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On the decay of singular inner functions

Thomas Ransford

Abstract. It is known that if $S(z)$ is a non-constant singular inner function defined on the unit disk, then $\min_{|z|\le r}|S(z)| \to 0$ as $r \to 1^-$. We show that the convergence can be arbitrarily slow.

A *singular inner function* on the unit disk $\mathbb{D}$ is a function of the form

$$S(z) := \exp\Big(-\int_{\mathbb{T}} \frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\mu(\theta)\Big) \quad (z \in \mathbb{D}), \tag{1}$$

where μ is a positive finite measure on the unit circle $\mathbb{T}$ that is singular with respect to linear Lebesgue measure. Such functions play an important role in the theory of Hardy spaces H^p. For example, according to the so-called canonical factorization theorem, every function in $H^p\,(0 < p \le \infty)$ that is not identically zero can be factorized in an essentially unique way as the product of an outer function in H^p, a Blaschke product and a singular inner function (see *e.g.,* [2, Theorem 1.12]).

As the name suggests, a singular inner function S is inner; *i.e.,* it is a bounded holomorphic function whose radial limits satisfy $|S^*(e^{i\theta})| = 1$ Lebesgue-a.e. on $\mathbb{T}$. In addition, the fact of being singular implies that S also satisfies $S^*(e^{i\theta}) = 0$ μ-a.e. on $\mathbb{T}$ (see *e.g.,* [2, Lemma 1.8]). In particular, if S is non-constant (so $\mu \not\equiv 0$), then S has at least one radial limit equal to zero, and hence,

$$\min_{|z|\le r}|S(z)| \longrightarrow 0 \quad (r \to 1^-). \tag{2}$$

Using formula (1), it is easy to show that

$$\inf_{0\le r<1}\Big(\min_{|z|\le r}|S(z)|\Big)^{1-r} > 0. \tag{3}$$

This imposes a constraint on how rapidly the minimum in (2) can converge to zero, and, in fact, this constraint is optimal (consider the case when μ is a point mass). In this note, we consider how *slowly* the minimum in (2) can converge to zero. The following theorem shows that the convergence can be arbitrarily slow.

Theorem *Let $\omega : [0,1) \to (0,1)$ be a function such that $\sup_{r\in[0,1)}\omega(r) < 1$ and $\lim_{r\to 1^-}\omega(r) = 0$. Then there exists a non-constant singular inner function S on $\mathbb{D}$*

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such that

$$\min_{|z|\le r} |S(z)| \ge \omega(r) \quad \big(r \in [0,1)\big). \tag{4}$$

Proof Replacing ω by a slightly larger, piecewise-linear function, if necessary, we can assume that, in addition to its other properties, ω is a continuous, decreasing function on $[0,1)$.

Define two auxiliary functions $g, h : (0,\pi] \to (0,\infty)$ by

$$g(t) := \frac{1}{3\pi} \log \frac{1}{\omega(1-t/\pi)} \quad \text{and} \quad h(t) := \min\Big\{\sqrt{t}, \inf_{s\in[t,\pi]} sg(s)\Big\}.$$

Then g is a positive, continuous, decreasing function on $(0,\pi]$ that satisfies $\lim_{t\to 0^+} g(t) = \infty$, and h is a positive, continuous, increasing function on $(0,\pi]$ such that $\lim_{t\to 0^+} h(t) = 0$. In addition, we clearly have $h(t) \le tg(t)$ for all $t \in (0,\pi]$. Notice also that, for $t \in (0,1]$,

$$\frac{h(t)}{t} = \min\left\{\frac{\sqrt{t}}{t}, \inf_{s\in[t,\sqrt{t}]} \frac{sg(s)}{t}, \inf_{s\in[\sqrt{t},\pi]} \frac{sg(s)}{t}\right\} \ge \min\left\{\frac{1}{\sqrt{t}}, g(\sqrt{t}), \frac{g(\pi)}{\sqrt{t}}\right\},$$

and so $\lim_{t\to 0^+} h(t)/t = \infty$.

The properties of h (continuous, increasing, $\lim_{t\to 0+} h(t) = 0$) make it a so-called measure function, which can be used to define an h-Hausdorff measure, Λ_h (see *e.g.*, [1, Chapter II]). We shall consider Λ_h on the unit circle $\mathbb{T}$, thought of as a metric space with the arclength distance.

