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Limits of commutative triangular systems on real and p-adic groups

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1. *Introduction*

A fundamental theorem of Khinchin says that every limit of an infinitesimal triangular system of probability measures on $\mathbf{R}$ is infinitely divisible. This was generalized to all divisible locally compact second countable abelian groups by Parthasarathy *et al.* (cf. [**PRV**]). Recently, Ruzsa eliminated the second countability condition and also proved the theorem for all Banach spaces (cf. [**R2**]). A similar theorem was also proved by Gangolli for certain symmetric spaces (cf. [**G**]). A result of Carnal shows that infinite divisibility of limits holds for commutative infinitesimal triangular systems on compact groups (cf. [**C**]). The same was recently proved by Neuenschwander for simply connected step-2 nilpotent Lie groups, provided the system is symmetric or supported on a discrete subgroup (cf. [**N**]).

Here we study the infinitesimal divisibility and embeddability of the limit μ of commutative infinitesimal triangular systems on more general locally compact groups G under certain conditions, such as when the system is symmetric or when G is a real or p-adic algebraic group and $\operatorname{supp}\mu$ is not contained in any proper algebraic subgroup of G (see Theorems 1·1, 1·2).

Let S be a Hausdorff second countable semigroup with identity e and let $\Delta = \{a_{ij} \in S : i \in \mathbf{N},\ 1 \leqslant j \leqslant n_i,\ n_i \to \infty\}$ be a triangular system. It is said to be *commutative* if for all $i \in \mathbf{N}$, $a_{ij}a_{ik} = a_{ik}a_{ij}$ for all $1 \leqslant j,\ k \leqslant n_i$. It is said to be *infinitesimal* if for every neighbourhood U of e in S, there exists an $i_0 \in \mathbf{N}$ such that $a_{ij} \in U$ for all $i > i_0$ and $1 \leqslant j \leqslant n_i$. Δ is said to be *convergent* if the sequence of row-wise product $\{a_i = \prod_{j=1}^{n_i} a_{ij}\}$ converges in S; if μ is its limit then we say that Δ *converges* to μ. In a topological semigroup S with identity e, any element $s \in S$ is said to be *infinitesimally divisible* if for every neighbourhood U of e, $s = s_1, \ldots, s_n$ for some $n = n(U) \in \mathbf{N}$ and $s_i \in U$ such that $s_i s_j = s_j s_i$ for all i, j.

Let G be a locally compact group and let $M^1(G)$ be the topological semigroup of all probability measures with weak topology and the convolution operation. A measure $\mu \in M^1(G)$ is said to be *infinitely divisible* if for every $n \in \mathbf{N}$, there exists $\mu_n \in M^1(G)$ such that $\mu_n^n = \mu$; and it is said to be *embeddable* if there exists a continuous one-parameter convolution semigroup $\{\mu_t\}_{t \geqslant 0}$ such that $\mu_1 = \mu$. Let $M_{\mathbf{C}}(G)$ be the complex Banach algebra of all bounded complex measures on G with the usual norm (cf. [**H**], p. 31). For $\lambda \in M_{\mathbf{C}}(G)$, the adjoint of λ is denoted by $\tilde{\lambda}$ and is defined as $\tilde{\lambda}(B) = \overline{\lambda(B^{-1})}$, (here $B^{-1} = \{g^{-1} : g \in B\}$), for all Borel sets B. We call a measure $\lambda \in M_{\mathbf{C}}(G)$ *normal* (resp. *symmetric*) if $\lambda\tilde{\lambda} = \tilde{\lambda}\lambda$ (resp. $\lambda = \tilde{\lambda}$). We call a triangular system $\Delta = (\mu_{ij})_{i \in \mathbf{N}, j=1}^{n_i}$ in $M^1(G)$ *normal* (resp. *symmetric*) if μ_{ij} is normal (resp. symmetric) for all i and j. Obviously any symmetric triangular system is normal.

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G is said to be 2-*root compact* if given any compact subset C of G there exists a compact subset C' of G, such that if for $x \in G$, $(xC)^2 \cap C \neq \varnothing$ then $x \in C'$. There is a large class of groups which are 2-root compact (see section 3·1 in [**H**]). In particular, compact groups, compactly generated nilpotent groups, solvable Lie groups whose Lie algebra admits only real roots and Zariski connected nilpotent p-adic algebraic groups are 2-root compact (for p-adic groups, see 3·1·9 of [**H**] and use lemma 2 of [**S1**] and induction).

Throughout, G is always locally compact, Hausdorff and second countable. For a $g \in G$, let δ_g in $M^1(G)$ denote the point mass at g and $g\lambda = \delta_g \lambda$ for any $\lambda \in M^1(G)$. For a $\lambda \in M^1(G)$, let $G(\lambda)$ denote the smallest closed subgroup containing supp λ in G. For a compact subgroup H in G, let ω_H be the Haar measure of H and let $M^1_H(G) = \omega_H M^1(G) \omega_H$. Then $M^1_H(G)$ is a closed subsemigroup of $M^1(G)$ with identity ω_H.

THEOREM 1·1. *Let G be a group and let $\Delta = (\mu_{ij})_{i \in \mathbf{N}, j=1}^{n_i}$ be a commutative infinitesimal symmetric triangular system in $M^1(G)$ converging to μ. Then*

(1) *if for all i, j, $\mu_{ij} = t_{ij}\tilde{t}_{ij}$ for some commutative normal system $(t_{ij})_{i \in \mathbf{N}, j=1}^{n_i}$ then μ is infinitesimally divisible in $M^1_H(G)$ for some compact group H in G. μ is embeddable if G is a simply connected solvable Lie group.*

(2) *if G is 2-root compact then for some $x \in G$, $x^2\mu = \mu$ and $x\mu$ is infinitesimally divisible; moreover, if G is a connected nilpotent Lie group or simply connected solvable Lie group, then μ itself is embeddable.*

Remark. The second part in the above theorem generalizes a theorem of Neuenschwander (cf. [**N**], theorem 1) about symmetric systems on simply connected step-2 nilpotent groups to all connected nilpotent Lie groups and all simply connected solvable Lie groups whose Lie algebra admits only real roots.

Let G be any locally compact group and let Z be the centre of G, let us define an equivalence relation '$\sim$' as follows: for $\lambda, \nu \in M^1(G)$, $\lambda \sim \nu$ if $\lambda = z\nu$ for some $z \in Z$. Then $M'(G) = M^1(G)/\sim$ is a semigroup and the map $\pi : M^1(G) \to M'(G)$ is a continuous open homomorphism. Let $M'_H(G) = \pi(M^1_H(G))$.

THEOREM 1·2. *Let G be a real or p-adic algebraic group and let Δ be a commutative infinitesimal triangular system in $M^1(G)$ converging to μ such that* supp μ *is not contained in any proper algebraic subgroup of G. Then $\pi(x\mu)$ is infinitesimally divisible in $M'_H(G)$ for some $x \in G$ and a compact group H. If G is a real algebraic group then $x\mu$ is infinitesimally divisible in $M^1_H(G)$ for some $x \in G$ such that $x^m = e$ for some m; moreover if G is a connected nilpotent group then μ itself is embeddable. If G is a simply connected solvable real algebraic group and Δ is normal then μ is embeddable.*

Remark. In this theorem the infinitesimal divisibility of $\pi(x\mu)$ will imply embeddability of $y\mu$ for some $y \in G$ provided one has a continuous λ-norm f_λ such that $f_\lambda(\lambda) > 0$, for all λ which 'divide' $\pi(x\mu)$ (see Section 2 for the definition of λ-norm).

