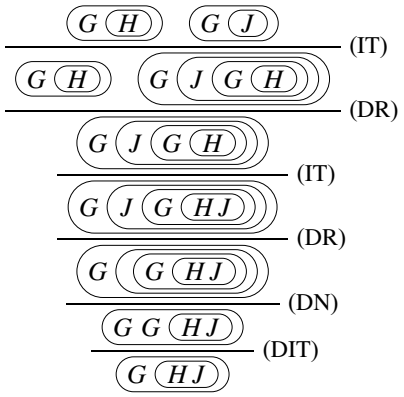
$$\frac{G_i \supset H}{G_1 G_2 \supset H}(\&\mathrm{L}) \quad \frac{G \supset H \quad G \supset J}{G \supset HJ}(\&\mathrm{R})$$

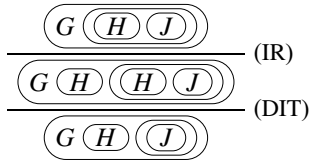
$$\frac{G \supset J \quad H \supset J}{G \circledcirc H \supset J}(\circledcirc\mathrm{L}) \quad \frac{G \supset H_i}{G \supset H_1 \circledcirc H_2}(\circledcirc\mathrm{L})$$

$$\frac{G \supset H \circledcirc J}{(H)\, G \supset J}(\mathrm{NL}) \quad \frac{GH \supset J}{H \supset (G) \circledcirc J}(\mathrm{NR}).$$

Proof. The rule (&L) is obtained by (IN) directly. The proofs of (&R) and (NL) are as follows:

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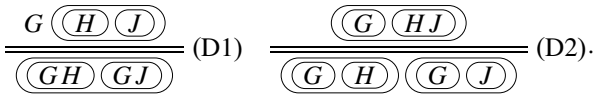


The remaining rules can be derived easily. □

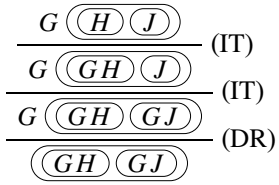
COROLLARY 4.9. *The following rules are provable in α*:

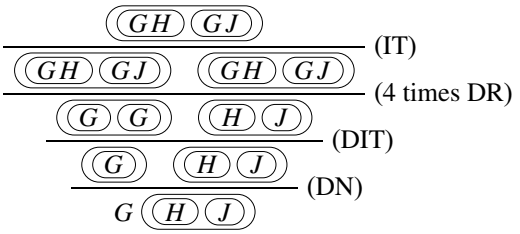
$$\frac{G \supset H \quad J \supset K}{GJ \supset HK}\ (\&) \qquad \frac{G \supset H \quad J \supset K}{(G \mathbin{\textcircled{v}} J) \supset (H \mathbin{\textcircled{v}} K)}\ (\textcircled{v}).$$

PROPOSITION 4.10. *The following distributivity rules are provable in α*:



Proof. For (D1), we have the following proofs:





(D2) is shown similarly. □

REMARK 4.11. The proof of distributivity in the alpha calculus is short. In the last proof above from $\textcircled{GH}\ \textcircled{GJ}$ to $G\ \textcircled{H}\ \textcircled{J}$, the premise is first duplicated by iteration. Peirce had noted in R 418 how this 'nonsyllogistic' operation of reusing the premise on the same area of the sheet which he had discovered in 1893 can be eminently useful: "The same premise may be written in more than once... the student of exclusively nonrelative logic is quite unaware that anything can be gained by bringing in again a premise already used. Ordinary syllogistic gives no hint of such a thing; nay seems rather opposed to it" (R 418 [29]). The alpha proof of (D1) is exactly the graphical version of his 1893 proof of distribution as given in R 418.

In the previous chapter (R 417), Peirce had produced a nearly equivalent proof of (D1), which does use the rule of iteration but without the initial duplication. There he stated that proof to have been what he always intended his demonstration of distributivity to be. He must also have meant his 1880 statement of the proof, namely the "easy but tedious" one. However, in the 1880 system he does not yet have the rule of iteration at work, which casts some doubt on the credibility of the claimed universality of his 1893 demonstration.

