THE WAVE EQUATION IN CONFORMAL SPACE

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Introduction. In a recent paper Dirac† has shown that by passing from the ordinary Euclidean space to a four-dimensional conformal space, some of the equations of physics can be written in a tensor form, the indices of which take on six values. Those equations which can be written in this form are then invariant under conformal transformations of the Euclidean space. Among the equations of physics which have this more general invariance are the Maxwell equations, as was proved by a direct transformation a long time ago by Cunningham‡, and Bateman§, so that Dirac's paper provides an alternative and more general proof of this result. Certain errors|| in Dirac's paper, however, necessitate a reformulation of the proof. Before we do this in § 2, we briefly recapitulate in § 1 some of the general results derived there. In § 3 we investigate further the conformal invariance of the wave equation for an electron in the presence of a general electromagnetic field.

1. The space which we treat is the four-dimensional surface of a hyperquadric in five-dimensional projective space, and we use homogeneous coordinates x_μ in this five-dimensional space, where the index μ takes on all values from one to six¶. We assume that the equation of the hyperquadric is

$$x_\mu x_\mu = 0. \tag{1}$$

Any equation which is invariant in the conformal space must be invariant under any linear homogeneous transformation which leaves equation (1) unaltered, i.e. any transformation of the type

$$x^*_\mu = b_{\mu\nu} x_\nu,$$

where

$$b_{\mu\nu} b_{\mu\rho} = k\delta_{\nu\rho}. \tag{2}$$

To connect this space with a four-dimensional Euclidean space, we fix our attention on some point on the hyperquadric, say the point $(0, 0, 0, 0, 1, i)$, and assume it to correspond to the point at infinity in the Euclidean space.

The largest subgroup of the transformations (2) which leaves the point $(0, 0, 0, 0, 1, i)$ on the hyperquadric invariant then corresponds to translations and rotations of the Euclidean space combined with uniform expansions and

† P. A. M. Dirac, *Ann. of Math.* 37 (1936), 429.

‡ E. Cunningham, *Proc. Lond. Math. Soc.* (2), 8 (1909), 77.

§ H. Bateman, *Proc. Lond. Math. Soc.* (2), 8 (1910), 223.

|| See footnote, p. 624 below.

¶ Greek indices take on all integral values from one to six, Latin indices from one to four. We always sum over repeated suffices.

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contractions. Dirac has shown that the most general transformation of this subgroup is given by

$$\left.\begin{aligned} x_r^* &= a_{rs}x_s + a_r(x_5+ix_6),\\ x_5^*+ix_6^* &= \lambda(x_5+ix_6),\\ x_5^*-ix_6^* &= \lambda^{-1}\{x_5-ix_6-2a_r a_{rs}x_s - a_r a_r(x_5+ix_6)\},\end{aligned}\right\} \tag{3}$$

where

$$a_{rs}a_{rl}=\delta_{sl},$$

combined with the trivial transformation in which all the x_μ are multiplied by a constant. If we connect the coordinates y_r of the Euclidean space with the x_μ by the relation

$$y_r = \frac{x_r}{x_5+ix_6}, \tag{4}$$

it can easily be shown that (3) then corresponds to a transformation of the y_r given by

$$y_r^* = \lambda^{-1}(a_{rs}y_s + a_r). \tag{5}$$

Any transformation of the type (2) which does not leave the point (0, 0, 0, 0, 1, i) invariant then corresponds to a conformal transformation of the Euclidean space.

If $A(x)$ and $A_\mu(x)$ are a conformal scalar and vector† respectively, which are homogeneous functions of the x_μ of degree n, then A_0 and A_{r_0} defined by‡

$$\left.\begin{aligned} A_0 &= (x_5+ix_6)^{-n}A\\ \text{and}\quad A_{r_0} &= (x_5+ix_6)^{-n}\{A_r - y_r(A_5+iA_6)\}\end{aligned}\right\} \tag{6}$$

behave respectively like a Euclidean scalar and vector. More precisely under transformations of the type (3) with $\lambda=1$ they transform thus

$$\left.\begin{aligned} A_0^*(y^*) &= A_0(y),\\ A_r^* &= a_{rs}A_s.\end{aligned}\right\} \tag{7}$$

Similarly it may be shown that, if $A_{\mu\nu}$ is a conformal tensor of the second rank and a homogeneous function of the x_μ of degree n,

$$A_{rs_0} = (x_5+ix_6)^{-n}[A_{rs} - y_r(A_{5s}+iA_{6s}) - y_s(A_{r5}+iA_{r6}) + y_r y_s(A_{55}-A_{66}+iA_{56}+iA_{65})] \tag{8}$$

is a Euclidean tensor of the second rank.

