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Two special subgroups of the universal sofic group

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Abstract. We define a subgroup of the universal sofic group, obtained as the normalizer of a separable abelian subalgebra. This subgroup can be obtained as an extension by the group of automorphisms on a standard probability space. We show that each sofic representation can be conjugated inside this subgroup.

1. *Introduction*

A well-known result due to Elek and Szabó is that a group is *sofic* if and only if it is a subgroup of the *universal sofic group* $\Pi_{k\to\omega} P_{n_k}$ [**ES05**]. Here $P_n \subset M_n(\mathbb{C})$ is the subgroup of permutation matrices, isomorphic to $\mathrm{Sym}(n)$, the symmetric group. Elements of the universal sofic group act on $\Pi_{k\to\omega} D_{n_k}$, where $D_n \subset M_n(\mathbb{C})$ is the subalgebra of diagonal matrices. As an abelian-type II_1 von Neumann algebra, $\Pi_{k\to\omega} D_{n_k}$ is isomorphic to the algebra of functions on a probability space, $L^\infty(X_\omega, \mu_\omega)$, where (X_ω, μ_ω) is the Loeb space.

The above picture has been fruitful in the study of soficity. It can be used to provide a proof of the fact that the free product of sofic groups amalgamated over amenable subgroups is still sofic [**Pău11**, Corollary 3.7]. It was also successfully used to provide a compact proof for stability of the commutant in permutations with respect to the Hamming distance [**AP15**]. It is therefore natural to wish for a better understanding of these objects and their interaction.

We will not go through the basics of ultraproducts with respect to ω, a non-principal ultrafilter on $\mathbb{N}$, and $\{n_k\}_k \subset \mathbb{N}$, an increasing sequence of natural numbers. The reader can consult the vast literature on the subject, including any of the previously cited articles or introductory texts such as [**Pes08**, **CL15**] or [**CSC10**].

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Let (X, μ) be the unit interval endowed with the Lebesgue measure. In this paper, in the second section we first construct a canonical embedding $L^\infty(X, \mu) \hookrightarrow L^\infty(X_\omega, \mu_\omega)$. In the third section we introduce the group $\mathcal{GA}$, the subgroup of the universal sofic group that normalizes $L^\infty(X, \mu)$. In the fourth section we prove that any sofic group is a subgroup of $\mathcal{GA}$. In the fifth section we use the *Coxeter length* to study this group and in the sixth section we obtain $\mathcal{GA}$ as an extension of $\mathrm{Aut}(X, \mu)$.

2. *The Loeb space*

We already defined the Loeb space $(X_\omega, \mathcal{B}_\omega, \mu_\omega)$ by the equation $\Pi_{k\to\omega} D_{n_k} \simeq L^\infty(X_\omega, \mu_\omega)$, but we need a better understanding of its structure. For this we have to enter the realm of non-standard analysis in more depth than just considering some metric ultraproducts. The Loeb space was introduced in [**Leo75**]. We follow here methods from [**ES**].

As a set, X_ω is the *algebraic ultraproduct* of sets $\{1, 2, \ldots, n_k\}$. From now on, $(x_k)_k$ or $(y_k)_k$ will denote elements in the Cartesian product $\Pi_k\{1, 2, \ldots, n_k\}$. On this set we define the equivalence relation $(x_k)_k \sim (y_k)_k \Leftrightarrow \{k : x_k = y_k\} \in \omega$, i.e. two sequences are equivalent if they are equal on a subset in the ultrafilter. The algebraic ultraproduct is defined as $X_\omega = \Pi_k\{1, 2, \ldots, n_k\}/\sim$. Not to overload notation, we still denote by $(x_k)_k$ its class in X_ω.

