How many real attractive fixed points can a polynomial have?

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1. *Introduction*

While the notion of roots of a quadratic polynomial is rudimentary in high school mathematics, that of its *fixed points* is uncommon. A real or complex number θ is a fixed point of a polynomial $p(x)$ if $p(\theta) = \theta$. The fact that the notion of fixed point of polynomials is not commonly covered in high school or undergraduate mathematics is surprising because the relevance of the fixed points of a quadratic can be demonstrated easily via iterative methods for the approximation of such numbers as $\sqrt{2}$, when the quadratic formula offers no remedy. In fact, in the context of fixed points one can also give a formal definition of the derivative of a quadratic and see its application in action. More precisely, if θ is a zero of $q(x)$, then it is a fixed point of $p(x) = cq(x) + x$, where c is any non-zero constant. A fixed point θ of $p(x) = a_2x^2 + a_1x + a_0$ is *attractive* if its *multiplier*, $p'(\theta) = 2a_2\theta + a_1$, has modulus less than 1.

The *fixed point iteration* refers to the sequence $x_{k+1} = p(x_k)$, starting with a given x_0 as *seed*. Assuming that θ is an attractive fixed point of $p(x)$, the fixed point iteration converges to θ when x_0 is sufficiently close to it. Specifically, it is straightforward to show

$$x_{k+1} - \theta = p(x_k) - p(\theta) = \tfrac{1}{2}\big(p'(x_k) + p'(\theta)\big)(x_k - \theta). \qquad (1)$$

When x_k is close to θ, $p'(x_k)$ is close to $p'(\theta)$ and from (1) it follows that $|x_{k+1} - \theta|$ is approximately $|p'(\theta)| \cdot |x_k - \theta|$ so that the fixed point iteration converges to θ. Such analysis can also help in the development of Newton's method and the study of its convergence and rate of convergence. These observations on a quadratic polynomial extend to a polynomial $p(x)$ of arbitrary degree n. Using the binomial expansion formula the reader can check that

$$x_{k+1} - \theta = p(x_k) - p(\theta) = (x_k - \theta)q_{n-1}(x_k),$$

where $q_{n-1}(x)$ is a polynomial of degree $n - 1$ satisfying $q_{n-1}(\theta) = p'(\theta)$. Thus the local convergence of the fixed point iteration to an attractive fixed point is valid for a polynomial of any degree, having real or complex coefficients.

A fixed point θ is called *repulsive* if $|p'(\theta)| > 1$. No matter how close to θ a seed $x_0 \neq \theta$ is selected, the next iterate gets farther away from θ. If $|p'(\theta)| = 1$, θ is an *indifferent* fixed point. The behaviour of the fixed point iteration is more complicated for indifferent fixed points, see e.g. [1, 2, 3]. A natural question arises: Can all fixed points of a complex polynomial be attractive? Anyone familiar with *Julia sets* of iterations of $z^2 + c$, where c is a complex number, popularised by Mandelbrot and the famous set that

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bears his name, must have noticed that not both fixed points of $p(z) = z^2 + c$ can be attractive. Surprisingly, despite the existence of many books on fractals, this aspect of iterations of a polynomial does not seem to be emphasised. The study of this question is one of the most basic attempts in the understanding of the dynamics of iterations of polynomials and rational functions (see e.g. [4, 5]), as well as in the study of polynomial root-finding methods, their fractal behaviour and visualisation of iterative methods (see e.g. [1, 2, 3]).

In this Article we first derive an explicit formula for a complex polynomial having a prescribed set of complex numbers as its fixed points and a prescribed set of complex numbers as the corresponding multipliers. We then use the explicit formula to prove a complex polynomial of degree n can have at most $\lceil \frac{1}{2}n \rceil$ attractive fixed points lying on any straight line in the complex plane. In particular, the bound applies to real polynomials. When the roots of a polynomial of degree $n > 1$ are not restricted to lie on a straight line, via a known but non-trivial result, the maximum number of attractive fixed points is $n - 1$ (see Section 7). For each $n > 1$, we exhibit a polynomial of degree n with $n - 1$ attractive fixed points. We make a conjecture that also prohibits all fixed points being attractive: Any polynomial of degree $n > 1$ has a fixed point where the real part of its multiplier is at least 1. We prove the conjecture for $n = 2, 3$.

2. *Interpolating polynomial and bound on attractive fixed points*

We first prove the existence of a unique polynomial H_n of degree at most $2n - 1$, having an arbitrary sequence of n complex numbers as its fixed points, with an arbitrary sequence of n complex numbers as the corresponding derivatives at the fixed points (Theorem 1).

Next, in a more technical theorem we derive an explicit formula for the recursive representation of H_n (Theorem 2).