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PROPOSITION 4.12. *The following rules are provable in* α:

$$\frac{GH \supset J}{G \supset (H \supset J)}(\text{RG1}) \quad \frac{G \supset (H \supset J)}{GH \supset J}(\text{RG2}).$$

Proof. (RG1) is easily obtained by (NR) and (DN). (RG2) is obtained by (DN). □

The rules (RG1) and (RG2) are the graphical correspondents of Peirce's leading principle. In the alpha system, Peirce's leading principle becomes provable. For (RG1), the proof only needs one application of (DN) that introduces a double cut. For (RG2), the proof needs one application of (DN) to delete a double cut. One can therefore conclude that the validity of Peirce's Rule is in the alpha system a matter of immediate observation.

4.3. Peirce's* PC *and the* α*-system. Peirce's 1880 sequent calculus **PC** is a calculus for Boolean algebras. If we take these terms x, y, z etc. to be formulas in the propositional language $\mathcal{L}_C$, we get a sequent calculus for classical propositional logic. Now we can prove the completeness of the system α with respect to **PC**.

Given any graph G, we have the sequent $\top \Rightarrow \pi(G)$. For any term x, we can define the graph $\rho(x)$ inductively as follows:

$$\rho(p) = p, \quad \rho(\top) = \mathsf{SA}, \quad \rho(\overline{x}) = \overline{(\rho(x))}, \quad \rho(x \wedge y) = \rho(x)\,\rho(y).$$

For any sequent $x \Rightarrow y$, we have the graph $\rho(x) \supset \rho(y)$.

THEOREM 4.13. *If* $\vdash_{\mathbf{PC}} x \Rightarrow y$, *then* $\vdash_\alpha \rho(x) \supset \rho(y)$.

Proof. By induction on the derivation of $x \Rightarrow y$ in **PC**. If $x \Rightarrow y$ is an axiom, obviously $\vdash_\alpha \rho(x) \supset \rho(y)$. Assume that $x \Rightarrow y$ is obtained by (R). We get the conclusion by induction hypothesis. For example, let (R) be ($\vee$I). By induction hypothesis, we have $\vdash_\alpha \rho(x_1) \supset \rho(z)$ and $\vdash_\alpha \rho(x_2) \supset \rho(z)$. By ($\otimledown$L), $\rho(x_1) \otimledown \rho(x_2) \supset \rho(z)$. Note that $\rho(x_1) \otimledown \rho(x_2) = \rho(x_1 \vee x_2)$. □

LEMMA 4.14. *For any graphs* G, H, J, *the following hold*:

(1) *if* $G^+[H]$ *and* $\vdash_\alpha H \supset J$, *then* $\vdash_\alpha G[H] \supset G[H/J]$.
(2) *if* $G^-[H]$ *and* $\vdash_\alpha H \supset J$, *then* $\vdash_\alpha G[H/J] \supset G[H]$.
(3) *if* $\vdash_\alpha G \equiv H$, *then* $\vdash_\alpha J \equiv J[G/H]$.

Proof. By induction on the construction of G we show (1) and (2) simultaneously. The case for primitive graphs is obvious because H is positive in G. Assume that $G = \overline{(G')}$. We have two cases according to the position of H in G.

Case 1. $G' = HG''$. Then $G^-[H]$. Assume $\vdash_\alpha H \supset J$. Then we have

$$\dfrac{\dfrac{\overline{(H\,\overline{(J)})}}{\overline{(HG''\,\overline{(JG'')})}}\,(\text{IN, IT})}{\overline{(\overline{(HG'')}\,\overline{(JG'')})}}\,(\text{DN}).$$

Case 2. $G' = G'[H]$. For (1), assume $G^+[H]$ and $\vdash_\alpha H \supset J$. Then $G'^-[H]$. By induction hypothesis, $\vdash_\alpha G'[H/J] \supset G'[H]$. By (CP), we get $G[H] \supset G[H/J]$. For (2), assume $G^-[H]$ and $\vdash_\alpha H \supset J$. Then $G'^+[H]$. By induction hypothesis, $\vdash_\alpha G'[H] \supset G'[H/J]$. By (CP), we get $G[H/J] \supset G[H]$.