Further, if A_0 is a Euclidean scalar of the type defined by (6), then Dirac has shown that

$$\frac{\partial A_0}{\partial y_r} = (x_5+ix_6)^{-n+1}\left[\frac{\partial A}{\partial x_r} - y_r\left(\frac{\partial A}{\partial x_5}+i\frac{\partial A}{\partial x_6}\right)\right]. \tag{9}$$

Since $\dfrac{\partial A}{\partial x_\mu}$ is a conformal vector, (6) shows that $\dfrac{\partial A_0}{\partial y_r}$ is a Euclidean vector as it must be.

† A conformal vector A_μ transforms by definition like x_μ, i.e. its transformation law is the same as (2). Tensors of higher rank transform correspondingly.

‡ Dirac, *loc. cit.*

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In a similar way it may be shown that

$$\frac{\partial^2 A_0}{\partial y_r^2} = (x_5 + ix_6)^{-n+2}\left[\frac{\partial^2 A}{\partial x_\mu^2} - \frac{2(n+1)}{x_5+ix_6}\left(\frac{\partial A}{\partial x_5} + i\frac{\partial A}{\partial x_6}\right)\right], \tag{10}$$

which holds only on the hyperquadric (1). Any given Euclidean scalar A_0 may be connected with a number of conformal scalars of different degree by the relation (6). If however we wish an equation of the type

$$\frac{\partial^2 A_0}{\partial y_r^2} = 0$$

to be conformally invariant, we must choose $n = -1$, since then the second term on the right-hand side of (10), which is not invariant under the most general transformation of the type (2), vanishes.

2. *The Maxwell equations.* 2.1. We introduce a vector potential A_μ to describe an electromagnetic field in the conformal space. It will appear below that we have to assume A_μ to be of degree -1. We assume further that the A_μ are such as to satisfy the equations

$$\frac{\partial}{\partial x_\nu}(x_\mu A_\mu) = 0 \tag{11}$$

on the hyperquadric.† It should be noticed that the equations (11) demand in general that the A_μ should be of degree -1, since it follows from Euler's equation that

$$x_\nu \frac{\partial}{\partial x_\nu}(x_\mu A_\mu) = (n+1)\, x_\mu A_\mu$$

and the left-hand side vanishes by (11). It follows that $n = -1$ except in the special case where $x_\mu A_\mu$ vanishes on the hyperquadric.

2.2. We introduce the electromagnetic field quantities defined in the usual way by

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}. \tag{12}$$

The antisymmetrical tensor $F_{\mu\nu}$ has fifteen different components. These satisfy the relations

$$\begin{aligned} x_\mu F_{\mu\nu} &= x_\mu \frac{\partial A_\nu}{\partial x_\mu} - x_\mu \frac{\partial A_\mu}{\partial x_\nu} \\ &= -A_\nu - x_\mu \frac{\partial A_\mu}{\partial x_\nu} \\ &= -\frac{\partial}{\partial x_\nu}(x_\mu A_\mu) = 0 \end{aligned} \tag{13}$$

by (11). There are only five such independent relations since $x_\mu x_\nu F_{\mu\nu}$ vanishes

† The equations (11) are necessary for the progress of our argument, as well as of Dirac's. Dirac does not assume the equations (11) explicitly, but proceeds to derive them from the two equations

$$x_\mu A_\mu = 0, \quad \frac{\partial A_\mu}{\partial x_\mu} = 0,$$

which he assumes to hold on the hyperquadric. An error has however occurred in his proof, and it may be shown that it is necessary to assume the conditions (11) explicitly.

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on account of symmetry, and hence any one of the relations (13) can be derived from the other five. The number of independent $F_{\mu\nu}$ is therefore only ten.