We now proceed to construct a measurable structure on X_ω. Let $A_k \subset \{1, 2, \ldots, n_k\}$ and construct $\Pi_{k\to\omega} A_k = \{(x_k)_k \in X_\omega : x_k \in A_k\}$. Let $\mathcal{B}^0_\omega$ be the collection of all such subsets of X_ω. Moreover, define $\mu_\omega : \mathcal{B}^0_\omega \to [0, 1]$ by $\mu_\omega(\Pi_{k\to\omega} A_k) = \lim_{k\to\omega} \mathrm{Card}(A_k)/n_k$. Then $\mathcal{B}^0_\omega$ is an algebra of sets and μ_ω is a pre-measure. Due to Caratheodory's extension theorem, μ_ω can be extended to $\mathcal{B}^1_\omega$, the σ-algebra generated by $\mathcal{B}^0_\omega$. Finally, we extend μ_ω to $\mathcal{B}_\omega$, the closure of $\mathcal{B}^1_\omega$ under the measure μ_ω.

2.1. *The universal sofic action.*

Let $p = \Pi_{k\to\omega} p_k \in \Pi_{k\to\omega} \mathrm{Sym}(n_k)$. Then $p((x_k)_k) = (p_k(x_k))_k$ defines an automorphism of (X_ω, μ_ω). Note the key role played by the measure μ_ω: the permutations p_k are determined up to some error in the Hamming distance. This translates to the fact that the above automorphism of X_ω is well defined up to a set of μ_ω-measure 0.

If $f \in L^\infty(X_\omega)$, we denote by $p(f) \in L^\infty(X_\omega)$ the function $p(f)(x) = f(p(x))$. If we identify $L^\infty(X_\omega)$ with $\Pi_{k\to\omega} D_{n_k}$ and $\Pi_{k\to\omega} \mathrm{Sym}(n_k)$ with $\Pi_{k\to\omega} P_{n_k}$, both of which are subsets in $\Pi_{k\to\omega} M_{n_k}$, then $p(f)$ can be written as $p^{-1} \cdot f \cdot p$.

2.2. *The standard part.*

The standard part function plays a key role in non-standard analysis. In our context it is defined as $\mathrm{St} : X_\omega \to [0, 1]$, $\mathrm{St}((x_k)_k) = \lim_{k\to\omega} (x_k/n_k)$. It is a measure-preserving function in the sense that $\mu_\omega(\mathrm{St}^{-1}(A)) = \mu(A)$ for every Lebesgue measurable set $A \subset [0, 1]$, where μ is the Lebesgue measure on the unit interval $X = [0, 1]$; see Proposition 2.2. As a consequence, we also have an inclusion of algebras $L^\infty([0, 1], \mu) \hookrightarrow L^\infty(X_\omega, \mu_\omega)$, by identifying χ_A with $\chi_{\mathrm{St}^{-1}(A)}$ (the characteristic function), as we now show.

Definition 2.1. Define on X_ω the equivalence relation $x \sim y \Leftrightarrow \mathrm{St}(x) = \mathrm{St}(y)$.

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PROPOSITION 2.2. [**Cut83**, Theorem 4.1] *The space with measure $X_\omega/\sim$, obtained by factoring X_ω by the above equivalence relation, is canonically isomorphic to the standard space (X, μ).*

Proof. Recall that (X, μ) is the unit interval $[0, 1]$ endowed with the Lebesgue measure. Firstly, we show the equality for the Borel structure. Let $\mathcal{B}^1$ be the σ-algebra on $[0, 1]$ generated by intervals, consisting of Borel sets. Then $\mathcal{B}$ is the completion of $\mathcal{B}^1$ under μ.

Let $\mathcal{B}^2$ be the σ-algebra on X induced by the map $\mathrm{St} : X_\omega \to X$, i.e. $\mathcal{B}^2 = \{Y : \mathrm{St}^{-1}(Y) \in \mathcal{B}^1_\omega\}$. We want to show that $\mathcal{B}^1 = \mathcal{B}^2$ and $\mu_\omega(\mathrm{St}^{-1}(Y)) = \mu(Y)$ for any such Y.

For $A \in \mathcal{B}^0_\omega$, the reader can check that $\mathrm{St}(A) \subset [0, 1]$ is a closed set. As $\mathcal{B}^0_\omega$ generates $\mathcal{B}^1_\omega$, it follows that $\mathrm{St}(A) \in \mathcal{B}^1$ for any $A \in \mathcal{B}^1_\omega$. As $Y = \mathrm{St}(\mathrm{St}^{-1}(Y))$, we get that $\mathcal{B}^2 \subset \mathcal{B}^1$.