THEOREM 1·3. *Let G be a group and let $\{\nu_n^{k_n}\}$ be a sequence in $M^1(G)$ converging to μ, where $\{k_n\}$ is an unbounded sequence in $\mathbf{N}$. If for each n, $\nu_n = \lambda_n \tilde{\lambda}_n$ for some normal measure $\lambda_n \in M^1(G)$ then μ is embeddable. If G is 2-root compact and each ν_n is symmetric then there exists $x \in G$ such that $x^2\mu = \mu$ and $x\mu$ $(= \mu x)$ is embeddable.*

In Section 2, we give some preliminary results and prove Theorem 1·3, whose proof is rather more elementary than that of 1·1 and 1·2. We also state a theorem (see

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Theorem 2·3), which is a generalization of a theorem of Ruzsa (cf. [**R1**, **R2**]) on the existence of one-parameter semigroups in certain abelian semigroups. In Section 3, we use the theory of decomposition of Hun semigroups, due to Ruzsa and Szekely (cf. [**RS**]) in our study of symmetric triangular systems and construct certain Hun semigroups containing the limit measure μ. In Section 4, we deal with triangular systems on algebraic groups and give an equivalence relation on $M^1(G)$ to get an abelian semigroup with desired properties. In Section 5, we construct partial homomorphisms on factors of a given element of an abelian semigroup of $M^1(G)$ into $\mathbf{R}_+$. In Section 6, we prove Theorems 1·1 and 1·2 using Theorem 2·3.

2. *Preliminaries*

Let S be a Hausdorff semigroup. Given $a, b \in S$, we say that b *divides* a and write $b \mid a$ if $a = tb = bt$ for some $t \in S$. If both $b \mid a$ and $a \mid b$, we say that a and b are *associates*. A subset A of S is said to be *associate-free* if any $a, b \in A$ are associates then $a = b$. Let $T_a = \{b \in S : b \mid a\}$.

For $\lambda \in M^1(G)$, let T'_λ be defined as above, (here T'_λ is same as $F(\lambda)$, the set of two-sided factors of λ in $M^1(G)$). Let $T''_\lambda = \{\nu \in F(\lambda) : \nu = \gamma\tilde{\gamma}$, for some $\gamma \in M^1(G)\}$ and let $T'''_\lambda = \{\nu \in F(\lambda) : \nu = \tilde{\nu}\}$. It is easy to show that both T''_λ and T'''_λ are closed.

LEMMA 2·1. *Let G be a group and let $\{\lambda_n\}$ be a relatively compact sequence in $M^1(G)$. Let $\{\nu_n\}$ and $\{\nu'_n\}$ be sequences in $M^1(G)$ such that $\nu_n \nu'_n = \lambda_n$ for all n. Then $\{\nu_n \tilde{\nu}_n\}$ is relatively compact. In particular, for any $\lambda \in M^1(G)$, T''_λ is compact.*

Proof. There exists a sequence $\{x_n\}$ in G such that $\{\nu_n x_n\}$ is relatively compact (cf. [**P**], chapter III, theorem 2·2). This implies that $\{\nu_n \tilde{\nu}_n\}$ $(= \{\nu_n x_n x_n^{-1} \tilde{\nu}_n\})$ is relatively compact. Let $\{\alpha_n = \nu_n \tilde{\nu}_n\}$ be a sequence in T''_λ for some $\lambda \in M^1(G)$. Then there exists $\{\beta_n\} \in M^1(G)$ such that $\alpha_n \beta_n = \nu_n \tilde{\nu}_n \beta_n = \lambda$ for all n. Now applying the above for $\lambda_n = \lambda$ for all n, we get that $\{\alpha_n\}$ is relatively compact, and all its limits are contained in T''_λ. Hence T''_λ is compact for all $\lambda \in M^1(G)$.

LEMMA 2·2. *Let G be a 2-root compact group. Let $\{\lambda_n\}$ be a relatively compact sequence in $M^1(G)$. Let $\{\nu_n\}$ be a sequence consisting of symmetric measures in $M^1(G)$ such that for all n, $\lambda_n = \nu_n * \nu'_n$ for some $\{\nu'_n\} \subset M^1(G)$. Then $\{\nu_n\}$ is relatively compact. In particular, for any $\lambda \in M^1(G)$, T'''_λ is compact.*

Proof. Since each ν_n is symmetric, by Lemma 2·1, $\{\nu_n^2\}$ is relatively compact. Since G is 2-root compact, theorem 3·1·4 of [**H**] implies that $\{\nu_n\}$ is relatively compact. Put $\lambda_n = \lambda$ for all n and the above will imply the second assertion.

Proof of Theorem 1·3. Let G be a group and let $\{\nu_n^{k_n}\}$ be a sequence in $M^1(G)$ converging to μ, where each ν_n is symmetric and $k_n \to \infty$. Let $A = \{\nu_n^m : m \leqslant k_n, n \in \mathbf{N}\}$. Let for each n, $\nu_n = \lambda_n \tilde{\lambda}_n$, $\lambda_n \in M^1(G)$ is normal. Then $\nu_n^m = (\lambda_n)^m (\tilde{\lambda}_n)^m$ and by Lemma 2·1, A is relatively compact. Hence by theorem 3·6 of [**S2**], there exists an $x \in G$ such that $x\mu$ is embeddable; here $x\omega_H \in \bar{A}$ for the Haar measure ω_H of some compact subgroup H of G, such that $\omega_H \mu = \mu$. Any element $\nu \in \bar{A}$ is of the form $\nu = \lambda\tilde{\lambda}$. In particular, the identity $e \in \operatorname{supp}(x\omega_H)$. Then $x \in H$, and μ is embeddable.

Now let G be 2-root compact. Then by Lemma 2·2, A is relatively compact. Hence by theorem 3·6 of [**S2**], there exists an $x \in G$, such that $x\mu$ is embeddable; as earlier

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$x\omega_H \in \bar{A}$ such that $\omega_H \mu = \mu$. Since all the elements of $\bar{A}$ are symmetric, we get that $x^2 \in H$ and hence $x^2\mu = \mu$.

Let S be a Hausdorff semigroup. An element $a \in S$ is said to be an *idempotent* if $a^2 = a$. An $s \in S$ is said to have an idempotent divisor if there exists an idempotent $a \in S$ such that $a \mid s$. An idempotent $h \in S$ is said to be a *maximal idempotent divisor of s in S* if $h \mid s$ and if for some idempotent $a \in S$, $a \mid s$ and $h \mid a$ then $a = h$. Here, if such an h is unique in S, then h is said to be the *hair* of s (in S), and it is denoted by $h(s)$. If $h(s) = e$, the identity in S, then s is said to be *bald*. Let ω_H denote the Haar measure of some compact subgroup H of G. Any idempotent in $M^1(G)$ is of the form ω_H, for some H. Any $\lambda \in M^1(G)$ has a unique maximal idempotent divisor.

For any s in a Hausdorff semigroup S, let I_s be the set of all idempotent divisors of s in S. Let $J_s = \{t \in S : t\lambda = \lambda t = \lambda \text{ for some } \lambda \in I_s\}$. For $\lambda \in M^1(G)$, let $I(\lambda) = \{x \in G : x\lambda = \lambda x = \lambda\}$, then $I(\lambda)$ is a compact group and if $\nu \in J_\lambda$, then $\operatorname{supp} \nu \subset I(\lambda)$ and $\nu h(\lambda) = h(\lambda)\nu = h(\lambda)$. For $\lambda \in M^1(G)$, I_λ and J_λ are compact.