Now all our quantities have a physical meaning only on the hyperquadric (1). We may change their values off the hyperquadric without changing any physical result. The arbitrariness in the A_μ consists therefore in the possibility of adding to them terms of the type $B_\mu x_\nu x_\nu$. It then follows from (11) that the B_μ must satisfy the equations

$$\frac{\partial}{\partial x_\lambda}(x_\mu B_\mu x_\nu x_\nu) = 0$$

on the hyperquadric, i.e.

$$2x_\lambda x_\mu B_\mu + \left(B_\lambda + x_\mu \frac{\partial B_\mu}{\partial x_\lambda}\right) x_\nu x_\nu = 0,$$

which requires that $x_\mu B_\mu$ should vanish on the hyperquadric. $x_\mu B_\mu$ must then be of the form

$$x_\mu B_\mu = B x_\nu x_\nu, \tag{14}$$

where B is an arbitrary function of the x's.

The $F_{\mu\nu}$ then contain an arbitrary term

$$\frac{\partial}{\partial x_\mu}(B_\nu x_\lambda x_\lambda) - \frac{\partial}{\partial x_\nu}(B_\mu x_\lambda x_\lambda)$$

which has the value

$$C_{\mu\nu} \equiv 2(B_\nu x_\mu - B_\mu x_\nu) \tag{15}$$

on the hyperquadric. The $C_{\mu\nu}$ satisfy the identities

$$x_\rho C_{\mu\nu} + x_\mu C_{\nu\rho} + x_\nu C_{\rho\mu} = 0 \tag{16}$$

with arbitrary B_μ. Now we may suppose that all the components $F_{\mu\nu}$ in which one of the indices has the value six are expressed in terms of those in which the indices only take on values from one to five by using the equations (13). The ten remaining $F_{\mu\nu}$ are then arbitrary to the extent of additive terms $C_{\mu\nu}$ of the type (15). If we let the indices μ, ν run from one to five, then there are only four independent $C_{\mu\nu}$, since the others can be expressed in terms of them by the equations (16). For example, we may take the independent components to be C_{12}, C_{13}, C_{14}, C_{15}, and it is easily seen that any of the others can be expressed in terms of them by using (16). Thus there is a fourfold arbitrariness in the $F_{\mu\nu}$, and the number of independent components reduces to six, as in the ordinary Maxwell theory. It is easily seen by substitution that the addition of terms of the type (15) to $F_{\mu\nu}$ makes no difference to the potentials F_{rs}, in Euclidean space, which are connected with $F_{\mu\nu}$ by a relation of the type (8).

2.3. Introducing a vector j_μ in conformal space to represent the charge-current density, we may write the Maxwell equations in the conformally invariant form

$$\left.\begin{aligned} \frac{\partial F_{\mu\nu}}{\partial x_\mu} &= j_\nu, \\ \frac{\partial F_{\mu\nu}}{\partial x_\lambda} + \frac{\partial F_{\nu\lambda}}{\partial x_\mu} + \frac{\partial F_{\lambda\mu}}{\partial x_\nu} &= 0. \end{aligned}\right\} \tag{17}$$

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By (12), the first equation gives

$$\frac{\partial^2 A_\nu}{\partial x_\mu^2} - \frac{\partial}{\partial x_\nu}\frac{\partial A_\mu}{\partial x_\mu} = j_\nu. \tag{18}$$

The equations (18) may now be simplified by a gauge transformation of the usual type. We introduce new potentials A'_μ defined by

$$A'_\mu = A_\mu + \frac{\partial S}{\partial x_\mu}, \tag{19}$$

where S is a scalar of degree zero satisfying†

$$\frac{\partial^2 S}{\partial x_\mu^2} = -\frac{\partial A_\mu}{\partial x_\mu}.$$

The equations (18) then reduce to

$$\frac{\partial^2 A'_\nu}{\partial x_\mu^2} = j_\nu, \quad \frac{\partial A'_\mu}{\partial x_\mu} = 0. \tag{20}$$

As usual, the potentials are still arbitrary to the extent that we may add the gradient of any function S which satisfies

$$\frac{\partial^2 S}{\partial x_\mu^2} = 0. \tag{21}$$

As Dirac has shown, with S quite arbitrary in the hyperquadric, we may choose its values off the hyperquadric in such a way that (21) is satisfied on the hyperquadric. In fact, with any arbitrary S' on the hyperquadric, we may take‡

$$S = S' - \frac{1}{4}\frac{\partial^2 S'}{\partial x_\mu^2} x_\nu x_\nu. \tag{22}$$

This means that the gauge transformations of the present theory form a wider group than the gauge transformations of the ordinary Maxwell theory. We shall show presently that this wider freedom in the gauge transformations is essential in order to preserve the invariance under conformal transformations of the equations satisfied by the potentials in Euclidean space.