Now let $\lambda_1, \lambda_2 \in [0, 1]$, $\lambda_1 < \lambda_2$. Let $\{x_k\}_k, \{y_k\}_k$ be sequences of natural numbers such that $\lim_{k\to\omega}(x_k/n_k) = \lambda_1$ and $\lim_{k\to\omega}(y_k/n_k) = \lambda_2$. Let $A = \Pi_{k\to\omega}\{x_k, x_k + 1, \ldots, y_k\} \in \mathcal{B}^0_\omega$. It can be checked that

$$\mathrm{St}^{-1}((\lambda_1, \lambda_2)) \subset A \subset \mathrm{St}^{-1}([\lambda_1, \lambda_2]).$$

Both of these inequalities are strict, but this is not important to the argument. Now fix $\lambda \in (0, 1]$ and let $\{\lambda_k\}_k$ be a strictly increasing sequence converging to λ, with $\lambda_0 = 0$. For every $j \in \mathbb{N}$, let $\{x^j_k\}_k$ be a sequence such that $\lim_{k\to\omega}(x^j_k/n_k) = \lambda_j$, with $x^0_k = 1$. Define $A_j = \Pi_{k\to\omega}\{x^j_k, \ldots, x^{j+1}_k - 1\} \in \mathcal{B}^0_\omega$. By the above inequalities,

$$\mathrm{St}^{-1}([0, \lambda)) = \bigcup_j A_j.$$

This implies that $[0, \lambda) \in \mathcal{B}^2$ for any λ. Moreover, $\mu_\omega(\mathrm{St}^{-1}([0, \lambda))) = \sum_j \mu_\omega(A_j) = \sum_j(\lambda_{j+1} - \lambda_j) = \lambda$. So, $\mathcal{B}^1 \subset \mathcal{B}^2$ and $\mathrm{St} : (X_\omega, \mathcal{B}^1_\omega, \mu_\omega) \to (X, \mathcal{B}^1, \mu)$ is measure preserving. The same is true for $\mathrm{St} : (X_\omega, \mathcal{B}_\omega, \mu_\omega) \to (X, \mathcal{B}, \mu)$. □

Notation 2.3. We denote by $\mathrm{St}^*(L^\infty(X, \mu))$ the subalgebra $\{f \circ \mathrm{St} : f \in L^\infty(X, \mu)\} \subset L^\infty(X_\omega, \mu_\omega)$.

2.3. *The order relation.*

OBSERVATION 2.4. *Throughout the paper, equality of subsets in X or X_ω is meant up to null sets with respect to μ or μ_ω, respectively.*

Definition 2.5. For $x = (x_k)_k$, $y = (y_k)_k \in X_\omega$, define $x \leqslant y$ if $\{k : x_k \leqslant y_k\} \in \omega$.

It can be checked that this is a total order relation, with antisymmetry following due to the algebraic ultraproduct construction ($x = y$ if and only if $\{k : x_k = y_k\} \in \omega$). We now define *initial segments*.

Definition 2.6. For $x \in X_\omega$, denote $I_x = \{y \in X_\omega : y \leqslant x\}$.

Note that $\mu_\omega(I_x) = \mathrm{St}(x)$. Moreover, we have the following proposition.

PROPOSITION 2.7. *For $x, y \in X_\omega$, $I_x = I_y$ if and only if $\mathrm{St}(x) = \mathrm{St}(y)$. Also, $I_x = \{y : \mathrm{St}(y) < \mathrm{St}(x)\}$.*

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Proof. It can be checked that $\mu_\omega(I_x \Delta I_y) = |\mathrm{St}(x) - \mathrm{St}(y)|$ for any $x, y \in X_\omega$. This implies the first statement. For the second part, notice that $\{y : \mathrm{St}(y) < \mathrm{St}(x)\} = \mathrm{St}^{-1}([0, \mathrm{St}(x)))$, so $\mu_\omega(\{y : \mathrm{St}(y) < \mathrm{St}(x)\}) = \mathrm{St}(x) = \mu_\omega(I_x)$. As $\{y : \mathrm{St}(y) < \mathrm{St}(x)\} \subset I_x$, the conclusion now follows. □

3. *Generalized maps*

We now proceed to the main definitions of this paper. Recall that X is the unit interval.