Let S be an abelian semigroup with identity e. Let $s \in S$ be such that $h(s)$ exists. Then by an *s-norm on S* we mean a function $f_s : T_s \to \mathbf{R}_+$ which is a partial homomorphism, that is, $f_s(ab) = f_s(a) + f_s(b)$ if $ab \mid s$, and is continuous at $h(s)$.

We now state a theorem which is a very slight variation of a theorem of Ruzsa (cf. [**R1**]).

THEOREM 2·3. *Let S be an abelian Hausdorff semigroup with identity e. Let a be a bald element in S such that T_a is compact and associatefree and a is the limit of an infinitesimal triangular system in S. Suppose for all $s \in T_a \setminus \{e\}$, there exists an s-norm f_s such that $f_s(s) > 0$. Then there exists a continuous homomorphism $\phi : R_+ \to S$ such that $\phi(1) = a$.*

3. *Symmetric triangular systems*

Given a triangular system $\Delta = (a_{ij})_{i \in \mathbf{N}, j=1}^{n_i}$ in a semigroup S, $\Delta(s) = (a_{s(i)\,s(j)})_{i \in \mathbf{N}, j=1}^{l_i}$ is said to be a *subsystem of* Δ if $1 \leqslant s(j) \leqslant s(l_i) \leqslant n_i$, and all $s(i)$s and all $s(j)$s are distinct. Δ is said to be *compact* if for any subsystem $\Delta(s)$ as above, the sequence of row-wise products $\{a(s)_i = \prod_{j=1}^{l_i} a_{s(i)\,s(j)}\}$ is relatively compact, and we call the limits of $\{a(s)_i\}$ as limits of the subsystem $\Delta(s)$. If Δ is compact then let $L = L(\Delta)$ be the limits of all the subsystems of Δ. Clearly, L is closed.

From now on let $\Delta = (\mu_{ij})_{i \in \mathbf{N}, j=1}^{n_i}$ be a commutative infinitesimal symmetric triangular system converging to μ in $M^1(G)$. Let for all i and j, $\mu_{ij} = t_{ij}\tilde{t}_{ij}$, $\mu_i = \prod_{j=1}^{n_i} \mu_{ij}$ and $t_i = \prod_{j=1}^{n_i} t_{ij}$, where $(t_{ij})_{i \in \mathbf{N}, j=1}^{n_i}$ is commutative and normal. If $\lambda, \nu \in M^1(G)$ are such that $\lambda\nu = \nu\lambda$ and $\nu\tilde{\nu} = \tilde{\nu}\nu$ then $\lambda\tilde{\nu} = \tilde{\nu}\lambda$. Hence we get that $\mu = \lim_{i \to \infty} \mu_i = \lim_{i \to \infty} t_i \tilde{t}_i$. By Lemma 2·1, Δ is compact. Let $L = L(\Delta)$ be the set of limits of all the subsystems of Δ. Then $L \subset T_\mu'$ and $\mu \in L$. Let $h(\mu)$ be the maximal idempotent divisor of μ in $M^1(G)$ and let $J_\mu = \{\lambda \in M^1(G) : \lambda h(\mu) = h(\mu)\lambda = h(\mu)\}$. Then J_μ is a compact semigroup.

Let S be a semigroup. For a $\lambda \in S$ and a $V \subset S$, we say that λ has a *V-decomposition* if for some $n \in \mathbf{N}$, $\lambda = \lambda_1 \ldots \lambda_n$ such that $\lambda_i \in V$ and $\lambda_i \lambda_j = \lambda_j \lambda_i$ for all i and j; and we will call $\lambda_1 \ldots \lambda_n$ a V-decomposition of λ; moreover, if for some $A \subset S$, $\lambda_i \in A$ for all i, then we say that λ has a V-decomposition in A.

LEMMA 3·1. *Let the notations be as above. Let U be any neighbourhood of δ_e in $M^1(G)$. Then μ has a $J_\mu U$-decomposition in L.*

The proof is similar to that of statement 6·1 in [**R1**].

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Proof. Step 1. First we show that for every neighbourhood W of δ_e in $M^1(G)$, there exists $n = n(W)$ such that if for $m \geqslant n$, $s_1, \ldots, s_m \in T'_\mu$, $s_1, \ldots, s_n \notin J_\mu W$ and $s_i s_j = s_j s_i$ for all i and j then $s_1 \ldots s_m \neq \mu$. Let $t \in T'_\mu \backslash J_\mu W$. There exists $n(t) \in \mathbf{N}$, such that $t^{n(t)} \notin T'_\mu$. Suppose $t^n \in T'_\mu$ for all n. Then $\{t^n\}$ is relatively compact and hence $\bigcap_{k \geqslant 1} \overline{\{t^n : n \geqslant k\}}$ is a compact abelian group with identity, (say) $a \in T'_\mu$ (cf. [**H**], section 2·1). Then $ts = st = a$ for some $s \in T'_\mu$. This implies that $\operatorname{supp} t \subset \operatorname{supp} a$. Therefore $ta = a$ and hence $t \in J_\mu$. This contradicts the fact that $t \notin J_\mu W$ and hence there exists $n(t)$ as above. Let $V(t^{n(t)})$ be a neighbourhood of $t^{n(t)}$ such that $T'_\mu \cap V(t^{n(t)}) = \varnothing$. Then there exists a neighbourhood $V(t)$ of t such that $(V(t))^{n(t)} \subset V(t^{n(t)})$. Now since $T'_\mu \backslash J_\mu W$ is compact, there exists $t_1, \ldots, t_k \in T'_\mu \backslash J_\mu W$ such that $T'_\mu \backslash J_\mu W \subset \bigcup_{i=1}^k V(t_i)$. Let $n = \sum_{i=1}^k n(t_i)$.

Step 2. Let V be a neighbourhood of δ_e in $M^1(G)$ such that $J_\mu \bar{V} \cap T'_\mu \subset J_\mu U$ (see [**R1**], lemma 6·3). Choose an open neighbourhood W of δ_e such that $WW \subset V$. Then we have $\mu_{ij} \in W$ for all $i > i_0$ and all j. Let $n = n(W)$ as in step 1.

For a fixed $i (> i_0)$, put $m_0 = 0$. Given m_k, let m_{k+1} be the first subscript m for which $x_{ik} = \prod_{j=m_k+1}^{m} \mu_{ij} \notin J_\mu W$ if such an m exists; otherwise put $m_{k+1} = \ldots = m_n = n_i$ and $x_{i(k+1)} = \ldots = x_{in} = \delta_e$. Finally, having defined $m_0, \ldots, m_n$ and $x_{i1}, \ldots, x_{in}$, we put $y_i = \prod_{j=m_n+1}^{n_i} \mu_{ij}$, if $m_n = n_i$ then we put $y_i = \delta_e$. Clearly, all $x_{ik} \in J_\mu WW \subset J_\mu V$ and either $y_i = \delta_e$ or all x_{ik} are outside $J_\mu W$. Since Δ is compact, one can choose the same subsequence of $\{i \in \mathbf{N}\}$ such that $\{x_{ik}\}$ and $\{y_i\}$ converges to (say) x_k and y in T'_μ respectively. We have $x_1 \ldots x_n y = \mu$ and all x_ks commute and they commute with y. $x_k \in J_\mu \bar{V} \cap T'_\mu \subset J_\mu U$. If $y = \delta_e$, then the above is a $J_\mu U$-decomposition of μ in L. If $y \neq \delta_e$, then the set $\{i : y_i = \delta_e\}$ cannot be cofinal. Thus $x_k = \lim x_{ik} \notin J_\mu W$ for all $1 \leqslant k \leqslant n$, contradiction to Step 1 and the choice of n.

