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Topological features of good resources for measurement-based quantum computation

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We study how graph states on fractal lattices can be used to perform measurement-based quantum computation, and investigate which topological features allow this application. We find fractal lattices of arbitrary dimension greater than one that all act as good resources for measurement-based quantum computation, and sets of fractal lattices with dimension greater than one that do not. The difference is put down to other topological factors such as ramification and connectivity. This is in direct analogy to the tendency of lattices to observe criticality in spin systems. We also discuss the analogy between thermodynamics and one-way computation in this context. This work adds confidence to the analogy and highlights new features of what we require for universal resources for measurement-based quantum computation. This paper is an extended version of Markham *et al.* (2010), which appeared in the proceedings of DCM 2010.

1. Introduction

Quantum computation offers a great potential benefit over conventional 'classical' computation: in particular for algorithms, where it gives exponential speed up and for the

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simulation of quantum systems – see, for example, Nielsen and Chuang (2000). But where exactly this apparent power comes from is a question subject to much debate, and is still largely unanswered. We focus on one particular model of quantum computation, measurement-based quantum computation, where evolution is driven through a series of measurements and corrections over an initial resource state. We then focus on the slightly narrower, but nevertheless very interesting question of what makes a good initial resource state. By 'good', we simply mean a state that will allow universal quantum computation, that is, one that can perform any computation allowed by a quantum computer. Various subtleties can be added to this definition, but our results here will also largely apply unaltered to these (see van den Nest *et al.* (2007) for a review).

Our approach is to use analogies between quantum computation and thermodynamics to inform our intuition. Drawing analogies is often a very powerful tool in science, which can not only deepen our understanding through the opening up of new perspectives, but can also allow technical tools developed for one discipline to be applied to another, often very fruitfully. Anders *et al.* (2008) uses an analogy between measurement-based quantum computation and thermodynamics, viewing the computation itself as a phase transition. This is, in spirit, the reverse of the thought-provoking analogy in Toffoli (1998), where physics is viewed as a computation, whereas Anders *et al.* (2008) tries to understand computation itself as a physical process. In doing so, key features of useful resources for one-way quantum computers were identified in direct analogy to the identification of critical systems in thermodynamics. In particular, some rather elegant and simple methods, which were first developed by Peierls (Peierls 1936) to show that while one-dimensional spin chains are not critical, two-dimensional chains are, were translated into arguments about how universal resources for one-way quantum computation depend on dimension. In the current paper, we extend this theme to look for other important features of universal resources, following the work on fractal lattices in Gefen *et al.* (1980), which showed that critical behaviour in spin systems is not solely dependent on the dimension of the lattice, but also on other features such as the order of ramification and connectivity. We again see an exact mirroring of results, highlighting these also as features crucial for universal resources for one-way quantum computation. As examples, we will show that there exists a set of fractal lattices (Sierpinski carpets) that for any dimension greater than one are guaranteed to be capable of acting as a universal resource. On the other hand, we will also provide examples with dimension greater than one that are not universal, thereby highlighting the importance of other topological features – such as ramification, connectivity and lacunarity.

2. Phase transitions and measurement-based quantum computation

2.1. *The analogy*

We will begin with a review of the analogy presented in Anders *et al.* (2008) and the results it supports with respect to good resources for measurement-based quantum computation. We start by reviewing the problems addressed in this analogy. In the case of thermodynamics and many-body physics, the problem of interest is the existence, or not, of some critical phenomena or phase transition. Simply put, a phase transition

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occurs when a small change in some parameters of a given system gives rise to a large macroscopic change of state, or phase. For example, at a temperature just below zero degrees Celsius, water becomes ice, and just above it, it becomes water again. These two phases of matter are clearly very different. In spin systems the macroscopic property of interest is whether the system is magnetised or not. This happens when sufficiently many spins point in the same direction – we call this the 'ordered state'. The effect is observable through the amount of magnetisation M present, which is called an 'order parameter'. In the Ising model, the ground state (corresponding to zero temperature) is ordered, and for high temperatures, the orientation of the spin becomes random and is disordered, and its magnetisation is zero. The question is then whether or not there is a finite, non-zero, temperature T_{crit} below which the system is ordered. If this is the case, we say there can be a phase transition from non-magnetised to magnetised at temperature T_{crit}[†]. It is known that for one-dimensional spin chains with nearest neighbour interactions only, there is no phase transition, but for two-dimensional lattices there are. This will be explained using Peierls' argument below.