† S may even be taken of the form $x_\mu x_\mu S'$, so that the equations can be brought to the form (20) without changing the values of the potentials in Euclidean space.

‡ The first of equations (20) only has a meaning on the hyperquadric, since j_ν is only defined on the hyperquadric. We may, if we choose, use (17) to define j_ν in the rest of space. In this case the first of the equations (20) automatically holds in all space, from (18), provided that the second of equations (20) holds in the whole of space. It should be noticed that the right-hand side of (22) only satisfies (21) on the hyperquadric, so that such a transformation leaves the equations (20) unchanged only on the hyperquadric. The divergence of the A_μ does not vanish in general off the hyperquadric. However, by adding terms of the type $B_\mu x_\nu x_\nu$ to the A_μ, which leave the potentials in Euclidean space unaltered, we can make the second of the equations (20) hold in the whole of space. To do this we choose $x_\mu B_\mu$ to be of the form $Bx_\nu x_\nu$, with a suitable choice of B and the B_μ.

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2.4. To investigate this point further, we notice first that if A_{r_0} are the potentials in Euclidean space, connected with the A_μ by relations of the type (6), namely

$$A_{r_0} = (x_5 + ix_6)\{A_r - y_r(A_5 + iA_6)\},$$

the new potentials which we get when we make a gauge transformation of the type (19), with S given by (22), differ from the old ones in having added terms of the type

$$(x_5 + ix_6)\left\{\frac{\partial S}{\partial x_r} - y_r\left(\frac{\partial S}{\partial x_5} + i\frac{\partial S}{\partial x_6}\right)\right\} = \frac{\partial S}{\partial y_r},$$

by (9). It follows from this that

$$\frac{\partial A'_{r_0}}{\partial y_r} = \frac{\partial A_{r_0}}{\partial y_r} + \frac{\partial^2 S}{\partial y_r^2}, \tag{23}$$

and since, by (10),

$$\frac{\partial^2 S}{\partial y_r^2} = (x_5 + ix_6)^2\left[\frac{\partial^2 S}{\partial x_\mu^2} - \frac{2}{x_5 + ix_6}\left(\frac{\partial S}{\partial x_5} + i\frac{\partial S}{\partial x_6}\right)\right],$$

which does not vanish in general on the hyperquadric by (21), a transformation of the type (19) with S satisfying (21) changes the divergence of the potentials in Euclidean space. This is obvious, since S which satisfies (21) may be arbitrary on the hyperquadric.

We may, therefore, by a suitable choice of the A_μ arrange that the equations

$$\frac{\partial^2 A_{r_0}}{\partial y_s^2} = j_{r_0} \quad \text{and} \quad \frac{\partial A_{r_0}}{\partial y_r} = 0 \tag{24}$$

hold in Euclidean space. Here j_{r_0} is connected with j_μ by relations of the type (6). A general transformation in conformal space leaves the equations (20) invariant, but they are not sufficient to require the fulfilment of equations (24) by the transformed potentials in Euclidean space. We should in general have to make another gauge transformation of the type (19) with a suitable S satisfying (21) in order to bring the transformed equations (24) back to their original form.

In fact, a simple calculation shows that

$$\frac{\partial A_{r_0}}{\partial y_r} = (x_5 + ix_6)^2\left[\frac{\partial A_\mu}{\partial x_\mu} - \frac{2}{x_5 + ix_6}(A_5 + iA_6)\right] \tag{25}$$

on the hyperquadric. Since the right-hand side of (25) is not invariant under a general conformal transformation, it follows that the left-hand side is also not conformally invariant. If the A_μ satisfy (20), and $A_5 + iA_6$ vanishes on the hyperquadric, then the potentials in Euclidean space satisfy (24). A general conformal transformation, however, does not leave $A_5 + iA_6$ invariant, and in the transformed system it does not in general vanish, so that we have

$$\frac{\partial A^*_{r_0}}{\partial y^*_r} = (x^*_5 + ix^*_6)^2\left[-\frac{2}{x^*_5 + ix^*_6}(A^*_5 + iA^*_6)\right] \neq 0. \tag{26}$$