Remark. In the hypothesis of Lemma 3·1, every $\lambda \in L$ has a $J_\mu U$-decomposition in L.

A Hausdorff semigroup S with identity e is said to be a *Hun semigroup* if S is associatefree and T_a is compact for all $a \in S$.

Proposition 3·2. *Let Δ, μ, L and J_μ be as above. Then there exists a Hun semigroup M such that $\mu \in M$ and if $\lambda \in T_\mu$ in M then λ is infinitesimally divisible in M.*

Proof. Let $\{U_k\}$ be a neighbourhood basis of identity δ_e in $M^1(G)$. Then μ has a $J_\mu U_1$-decomposition in L as in Lemma 3·1. Now for $k \in \mathbf{N}$, let $\mu = s_1 \ldots s_{m(k)}$ be the $J_\mu U_k$-decomposition of μ in L, where each s_l is a limit of a subsystem of Δ, that is, $s_l = \lim_{i \to \infty} \prod_{j \in A(i,l)} \mu_{k(i)j}$, where $\{k(i)\} \subset \mathbf{N}$ is a fixed sequence for all l and $A(i,l) \subset \{1, \ldots, n_i\}$ depends on i and l. Hence using the fact that $T'_{s_l} \subset T'_\mu$ is compact, and that all s_ls commute, one can get a $J_\mu U_{k+1}$-decomposition of s_l in L for all l, (say) $s_l = s_{l1} \ldots s_{lm(l)}$, such that $s_{ln} s_{pq} = s_{pq} s_{ln}$ for all l, n, p, q. Here, for all l and n, $s_{ln} = \lim_{i \to \infty} \prod_{j \in A(i,l,n)} \mu_{(k+1)(i)j}$, where $\{(k+1)(i)\}$ is a fixed subsequence of $\{k(i)\}$ and $A(i,l,n) \subset A(i,l)$ depends on i, l, n. We continue this process.

For $k \in \mathbf{N}$, let $\mu = s_{k1} \ldots s_{km(k)}$ be a $J_\mu U_k$-decomposition of μ in L obtained in the above manner. Let M_k be the semigroup generated by $\{s_{k1}, \ldots, s_{km(k)}\}$ in $M^1(G)$ for every k. Then for all k, $M_k \subset M_{k+1}$, M_k is abelian and $\mu \in M_k$. Let $S_1 = \overline{\bigcup_k M_k}$. Then S_1 is a (metrizable) abelian semigroup containing μ. Since S_1 is the closure of the semigroup generated by elements of L, for any $\lambda \in S_1$, T_λ in S_1 is contained in T'_λ, and

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hence it is compact (cf. Lemma 2·1). For $\lambda, \nu \in S_1$, if $\lambda | \nu$ and $\nu | \lambda$ then using the fact that the identity e of group G is contained in the support of every measure in S_1, one can easily show that $\lambda = \nu$. Now let $J = J_\mu \cap S_1$. As each M_k is generated by $J_\mu U_k$-decomposition of μ, J is a non-empty compact abelian semigroup. Hence there exists an idempotent in J. Let I_μ be the set of idempotent divisors of μ in S_1. Then I_μ is a non-empty compact abelian semigroup. Hence there exists a maximal idempotent divisor $h = \omega_H$ of μ in S_1 such that if $a \in I_\mu$, then $ah = h$ (cf. [**R1**], lemma 5·1). Now if $t \in J$, then there exists an idempotent $a \in I_\mu$ such that $t^{n_k} \to a$ and $ta = a$. Then $Jh = h$. Let $M = hS_1$. Then M is a closed subsemigroup of S_1 with identity h. Hence M is a Hun semigroup.

Let U be any neighbourhood of identity h in M. Then $hU' \cap M \subset U$, for some neighbourhood U' of δ_e in $M^1(G)$. Let $\lambda \in T_\mu$ in M. Then $\lambda = \lambda' h$ for some $\lambda' \in S_1$. If $\lambda' = s_1^{k_1} \dots s_n^{k_n}$ where s_ls are generators of M_k for some k then using the fact that $J = J_\mu \cap S_1$ is a semigroup, it is easy to show that λ' has a JU'-decomposition in S_1, say $\lambda' = u_1 \dots u_m$, where each $u_i \in JU'$. Since $Jh = h$, $u_i h \in hU' \cap M \subset U$, for each i, and $\lambda = \lambda' h = (u_1 h) \dots (u_m h)$ is a U-decomposition of λ in T_μ. If $\lambda' = \lim \lambda'_n = s_{n1}^{n} s_{n2}^{n_2} \dots s_{nk(n)}^{n_k}$, where $\{s_{ni}\} \subset T_\mu$ are generators of S_1. Then using the U-decomposition of $\lambda'_n h$s we can show that λ is a limit of an infinitesimal triangular system in M. Since T_λ in M is compact and $h(\lambda) = h$ in M, this implies that λ has a U-decomposition in T_λ for every neighbourhood U of h in M (cf. [**R1**], statement 6·1).

Now let G be a 2-root compact group and $\Delta = (\mu_{ij})_{i \in \mathbf{N}, j=1}^{n_i}$ be a symmetric system converging to μ. Then Lemma 2·2 implies that Δ is compact. Let $L' = L(\Delta)$ be the set of limits of all subsystems of Δ in $M^1(G)$. $L' \subset T''_\mu$ and $\mu \in L'$. Let $h(\mu)$ be the maximal idempotent divisor of μ in $M^1(G)$ and let $K_\mu = \{\lambda \in T''_\mu : \lambda h(\mu) = h(\mu) \lambda = \delta_g h(\mu)$ for some $g = g(\lambda) \in G\}$. K_μ is compact. Note that if $t^n \in T'_\mu$ for all n, then there exists an idempotent $a \in T'_\mu$, such that $t^{n_k} \to a$ and hence $ta = at = \delta_g a$ and $t \in K_\mu$.

LEMMA 3·3. *Let G, Δ, μ, L' and K_μ be as above. Let U be any neighbourhood of δ_e in $M^1(G)$. Then μ has a $K_\mu U$-decomposition in L'.*

The proof is similar to that of Lemma 3·1 if we use K_μ instead of J_μ and the fact that if $\{t^n\} \subset T'_\mu$ then $t \in K_\mu$. One can also show that every element in L' has a $K_\mu U$-decomposition in L'.