In the case of measurement-based quantum computation, the problem of interest is the ability, or not, to perform universal quantum computation. In one-way quantum computation[‡] (Raussendorf and Briegel 2001), the first step in carrying out a computation is to prepare a highly entangled multipartite quantum state. This state, which is called a 'resource state', is independent of the actual computation to be performed. The computation is then effected through local measurements and local corrections at individual sites. The choice of measurements, and how they depend on each other, determines the computation performed (including the possible entry of a classical input). During the process of measurement, entanglement is destroyed, and in this sense it is consumed by the computation. At the end of the computation, the classical information I of the solution is obtained as the measurement outcomes of the final few measurements. Since its invention, considerable effort has gone into finding out what constitutes a good initial resource state (see, for example, van den Nest *et al.* (2007) and Gross *et al.* (2009)). Given a particular set of states, the question is, whether or not it can act as a universal resource for quantum computation. The analogy we will now draw takes some steps towards answering this question. Note that in this paper we will always consider resource states as graph states.

Table 1 lists analogous quantities in thermodynamics on the one hand and one-way quantum computation on the other.

The second law of thermodynamics states that systems interacting with a thermal bath will always conspire to minimise the free energy F, which is given by

$$F = U - TS. \tag{1}$$

[†] Indeed, for the purposes of the current paper, we define a phase transition as the process of going from some defined ordered state to a disordered state due to some parameter change (in our case this is temperature or inverse time). Various mathematical definitions of what a phase transition is exist within physics, and these depend on the context and physical systems being considered, though all have the same idea at their heart.

[‡] We abbreviate this to 1-way QC, and in the current paper we take it to be synonymous with measurement-based quantum computation.

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Table 1. *(This table is borrowed from Anders et al. (2008).) The first column lists a set of quantities from thermodynamics; the second column lists their analogues in one-way quantum computation. In the examples considered in Anders et al. (2008), we took the entanglement E to mean the relative entropy of entanglement (Vedral et al. 1998), but other measures can also be applied, and may be more appropriate in different circumstances. The computational capacity C is the number of ways entanglement can be used, which is essentially the number of different measurement patterns (Danos and Kashefi 2006) that can be carried out using the given amount of entanglement. Details for the other quantities can be found in the text.*

Thermodynamics	**One-way computing**
Energy U	Entanglement E
Entropy S	Computational Capacity C
Temperature T	Inverse Time $1/t$
Free Energy F	Computational Potential P
Thermalisation	Computation
Magnetisation M	Extracted Information I

Intuitively, we can understand the second law as saying that through the process of thermalisation, nature insists that at a given temperature T, the energy U is spread out as much as possible by maximising the entropy S.

When making our analogy, the quality we require for our 1-way QC computation is that it should be universal. To this end, we postulate a kind of 'law of 1-way QC' whereby we insist (acting as 'mother nature' for the 1-way QC – it is, after all, we who design and control it) that the quantum computer be as universal as possible at each step. That is, we insist that, at each time, the computation is carried out in such a way as to maximise the potential. In terms of quantities, we say that for a given amount of entanglement E, we insist that any computation at any time t maximises the number of ways it can be used – we call this the computational capacity C. We thus phrase our 'second law of universal 1-way QC' as follows: at each time t, the potential P given by

$$P = E - 1/tC \tag{2}$$

must be minimised (that is, the potential should be consumed as fully as possible).

Following an argument first put forward in Peierls (1936), and cleaned up in Griffiths (1964), which shows that one-dimensional spin chains are not critical, but two-dimensional spin chains are, Anders *et al.* (2008) gave an intuitive argument for why a one-dimensional cluster state is not a universal resource for 1-way QC, whereas a two-dimensional cluster state is a universal resource. Peierls' argument goes as follows. If we want to test whether an 'ordered state' (that is, one with a large number of spins pointing in the same direction such that there is a positive overall magnetisation) is possible for some non-zero temperature T, we simply check wether small perturbations to this state will raise or lower the free energy of that state (the very physicsy 'shake it