We then use this explicit formula to prove that if the fixed points all lie on a straight line and the derivative at each fixed point has modulus less than 1, the degree of H_n is exactly $2n - 1$ (Theorem 3).

Finally, we use the previous results to prove that a complex polynomial of degree n can have at most $\lceil \frac{1}{2}n \rceil$ attractive fixed points on a straight line (Theorem 4). In particular, a real polynomial of degree n can have at most $\lceil \frac{1}{2}n \rceil$ real attractive fixed points.

3. *Existence of interpolating polynomial with prescribed fixed points*

Theorem 1: Given distinct complex numbers $z_1, \ldots, z_n$, and arbitrary complex numbers $\alpha_1, \ldots, \alpha_n$, there exists a unique complex polynomial $H_n(z)$ of degree at most $2n - 1$ such that

$$H_n(z_i) = z_i, \qquad H_n'(z_i) = \alpha_i. \qquad \text{for all } i = 1, \ldots, n. \tag{2}$$

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Specifically, for $n = 1$

$$H_1(z) = z_1 + \alpha_1(z - z_1).$$

For $n \geqslant 2$, if $H_{n-1}(z)$ satisfies (2) for $i = 1, \ldots, n-1$, then

$$H_n(z) = H_{n-1}(z) + \big(a_{2n-2} + a_{2n-1}(z - z_n)\big)\pi_{n-1}^2(z) \tag{3}$$

where

$$\pi_{n-1}(z) = \prod_{i=1}^{n-1}(z - z_i) = (z - z_1)\ldots(z - z_{n-1}),$$

and

$$a_{2n-2} = \frac{z_n - H_{n-1}(z_n)}{\pi_{n-1}^2(z_n)}, a_{2n-1} = \frac{\alpha_n - H'_{n-1}(z_n) - 2a_{2n-2}\pi_{n-1}(z_n)\pi'_{n-1}(z_n)}{\pi_{n-1}^2(z_n)}. \tag{4}$$

Proof: We first prove the uniqueness of $H_n(z)$ satisfying (2). Suppose there is another polynomial $p(z)$ of degree at most $2n-1$, distinct from $H_n(z)$, satisfying (2). Set $Q(z) = H_n(z) - p(z)$ and note that $Q(z)$ is not identically zero. On the other hand, for all $i = 1, \ldots, n$ both $Q(z_i) = 0$ and $Q'(z_i) = 0$. This implies $Q(z)$ has $2n$ roots, counted with multiplicity, while its degree is at most $2n-1$. This contradicts the fundamental theorem of algebra which states that the number of roots of a complex polynomial, counting with multiplicity, is equal to its degree. Hence $H_n(z)$ is unique.

Next we prove by induction on n that H_n satisfies (2). Clearly, for $n = 1$, $H_1(z)$ satisfies (2). By the induction hypothesis H_{n-1} satisfies (2) for all $i = 1, \ldots, n-1$. Set

$$G_n(z) = H_n(z) - H_{n-1}(z) = \big(a_{2n-2} + a_{2n-1}(z - z_n)\big)\pi_{n-1}^2(z).$$

Since each z_i is a double root of $\pi_{n-1}^2(z)$, for $i = 1, \ldots, n-1$, $G_n(z_i) = 0$ and $G'_n(z_i) = 0$. Thus (2) holds for $i = 1, \ldots, n-1$. To complete the proof of the theorem, it suffices to prove $H_n(z_n) = z_n$ and $H'_n(z_n) = \alpha_n$. This is proved next.

From (3) $H_n(z_n) = z_n$ is equivalent to

$$z_n = H_{n-1}(z_n) + a_{2n-2}\pi_{n-1}^2(z_n). \tag{5}$$

Solving for a_{2n-2} in (5) results in the first equation in (4). Since the z_i are distinct, a_{2n-2} is well-defined.

Differentiating (3) and evaluating at z_n, $H'_n(z_n) = \alpha_n$ is equivalent to the equation

$$\alpha_n = H'_{n-1}(z_n) + G'_n(z_n). \tag{6}$$

Now observe

$$G'_n(z_n) = 2a_{2n-2}\pi_{n-1}(z_n)\pi'_{n-1}(z_n) + a_{2n-1}\pi_{n-1}^2(z_n). \tag{7}$$

Substituting (7) into (6) and solving for a_{2n-1} gives the desired expression in (4).

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Next we state two corollaries of Theorem 1. These will be used to prove the explicit representation of the a_i in Theorem 2.