Definition 3.1. Let $p \in \Pi_{k\to\omega} \mathrm{Sym}(n_k)$. We say that $p|_X$ *exists* if there is $\varphi : X \to X$ such that $\mathrm{St}(p(x)) = \varphi(\mathrm{St}(x))$ for μ_ω-almost all $x \in X_\omega$. In this case we write $p|_X = \varphi$.

Definition 3.2. Denote by $\mathcal{GM}$ the set of elements $p \in \Pi_{k\to\omega} \mathrm{Sym}(n_k)$ such that $p|_X$ exists and, by $\mathcal{GA}$, the set of such elements for which $p|_X$ is an automorphism of (X, μ).

PROPOSITION 3.3. *Let $p \in \Pi_{k\to\omega} \mathrm{Sym}(n_k)$. If $p|_X = \varphi$ exists, then φ is a measurable, measure-preserving map.*

Proof. Let $A \subset X$ be a measurable set. The equality $\varphi(\mathrm{St}(x)) = \mathrm{St}(p(x))$ for μ_ω-almost all $x \in X_\omega$ implies that $\mathrm{St}^{-1}(\varphi^{-1}(A)) = p^{-1}(\mathrm{St}^{-1}(A))$:

$$\begin{aligned}\mathrm{St}^{-1}(\varphi^{-1}(A)) &= \{x : \mathrm{St}(x) \in \varphi^{-1}(A)\} = \{x : \varphi(\mathrm{St}(x)) \in A\} = \{x : \mathrm{St}(p(x)) \in A\} \\ &= \{x : p(x) \in \mathrm{St}^{-1}(A)\} = \{x : x \in p^{-1}(\mathrm{St}^{-1}(A))\} = p^{-1}(\mathrm{St}^{-1}(A)).\end{aligned}$$

As $p^{-1}(\mathrm{St}^{-1}(A)) \in \mathcal{B}_\omega$, this equality implies that $\varphi^{-1}(A)$ is measurable in X. Moreover,

$$\mu(\varphi^{-1}(A)) = \mu_\omega(\mathrm{St}^{-1}(\varphi^{-1}(A))) = \mu_\omega(p^{-1}(\mathrm{St}^{-1}(A))) = \mu_\omega(\mathrm{St}^{-1}(A)) = \mu(A). \quad \square$$

We show later that every measure-preserving map can be obtained as $p|_X$ for some p (Theorem 3.6).

PROPOSITION 3.4. *Let $p \in \Pi_{k\to\omega} \mathrm{Sym}(n_k)$. If p and p^{-1} are elements of $\mathcal{GM}$, then $p \in \mathcal{GA}$.*

Proof. As p and p^{-1} are in $\mathcal{GM}$, there exist $\varphi : X \to X$ and $\psi : X \to X$ such that $p|_X = \varphi$ and $p^{-1}|_X = \psi$. Then, for almost all $x \in X_\omega$, we have

$$\psi \circ \varphi(\mathrm{St}(x)) = \psi(\mathrm{St}(p(x))) = \mathrm{St}(p^{-1}(p(x))) = \mathrm{St}(x).$$

It follows that $\psi \circ \varphi = \mathrm{Id}$. These are measure-preserving maps, so φ is an automorphism of (X, μ) and therefore $p \in \mathcal{GA}$. □

Before proving that each measure-preserving function is obtained as a $p|_X$, for a $p \in \Pi_{k\to\omega} \mathrm{Sym}(n_k)$, we need the following well-known lemma. It contains a basic principle of measure theory: by controlling the behaviour on sets, we control the behaviour on points almost everywhere.