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and see' approach). Any such perturbation will change the free energy:

$$\Delta F = \Delta U - T\Delta S. \tag{3}$$

If by perturbing it we can *reduce* the free energy, the state clearly is not a valid thermal state, by the 2nd law. So, in terms of equation (3), this becomes a question of the balance between the change in energy and the change in entropy. If perturbing the system increases the entropy more than the energy, the state before perturbation was not a valid thermal state. In the case of a one-dimensional spin chain, the cost of any perturbation in terms of entropy is much greater (it scales with the number of spins n) than the cost in energy (which is fixed). In the case of a two-dimensional spin, they both scale with n in the same way, hence a balance can be found. By finding the fixed point of the free energy (that is, the point where the perturbation has no effect on it), we can set ΔF to zero in (3) to find a critical temperature above which the system is not ordered. Remarkably, given the simplicity of this approach, this is very close to the actual critical temperature, below which it can also be shown that the system is ordered.

When testing if a system can be used for 1-way QC, the ordered state is the 'solution state' (the state reached after all measurements have been made in the 1-way QC) and the test is whether it can be reached for some finite time t. Again, we test this by perturbing it and seeing if it violates our 2nd law of 1-way QC. If it does, it is definitely not a valid state according to our second law – that is, no computation satisfying our 'law of 1-way QC' can find such a state within a time t. Any perturbation results in a change in computational potential

$$\Delta P = \Delta E - 1/t\Delta C. \tag{4}$$

This then comes down to a balance between the entanglement E and the number of ways of using it C, for a given t. As above, we cannot achieve this balance in the case of a one-dimensional cluster state – the number of ways of using the entanglement is larger than the entanglement available, hence some choice must be made about its use, thereby sacrificing universality. Alternatively, it says that there is no finite length of time within which it could be achieved, so if it were possible, it would take an infinite amount of time. On the other hand, as in the spin case, these quantities do balance for a two-dimensional lattice. Again as above, it is possible to approximate a critical time t_{crit} below which the computation cannot be completed in a universal fashion by setting (4) to zero. In fact, by comparing how these two factors scale with dimension D, we can arrive at the formula

$$t_{crit} \propto \frac{lnD}{D}, \tag{5}$$

which agrees with both our intuition and examples showing that higher dimensional states can allow greater speed in computation.

2.2. *Computing on fractal lattices*

We can now extend this analogy to cover another interesting set of examples from many-body physics, where it is shown that not only does dimension play a role in spin criticality, but other topological features are also relevant. Gefen *et al.* (1980) used similar techniques

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to those of Peierls and Griffiths described above to test the criticality of spin systems, but this time based over several self-similar fractal lattices. Again the arguments test the ability of a lattice to balance the change in energy and entropy for small perturbations. Examples were given that either allow or do not allow criticality for all (fractal) dimensions between one and two. The additional features that capture the existence of criticality are shown to be topological, and include ramification, connectivity and lacunarity.

The term fractal has now come to encompass many different types of mathematical objects (see, for example, Falconer (1990)). In this paper, we will consider fractals based on self-similar iterative constructions, that is, where the lattice is constructed by replacing a segment, such as a line, triangle or square, by N copies of itself in a specified manner, *ad infinitum*. If the relative linear scale is b (that is, how much smaller the replacement parts are), the fractal dimension (also called the 'similarity dimension') is defined by

$$D = \frac{\ln N}{\ln b}, \tag{6}$$

which is equal to the Haussdorff dimension (Falconer 1990), which is defined for more general fractals. See Figures 1, 2 and 3 for examples.

We will follow the same analogy as before to show exact analogues of these results in 1-way QC. We will show that graph states of the fractal lattices of Koch curves and Sierpinski gaskets are not universal resources for 1-way QC, but Sierpinski carpets are, independent of dimension. As in Gefen *et al.* (1980), we can interpret this as the effect of other topological factors, including the ramification.

The Koch curve is illustrated in Figure 1. It is constructed iteratively, starting from a (horizontal) line and at each stage replacing the middle third of each segment by the top parts of an equilateral triangle based there. So, at each stage, a line/segment is replaced by four lines ($N = 4$), each of which is $1/3$ the original length ($b = 3$). The dimension is thus

$$D = \frac{\ln 4}{\ln 3}.$$

For the purposes of our analogy (and in order to test universality), this behaves exactly as in the one-dimensional case in the previous section, where we argue that it is indeed not a good universal resource (Anders *et al.* 2008). Proofs to this effect can also be found in the literature (van den Nest *et al.* 2007).