PROPOSITION 3·4. *Let Δ, μ and K_μ be as above. Then there exist a closed subsemigroup M of $M^1(G)$ consisting of symmetric measures with identity h and an equivalence relation '$\sim$' on M such that $\mu \in M$, $h = h(\mu)$ in M and if $\psi : M \to M' = M/\sim$ is the natural map then M' is a Hausdorff semigroup and ψ is an open homomorphism. Moreover, for every element $\lambda \in M'$, T_λ in M' is compact, $T_{\psi(\mu)}$ is associatefree and $\psi(\mu)$ is bald and infinitesimally divisible in M'.*

Proof. Let $\{U_k\}$ be a neighbourhood basis for δ_e in $M^1(G)$. Let S_2 be the closed abelian subsemigroup of $M^1(G)$ containing μ constructed in the same manner as Proposition 3·2 using Lemma 3·3 instead of Lemma 3·1. That is $S_2 = \overline{\bigcup_k M_k}$ where M_k is a semigroup generated by some $K_\mu U_k$-decomposition of μ in L' and $M_k \subset M_{k+1}$ for all k. Any element $\lambda \in S_2$ is of the form $\lambda = \lim_{i \to \infty} \Pi_{j=a_i}^{b_i} \mu_{\lambda(i)j}^{c_{ij}}$, where $c_{ij} > 0$ or $\mu_{\lambda(i)j} = \delta_e$ and all a_i, b_i and c_{ij} depend on λ; λ is symmetric and T_λ in S_2 is compact as it is contained in T''_λ. Let $K = K_\mu \cap S_2$. Then it is easy to see that K is a compact abelian semigroup contained in T''_μ. Then as in the proof of Proposition 3·2, the set I_μ of

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idempotent divisors of μ in S_2 is a non-empty compact abelian semigroup and there exists $h = \omega_H$, the hair of μ in S_2. Then $I_\mu h = h$ and if $t \in K$, then there exists $a \in I_\mu$ such that $t^{n_k} \to a$ and $ta = at = \delta_g a$ for some $g \in G$. Hence $th = \delta_g h$ and since t is symmetric, $t^2 h = h$. This implies that $K' = Kh$ is a compact abelian group. Let $M = S_2 h$. Then M is an abelian semigroup with identity h, $\mu \in M$ and $K' \subset T_\mu$ in M. Let U be a neighbourhood of h in M. Then there exists a neighbourhood U' of δ_e in $M^1(G)$, such that $hK_\mu U' \cap M \subset K'U$. Now from the construction of S_2 and M, it is easy to see that μ has a $K'U$-decomposition in M.

Let $\lambda \mid \lambda'$ and $\lambda' \mid \lambda$ for some $\lambda, \lambda' \in T_\mu$ in M. Then there exists ν, $\nu' \in T_\mu$ in M such that $\lambda\nu = \lambda'$ and $\lambda'\nu' = \lambda$. Then $\lambda = \lambda\nu\nu'$. This implies that $\nu^n \mid \lambda$ and hence $\nu^n \mid \mu$ for all $n \in \mathbf{N}$. Thus $\nu \in K_\mu \cap M = K'$. Similarly, $\nu' \in K'$. That is $\lambda = k\lambda'$ for some $k \in K'$.

Let us define an equivalence relation '$\sim$' on M as follows:

$$\lambda, \nu \in M, \quad \lambda \sim \nu \quad \text{if} \quad \lambda = k\nu \quad \text{for some} \quad k \in K'.$$

Let $M' = M/\sim$ and $\psi : M \to M'$ be the natural map. It is easy to check that M' is a Hausdorff semigroup, ψ is an open homomorphism and for every $k \in K'$, $\psi(k)$ is identity in M'. Hence $\psi(\mu)$ has a U-decomposition for every neighbourhood U of identity in M'. From the above discussion, it is clear that $T_{\psi(\mu)}$ is associatefree. For every $\lambda \in M'$, $\psi^{-1}(T_\lambda) \subset T''_\nu$, where $\psi(\nu) = \lambda$, and hence T_λ is compact. It is easy to check that $\psi(\mu)$ is bald.

4. *Triangular systems on algebraic groups*

Let $\mathbf{Q}_p$, p a prime number, be the field of p-adic numbers equipped with the p-adic topology. In this section, let G be a real or p-adic algebraic group. That is, G is an algebraic subgroup of $GL(n, \mathbf{R})$ or $GL(n, \mathbf{Q}_p)$ for some $n \in \mathbf{N}$ (cf. [**B**], [**Ho**] or [**Hu**] for generalities on algebraic groups). A measure $\mu \in M^1(G)$ is said to have *full algebraic support* if there is no proper algebraic subgroup of G containing $\operatorname{supp} \mu$. Let Z be the centre of G and let Z^0 be the Zariski connected component of identity in Z. Since Z has finitely many Zariski connected components, Z/Z^0 is finite. Let $\pi : G \to G/Z^0$ be the natural projection.

THEOREM 4·1. *Let $\{\lambda_n\}$ be a convergent sequence in $M^1(G)$ converging to λ such that λ has full algebraic support on G. Let $\{\nu_n\}$ and $\{\nu'_n\}$ be sequences in $M^1(G)$ such that $\nu_n \nu'_n = \nu'_n \nu_n = \lambda_n$ for all n. Then there exists a sequence $\{z_n\} \subset Z^0$ such that $\{\nu_n z_n\}$ is relatively compact, in particular, $\{\pi(\nu_n)\}$ is relatively compact.*

The proof is same as theorem 3·2 of [**DM**] in the real case and theorem 2 of [**S1**] or theorem 3·2 of [**S2**] in the p-adic case. We will not go into details here.

In this section $\Delta = (\mu_{ij})_{i \in \mathbf{N}, j=1}^{n_i}$ be a commutative infinitesimal triangular system converging to μ such that μ has full algebraic support. Let us define an equivalence relation '$\approx$' on $M^1(G)$ as follows:

$$\lambda, \nu \in M^1(G), \quad \lambda \approx \nu \quad \text{if} \quad \lambda = z\nu \quad \text{for some} \quad z \in Z^0.$$

Let $M^*(G) = M^1(G)/\approx$ and $\psi_1 : M^1(G) \to M^*(G)$ be the natural map. It is easy to check that $M^*(G)$ is a Hausdorff second countable semigroup with $\psi_1(\delta_e)$ as identity and ψ_1 is an open homomorphism. Let $\psi_1(\lambda) = \bar{\lambda}$ for convenience. $\bar{\Delta} = (\bar{\mu}_{ij})_{i \in \mathbf{N}, j=1}^{n_i}$ is a commutative infinitesimal triangular system converging to $\bar{\mu}$ in $M^*(G)$. $\bar{\Delta}$ is

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infinitesimal in $M^*(G)$. Moreover, the above proposition implies that $\bar{\Delta}$ is compact in $M^*(G)$. Let $T_{\bar{\mu}}$ be the set of divisors of $\bar{\mu}$ in $M^*(G)$. Then $\psi_1^{-1}(T_{\bar{\mu}}) \subset F(\mu)$ and hence $T_{\bar{\mu}}$ is compact (cf. Proposition 4·1). Let $L_1 = L(\bar{\Delta})$ be the set of all the limits of subsystems of $\bar{\Delta}$. Then $L_1 \subset T_{\bar{\mu}}$ is compact and $\bar{\mu} \in L_1$. Any idempotent $a \in M^*(G)$ is of the form $\bar{\omega}_H = \psi_1(\omega_H)$. Let $h(\bar{\mu})$ be the maximal idempotent divisor of $\bar{\mu}$ in $M^*(G)$ and let $K_{\bar{\mu}} = \{\lambda \in T_{\bar{\mu}} : \lambda h(\bar{\mu}) = h(\bar{\mu})\,\lambda = \bar{\delta}_g\, h(\bar{\mu})$ for some $g = g(\lambda) \in G\}$. Then $K_{\bar{\mu}}$ is compact. Moreover, if $t^n \in T_{\bar{\mu}}$ for all n, then it is easy to see that $t \in K_{\bar{\mu}}$.