Corollary 1: Under the assumptions of Theorem 1, for each $n \geqslant 2$,

$$H_{n-1}(z_n) = z_n - a_{2n-2}\pi_{n-1}^2(z_n), \tag{8}$$

$$H'_{n-1}(z_n) = \alpha_n - 2a_{2n-2}\pi_{n-1}(z_n)\pi'_{n-1}(z_n) - a_{2n-1}\pi_{n-1}^2(z_n). \tag{9}$$

Proof: To prove (8), set $z = z_n$ in (3) and use $H_n(z_n) = z_n$. To prove (9), differentiate (3), then set $z = z_n$ and use $H'_n(z_n) = \alpha_n$.

Definition 1: For each $n \geqslant 3$, let $\hat{a}_{2n-3}$ and $\hat{a}_{2n-4}$ be the expressions obtained from a_{2n-3} and a_{2n-4}, respectively, by replacing z_{n-1} by z_n and α_{n-1} by α_n.

Corollary 2. Under the assumptions of Theorem 1, for each $n \geqslant 3$,

$$H_{n-2}(z_n) = z_n - \hat{a}_{2n-4}\pi_{n-2}^2(z_n), \tag{10}$$

$$H'_{n-2}(z_n) = \alpha_n - 2\hat{a}_{2n-4}\pi_{n-2}(z_n)\pi'_{n-2}(z_n) - \hat{a}_{2n-3}\pi_{n-2}^2(z_n). \tag{11}$$

Proof: Begin by replacing n with $n-1$ everywhere in Corollary 1. Then evaluate at z_n instead of z_{n-1} and use Definition 1; this proves both (10) and (11).

4. *Explicit representation of interpolating polynomial*

Theorem 2: Under the assumptions of Theorem 1, for each $n \geqslant 2$,

$$a_{2n-2} = \sum_{i=1}^{n-1} \frac{(\alpha_i - 1)(z_i - z_n)}{\prod_{j=1, j\neq i}^{n}(z_i - z_j)^2}, \qquad a_{2n-1} = \sum_{i=1}^{n} \frac{(\alpha_i - 1)}{\prod_{j=1, j\neq i}^{n}(z_i - z_j)^2}. \tag{12}$$

In order to prove Theorem 2, we first prove two lemmas. Lemma 1 represents a_{2n-2} in terms of a_{2n-4}, $\hat{a}_{2n-4}$ and a_{2n-3}. Subsequently, using this representation and the induction hypothesis we prove the first formula in Theorem 2. Lemma 2 represents a_{2n-1} in terms of a_{2n-3}, $\hat{a}_{2n-3}$ and a_{2n-2}. This will be used to prove the second formula in Theorem 2.

Lemma 1: Under the assumptions of Theorem 1, for each $n \geqslant 3$,

$$a_{2n-2} = -\frac{a_{2n-4} - \hat{a}_{2n-4}}{(z_{n-1} - z_n)^2} + \frac{a_{2n-3}}{(z_{n-1} - z_n)}. \tag{13}$$

Proof: From the recursive equation (3), we can express $H_n(z)$ in terms of $H_{n-2}(z)$ to get

$$\begin{aligned} H_n(z) = H_{n-2}(z) &+ (a_{2n-4} + a_{2n-3}(z - z_{n-1}))\pi_{n-2}^2(z) \\ &+ (a_{2n-2} + a_{2n-1}(z - z_n))\pi_{n-1}^2(z). \end{aligned} \tag{14}$$

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Substituting for $H_{n-2}(z_n)$ from (10) in Corollary 2, also using $\pi_{n-1}(z) = \pi_{n-2}(z)(z - z_{n-1})$,

$$H_n(z_n) = z_n + \left(\left(a_{2n-4} - \hat{a}_{2n-4}\right) + a_{2n-3}(z_n - z_{n-1}) + a_{2n-2}(z_n - z_{n-1})^2\right)\pi_{n-2}^2(z_n).$$

Since $H_n(z_n) = z_n$ and $\pi_2(z_n) \neq 0$, from the above we can solve for a_{2n-2} to get (13).

Lemma 2: Under the assumptions of Theorem 1, for each $n \geqslant 3$,

$$a_{2n-1} = -\frac{a_{2n-3} - \hat{a}_{2n-3}}{(z_{n-1} - z_n)^2} + \frac{2a_{2n-2}}{(z_{n-1} - z_n)}. \tag{15}$$