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For (3), we prove it by induction on the construction of J. If J is primitive, the conclusion is obvious. Suppose that $J = \overbrace{J'[G]}$. By induction hypothesis, $\vdash_\alpha J'[G] \equiv J'[G/H]$. By (DN), $\vdash_\alpha J \equiv J[G/H]$. □

By the translation π, if $G^+[H]$, then $\pi(G[H]) = \pi(G)^+[\pi(H)]$, namely, the term $\pi(H)$ is positive in $\pi(G)$. Similarly, if $G^-[H]$, then $\pi(G[H]) = \pi(G)^-[\pi(H)]$, namely, the term $\pi(H)$ is negative in $\pi(G)$.

For any sequent $x \Rightarrow y$, we stipulate that x is negative and y is positive. Then the polarity of each occurrence of a subformula in the sequent can be calculated according to the following rules:

(1) If $x \wedge y$ is positive (negative), then x and y are positive (negative).
(2) If $\overline{x}$ is positive (negative), then x is negative (positive).

By $x^+[y]$ ($x^-[y]$) we mean that y is positive (negative) in x. Then we have the following result about the polarity of terms.

LEMMA 4.15. *For any sequent $x \Rightarrow y$, the following hold*:

(1) *If $x^+[z]$ and $\vdash z \Rightarrow u$, then $\vdash_{\mathbf{PC}} x[z] \Rightarrow x[z/u]$.*
(2) *If $x^-[z]$ and $\vdash z \Rightarrow u$, then $\vdash_{\mathbf{PC}} x[z/u] \Rightarrow x[z]$.*
(3) *If $\vdash_{\mathbf{PC}} x = y$, then $\vdash_{\mathbf{PC}} z = z[x/y]$.*
(4) $\vdash_{\mathbf{PC}} x \wedge y[z] = x \wedge y[x \wedge z]$.

Proof. By induction on x we show (1) and (2) simultaneously. (3) is easily shown by induction on z. □

THEOREM 4.16. *If $\vdash_\alpha G \supset H$, then $\vdash_{\mathbf{PC}} \pi(G) \Rightarrow \pi(H)$.*

Proof. By induction on the proof of $G \supset H$ in α. If it is the blank, obviously $\vdash_{\mathbf{PC}} \top \Rightarrow \top$. Assume that $G \supset H$ is obtained by (R). We have the following cases:

(1) (R) is (DR). Let the deleted graph be J. There are only two cases according to the position of J in $G \supset H$.

(1.1) $G^-[J]$. By induction hypothesis, $\vdash_{\mathbf{PC}} \pi(G)^-[\pi(J)]) \Rightarrow \pi(H)$. Clearly $\vdash_{\mathbf{PC}} \pi(J) \Rightarrow \top$. By Lemma 4.15(2), $\vdash_{\mathbf{PC}} \pi(G) \Rightarrow \pi(G)^-[\pi(J)])$. By (TR), $\vdash_{\mathbf{PC}} \pi(G) \Rightarrow \pi(H)$.

(1.2) $H^+[J]$. By induction hypothesis, $\vdash_{\mathbf{PC}} \pi(G) \Rightarrow \pi(H)^+[\pi(J)]$. Clearly $\vdash_{\mathbf{PC}} \pi(J) \Rightarrow \top$. By Lemma 4.15(1), $\vdash_{\mathbf{PC}} \pi(H) \Rightarrow \pi(H)^+[\pi(J)]$. By (TR), $\vdash_{\mathbf{PC}} \pi(G) \Rightarrow \pi(H)^+[\pi(J)]$.

(2) (R) is (DR). Let the inserted graph be J. Then we can get the conclusion by induction hypothesis according to the polarity of J. The proof is similar with (1).

(3) (R) is (DN). We get the result by induction hypothesis and Lemma 4.15(3).