LEMMA 4·2. *Let G, Δ, μ, $\bar{\Delta}$, $\bar{\mu}$, $K_{\bar{\mu}}$ and L_1 be as above. Then given any neighbourhood U of the identity $\bar{\delta}_e$ in $M^*(G)$, $\bar{\mu}$ has a $K_{\bar{\mu}}U$-decomposition in L_1.*

The proof is similar to that of statement 6·1 of [**R1**] or Lemma 3·1.

PROPOSITION 4·3. *Under the hypothesis of the above lemma, there exists an abelian semigroup $S_3(h)$ with identity h, such that $\bar{\mu} \in S_3(h)$ and for every neighbourhood U of h in $S_3(h)$, $\bar{\mu}$ has a $K_1'U$-decomposition in $S_3(h)$, where $K_1' = K_{\bar{\mu}}h \cap S_3(h)$ is the compact group of all invertible elements in $S_3(h)$. Also $T_{\bar{\mu}}$ is compact in $S_3(h)$.*

Proof. Let $\{U_k\}$ be the neighbourhood basis for the identity $\bar{\delta}_e$ in $M^*(G)$. As in Proposition 3·2, for $n \in \mathbf{N}$, one can construct an increasing sequence of abelian semigroups M_k generated by the $K_{\bar{\mu}}U_k$-decomposition of $\bar{\mu}$. Let $S_3 = \overline{\bigcup_k M_k}$. S_3 is an abelian semigroup containing $\bar{\mu}$ and $T_{\bar{\mu}}$ in S_3 is compact. Let $K_1 = K_{\bar{\mu}} \cap S_3$. Then $K_1 \subset T_{\bar{\mu}}$ is a non-empty compact abelian semigroup and hence the set $I_{\bar{\mu}}$ of idempotent divisors of $\bar{\mu}$ in S_3 is also a non-empty compact abelian semigroup. Let $h = \bar{\omega}_H$ be the hair of $\bar{\mu}$ in S_3 and let $S_3(h) = S_3 h$. Then $S_3(h)$ is an abelian semigroup with identity h and $\bar{\mu} \in S_3(h)$. Let $K_1' = K_1 h = K_{\bar{\mu}}h \cap S_3(h)$. Then as in the proof of Proposition 3·4, $K_1' \subset T_{\bar{\mu}}$ is the compact abelian group of invertible elements in $S_3(h)$. Let U be any neighbourhood of h in $S_3(h)$. Then there exists a neighbourhood U' of $\bar{\delta}_e$ such that $hK_{\bar{\mu}}U' \cap S_3(h) \subset K_1'U$. Then from the definition of S_3 and $S_3(h)$, it is clear that $\bar{\mu}$ has a $K_1'U$-decomposition in $S_3(h)$.

Remark. Let $\{V_n \mid n \in \mathbf{N}\}$ be the neighbourhood basis of identity ω_H in $\psi_1^{-1}(S_3(h))$ where $h = \psi_1(\omega_H)$. Fix an $n \in \mathbf{N}$. $\psi_1(V_n)$ is the neighbourhood of the identity in $S_3(h)$ and we have that $\psi_1(\mu) = a_n v_1 v_2 \dots v_{m(n)}$, where $v_i \in \psi_1(V_n)$ for all i, and $a_n \in K_1'$. Since K_1' is a compact group, $\{a_n\}$ is relatively compact, let a be the limit of a convergent subsequence $\{a_{n_k}^{-1}\}$. Then $a\psi_1(\mu) = aa_{n_k} v_1 \dots v_{m(n_k)}$. Now $aa_{n_k} \to h$ as $n_k \to \infty$. Hence it is easy to show that $a\psi_1(\mu)$ has a $\psi_1(U)$-decomposition for all neighbourhood U of ω_H. That is $x\mu = zu_1 \dots u_n$ for some $n \in \mathbf{N}$, $z = z(U) \in Z^0$, $x \in \psi_1^{-1}(a)$ and $u_i \in U$ for all i. Now if π is defined as in Section 1, then $\pi(x\mu)$ is infinitesimally divisible in $M_H'(G)$. If G is a real algebraic group then any element in Z^0 is infinitesimally divisible in Z^0. Hence the above implies that $x\mu$ is infinitesimally divisible in $M_H^1(G)$. Moreover, if G_x is the closed (abelian) group generated by x and Z^0, then G_x has finitely many connected components and $G_x = G_x^0 \times F$ for some finite group F. Since any element in G_x^0 is infinitesimally divisible, we can choose x such that $x^m = e$, where m is the order of F.

Let $a \mid b$ and $b \mid a$ for some $a, b \in T_{\bar{\mu}}$ in $S_3(h)$. Then $ac = b$ and $bd = a$ for some $c, d \in T_{\bar{\mu}}$. Therefore, $a = acd$ and hence $c^n \mid a$ and $c^n \in T_{\bar{\mu}}$ for all n. This implies that $c \in K_{\bar{\mu}} \cap S_3(h) = K_1'$. Similarly, $d \in K_1'$. Hence $a = kb$ for some $k \in K_1'$.

Let us define an equivalence relation '$\cong$' on $S_3(h)$ as follows:

$$a, b \in S_3(h), \quad a \cong b \quad \text{if} \quad a = kb \quad \text{for some} \quad k \in K_1'.$$

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Let $S^* = S_3(h)/\cong$ and let $\psi_2 : S_3(h) \to S^*$ be the natural homomorphism then S^* is an abelian group with identity $h' = \psi_2(h)$ and ψ_2 is open. Let $\mu' = \psi_2(\bar{\mu}) = \psi_2(\psi_1(\mu))$.

From the discussion above, we get

PROPOSITION 4·4. *Let the notation be as above. Then $h(\mu') = h', T_{\mu'}$ (in S^*) is compact and associatefree and μ' is infinitesimally divisible in S^*.*

5. *Construction of homomorphisms*

In this section we construct s-norm (defined in Section 2) on a given abelian semigroup of $M^1(G)$ when G is a connected nilpotent Lie group or simply connected solvable Lie group.

THEOREM 5·1. *Let G be a connected nilpotent Lie group. Let S be a closed abelian subsemigroup of $M^1(G)$ with identity $h = \omega_H$. If λ is not a translate of an idempotent then there exits a continuous λ-norm f_λ on S such that $f_\lambda(\lambda) > 0$.*

Proof. Let G be a connected nilpotent Lie group. Let λ be not a translate of an idempotent. Let $N = G(\lambda\tilde{\lambda})$, the closed subgroup generated by $\operatorname{supp}(\lambda\tilde{\lambda})$ in G. Then $\operatorname{supp}(\nu\tilde{\nu}) \subset N$ for all $\nu \mid \lambda$. Let $N \subset Z$, the centre of G. Then if $\nu \mid \lambda$, $\nu\tilde{\nu} = \tilde{\nu}\nu$. This implies that for any $\nu' \in M^1(G)$, if $\nu\nu' = \nu'\nu$ then $\tilde{\nu}\nu' = \nu'\tilde{\nu}$. Therefore, $\nu\tilde{\nu} \mid \lambda\tilde{\lambda}$. Since Z is abelian $\lambda\tilde{\lambda}$ is not an idempotent and there exists a continuous $\lambda\tilde{\lambda}$-norm on Z such that $f_{\lambda\tilde{\lambda}}(\lambda\tilde{\lambda}) > 0$, (cf. [**R2**], statement 6·1). Now define $f_\lambda : T_\lambda \to \mathbf{R}_+$ by $f_\lambda(\nu) = f_{\lambda\tilde{\lambda}}(\nu\tilde{\nu})$ for all $\nu \in T_\lambda$.