The Sierpinski gasket is shown in Figure 2. Again, the same Peierls-like arguments as those made above show in the same way that it is not a valid possibility for a universal resource. That is, it is not possible to find a balance for some finite t between the entanglement present and the number of ways of using the entanglement. By analogy with the spin case (Gefen *et al.* 1980), a significant perturbation of the solution state by adding entanglement can be done in many more ways than the amount of entanglement added, causing a negative change in P (equation (4)). Unfortunately, this is not a rigorous proof of non-universality since our analogy (and, in particular, our 'law of 1-way QC') has only been justified and not proved. However, we can use methods introduced in van den Nest *et al.* (2006) to prove that the Sierpinski gasket is indeed not a universal resource, but we will leave this until Section 3.

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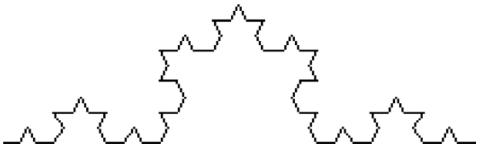


Fig. 1. The Koch curve. The dimension of the lattice is $D = \frac{\ln 4}{\ln 3} = 1.2619$. It has ramification $R = 2$.

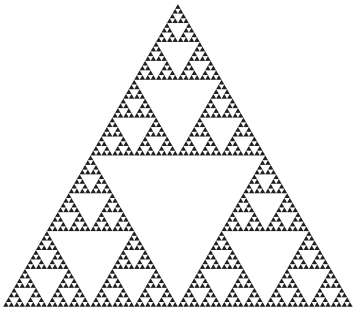


Fig. 2. A sierpinski gasket is constructed by subdividing an equilateral triangle into four sub-triangles, then removing the central triangle and repeating the process. The dimension of the lattice is $D = \frac{\ln 3}{\ln 2} = 1.585$, and it has ramification $R = 3$ or $R = 4$.

On the other hand, Sierpinski carpets (see Figure 3) are universal for all dimensions greater than one. The arguments to show it is a valid possibility for a universal resource follow along the same lines as the Peierls-like argument made above. That is, we can always find a balance for some finite t between the entanglement present and the number of ways it can be used. Of course, this is not a proof that it is a universal resource since even if we assume our 'law of 1-way QC', it only shows that it is a possible resource, that is, that it does not violate the law of 1-way QC. However, we can construct exact proofs for all cases. To show explicitly that these are universal, we adopt a similar technique to that used in Browne *et al.* (2008) and actively construct a standard two-dimensional lattice by taking out vertices using local Z and local X measurements, resulting in what is known to be a universal resource (Raussendorf and Briegel 2001). The idea is that, given an arbitrary lattice (which may even be irregular, as in the case of Browne *et al.* (2008)), if we can draw a standard two-dimensional grid over this lattice, we can measure away the extra qubits to leave only the ideal two-dimensional lattice. This is possible because of the way X and Z measurements convert one graph state into another (see, for example, Hein *et al.* (2005)). It is easy to see by inspection of the Sierpinksi carpet in Figure 3 that we can always draw a two-dimensional grid that grows with the size of the carpet. Thus there is always a way to get a known universal resource for any dimension greater than one.

We can thus see that the ability of a lattice to act as a universal resource for 1-way QC does not just depend on dimension. In particular, for any dimension between 1 and 2,

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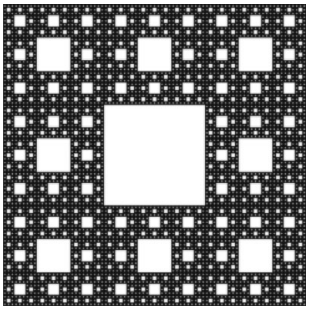


Fig. 3. A Sierpinski carpet is constructed by subdividing a square into b^2 sub-squares, then cutting out l^2 of these sub-squares, and repeating the process. The dimension of the lattice is defined as $D = \frac{\ln(b^2-l^2)}{\ln b}$. Here we illustrate a Sierpinski carpet with $b = 3$, $l = 1$. The ramification is infinite.