(4) (R) is (IT) or (DIT). We show the case of (IT). The proof of the case (DIT) is similar. Let the iterated graph be J. Assume that J is iterated in G and $G = G_1[JK_2[J']]$. By induction hypothesis, $\vdash_{\mathbf{PC}} \pi(K_1)[\pi(J) \wedge \pi(K_2)[\pi(J')]] \Rightarrow \pi(H)$. By Lemma 4.15(3), $\vdash_{\mathbf{PC}} \pi(J) \wedge \pi(K_2)[\pi(J')] = \pi(J) \wedge \pi(K_2)[\pi(J) \wedge \pi(J')]$. Hence $\vdash_{\mathbf{PC}} \pi(K_1)[\pi(J) \wedge \pi(K_2)[\pi(J) \wedge \pi(J')]] \Rightarrow \pi(H)$. □

THEOREM 4.17. *For any graph G, $\vdash_\alpha G$ if and only if $\vdash_{\mathbf{PC}} \top \Rightarrow \pi(G)$.*

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Proof. Assume $\vdash_\alpha G$. Then $\vdash_\alpha \top \supset G$. By Theorem 4.16, $\vdash_{\mathbf{PC}} \top \Rightarrow\supset \pi(G)$. Conversely, assume $\vdash_{\mathbf{PC}} \top \Rightarrow\supset \pi(G)$. By Theorem 4.13, $\vdash_\alpha \top \supset \rho(\pi(G))$. Then $\vdash_\alpha \top \supset G$. By (DN), $\vdash_\alpha G$. □

4.4. Algebraic semantics for α. In this subsection, we present an algebraic semantics for the system α. First of all, the language $\mathfrak{G}_\alpha$ can be interpreted in Boolean algebras. For any assignment θ in a Boolean algebra B, define $\widehat{\theta}(G) = \widehat{\theta}(\pi(G))$. We say that a graph G is *valid* in a Boolean algebra $\mathbf{B}$, notation $\mathbf{B} \models G$, if $\mathbf{B} \models \pi(G)$. The notation $\mathbf{BA} \models G$ means that G is valid in all Boolean algebras.

THEOREM 4.18. *For any graph G, $\vdash_\alpha G$ if and only if $\mathbf{BA} \models G$.*

Proof. The soundness part is shown by induction on a proof of G in α. Conversely, assume $\nvdash_\alpha G$. By Theorem 4.17, $\nvdash_{\mathbf{PC}} \top \Rightarrow \pi(G)$. Then by the completeness of **PC**, we have $\mathbf{B} \not\models \top \Rightarrow \pi(G)$ for some Boolean algebra $\mathbf{B}$. Then $\mathbf{B} \not\models G$. □

It is well-known that Boolean algebras characterize classical propositional logic. It follows that the system α axiomatizes classical propositional logic. There is a direct algebraic analysis of the system α. We now define *alpha algebras* for the language $\mathfrak{G}_\alpha$.

DEFINITION 4.19. *An* alpha algebra *is an algebra $\mathbf{A} = (A, \wedge, \neg, 1)$ satisfying the following axioms: for any $a, b \in A$,*

(*1*) $a \wedge b \leq a$	(*Deletion*)
(*2*) $\neg a \leq \neg(a \wedge b)$	(*Insertion*)
(*3*) $\neg\neg a = a$	(*Double negation*)
(*4*) $a \wedge \neg b = a \wedge \neg(a \wedge b)$	(*Iteration*)
(*5*) $a \wedge \neg(\neg b \wedge \neg c) \leq a \wedge \neg(\neg(a \wedge b) \wedge \neg(a \wedge c))$	(*Distributivity*)
(*6*) $a \wedge b = b \wedge a$	(*Commutativity*)
(*7*) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$	(*Associativity*)
(*8*) $a \wedge \neg a = \neg 1$	(*The law of contradiction*)

where $\leq$ is a binary relation on A defined as follows: $a \leq b$ if and only if $a \wedge b = a$. For any alpha algebra $\mathbf{A}$, define $a \vee b := \neg(\neg a \wedge \neg b)$ and $0 := \neg 1$.