Now suppose that N is not central in G. Let $Z_0 = Z$ and let $Z_{i+1} = \{x \in G \mid xgx^{-1}g^{-1} \in Z_i \text{ for all } g \in G\}$ for all $i \geqslant 0$. Then for each i, $Z_i \subset \mathbf{Z}_{i+1}$, Z_i is a closed, connected normal subgroup in G and Z_{i+1}/Z_i is the centre of G/Z_i. Moreover, there exists n, such that $Z_n = G$. Let $1 \leqslant m \leqslant n$ be such that $N \subset Z_m$ and $N \not\subset Z_{m-1}$. Let $\pi : G \to G/Z_{m-1}$ be the natural projection. Then G/Z_{m-1} is a unipotent group without any non-trivial compact subgroups (cf. [**V**], section 3·6). Then since $\pi(N)$ is non-trivial, $\pi(\lambda)$ is not a translate of an idempotent. In particular, $\pi(\lambda\tilde{\lambda})$ is not an idempotent. Also Z_m/Z_{m-1} is central in G/Z_{m-1}. Arguing as above, we get that there exists a $\pi(\lambda\tilde{\lambda})$-norm $f_{\pi(\lambda\tilde{\lambda})}$ on $M^1(G/Z_{m-1})$ such that $f_{\pi(\lambda\tilde{\lambda})}(\pi(\lambda\tilde{\lambda})) > 0$. Now we define $f_\lambda : T_\lambda \to \mathbf{R}_+$ as $f_\lambda(\nu) = f_{\pi(\lambda\tilde{\lambda})}(\pi(\nu\tilde{\nu}))$. Clearly, f_λ is a continuous λ-norm on S and $f_\lambda(\lambda) > 0$.

PROPOSITION 5·2. *Let G be a simply connected solvable Lie group. Let S be a closed abelian subsemigroup of $M^1(G)$ with identity $h = \omega_H$ such that S consists of normal measures. If λ is not a translate of an idempotent then there exists a continuous λ-norm f_λ on S such that $f_\lambda(\lambda) > 0$.*

Proof. Let $\lambda \in S$. If $\nu \in T_\lambda$ in S then $\nu\tilde{\nu} = \tilde{\nu}\nu$ and hence $\nu\tilde{\nu} \mid \lambda\tilde{\lambda}$. Let λ be such that λ is not a translate of an idempotent. Since any simply connected solvable group has no non-trivial compact subgroup (cf. [**V**]), this implies that $\lambda \neq \delta_x$ for any $x \in G$, i.e. $\lambda\tilde{\lambda} \neq \delta_e$. Now let $\operatorname{supp}(\lambda\tilde{\lambda}) \subset [G, G]$. Then if $\nu \in T_\lambda$ in S, then $\operatorname{supp}(\nu\tilde{\nu}) \subset G(\lambda\tilde{\lambda})$. $[G, G]$ is a closed simply connected nilpotent Lie subgroup of G. Now the assertion can be deduced along the same lines as Theorem 5·1. Now let $\operatorname{supp}(\lambda\tilde{\lambda})$ is not contained in $[G, G]$. Let $\psi : G \to G/[G, G]$ be the natural projection. Here $G/[G, G]$ is isomorphic to $\mathbf{R}^n$ for some $n \in \mathbf{N}$ and $\psi(\lambda\tilde{\lambda}) \neq \delta_e$. Hence there exists a continuous $\psi(\lambda\tilde{\lambda})$-norm $f_{\pi(\lambda\tilde{\lambda})}$ on $M^1(G/[G, G])$ such that $f_{\psi(\lambda\tilde{\lambda})}(\psi(\lambda\tilde{\lambda})) > 0$ (cf. [**R2**], statement 6·1). Let $\nu \in T_\lambda$ in S.

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Then ν is normal and hence $\nu\tilde{\nu}|\lambda\tilde{\lambda}$. Now we define $f_\lambda: T_\lambda \to \mathbf{R}_+$ as $f_\lambda(\nu) = f_{\psi(\lambda\tilde{\lambda})}(\psi(\nu\tilde{\nu}))$. Clearly, f_λ is a continuous λ-norm on S and $f_\lambda(\lambda) > 0$.

6. *Proofs of the main theorems*

Proof of Theorem 1·1. Let G be any group. Let $\Delta = (\mu_{ij})_{i\in\mathbf{N}, j=1}^{n_i}$ be a commutative infinitesimal symmetric triangular system converging to μ in $M^1(G)$. For each $i, j, \mu_{ij} = t_{ij}\tilde{t}_{ij}$, where $(t_{ij})_{i\in\mathbf{N}, j=1}^{n_i}$ is commutative and normal. Then by Proposition 3·2, there exists a Hun semigroup $M \subset M_H^1(G)$ containing μ such that the identity $h = \omega_H$ in M is the hair of μ and μ is infinitesimally divisible in M.

Now let G be a simply connected solvable Lie group. Let $\lambda \in T_\mu$ in M be not an idempotent. Then by Proposition 5·2, there exists a continuous λ-norm f_λ on M such that $f_\lambda(\lambda) > 0$. Hence by a theorem of [**R1**], there exists a continuous homomorphism $\phi: \mathbf{R}_+ \to M$, such that $\phi(1) = \mu$.

Now we prove the second assertion. Let G be a 2-root compact group and let Δ be a symmetric triangular system in $M^1(G)$ converging to μ. Let M, K', M' and $\psi: M \to M'$ be as in Proposition 3·4. Then M is an abelian semigroup consisting of symmetric measures with identity h, $\mu \in M$ and $h = h(\mu)$ in M. M' is an abelian semigroup such that $T_{\psi(\mu)}$ is compact and associatefree. Now let $\{U_n | n \in \mathbf{N}\}$ be the neighbourhood basis of identity $\omega_H = h(\mu)$ in M. Fix an $n \in \mathbf{N}$. $\psi(U_n)$ is the neighbourhood of the identity in M' and we have that $\psi(\mu) = u_1 u_2 \dots u_{m(n)}$, where $u_i \in \psi(U_n)$ for all i. This implies that $\mu = a_n \lambda_1 \lambda_2 \dots \lambda_{m(n)}$, where $a_n \in K'$ and $\lambda_i \in U_n$ for each i. Since K' is a compact group, $\{a_n\}$ is relatively compact, let a be the limit of a convergent subsequence $\{a_{n_k}^{-1}\}$. Then $a\mu = aa_{n_k}\lambda_1\lambda_2 \dots \lambda_{m(n_k)}$. Since $aa_{n_k} \to \omega_H$ as $n_k \to \infty$, it is easy to show that $a\mu$ has a U-decomposition for all neighbourhood U of ω_H. Now choose $x \in G$, such that $x\omega_H = a \in K'$. Then $x^2\mu = \mu$ and $x\mu$ is infinitesimally divisible.