Proof: To prove the lemma we differentiate $H_n(z)$ in (14), set $z = z_n$, then use the representation of $H'_{n-2}(z_n)$ in Corollary 2 and simplify. This gives

$$H'_n(z_n) = \alpha_n + A + B,$$

where $A + B = 0$ and A and B are given as

$$A \equiv 2\left(\left(a_{2n-4} - \hat{a}_{2n-4}\right) + a_{2n-3}(z_n - z_{n-1})\right)\pi_{n-2}(z_n)\pi'_{n-2}(z_n),$$

$$B \equiv \left(a_{2n-3} - \hat{a}_{2n-3}\right)\pi_{n-2}^2(z_n) + 2a_{2n-2}\pi_{n-1}(z_n)\pi'_{n-1}(z_n) + a_{2n-1}\pi_{n-1}^2(z_n).$$

From Lemma 1 we can simplify A to get

$$A = -2a_{2n-2}(z_n - z_{n-1})^2\pi_{n-2}(z_n)\pi'_{n-2}(z_n).$$

To simplify $A + B$, we need to express $\pi_{n-1}(z_n)\pi'_{n-1}(z_n)$ in terms of $\pi_{n-2}(z_n)\pi'_{n-2}(z_n)$. From the fact that $\pi_{n-1}(z) = (z - z_{n-1})\pi_{n-2}(z)$,

$$\pi'_{n-1}(z) = \pi_{n-2}(z) + (z - z_{n-1})\pi'_{n-2}(z).$$

Thus

$$\pi_{n-1}(z)\pi'_{n-1}(z) = (z - z_{n-1})\pi_{n-2}^2(z) + (z - z_{n-1})^2\pi_{n-2}(z)\pi'_{n-2}(z).$$

Using the above, evaluated at z_n, then substituted into $A + B = 0$, it is easy to derive

$$\left(a_{2n-3} - \hat{a}_{2n-3}\right) + 2a_{2n-2}(z_n - z_{n-1}) + a_{2n-1}(z_n - z_{n-1})^2 = 0.$$

Solving for a_{2n-1} gives (15).

5. *Proof of Theorem* 2

The theorem will be proved by induction on $n \geqslant 2$. From (3), written for $n = 2$,

$$H_2(z) = H_1(z) + \left(a_2 + a_3(z - z_2)\right)(z - z_1)^2. \tag{16}$$

Since $H_1(z) = z_1 + \alpha_1(z - z_1)$, substituting $z = z_2$ in (16), then using $H_2(z_2) = z_2$,

$$z_2 = z_1(z_2 - z_1) + a_2(z_2 - z_1)^2.$$

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Solving for a_2 gives $a_2 = \dfrac{\alpha_1 - 1}{z_1 - z_2}$. This coincides with a_{2n-2} in (12) when $n = 2$. Differentiating $H_2(z)$, setting $z = z_2$ and using $H_2'(z_2) = \alpha_2$,

$$\alpha_2 = \alpha_1 + a_3(z_2 - z_1)^2 + 2a_2(z_2 - z_1).$$

Substituting for a_2 and solving for a_3, proves the formula for a_{2n-1} in (12) for $n = 2$,

$$a_3 = \frac{\alpha_1 - 1}{(z_1 - z_2)^2} + \frac{\alpha_2 - 1}{(z_1 - z_2)^2}.$$

From (12) and the induction hypothesis, the expression for $a_{2(n-1)-2} = a_{2n-4}$ is

$$a_{2n-4} = \sum_{i=1}^{n-2} \frac{(\alpha_i - 1)(z_i - z_{n-1})}{\prod_{j=1, j\neq i}^{n-1} (z_i - z_j)^2}, \tag{17}$$

Multiplying and dividing each term of the summation by $(z_i - z_n)^2$ and then simplifying, we get

$$a_{2n-4} = \sum_{i=1}^{n-2} \frac{(\alpha_i - 1)(z_i - z_{n-1})(z_i - z_n)^2}{\prod_{j=1, j\neq i}^{n} (z_i - z_j)^2}. \tag{18}$$

From (17) and Definition 1 the expression for $\hat{a}_{2n-4}$ can be obtained by replacing (z_{n-1}, α_{n-1}) with (z_n, α_n). Subsequently multiplying and dividing each term of the summation in this expression by $(z_i - z_{n-1})^2$, we get

$$\hat{a}_{2n-4} = \sum_{i=1}^{n-2} \frac{(\alpha_i - 1)(z_i - z_n)(z_i - z_{n-1})^2}{\prod_{j=1, j\neq i}^{n} (z_i - z_j)^2}. \tag{19}$$

Subtracting (19) from (18), simplifying and then dividing by $-(z_{n-1} - z_n)^2$,

$$-\frac{(a_{2n-4} - \hat{a}_{2n-4})}{(z_{n-1} - z_n)^2} = -\frac{1}{(z_{n-1} - z_n)} \sum_{i=1}^{n-2} \frac{(\alpha_i - 1)(z_i - z_{n-1})(z_i - z_n)}{\prod_{j=1, j\neq i}^{n} (z_i - z_j)^2}. \tag{20}$$