LEMMA 4.20. *Let $\leq$ be the order on A in an alpha algebra $\mathbf{A}$. The following hold for all $a, b, c, d \in A$:*

(1) *$\leq$ is a partial order on A.*
(2) *If $a \leq b$ and $a \leq c$, then $a \leq b \wedge c$.*
(3) *$a = a \wedge a$ (Idempotency).*
(4) *If $a \leq b$ and $c \leq d$, then $a \wedge c \leq b \wedge d$.*
(5) *$a \vee b = b \vee a$, $a \vee (b \vee c) = (a \vee b) \vee c$ and $a \vee a = a$.*
(6) *$a \wedge 1 = a$, $a \wedge 0 = 0$, $a \vee 0 = a$ and $a \vee 1 = 1$. Hence $a \leq 1$ and $0 \leq a$.*
(7) *$a = a \wedge (a \vee b)$ and $a = a \vee (a \wedge b)$. Hence $a \leq a \vee b$ and $b \leq a \vee b$.*
(8) *$a \leq b$ if and only if $\neg b \leq \neg a$.*
(9) *If $a \leq b$ and $c \leq d$, then $a \vee c \leq b \vee d$.*

Proof. We show only (1), (7), and (8). The remaining items are shown easily.

(1) Obviously $\leq$ is reflexive. For transitivity, assume $a \leq b$ and $b \leq c$. Then we have $a \wedge b = a$ and $b \wedge c = b$. Then $a \wedge c = (a \wedge b) \wedge c = a \wedge (b \wedge c) = a \wedge b = a$. Hence

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$a \leq c$. For anti-symmetry, assume $a \leq b$ and $b \leq a$. Then $a \wedge b = a$ and $b \wedge a = b$. Hence $a = b$.

(7) We have $a \wedge (a \vee b) = a \wedge \neg(\neg a \wedge \neg b) = a \wedge \neg(a \wedge \neg a \wedge \neg b) = a \wedge \neg(\neg 1 \wedge b) = a \wedge \neg\neg 1 = a \wedge 1 = a$. Second, $a \vee (a \wedge b) = \neg(\neg a \wedge \neg(a \wedge b)) = \neg(\neg a \wedge \neg(\neg a \wedge a \wedge b)) = \neg(\neg a \wedge \neg(\neg 1 \wedge b)) = \neg(\neg a \wedge \neg\neg 1) = \neg\neg a = a$.

(8) Assume $a \leq b$. Then $\neg b \wedge \neg a = \neg b \wedge \neg(b \wedge a) = \neg b \wedge \neg(\neg b \wedge b \wedge a) = \neg b \wedge \neg\neg 1 = \neg b$. Hence $\neg b \leq \neg a$. Conversely, assume $\neg b \leq \neg a$. By the 'only if' part, $\neg\neg a \leq \neg\neg b$. By double negation, $a \leq b$. □

THEOREM 4.21. *For any alpha algebra* $\mathbf{A} = (A, \wedge, \neg, 1)$*, the algebra* $(A, \wedge, \vee, \neg, 0, 1)$ *is a Boolean algebra. Conversely, for any Boolean algebra* $\mathbf{B} = (B, \wedge, \vee, \neg, 0, 1)$*, the* $(\wedge, \neg, 0, 1)$*-reduct of* $\mathbf{B}$ *is an alpha algebra.*

Proof. By Lemma 4.20, the algebra $(A, \wedge, \vee, \neg, 0, 1)$ is a Boolean algebra. □

The completeness of α with respect to alpha algebras can be obtained by the Lindenbaum–Tarski construction. Define the relation $\sim_\alpha$ on $\mathcal{G}_\alpha$ by:

$$G \sim_\alpha H \text{ if and only if } \vdash_\alpha G \equiv H.$$

The relation $\sim_\alpha$ is a congruence relation. Then we can define the Lindenbaum–Tarski algebra and show the following completeness theorem:

THEOREM 4.22. *For any graph* G*,* $\vdash_\alpha G$ *iff* $\mathbf{A} \models G$ *for all alpha algebras* $\mathbf{A}$.