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To bring (26) back to the form (24), we choose S', a function of the y's, so as to make the left-hand side of (23) vanish by adding $\frac{\partial S'}{\partial y_r}$ to A_{r_0}, thus making the left-hand side of (26) vanish. In fact, S' is just a gauge transformation of the usual type in Euclidean space. We now form S from S' by (22). If we make a gauge transformation of the type (19) with this S, the left-hand side of (26) is reduced to zero, while $\frac{\partial A^*_\mu}{\partial x^*_\mu}$ remains zero on the hyperquadric by (21), so that we have in fact reduced $A^*_5 + iA^*_6$ to zero on the hyperquadric.

2.5. The Maxwell equations in Euclidean space, namely

$$\left.\begin{aligned} \frac{\partial F_{rs_0}}{\partial y_r} &= j_{s_0}, \\ \frac{\partial F_{rs_0}}{\partial y_t} + \frac{\partial F_{st_0}}{\partial y_r} + \frac{\partial F_{tr_0}}{\partial y_s} &= 0, \end{aligned}\right\} \tag{27}$$

are unaffected by the gauge transformations which we have been considering above, and their invariance under a conformal transformation follows at once from the conformal invariance of the equations (17). In fact, on the hyperquadric,

$$\begin{aligned} \frac{\partial F_{rs_0}}{\partial y_r} &= \frac{\partial^2 A_{s_0}}{\partial y_r^2} - \frac{\partial}{\partial y_s}\frac{\partial A_{r_0}}{\partial y_r} \\ &= (x_5 + ix_6)^3 \left[\frac{\partial^2 A_s}{\partial x_\mu^2} - y_s \frac{\partial^2}{\partial x_\mu^2}(A_5 + iA_6) - \frac{\partial}{\partial x_s}\frac{\partial A_\mu}{\partial x_\mu} + y_s \left(\frac{\partial}{\partial x_5} + i\frac{\partial}{\partial x_6}\right)\frac{\partial A_\mu}{\partial x_\mu}\right] \\ &= (x_5 + ix_6)^3 \left[\frac{\partial}{\partial x_\mu}\left\{\frac{\partial A_s}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_s}\right\} - y_s \frac{\partial}{\partial x_\mu}\left\{\frac{\partial}{\partial x_\mu}(A_5 + iA_6) - \left(\frac{\partial}{\partial x_5} + i\frac{\partial}{\partial x_6}\right)A_\mu\right\}\right] \\ &= (x_5 + ix_6)^3 \left[\frac{\partial F_{\mu s}}{\partial x_\mu} - y_s \frac{\partial}{\partial x_\mu}(F_{\mu 5} + iF_{\mu 6})\right]. \end{aligned} \tag{28}$$

(The term in $\frac{\partial}{\partial y_s}\frac{\partial A_{r_0}}{\partial y_r}$ which derives from the second term on the right-hand side of (25) just cancels a similar term which occurs in $\frac{\partial^2 A_{s_0}}{\partial y_r^2}$.) Since (28) holds on the hyperquadric in all systems, quite irrespective of whether the potentials satisfy (24) or not, we get in all systems, using (17), that

$$\frac{\partial F_{rs_0}}{\partial y_r} = (x_5 + ix_6)^3 \{j_s - y_s (j_5 + ij_6)\} = j_{s_0}.$$

2.6. As we have already remarked, we may add terms of the type $B_\mu x_\nu x_\nu$ to the A_μ. Since (11) must be satisfied, it follows that $x_\mu B_\mu$ must be of the form $Bx_\nu x_\nu$. This introduces an uncertainty in $F_{\mu\nu}$ of a term

$$\left(\frac{\partial B_\nu}{\partial x_\mu} - \frac{\partial B_\mu}{\partial x_\nu}\right) x_\lambda x_\lambda + 2(B_\nu x_\mu - B_\mu x_\nu).$$

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The field quantities in Euclidean space are naturally unaltered. Since the current j_ν is given by (17), the uncertainty in j_ν is

$$\frac{\partial}{\partial x_\nu}\left\{\left(\frac{\partial B_\nu}{\partial x_\mu}-\frac{\partial B_\mu}{\partial x_\nu}\right)x_\lambda x_\lambda+2\left(B_\nu x_\mu-B_\mu x_\nu\right)\right\}=-2\frac{\partial x_\mu B_\mu}{\partial x_\nu}-2\frac{\partial B_\mu}{\partial x_\mu}x_\nu$$

on the hyperquadric. It may easily be seen that it introduces no uncertainty in the current j_{r_0} in Euclidean space.