On the other hand, from (12) and the induction hypothesis $a_{2(n-1)-1} = a_{2n-3}$ can be written

$$a_{2n-3} = \sum_{i=1}^{n-1} \frac{(\alpha_i - 1)}{\prod_{j=1, j\neq i}^{n-1} (z_i - z_j)^2}.$$

Multiplying and dividing each term in the summation by $(z_i - z_n)^2$, then isolating the last term,

$$a_{2n-3} = \sum_{i=1}^{n-1} \frac{(\alpha_i - 1)(z_i - z_n)^2}{\prod_{j=1, j\neq i}^{n} (z_i - z_j)^2}$$

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$$= \sum_{i=1}^{n-2} \frac{(\alpha_i - 1)(z_i - z_n)^2}{\prod_{j=1, j\neq i}^{n} (z_i - z_j)^2} + \frac{(\alpha_{n-1} - 1)(z_{n-1} - z_n)^2}{\prod_{j=1, j\neq n-1}^{n} (z_{n-1} - z_j)^2}. \quad (21)$$

Dividing each side of (21) by $(z_{n-1} - z_n)$,

$$\frac{a_{2n-3}}{(z_{n-1} - z_n)} = \frac{1}{(z_{n-1} - z_n)} \sum_{i=1}^{n-2} \frac{(\alpha_i - 1)(z_i - z_n)^2}{\prod_{j=1, j\neq i}^{n} (z_i - z_j)^2} + \frac{(\alpha_{n-1} - 1)(z_{n-1} - z_n)}{\prod_{j=1, j\neq n}^{n} (z_{n-1} - z_j)^2}. \quad (22)$$

Adding (20) to (22) and using (13), the left-hand side of the sum is a_{2n-2}. Combining terms of summations in the right-hand side of the sum gives

$$a_{2n-2} = \frac{1}{(z_{n-1} - z_n)} \sum_{i=1}^{n-2} \frac{(\alpha_i - 1)\left(-(z_i - z_{n-1})(z_i - z_n) + (z_i - z_n)^2\right)}{\prod_{j=1, j\neq n-1}^{n} (z_{n-1} - z_j)^2}$$

$$+ \frac{(\alpha_{n-1} - 1)(z_{n-1} - z_n)}{\prod_{j=1, j\neq n-1}^{n} (z_{n-1} - z_j)^2}.$$

Since $-(z_i - z_{n-1})(z_i - z_n) + (z_i - z_n)^2 = (z_i - z_n)(z_{n-1} - z_n)$, simplifying the above gives

$$a_{2n-2} = \sum_{i=1}^{n-2} \frac{(\alpha_i - 1)(z_i - z_n)}{\prod_{j=1, j\neq i}^{n} (z_i - z_j)^2} + \frac{(\alpha_{n-1} - 1)(z_{n-1} - z_n)}{\prod_{j=1, j\neq n-1}^{n} (z_{n-1} - z_j)^2}$$

$$= \sum_{i=1}^{n-1} \frac{(\alpha_i - 1)(z_i - z_n)}{\prod_{j=1, j\neq i}^{n} (z_i - z_j)^2}. \quad (23)$$

This proves the first equation in (12).

Next we derive the closed form for a_{2n-1} in (12). From the definition of $\hat{a}_{2n-3}$ in Definition 1, replacing the pair (z_{n-1}, α_{n-1}) with (z_n, α_n) in the expression for a_{2n-3} in (21) gives

$$\hat{a}_{2n-3} = \sum_{i=1}^{n-2} \frac{(\alpha_i - 1)(z_i - z_{n-1})^2}{\prod_{j=1, j\neq i}^{n} (z_i - z_j)^2} + \frac{(\alpha_n - 1)(z_n - z_{n-1})^2}{\prod_{j=1, j\neq n-1}^{n} (z_n - z_j)^2}. \quad (24)$$

Subtracting (24) from (21), combining summations and simplifying, then dividing by $-(z_{n-1} - z_n)^2$, gives

$$-\frac{(a_{2n-3} - \hat{a}_{2n-3})}{(z_{n-1} - z_n)^2} = -\frac{1}{(z_{n-1} - z_n)} \sum_{i=1}^{n-2} \frac{(\alpha_i - 1)(2z_i + z_{n-1} + z_n)}{\prod_{j=1, j\neq i}^{n} (z_i - z_j)^2}$$

$$- \frac{(\alpha_{n-1} - 1)}{\prod_{j=1, j\neq n-1}^{n} (z_{n-1} - z_j)^2} + \frac{(\alpha_n - 1)}{\prod_{j=1, j\neq n}^{n} (z_n - z_j)^2}. \quad (25)$$