Since $\lim_{t\to 0^+} h(t)/t = \infty$ and $\mathbb{T}$ has positive linear measure, we have $\Lambda_h(\mathbb{T}) > 0$ (in fact even $\Lambda_h(\mathbb{T}) = \infty$). By [1, Chapter II, Theorem 3], there exists a closed subset E of $\mathbb{T}$ such that $0 < \Lambda_h(E) < \infty$. Again using the fact that $\lim_{t\to 0^+} h(t)/t = \infty$, the inequality $\Lambda_h(E) < \infty$ implies that E is of linear measure zero.

Now we apply a theorem of Frostman [1, Chapter II, Theorem 1]. The fact that $\Lambda_h(E) > 0$ implies that there is a finite positive measure μ on E such that $\mu(E) > 0$ and $\mu([\theta - t, \theta + t]) \le h(t)$ for all $\theta \in [-\pi,\pi]$ and $t \in (0,\pi]$. Since E is of linear measure zero, μ is singular with respect to Lebesgue measure. Define S by (1). Then S is a non-constant singular inner function, and we will show that it satisfies the estimate (4).

Fix $r \in [0,1)$ and $\theta \in [-\pi,\pi]$. We have

$$-\log|S(re^{i\theta})| = \int_{\mathbb{T}} P_r(t-\theta)\, d\mu(t),$$

where $P_r(\cdot)$ denotes the Poison kernel, namely,

$$P_r(t-\theta) = \frac{1-r^2}{1+r^2-2r\cos(t-\theta)}.$$

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We estimate the integral by splitting it into two, according to whether $|t-\theta|$ is smaller than or greater than ε. Here, $\varepsilon \in (0,\pi]$ is a constant, to be chosen later. First, we have

$$\begin{aligned}\int_{|t-\theta|\le\varepsilon} P_r(t-\theta)\,d\mu(t) &\le \int_{|t-\theta|\le\varepsilon} \frac{1-r^2}{(1-r)^2}\,d\mu(t)\\ &\le \frac{1-r^2}{(1-r)^2}\mu\big([\theta-\varepsilon,\theta+\varepsilon]\big)\\ &\le \frac{1+r}{1-r}h(\varepsilon) \le \frac{2}{1-r}\varepsilon g(\varepsilon).\end{aligned}$$

Secondly, using integration by parts, we have

$$\begin{aligned}\int_{|t-\theta|>\varepsilon} (P_r(t-\theta)-P_r(\pi))\,d\mu(t) &= \int_{|t-\theta|>\varepsilon}\int_{s=|t-\theta|}^{\pi} -P_r'(s)\,ds\,d\mu(t)\\ &= \int_{s=\varepsilon}^{\pi}\int_{|t-\theta|\le s} -P_r'(s)\,d\mu(t)\,ds\\ &= \int_{s=\varepsilon}^{\pi} \mu\big([\theta-s,\theta+s]\big)\big(-P_r'(s)\big)\,ds\\ &\le \int_{s=\varepsilon}^{\pi} h(s)\big(-P_r'(s)\big)\,ds\\ &\le \int_{s=\varepsilon}^{\pi} g(s)\big(-sP_r'(s)\big)\,ds\\ &\le g(\varepsilon)\int_{s=0}^{\pi} -sP_r'(s)\,ds\\ &= -\pi P_r(\pi)g(\varepsilon) + g(\varepsilon)\int_0^{\pi} P_r(s)\,ds\\ &= -\pi P_r(\pi)g(\varepsilon) + \pi g(\varepsilon),\end{aligned}$$

whence

$$\begin{aligned}\int_{|t-\theta|>\varepsilon} P_r(t-\theta)\,d\mu(t) &\le -\pi P_r(\pi)g(\varepsilon) + \pi g(\varepsilon) + P_r(\pi)\mu(\mathbb{T})\\ &\le -\pi P_r(\pi)g(\varepsilon) + \pi g(\varepsilon) + P_r(\pi)\pi g(\pi) \le \pi g(\varepsilon).\end{aligned}$$

Putting these estimates together, we obtain

$$-\log|S(re^{i\theta})| \le \Big(\frac{2\varepsilon}{1-r}+\pi\Big)g(\varepsilon).$$

In particular, taking $\varepsilon = \pi(1-r)$, we get

$$-\log|S(re^{i\theta})| \le 3\pi g(\pi(1-r)) = \log\frac{1}{\omega(r)}.$$

Rearranging this equality and minimizing over θ yields

$$\min_{|z|=r}|S(z)| \ge \omega(r),$$

which is equivalent to (4), by the minimum principle. ∎

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Département de mathématiques et de statistique, Université Laval, Québec, QC G1V 0A6, Canada
e-mail: ransford@mat.ulaval.ca

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