2.7. We may, if we choose, in addition to the condition (11), impose on our potentials the condition

$$x_\mu A_\mu=0 \tag{29}$$

on the hyperquadric, which, with (11), requires that $x_\mu A_\mu$ should be of the form

$$x_\mu A_\mu=A\,(x_\nu x_\nu)^2. \tag{30}$$

We have already shown that by a suitable gauge transformation, with S arbitrary on the hyperquadric, we may make

$$\frac{\partial A_\mu}{\partial x_\mu}=0 \tag{31}$$

on the hyperquadric, i.e. reduce it to the form

$$\frac{\partial A_\mu}{\partial x_\mu}=Dx_\nu x_\nu. \tag{32}$$

We now wish to show that by changing the values of the A_μ off the hyperquadric, i.e. by adding terms of the type $K_\mu x_\nu x_\nu$, (29) and (31) can be made to hold in the whole of space. For if

$$A'_\mu=A_\mu+K_\mu x_\nu x_\nu,$$

then

$$\frac{\partial A'_\mu}{\partial x_\mu}=Dx_\nu x_\nu+\frac{\partial K_\mu}{\partial x_\mu}x_\nu x_\nu+2K_\mu x_\mu$$

$$=x_\nu x_\nu\left(D+\frac{\partial K_\mu}{\partial x_\mu}+2K\right), \tag{33}$$

where we have written

$$x_\mu K_\mu=Kx_\nu x_\nu, \tag{34}$$

and

$$x_\mu A'_\mu=(A+K)\,(x_\nu x_\nu)^2. \tag{35}$$

By choosing K and K_μ suitably, (33) and (35) can be reduced to zero in the whole of space. Since $x_\mu K_\mu$ must be of the form (34), the reduction of $x_\mu A_\mu$ to zero in the whole of space is only possible if it is of the form (30). This last calculation shows, as was mentioned in the footnote on page 624, that it is not sufficient merely to assume that $x_\mu A_\mu$ vanishes on the hyperquadric, as Dirac does.

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3. *The spin-wave equation.* Dirac has shown that the only simple conformally invariant wave equation is

$$\beta_\mu x_\mu \gamma_\nu \frac{\partial}{\partial x_\nu} \psi = 0, \tag{36}$$

where the β's and γ's are two sets of matrices satisfying the relations

$$\beta_\mu \gamma_\nu + \beta_\nu \gamma_\mu = 2\delta_{\mu\nu} \tag{37}$$

and the relations which follow from these, namely

$$\gamma_\mu \beta_\nu + \gamma_\nu \beta_\mu = 2\delta_{\mu\nu}. \tag{37 (a)}$$

A special set of β's and γ's satisfying (37) is given by

$$\left.\begin{aligned} \beta_\mu &= \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, i, \\ \gamma_\mu &= \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, -i, \end{aligned}\right\} \tag{38}$$

where the α's are the usual five anticommuting matrices whose square is unity. (The α's may be represented as matrices of four rows and columns.)

Using the relation

$$\frac{2\beta_\mu x_\mu}{x_5 + ix_6} = [1 + \tfrac{1}{2} y_r \beta_r (\gamma_5 + i\gamma_6)] (\beta_5 - i\beta_6) [1 - \tfrac{1}{2} y_s \gamma_s (\beta_5 + i\beta_6)], \tag{39}$$

which holds on the hyperquadric (1), Dirac has shown that (36) may be reduced to

$$(\beta_5 - i\beta_6) \left[\gamma_s \frac{\partial}{\partial y_s} + (\gamma_5 + i\gamma_6)(n+2) \right] \phi = 0, \tag{40}$$

where ϕ is connected with ψ by

$$\phi = [1 - \tfrac{1}{2} y_t \beta_t (\gamma_5 + i\gamma_6)] \psi (x_5 + ix_6)^{-n}, \tag{41}$$

and ψ is of degree n. Using the representation (38), (40) becomes

$$(1 + \alpha_5) \left[\alpha_s \frac{\partial}{\partial y_s} + 2(n+2) \right] \phi = 0. \tag{42}$$

Since the matrix $1 + \alpha_5$ is degenerate, i.e.

$$(1 + \alpha_5)(1 - \alpha_5) = 0, \tag{43}$$

(42) is just the ordinary relativistic wave equation for an electron in a degenerate form.