This is our interpretation of what Peirce meant when he told Ladd–Franklin that logical graphs have a bearing on nonrelative logic "in one highly important particular, that they supply an entirely new system of fundamental assumptions to logical algebra" (Peirce to Ladd–Franklin, November 9, 1900). Our alpha algebra $\mathbf{A}$ presents a new system of such fundamental assumptions.

§5. Conclusions. During his long career, Peirce created a number of logical systems that coincide with Boole's algebra. One direction which we leave for the future work is to investigate under which minimal modifications his principles and rules would permit nonclassical calculi and graphs, such as intuitionistic and bi-intuitionistic ones, closure algebras, positive implicational fragments of propositional logic and other sub-propositional logics, or even those of substructural systems. A closure algebra, for example, has an operator which arises from having another, strong cut in the system to interact with the classical, contradictory negation as a modal closure operator. Implicational fragments arise from omitting the fourth icon that Peirce used in 1885 in order to introduce a negation to the implicational system. Substructural logics, such as resource-bounded linear logics, would necessitate specific and so far unknown modifications to the structural rules of the graphical sequent calculus, such as taking the sheet of assertion to be a vector space.

As far as the basic, classical calculi are concerned, Peirce repeatedly credited his former student Oscar H. Mitchell for making him see that logical inferences are fundamentally based on only two kinds of operations, insertions and omissions:

> The RULE OF OMISSION, in the form which it takes in the Universal Algebra of Logic, is that *Any component of any term of the Boolian can be struck out, provided some component remain*,—a rule first given by O. C. [sic., H.] Mitchell. The correlative RULE OF INSERTION is that

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> *Any term may have any term inserted as an integrant part of it*. Perhaps this had likewise never been stated before O. C. Mitchell. (R 253, 1905 [46])

Mitchell's doctoral thesis, which Peirce supervised, was published in *Studies in Logic* in 1883 [19]. Peirce admitted that he had not yet grasped its full import when working on his 1885 article.[19]

The major transition away from his investigations on the algebraic aspects of logic thus took place not when he set out to produce logical calculi as such, but when he had achieved the most *analytic* versions of those systems. The most analytic systems are those of logical graphs:

> That I never considered that algebra to be a great masterpiece is sufficiently shown by my giving my exposition of it no other title than 'Note B'. The perusal of Schröder's book convinced me that the algebra was not what was wanted, and in the Monist for January 1897 I produced a system of graphs which I now term Entitative Graphs. I shortly after abandoned that and took up Existential Graphs. (R 467, 1903 [43])

It is these graphical systems that eventually carry out the job of logical analysis, as they can represent the processes of reasoning with the minimal amount of operations: generically just the operations of insertions and omissions suffice for completeness. "My system of 'Existential Graphs"', Peirce explains in hindsight, "puts in a clearer light a truth first virtually enunciated by my student (afterward professor) O. H. Mitchell". Mitchell showed that deductive reasoning "can always be reached by *adding to the stated antecedents and subtracting from stated consequents*, being understood that if an antecedent be itself a conditional proposition, its antecedent is of the nature of a consequent" (R 905, 1908 [47]). These two operations that adequately characterize logically necessary reasoning are much exploited in modern systems of proof sequents.[20]

The invention of graphical systems of logic was preset by Peirce's calculi for the algebra of the copula. A further push took place when Peirce switched to $\therefore$ as the sign of illation. The latter denoted the scope of the antecedent of the copula. From this innovation it was

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[19] He did acknowledge, in 1884, that in the elementary operation of logical thought it is important to have "two modes of changing the assertions" (W4, p. 107). On the other hand, Mitchell studied closely Peirce's 1880 Algebra of Logic article. A reprint with both Peirce's and Mitchell's abundant corrections and additions to it is preserved at Marietta College Special Collections.