LEMMA 3.5. *Let (Y, ν) be a σ-finite measure space and $f, g : Y \to \mathbb{R}$ be measurable functions such that $f^{-1}(B) = g^{-1}(B)$ almost everywhere for every measurable set $B \subset \mathbb{R}$. Then $f(y) = g(y)$ for ν-almost every $y \in Y$.*

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Proof. Let us assume that $\nu(\{y \in Y | f(y) \neq g(y)\}) > 0$. If $\Delta = \{(x, x) : x \in \mathbb{R}\}$, one can find closed–open intervals I_n, J_n, $n \geq 1$, such that $I_n \cap J_n = \emptyset$ and

$$\mathbb{R} \times \mathbb{R} \backslash \Delta = \bigcup_{n=1}^{\infty} I_n \times J_n.$$

Then there exist a measurable set $A \subset Y$, $\nu(A) > 0$ and two closed–open intervals I and J with $I \cap J = \emptyset$ and such that

$$\{(f(y), g(y)) : y \in A\} \subset I \times J.$$

Thus, $f(A) \subset I$ and $g(A) \subset J$, so $A \subseteq f^{-1}(I)$ and $A \subseteq g^{-1}(J)$. On the other hand,

$$g^{-1}(I) \cap A \subseteq g^{-1}(I) \cap g^{-1}(J) = g^{-1}(I \cap J) = \emptyset.$$

As $\nu(A) > 0$, this implies that $f^{-1}(I) \neq g^{-1}(I)$, which is a contradiction. □

THEOREM 3.6. *Let $\varphi : X \to X$ be a measure-preserving map. Then there exists $p \in \Pi_{k \to \omega} \mathrm{Sym}(n_k)$ such that $p|_X = \varphi$.*

Proof. We need to construct $p \in \Pi_{k \to \omega} \mathrm{Sym}(n_k)$ such that $\mathrm{St}^{-1}(\varphi^{-1}(A)) = p^{-1}(\mathrm{St}^{-1}(A))$ for any $A \subset X$. The proof is similar to the one in [**Pău11**, Proposition 3.3]. Choose $\{A_i\}_i$, a finite partition of X. Then $\{\mathrm{St}^{-1}(A_i)\}_i$ and $\{\mathrm{St}^{-1}(\varphi^{-1}(A_i))\}_i$ are two partitions of X_ω with $\mu_\omega(\mathrm{St}^{-1}(A_i)) = \mu_\omega(\mathrm{St}^{-1}(\varphi^{-1}(A_i)))$ for any i. By [**Pău11**, Lemma 3.2], there exists $q_1 \in \Pi_{k \to \omega} \mathrm{Sym}(n_k)$ such that $q_1(\mathrm{St}^{-1}(A_i)) = \mathrm{St}^{-1}(\varphi^{-1}(A_i))$ for any i. We construct a sequence of elements q_j that work for finer and finer partitions of X. By a diagonal argument, we construct q such that $q(\mathrm{St}^{-1}(A)) = \mathrm{St}^{-1}(\varphi^{-1}(A))$ for any $A \subset X$. Define $p = q^{-1}$. By the previous lemma, applied for the functions $\mathrm{St} \circ p$, $\varphi \circ \mathrm{St} : X_\omega \to X$, we get that $\mathrm{St}(p(x)) = \varphi(\mathrm{St}(x))$ for almost all $x \in X_\omega$. □

3.1. *Generalized automorphisms.*

PROPOSITION 3.7. *For $p \in \Pi_{k \to \omega} \mathrm{Sym}(n_k)$, $p|_X = \mathrm{Id}$ if and only if $p(I_x) = I_x$ for (μ_ω-almost) all $x \in X_\omega$.*

Proof. Without any hypothesis on $p \in \Pi_{k \to \omega} \mathrm{Sym}(n_k)$, we have

$$\begin{aligned}
&\int_{X_\omega} \mu_\omega(p(I_x) \Delta I_x)\, d\mu_\omega(x) \\
&\quad = \int_{X_\omega} \mu_\omega(\{y : y \leqslant x \text{ and } p^{-1}(y) > x \text{ or } y > x \text{ and } p^{-1}(y) \leqslant x\})\, d\mu_\omega(x) \\
&\quad = \int_{X_\omega} \mu_\omega(\{x : y \leqslant x \text{ and } p^{-1}(y) > x \text{ or } y > x \text{ and } p^{-1}(y) \leqslant x\})\, d\mu_\omega(y).
\end{aligned}$$