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On the other hand, by multiplying each side of (23) by $2/(z_{n-1} - z_n)$ we may write

$$\frac{2a_{2n-2}}{(z_{n-1} - z_n)} = \frac{1}{(z_{n-1} - z_n)} \sum_{i=1}^{n-2} \frac{2(\alpha_i - 1)(z_i - z_n)}{\prod_{j=1, j\neq i}^{n} (z_i - z_j)^2} + \frac{2(\alpha_{n-1} - 1)}{\prod_{j=1, j\neq i}^{n} (z_{n-1} - z_j)^2}. \quad (26)$$

From Lemma 2, the sum of the left-hand sides of (25) and (26) is a_{2n-1}. Simplifying the sum of the right-hand sides, and using $-(2z_i + z_{n-1} + z_n) + 2(z_i - z_n) = (z_{n-1} - z_n)$, we get

$$\sum_{i=1}^{n-2} \frac{(\alpha_i - 1)}{\prod_{j=1, j\neq i}^{n} (z_i - z_j)^2} + \frac{(\alpha_{n-1} - 1)}{\prod_{j=1, j\neq n-1}^{n} (z_{n-1} - z_j)^2} + \frac{(\alpha_n - 1)}{\prod_{j=1, j\neq n}^{n} (z_n - z_j)^2} = \sum_{i=1}^{n} \frac{(\alpha_i - 1)}{\prod_{j=1, j\neq i}^{n} (z_i - z_j)^2}.$$

This verifies the second equation in (12) and completes the proof of Theorem 2.

6. *Bound on number of attractive fixed point lying on a line*

Theorem 3: Suppose the fixed points z_j, $j = 1, \ldots, n$ all lie on a straight line in the complex plane, and the corresponding multipliers satisfy $|\alpha_j| < 1$, $j = 1, \ldots, n$. Then the degree of $H_n(z)$ is exactly $2n - 1$.

Proof: To prove the theorem is to show a_{2n-1} in $H_n(z)$ is non-zero. First, assume the points do not lie on a vertical line. Then for some m, b real, all the z_j lie on the line $y = mx + b$. For $j = 1, \ldots, n$ write $z_j = x_j + \sqrt{-1}y_j$, where x_j and y_j are the real and imaginary parts of z_j, respectively. Then, for each $i = 1, \ldots, n$, $z_i - z_j = (x_i - x_j)(1 + m\sqrt{-1})$. Thus, for each $i = 1, \ldots, n$,

$$\prod_{j=1, j\neq i}^{n} (z_i - z_j) = c \prod_{j=1, j\neq i}^{n} (x_i - x_j),$$

where $c = (1 + m\sqrt{-1})^{n-1} \neq 0$.

For each $i = 1, \ldots, n$, $r_i = \prod_{j=1, j\neq i}^{n} (x_i - x_j)$ is a non-zero real number. From Theorem 2 we may write,

$$a_{2n-1} = \sum_{i=1}^{n} \frac{(\alpha_i - 1)}{c^2 r_i^2} = \frac{1}{c^2} \sum_{i=1}^{n} \frac{(\alpha_i - 1)}{r_i^2}.$$

Since c is a non-zero complex number, c^2 is non-zero. Thus to prove a_{2n-1} is non-zero is equivalent to showing $\sum_{i=1}^{n} \frac{(\alpha_i - 1)}{r_i^2}$ is non-zero. Each r_i^2 is a positive number. Furthermore, for each i the assumption that $|\alpha_i| < 1$ implies the real part of $(\alpha_i - 1)$ is negative. Hence a_{2n-1} is non-zero. When all the fixed points lie on a vertical line, then $z_i - z_j = \sqrt{-1}(y_i - y_j)$. It is easy to verify that again a_{2n-1} is non-zero.

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Theorem 4: Suppose $p(z)$ is a complex polynomial having n distinct attractive fixed points lying on a line, then its degree must be at least $2n - 1$.

Proof: Let these fixed points and their corresponding multipliers be z_j and α_j, $j = 1, \ldots, n$. Suppose the degree of $p(z)$ is less than $2n - 1$. Let $q(z) = H_n(z) - p(z)$, where $H_n(z)$ is as in Theorem 1. On the one hand, $q(z)$ is a polynomial of degree $2n - 1$. On the other hand, since $q'(z_j) = 0$, $(z - z_j)^2$ must divide $q(z)$ for each $j = 1, \ldots, n$. This implies the degree of $q(z)$ is at least $2n$, a contradiction.