we can find a Sierpinski carpet that is a universal resource, but we have also shown two examples with dimension in this range that are not universal. As in Gefen *et al.* (1980), we can then infer that it is down to other topological properties. One such property that resonates in the case of 1-way QC is *ramification*. The ramification R is the minimal number of edges that must be removed to separate a part of the lattice of arbitrary size. It tells us something about how globally connected the lattice is, or how easily the lattice can be separated into chunks. The lower the ramification, the easier it is to separate parts of the lattice off and the less globally connected it is. From our examples, those lattices with finite ramification are not universal resources, whereas those with infinite ramification are (the two-dimensional lattice also has infinite ramification). This is very similar in flavour to the idea behind the entanglement width introduced in van den Nest *et al.* (2006) to check for the universality of resource states. In a sense, this also looks for some global connectedness. Here too an infinite scaling is required for universality. In the next section, we calculate the entanglement width of the Sierpinski gasket state to prove the result we obtained above intuitively, and then go on to discuss the connection to ramification.

3. Non-universality and entanglement

In the previous section we argued for the inability of certain lattices to be universal resources based on our analogy. Unfortunately, this does not constitute a proof at this point. In this section, we will give proofs based on the connection with entanglement.

Intuitively, it is clear that some relationship between the entanglement of the resource state and its ability to perform universal computation may well be expected. This is because, after the preparation of the resource state, all operations are local, that is, local measurements and local corrections. Under this set of operations, it is impossible to increase the entanglement (this is indeed the main property of any good quantity we call entanglement). Thus, whatever the initial resource state $|\psi\rangle$, the state at any subsequent point, say $|\psi'\rangle$, must satisfy

$$E(|\psi\rangle) \geqslant E(|\psi'\rangle), \tag{7}$$

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where the entanglement E is a map from states to real numbers satisfying certain axioms (which essentially capture the above idea) – see Virmani and Plenio (2005) and Horodecki *et al.* (2009). It turns out that many conditions can be formed with respect to the entanglement features of good resources. In particular, it is known that if the entanglement (defined in the correct way) does not scale with a family of resource states (such as our lattices), then it cannot be a universal resource for 1-way QC (van den Nest *et al.* 2006; van den Nest *et al.* 2007).

In fact, the choice of which entanglement measure to use is not obvious (there are many possible measures) – see Virmani and Plenio (2005) and Horodecki *et al.* (2009). The main difference between an arbitrary entanglement measure and one that can be used to indicate the ability of a state (or set of states) to behave as a good resource is that we also require that the addition of a system does not prevent it from being a good resource. That is, if a system, of n qubits say, in a certain state $|\psi\rangle$ is a good resource, then adding one (or more) extra qubits on an arbitrary state, say $|0\rangle$, should not stop if from being a good resource – one can either just ignore the qubit, or measure it and throw away the outcome. Thus, for a quantity to test the quality of a state, we would like it to be defined over an arbitrary number of systems, and have the property

$$E(|\psi\rangle|0\rangle) = E(|\psi\rangle). \tag{8}$$

One such measure, called the *entanglement width* E_{wd}, was introduced in van den Nest *et al.* (2006) and defined by

$$E_{wd}(|\psi\rangle) := \min_{T} \max_{e} E^{bi}_{T,e}(|\psi\rangle), \tag{9}$$

where $E^{bi}_{T,e}(|\psi\rangle)$ is the bipartite entropy of entanglement across the bipartite cut defined by T and e (in fact, the entanglement width also works for different bipartite measures, but here we will stick to the entropy of entanglement whenever we refer to bipartite entanglement). T is a sub-cubic graph with n leaves (edges not leading to a vertex at one end), and e is an edge of T. Each leaf corresponds to a qubit. The bipartite cut is defined by removing edge e to give two separate trees. The leaves of one tree correspond to one side of the cut, and the other tree corresponds to the other side of the cut. van den Nest *et al.* (2006) and van den Nest *et al.* (2007) showed that if the entanglement width does not scale with a family of resource states, then it cannot be a universal resource, and we will use this to show that the Sierpinski gasket is not a universal resource.