We wish to consider the introduction of the potentials into equation (36). We take as our conformally invariant wave equation in the presence of an arbitrary field

$$\beta_\mu x_\mu \gamma_\nu \left(\frac{\partial}{\partial x_\nu} + A_\nu \right) \psi = 0, \tag{44}$$

which may be written also in the form

$$\left[\tfrac{1}{2} \beta_\mu \gamma_\nu \left\{ x_\mu \frac{\partial}{\partial x_\nu} - x_\nu \frac{\partial}{\partial x_\mu} + (x_\mu A_\nu - x_\nu A_\mu) \right\} + x_\mu A_\mu \right] \psi = 0. \tag{45}$$

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By (11), $x_\mu A_\mu$ is a constant on the hyperquadric, and it may be shown by an argument similar to one used by Dirac that one of the cases in which (45) is of use as a wave equation is when $x_\mu A_\mu$ is zero. This is consistent with condition (29).

By a simple algebraic calculation, we may prove the equation

$$\begin{aligned} x_\mu A_\mu (x_5 + ix_6) = [1 + \tfrac{1}{2} y_r \gamma_r (\beta_5 + i\beta_6)] \\ \times \{\gamma_s A_{s_0} + \tfrac{1}{2}(\gamma_5 - i\gamma_6)(A_5 + iA_6)(x_5 + ix_6) + (\gamma_5 + i\gamma_6) x_\mu A_\mu\} \\ \times [1 - \tfrac{1}{2} y_l \beta_l (\gamma_5 + i\gamma_6)], \end{aligned} \tag{46}$$

which holds on the hyperquadric. Using (46), (39) and (41), introducing the representation (38) for the β's and γ's, and remembering (43), we may reduce (44) to

$$(1 + \alpha_5)\left[\alpha_s\left(\frac{\partial}{\partial y_s} + A_{s_0}\right) + 2\{n + 2 + x_\mu A_\mu\}\right]\phi = 0. \tag{47}$$

We see that the potentials A_μ occur in (47), where the mass usually occurs. By (11), $x_\mu A_\mu$ is a constant on the hyperquadric. It is clear that a gauge transformation of the type (19) does not change the value of $x_\mu A_\mu$, since it only adds a term which vanishes by Euler's equation, S being a homogeneous function of the x's of degree zero.

But although $x_\mu A_\mu$ is a constant on the hyperquadric, its particular value is arbitrary. Any given field described by the Euclidean potentials A_{r_0} does not determine the A_μ uniquely, and the value of $x_\mu A_\mu$ may be chosen arbitrarily. We now proceed to show that there are certain values which may not be assigned to $x_\mu A_\mu$, if the wave equation (44) is to have a meaning in conformal space. It can be shown after some calculation that on the hyperquadric

$$\left\{\beta_\rho x_\rho \gamma_\sigma \left(\frac{\partial}{\partial x_\sigma} + A_\sigma\right)\right\} \beta_\mu x_\mu \chi = 2\{n + 2 + x_\mu A_\mu\} \beta_\mu x_\mu \chi, \tag{48}$$

so that, if

$$n + 2 + x_\mu A_\mu = 0 \tag{49}$$

on the hyperquadric, the right-hand side of (48) vanishes, and any ψ of the type

$$\beta_\mu x_\mu \chi,$$

with arbitrary χ, is a solution of the wave equation (44). Therefore we may not assign a value to $x_\mu A_\mu$ which would make the left-hand side of (49) vanish. We can still demand that $x_\mu A_\mu$ should vanish in the hyperquadric, which is always permissible as was shown in § 2·7, and does not make the differential equation (44) illusory. Equation (47) is Dirac's equation for a particle in an arbitrary field, multiplied by the factor $1 + \alpha_5$, which makes it degenerate.

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