Remark 1: The polynomial $H_n(z)$ derived in the Theorem 1 is in fact a special case of a *Hermite interpolating polynomial* (see e.g. [6] for a general definition), defined in terms of the *divided differences*: Set $f[z_i] = z_i$ and $f[z_i, z_i] = \alpha_i$; then recursively compute $f[z_1, z_1, z_2]$, $f[z_1, z_1, z_2, z_2]$, …, using the formula

$$f[z_{i_1}, \ldots, z_{i_k}] = \frac{(f[z_{i_2}, \ldots, z_{i_k}] - f[z_{i_1}, \ldots, z_{i_{k-1}}])}{z_{i_k} - z_{i_1}}.$$

It turns out that the coefficients defined in (12) coincide with the divided differences as follows:

$$a_{2k-2} = f[z_1, z_1, \ldots, z_{k-1}, z_{k-1}, z_k],\ a_{2k-1} = f[z_1, z_1, \ldots, z_k, z_k],\ k = 2, \ldots, n.$$

While our derivation of the explicit formula for H_n is inspired by Hermite interpolation formula, its proof is an independent result in the sense that it is complete in itself and does not rely on any previous results.

Furthermore, the divided difference formula does not by itself give the explicit formula for H_n proved in Theorem 2.

7. *Conjecture on non-attractive fixed points*

Theorem 4 implies that a polynomial of degree n can have at most $\lceil \frac{1}{2}n \rceil$ attractive fixed points on a line. This upper bound on the number of attractive fixed points is no longer valid when the roots are not restricted to lie on a line. Consider the polynomial

$$p(z) = \frac{-z^{n+1} + (n+1)z}{n}.$$

It can be checked that the fixed points of $p(z)$ are 0 and the n-th roots of unity. Because $p'(z) = (n+1)(1 - z^n)$, the n th roots of unity are the zeros of $p'(z)$ and hence they are attractive fixed points of $p(z)$. Thus $p(z)$ is an example of a polynomial of degree $n + 1$ with n attractive fixed points.

The question arises whether all the fixed points of a polynomial can be attractive. The answer is 'no'. This can be proved as follows. Suppose $p(z)$ is a polynomial of degree n. The *basin of attraction* of a fixed point θ of $p(z)$ is the set of all complex numbers, z_0, such that the iteration given by $z_{k+1} = p(z_k)$, when started at z_0, converges to θ. It is known that the basin

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of attraction of each attractive fixed point of a polynomial must contain a *critical point*; that is, a point where $p'(z) = 0$ (see, for example, Theorem 5.32 in [3], or [4]). However, $p'(z)$ is a polynomial of degree $n - 1$ and hence, by the fundamental theorem of algebra it has only $n - 1$ zeros. So not all the fixed points of $p(z)$ are attractive. By the way, we can ignore the case where the fixed point equation $p(z) = z$ has a zero of multiplicity greater than 1, as it can be checked that in this case the fixed point is not attractive.

In fact we make a conjecture (*the real non-attractive fixed point conjecture*): a complex polynomial of degree $n > 1$, has at least one fixed point whose multiplier has real part at least 1. We prove this conjecture for $n = 2$ and $n = 3$.

The proof for $n = 2$ is straightforward. Let $p(z)$ be a polynomial of degree 2 with fixed points z_1 and z_2. It follows that z_1 and z_2 are zeros of the quadratic polynomial $p(z) - z$, and hence $p(z) = c(z - z_1)(z - z_2) + z$ for some non-zero constant c. Then $p'(z_1) = c(z_1 - z_2) + 1$ and $p'(z_2) = c(z_2 - z_1) + 1$. The real parts of $c(z_1 - z_2)$ and $c(z_2 - z_1)$ have oppositive signs. Hence at least one of the real parts of $p'(z_1)$ and $p'(z_2)$ is at least 1.

Next we consider the case where $n = 3$. Let $p(z)$ be a polynomial of degree 3 with fixed points z_1, z_2 and z_3. It follows that, for some non-zero constant c, $p(z) = c(z - z_1)(z - z_2)(z - z_3) + z$. Then

$$p'(z) = c\big((z - z_2)(z - z_3) + (z - z_1)(z - z_3) + (z - z_1)(z - z_2)\big) + 1.$$

Hence the multipliers are given by $p'(z_1) = c(z_1 - z_2)(z_1 - z_3) + 1$, $p'(z_2) = c(z_2 - z_1)(z_2 - z_3) + 1$ and $p'(z_3) = c(z_3 - z_1)(z_3 - z_2) + 1$. Let the complex numbers α_1, α_2 and α_3 be defined as

$$\alpha_1 = a(z_2 - z_3) = a_1 + \sqrt{-1}b_1,\ \alpha_2 = a(z_3 - z_1) = a_2 + \sqrt{-1}b_2,$$
$$\alpha_3 = a(z_1 - z_2) = a_3 + \sqrt{-1}b_3, \tag{27}$$

where a is chosen so that $a^2 = c$, and a_1, a_2, a_3, b_1, b_2, b_3 are all real. We note that $\alpha_1 + \alpha_2 + \alpha_3 = 0$ and the multipliers $p'(z_1)$, $p'(z_2)$ and $p'(z_3)$ are $1 - \alpha_2\alpha_3$, $1 - \alpha_1\alpha_3$ and $1 - \alpha_1\alpha_2$, respectively.