3.1. *Proof that the Sierpinski gasket is not a universal resource*

It is easy to see that a tree can be defined for the Sierpinski gasket $|\psi_{SG}\rangle$ with the same self-similar properties such that the best cut e also has self-similar properties and gives entanglement $E^{bi}_{T,e}(|\psi_{SG}\rangle) = 3$ that does not grow. To find this tree, we start with a triangle with three qubits. We associate this with the tree with four vertices, one top vertex connected to three leaf vertices (one for each qubit). We will call this the $\alpha = 1$ tree. In the next iteration, we take the triangle with three sub-triangles and replace each leaf vertex with one of the $\alpha = 1$ trees. This is then the $\alpha = 2$ tree. We number the leaves such that each final sub-tree, corresponds to a different sub-triangle. Iterating like this

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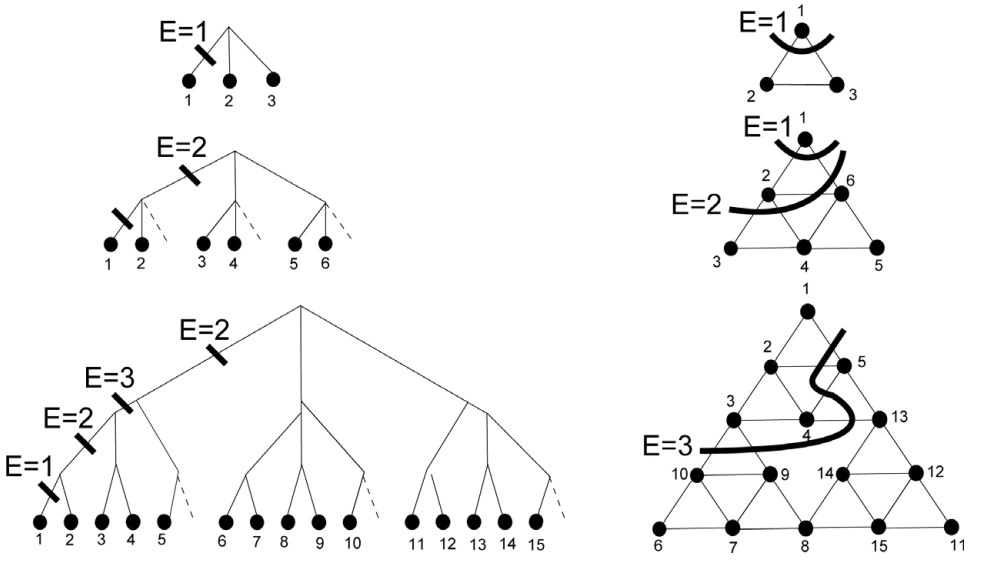


Fig. 4. The trees (left-hand side) bounding the entanglement width for the Sierpinski Gasket (right-hand side), for the first three rounds of iteration, with bold lines indicating the cuts. It is clear that when continuing the pattern, the entanglement of any cut will be less than or equal to three.

we get a tree where all cuts made are the same as cuts separating the sub-triangles in the lattice, which has entanglement $E^{bi}_{T,e}(|\psi_{SG}\rangle) = 3$ at most. This can be shown using standard techniques for calculating the bipartite entanglement in a graph state – see, for example, Hein *et al.* (2005) and Markham *et al.* (2007). We are not quite there yet since these trees have vertices with four incident edges (sub-cubic trees only have exactly one or three). It can be seen that such trees can be easily changed to give sub-cubic trees without changing the self-similar properties (see Figure 4). Hence the entanglement width is bounded, $E_{wd}(|\psi_{SG}\rangle) \leqslant 3$, for any lattice size. Since it does not scale with the size of the lattice, it cannot be a universal resource.

3.2. *Ramification*

We proved in the last section that ramification does indeed seem to be an important feature when considering lattices for resources for measurement-based quantum computation. We have confirmed that there are examples with between dimension one and two with finite ramification that are not good resources, and that there are also examples with infinite ramification that are good resources. The proofs were based on the finiteness of the entanglement of said resources (which either scales, or not, in the same way as the ramification). We are then led to ask whether there could be a connection between the two. We may also wonder whether an alternative entanglement measure can be defined with respect to ramification (or indeed other topological properties) that could be used to determine their usefulness as resources for 1-way QC.