One can check that, for $a, b \in [0, 1]$, $\mu(\{x : a \leqslant x \text{ and } b > x \text{ or } a > x \text{ and } b \leqslant x\}) = |a - b|$. In the Loeb measure this translates to $\mu_\omega(\{x : y \leqslant x \text{ and } p^{-1}(y) > x \text{ or } y > x \text{ and } p^{-1}(y) \leqslant x\}) = |\mathrm{St}(y) - \mathrm{St}(p^{-1}(y))|$. We reach the following equation, interesting in its own right:

$$\int_{X_\omega} \mu_\omega(p(I_x) \Delta I_x)\, d\mu_\omega(x) = \int_{X_\omega} |\mathrm{St}(y) - \mathrm{St}(p^{-1}(y))|\, d\mu_\omega(y).$$

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Assume now that $p(I_x) = I_x$ almost everywhere. Then $\int_{x\in X_\omega} \mu_\omega(p(I_x)\Delta I_x)d\mu_\omega(x) = 0$. It follows that $\mathrm{St}(y) = \mathrm{St}(p^{-1}(y))$ for μ_ω-almost all $y \in X_\omega$ and hence the conclusion.

For the reverse implication, assume that $\mathrm{St}(p(x)) = \mathrm{St}(x)$ almost everywhere. By Proposition 2.7, $p(I_x) = \{p(y) : \mathrm{St}(y) < \mathrm{St}(x)\} = \{p(y) : \mathrm{St}(p(y)) < \mathrm{St}(x)\} \subset I_x$. Hence, $\mu_\omega(p(I_x)\backslash I_x) = 0$ for any $x \in X_\omega$. As these sets are of equal measure, they must be equal. □

Remember that $\mathrm{St}^*(L^\infty(X, \mu)) = \{f \circ \mathrm{St} : f \in L^\infty(X, \mu)\} \subset L^\infty(X_\omega, \mu_\omega)$.

THEOREM 3.8. *Let $p \in \Pi_{k\to\omega} \mathrm{Sym}(n_k)$. Then $p \in \mathcal{GA}$ if and only if $p(\mathrm{St}^*(L^\infty(X))) = \mathrm{St}^*(L^\infty(X))$, i.e. p is in the normalizer of $\mathrm{St}^*(L^\infty(X))$ when ultraproducts are identified with subsets in $\Pi_{k\to\omega} M_{n_k}$.*

Proof. Let p be in the normalizer of $\mathrm{St}^*(L^\infty(X, \mu))$. For $f \in L^\infty(X, \mu)$, we define $\Phi(f)$ to be the unique function in $L^\infty(X, \mu)$ such that $p(f \circ \mathrm{St}) = \Phi(f) \circ \mathrm{St}$. Hence, $f \to \Phi(f)$ defines an automorphism of $L^\infty(X, \mu)$ and, then, there exists a non-singular automorphism φ of (X, μ) such that $\Phi(f) = f \circ \varphi$ for all $f \in L^\infty(X, \mu)$. Thus,

$$p(f \circ \mathrm{St}) = f \circ \varphi \circ \mathrm{St}$$

for all $f \in L^\infty(X, \mu)$. It follows that for μ_ω-almost all $x \in X_\omega$, we have

$$p(f \circ \mathrm{St})(x) = f \circ \mathrm{St}(p(x)) = f \circ \varphi(\mathrm{St}(x)).$$

By Lemma 3.5, it follows that

$$\mathrm{St}(p(x)) = \varphi(\mathrm{St}(x)).$$

By Definition 3.1, this translates to $p|_X = \varphi$. As $\varphi \in \mathrm{Aut}(X, \mu)$, we get $p \in \mathcal{GA}$.