[20] That Peirce's example of logically necessary reasoning was indeed a manipulation of sequents comes clear in the related passage:

> I mean that what is ordinarily called a "logically necessary reasoning" is one in which the substance of the conclusion can be reached from the copulate premiss by such changes as those which convert "If A be true and if it be true that if C and D are true, E is true; then it is true that F and G are true", or, what is the same thing, by changes that convert "If A is true then either C and D may be true though E is false or else F and G are true", into "If A and X is true then either C may be true though E and Y are not both true or else F is true. 
>
> (R 905 [47])

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then only a stone's throw to the idea of the cuts, the ovals that encircle graph-instances, and thus to the denial. Even before all this, he had proposed the iteration/deiteration rules in the 1893 algebra of the copula in which $\dashv$ was used as the sign of consequence.

Conversely, one can ask how do the logical graphs affect algebras. As we have shown here, graphical systems admit of a reinterpretation in terms of algebraic rules. This answers the enigmatic remark in Peirce's reply to Ladd–Franklin's letter, where he wrote that logical graphs have not much bearing on nonrelative logic, "*except* in one highly important particular, that they supply an entirely new system of fundamental assumptions to logical algebra" (Peirce to Ladd–Franklin, November 9, 1900). On this, Roberts [58] asked: Was justified Peirce to have such high confidence in the alpha system? Our answer is that besides being the most analytic of the logical systems ever devised, they *can* be reinterpreted as a complete algebraic system (our **A**), which contains the important rule of iteration/deiteration. As we have shown, **A** is nevertheless best to be understood and studied in semantical terms, wherefore its consequence relation agrees with the Boolean lattice order.

To summarize, our article presents Peirce's 1880 and 1885 logical calculi and showed that they are Boolean algebras. The full law of distributivity can be proved by using what we term Peirce's Rule, which is residuation. We emphasize that Peirce's calculi are those of sequents and not of natural deduction with hypotheses that need to be discharged. The central notion that characterizes his logical investigations is the relation of a consequence. A sequent calculus is a theory about the consequence relation. It was such sequent calculus that Peirce was developing since at least 1880 for systems that agree with Boolean algebras.

We also presented some of the key ideas concerning his transitory 1886–1893 algebras of the copula, and provided a new description of his 1896 alpha calculi for propositional logical graphs, showing how the rules of the latter reinterpret Boolean algebras. This completes Peirce's unfulfilled remark as to how the alpha system supplies an entirely new system of fundamental assumptions to logical algebra.

While Peirce managed to create and experiment with a number of interesting logical calculi, it is important in conclusion to recall that his overall aim was not to invent logical systems solely for their calculus-like or pasigraphic aspects that deal with notions of formal truth and validity. After all, logically necessary reasoning is not the be-all and end-all of all logical reasoning; it is not even all of what deductive reasoning encompasses. Deductive reasoning involves, among others, an application of the method of logical analysis. Logical analysis requires, according to Peirce, experimentation with systems of graphs. It may also involve altogether different, *abductive* principles of inference [55]. Peirce's ultimate goal was to erect a system of logic as a system of semeiotic that respects philosophical and ethical maxims of notation (R 253 [46], R 516 [36], R 530 [44]). It was to invent exact instruments that are historically conscious as well as the most appropriate for the purposes of the logical analysis of meanings. Indeed Peirce would soon move on to the graphical analysis of meanings in the modal part of the gamma graphs ([15, 16]).

Peirce's central method to perform logical analysis is to define the truth-conditions of signs (the logic proper, scriptibility). Yet the method must be allied with the formal study that underpins those conditions, namely to map out the exact behavior of the illative consequence relation. This surprisingly modern synergy is summarized in this paragraph:

> Logic, in the general sense, is another name for *semeiotic*, the necessary doctrine of signs; and *semiotic* forms a trivium, universal speculative grammar, logic, and universal speculative rhetoric. The purpose of signs is to convey the *truth*, and hence it is that logic proper, or the

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> science of the necessary conditions of the *truth* of signs, is the central and paramount doctrine of the trivium of semiotic. Now a representation, to be true by formal necessity, must *follow* from some other representation; and thus the illative relation becomes the relation of paramount importance in logic. (R 737, 1897 [32])

Calculi that agree with Boolean algebra are the simplest systems in which the conditions for a representation necessarily following from another can be completely characterized.

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