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The difficulty comes in a subtlety in the definition of ramification. Initially, ramification was defined on self-similar fractals (which was later extended to more general cell-like objects – see, for example, Teplyaev (2005)) as the smallest number of cuts needed to separate off a 'cell' (with an appropriate definition, see, for example, Teplyaev (2005)) of arbitrary size, in the asymptotic setting. The entanglement ramification would then look something like the bipartite entanglement over cut A, B maximised over all cuts of size $|A|$ such that A is made of a cell (with appropriate definition). It is not difficult to see that such a measure would be bounded by the actual ramification, and therefore would be finite on the Sierpinski gasket. This is because for a graph state, the bipartite entanglement across a cut A, B that cuts through C edges of the graph is bounded by

$$E_{A,B}(|G\rangle) \leqslant C. \tag{10}$$

This follows because a graph state is constructed by applying a control phase gate across each edge, on an initially unentangled state (see, for example, Hein *et al.* (2005)). The maximum entanglement that can be generated by C control phase gates, starting from nothing, is C (this is clear from dimensionality arguments – see, for example, Virmani and Plenio (2005) and Horodecki *et al.* (2009) for clarification on entanglement). Note that this can sometimes be a large overestimate – for example, the fully connected graph has many edges between any partition, but the entanglement is equal to one (Hein *et al.* 2005). Thus, the cut that achieves the ramification gives entanglement less than or equal to the ramification.

However, there are several difficulties with this approach. First, difficulties arise, in particular, in guaranteeing the properties (7) and (8) of a good entanglement measure. When considering states $|\psi'\rangle$, if size $m < n$, such a size may not naturally fit into the cell structure, and we would have to adapt the definition of the cell or the definition of the cell must be chosen to accommodate this. The same goes for the addition of systems required to test the ability of a resources state. Second, such measures would only work for families of states where such cell structures could be applied. Whilst this may be large, and indeed it may often be possible to adapt the definition of the cell structures in the correct way, it seems it would be a complicated approach and is beyond the scope of this paper. However, the similarity of the entanglement width to such a possible measure is noteworthy, and it does indeed capture similar properties.

4. Conclusions

In this paper we have shown that the analogy developed in Anders *et al.* (2008) can be used to argue that fractal lattices can also act as universal resources for 1-way QC, and that it is not just dimensionality that is important, but also other topological features such as ramification. By providing further examples where the analogy succeeds, we have given further support for its validity. It also highlights new features that we can expect good resources should posses for 1-way QC. We can also ask how this corresponds to known conditions for good resources, such as the entanglement conditions in van den Nest *et al.* (2007). In this context, it may be that the important topological features could

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also correspond to particular entanglement features. We have discussed the relationship between ramification and entanglement width in this context.

In many-body physics, the finiteness of ramification also allows for efficient classical simulations of spin systems. Indeed, we may expect that finite ramification makes simulation of the quantum computation over such lattices similarly easy – since effectively it means that each sub-part can be considered separately, or that in some sense its effects are only finitely spread. It is perhaps then not surprising that lattices with finite ramification do not seem to be universal.

We can also consider another possible connection to existing conditions for good resources for measurement-based quantum computation, namely the existence of Flow on these lattices. Flow (and gFlow) are known sufficient conditions for a lattice or graph to allow 1-way QC (Danos and Kashefi 2006; Browne *et al.* 2007). The fact, for example, that we can always reduce the Sierpinski carpet case to a two-dimensional lattice implies that we can always extract a Flow for it in some sense. Perhaps the topological features presented here are also important for the existence of Flow. We also note that the techniques, and indeed the lattices used, are very similar to those in Browne *et al.* (2008), which arise in the context of two-dimensional lattices with noise. In a sense, this is no surprise since it is similar situations that may also give rise to fractal lattices in many-body physics. But it may also indicate that the analogy used could be useful in treating noise over fixed lattices. At a foundational level, this analogy, and the support it gains from the work in the current paper, opens up many interesting questions and possibilities. For example, we can ask whether these analogies can be made more solid by a kind of path integral approach to 1-way QC. And how deep can we take these analogies beyond 1-way QC – for example, does it work for other models of computation? We hope this work will stimulate further research in these areas.

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