Now let G be a simply connected solvable Lie group which is 2-root compact or connected nilpotent Lie group. Let $\lambda \in M$ be such that $\psi(\lambda) = \lambda' \in T_{\psi(\mu)} \setminus \{h'\}$, where $h' = \psi(h)$ is the identity of M'. Then λ is not a translate of an idempotent and by Theorem 5·1 and Proposition 5·2, there exists a continuous λ-norm f_λ on M such that $f_\lambda(\lambda) > 0$. Moreover, for any $\nu, \gamma \in T_\lambda$ if $\psi(\nu) = \psi(\gamma)$ then $\nu^2 = \gamma^2$ and hence from the definition of f_λ in Theorem 5·1 and Proposition 5·2, we get that $f_\lambda(\nu) = f_\lambda(\gamma)$. And since $\psi(T_\lambda) = T_{\lambda'}$, we define $f_{\lambda'}: T_{\lambda'} \to \mathbf{R}_+$ as $f_{\lambda'}(\psi(\gamma)) = f_\lambda(\gamma)$ for $\gamma \in T_\lambda$. It is clear that, $f_{\lambda'}$ is well defined and it is a continuous λ'-norm on M' such that $f_{\lambda'}(\lambda') > 0$. Hence by Theorem 2·3, there exists a continuous homomorphism $\phi: \mathbf{R}_+ \to M'$, such that $\phi(1) = \psi(\mu)$. This implies that for all $n \in \mathbf{N}$, there exists $x_n \in G$ and $\mu(1/n) \in T_\mu$ (in M) $\subset T_\mu''$ such that $x_n h = x_n \omega_H \in K'$ (where K' is as in the proof of Proposition 3·4) and $\mu = \mu(1/n)^n x_n$. That is, μ is weakly infinitely divisible in $M^1(G)$. Any non-trivial compact subgroup of G is either trivial or central, the element x_n as above is either trivial or central; and in the second case it is infinitely divisible in the centre of G. This implies that μ is infinitely divisible in T_μ'' which is compact. Now we can apply theorem 3·6 of [**S2**], and we get that there exists $x \in G$, such that $xh \in K'$, such that $x\mu$ is embeddable. Here $xh = x\omega_H$ is symmetric and $x^2\mu = \mu$. Now if G is simply connected and solvable then $x = e$. If G is a connected nilpotent Lie group, then $x \in Z$, the centre of G, which is also connected. Hence as in the remark after theorem 3·7 of [**S2**], we get that μ itself is embeddable.

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Proof of Theorem 1·2. Let G be a real or p-adic algebraic group and let μ be the limit of Δ such that μ has full algebraic support. Let π be defined as in Theorem 1·2. Then from the remark following Proposition 4·3, we know that $\pi(x\mu)$ is infinitesimally divisible in $M'_H(G)$ for some x in G and $H \subset G$. Also if G is a real algebraic group, then $x\mu$ is infinitesimally divisible in $M^1_H(G)$, for some $x \in G$ such that $x^m = e$.

Now let $S_3(h)$, S^*, ψ_1 and ψ_2 be as in Section 4. Then $\mu' = \psi_2(\psi_1(\mu)) \in S^*$ is bald and $T_{\mu'}$ in S^* is compact and associatefree. Moreover, μ' is infinitesimally divisible in S^* (cf. Proposition 4·4).

Let $\lambda' \in T_{\mu'} \backslash \{h'\}$. Let $M'' = \psi_1^{-1}(\psi_2^{-1}(S^*)) = \psi_1^{-1}(S_3(h))$. From the construction of $S_3(h) = \overline{\cup M_k}$, it is clear that M'' is a closed abelian subsemigroup of $M^1_H(G)$, where $\psi_1(\omega_H) = h$. In fact for any $\lambda \in \psi_1^{-1}(M_k)$ there exist sequences $\{z(\lambda)_i\} \subset Z^0$ such that $\lambda = \lim_{i \to \infty} \Pi_{j=a_i}^{b_i} \mu_{k(i)j}^{c_{ij}} z(\lambda)_i$, where $1 \leqslant a_i \leqslant b_i \leqslant n_{k(i)}, c_{ij} > 0$ or $\mu_{k(i)j} = \delta_e$, (all a_i, b_i and c_{ij} depend on λ) and $\{k(i)\} \subset \mathbf{N}$ is a sequence depending only on k, and $\{(k+1)(i)\} \subset \{k(i)\}$ for all k.

Let $\lambda \in M''$ be such that $\lambda' = \psi_2(\psi_1(\lambda))$. Then λ is not a translate of an idempotent and $\psi_1^{-1}(\psi_2^{-1}(T_{\lambda'})) \subset T_\lambda$ in M''.

From now on let G be a real algebraic group. Let G be simply connected and solvable and let Δ be normal. Then $\nu\tilde{\nu} = \tilde{\nu}\nu$ for all $\nu \in M''$. By Proposition 5·2, there exists a continuous λ-norm f_λ on M'' such that $f_\lambda(\lambda) > 0$. Let $\gamma' \in T_{\lambda'}$ be any element and let $\gamma \in T_\lambda$ be such that $\psi_2(\psi_1(\gamma)) = \gamma'$. Let $f_{\lambda'} : T_{\lambda'} \to \mathbf{R}_+$ be defined as $f_{\lambda'}(\gamma') = f_\lambda(\gamma)$. If $\nu \in T_\lambda$ be such that $\psi_2(\psi_1(\nu)) = \gamma'$, then it is easy to see that $\nu\tilde{\nu} = \gamma\tilde{\gamma}$ and hence $f_\lambda(\gamma) = f_\lambda(\nu)$ from the definition of f_λ. Thus $f_{\lambda'}$ is well defined and it is a continuous λ'-norm on S^* such that $f_{\lambda'}(\lambda') = f_\lambda(\lambda) > 0$.

Now let G be connected and nilpotent. Then by Theorem 5·1, there exists a continuous λ-norm f_λ on M'' such that $f_\lambda(\lambda) > 0$. Define $f_{\lambda'}$ as above.

Now we apply Theorem 2·3, and get that there exists a continuous homomorphism $\phi : \mathbf{R}_+ \to S^*$ such that $\phi(1) = \mu'$. This implies that for all $n \in \mathbf{N}$, there exists $x_n \in G$ and $\mu(1/n) \in T_\mu$ in $M^1(G)$, such that $\mu = \mu(1/n)^n x_n$ and all $\mu(1/n)$, $x_n \omega_H \in M''$. That is, μ is weakly infinitely divisible in $M^1(G)$.

Consider $\mu = y_n \mu(1/n!)^{n!}$, (where $y_n = x_{n!}$) for all n. $\{y_n\}/Z^0$ is relatively compact and $y_n = z_n y'_n$ for each n such that $\{y'_n\}$ is a relatively compact sequence and $\{z_n\} \subset Z^0$.

Since G is a real algebraic group, Z^0 is divisible and we have $\mu = y'_n((z_n/n!)\mu(1/n!))^{n!}$. Taking a limit y' of $\{y'^{-1}_n\}$, we have $y\omega_H \in M''$ and $y\mu$ is the limit of $\{((z_n/n!)\mu(1/n!))^{n!}\}$. Since $\{\mu(1/n!)^{n!}\} \subset F(\mu)$ and $F(\mu)/Z^0$ is relatively compact (cf. Theorem 4·1), $A = \{((z_n/n!)\mu(1/n!))^m : m \leqslant n!, n \in \mathbf{N}\}$ is relatively compact (cf. [**S2**], lemma 3·2). Then $y\mu$ is infinitely divisible in A and hence embeddable (see [**H**], 3·1·32 and 3·5·1 or [**DM**]).

Now let G_y be the (abelian) subgroup of G generated by y and Z^0. Then G_y/Z^0 is compact. Since G/Z^0 is simply connected in both the above cases of G, we have that $y \in Z$. Hence as in the remark after theorem 3·7 of [**S2**], we get that μ itself is embeddable.

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