At least two of b_1, b_2, b_3 have the same sign and hence their product is non-negative. We assume, without loss of generality, that

$$b_1b_2 \geqslant 0. \tag{28}$$

We need to show that the real part of at least one of $1 - \alpha_2\alpha_3$, $1 - \alpha_1\alpha_3$, $1 - \alpha_1\alpha_2$ is at least 1, that is, that the real part of at least one of $\alpha_2\alpha_3$, $\alpha_1\alpha_3$, $\alpha_1\alpha_2$ is at most 0.

Suppose not. Then by (27), the real parts of these products are $a_2a_3 - b_2b_3$ and so on. Hence, by our assumption that none of these is at most 0,

$$a_2a_3 > b_2b_3 \tag{29}$$
$$a_1a_3 > b_1b_3 \tag{30}$$

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and

$$a_1a_2 > b_1b_2. \tag{31}$$

Because $\alpha_1 + \alpha_2 + \alpha_3 = 0$, $a_3 = -a_1 - a_2$ and $b_3 = -b_1 - b_2$. Hence, by (29) and (30),

$$a_1^2 + a_1a_2 < b_1^2 + b_1b_2, \qquad a_2^2 + a_1a_2 < b_2^2 + b_1b_2. \tag{32}$$

It follows from (31) and (32) that

$$a_1^2 < b_1^2, \qquad a_2^2 < b_2^2$$

and hence that

$$a_1^2a_2^2 < b_1^2b_2^2. \tag{33}$$

The inequality (33) contradicts the fact that, by (28) and (31), $a_1a_2 > b_1b_2 \geqslant 0$.

This contradiction shows that the real part of at least one of $1 - \alpha_2\alpha_3$, $1 - \alpha_1\alpha_3$, $1 - \alpha_1\alpha_2$ is at least 1, and hence completes the proof of the conjecture for the case $n = 3$.

Final remarks

In this Article we have presented a closed formula for a polynomial with prescribed set of fixed points and prescribed multipliers. A complex polynomial of degree n can have at most $n - 1$ attractive fixed points. However, our closed formula implies a complex polynomial of degree n can have at most $\lceil \frac{1}{2}n \rceil$ attractive fixed points lying on any line in the complex plane. The question may arise if this property of polynomials can also be proved from existing results in dynamical systems. This is not known to us; even if this is possible, it is doubtful that such a proof can be established via the elementary technique given here. Indeed it is interesting to note that our formula for a_{2n-1} implies there are other configuration of attractive fixed points that may still make this coefficient to be non-zero. This Article also introduced the 'real non-attractive fixed point conjecture' and proved it for $n = 2, 3$. It would be interesting to prove this for all n, and, if true, via elementary techniques.

Finally, we feel the analysis helps introduce the notion of fixed points of a polynomial in basic mathematics courses in high school and college, promoting algorithmic methods for solving polynomial equations, as well as concepts in discrete dynamical systems, aided by the visualisation of iterative techniques (see e.g. [3]) which gives rise to spectacular fractal and non-fractal images, *polynomiographs*. These in turn will help student and teacher appreciation of polynomials and will help promote novel applications. We end by suggesting a related project for the interested reader. Consider constructing $H_n(z)$. For example, consider the case where z_i, $i = 1, \ldots, 5$ are $-2, -1, 0, 1, 2$ with respective multipliers

$$\frac{-1}{5}, \frac{13}{10}, \frac{4}{5}, \frac{13}{10}, \frac{-1}{5}.$$

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The corresponding polynomial, H_5, has attractive fixed points at -2, 0, 2 and repulsive ones at -1, 1. It is not difficult to compute $H_5(z)$ to get

$$z - \frac{z(z^2 - 1)(z^2 - 4)}{20}.$$

More generally, the formula for H_n can be derived by using the divided differences. For larger degree polynomials it is advisable to write a computer code. Additionally, a code can be written to produce the fractal polynomigraphs that would result from the fixed point iterations applied to H_n. This will visually reveal the nature of fixed points and fixed point iterations, as well as leading to interesting images, even for a small degree polynomial.

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