Conversely, if there exists φ, an automorphism of (X, μ) such that $p|_X = \varphi$, then

$$p(f \circ \mathrm{St})(x) = f \circ \mathrm{St}(p(x)) = (f \circ \varphi) \circ \mathrm{St}(x) \in \mathrm{St}^*(L^\infty(X, \mu))$$

for all $f \in L^\infty(X, \mu)$. This implies that $p(\mathrm{St}^*(L^\infty(X))) \subset \mathrm{St}^*(L^\infty(X))$ and, by replacing f with $f \circ \varphi^{-1}$ in the above equality, we get the reverse inclusion. □

4. *Sofic representations*

The purpose of this section is to prove that any sofic representation of any group $\theta : G \to \Pi_{k\to\omega} \mathrm{Sym}(n_k)$ can be conjugated inside $\mathcal{GA}$, i.e. there exists $p \in \Pi_{k\to\omega} \mathrm{Sym}(n_k)$ such that $p\theta p^* \subset \mathcal{GA}$. Recall that a *sofic representation* is a group morphism $\theta : G \to \Pi_{k\to\omega} \mathrm{Sym}(n_k)$ such that $\ell_H(\theta(g)) = 1$ for any $g \neq e$, where ℓ_H is the normalized Hamming length.

We do this with the help of a theorem by Elek and Lippner [**EL10**], saying that a Bernoulli shift action of a sofic group is sofic. Ozawa has a nice proof of this result (see [**Oza09**], see also [**Pău11**, Theorem 3.5]), but it uses an amplification, which would provide a weaker result in our context. This is why we need to inspect the original proof of Elek and Lippner.

In the following theorem, the space $Y = \{0, 1\}^G$ is the product space of $\{0, 1\}$ endowed with the normalized counting measure, indexed by the countable group G. We denote by $\beta : G \to \mathrm{Aut}(Y)$ the Bernoulli shift action.

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THEOREM 4.1. [**EL10**, Proposition 7.1] *Let $\theta : G \to \Pi_{k\to\omega} P_{n_k}$ be a sofic representation. There exists $\overline{\theta} : L^\infty(Y) \rtimes_\beta G \to \Pi_{k\to\omega} M_{n_k}$, an embedding of the crossed product that extends θ, such that $\overline{\theta}(L^\infty(Y)) \subset \Pi_{k\to\omega} D_{n_k}$.*

Proof. For the reader's convenience, we outline here the main ideas in the original proof, adapted to our language and notation. The σ-algebra on $Y = \{0, 1\}^G = \{f : G \to \{0, 1\}\}$ is generated by the cylinder sets

$$c^{i_1,i_2,\dots,i_m}_{g_1,g_2,\dots,g_m} = \{f \in Y : f(g_j) = i_j,\ j = 1, \dots, m\},$$

where $g_1, \dots, g_m$ are distinct elements of G and $i_j \in \{0, 1\}$ for $j = 1, 2, \dots, m$. The measure of such a cylinder is $1/2^m$. We denote by $Q^{i_1,i_2,\dots,i_m}_{g_1,g_2,\dots,g_m} \in L^\infty(Y)$ the projection onto this set.

The key observation is that we only need to construct $\overline{\theta}(Q^1_e)$, as the rest of the embedding is generated by the following relations:

$$Q^1_g = u_g Q^1_e u^*_g, \tag{1}$$

$$Q^0_g = \mathrm{Id} - Q^1_g, \tag{2}$$

$$Q^{i_1,i_2,\dots,i_m}_{g_1,g_2,\dots,g_m} = Q^{i_1}_{g_1} \cdot Q^{i_2}_{g_2} \cdot \dots \cdot Q^{i_m}_{g_m}. \tag{3}$$

These relations are written in $L^\infty(Y) \rtimes_\beta G$ and u_g is the unitary corresponding to $g \in G$. All we need is to construct $a \in \Pi_{k\to\omega} D_{n_k}$ such that $\mathrm{Tr}(\theta(g_1)a\theta(g_1)^* \cdot \theta(g_2)a\theta(g_2)^* \cdot \dots \cdot \theta(g_m)a\theta(g_m)^*) = 1/2^m$ for each m and $g_1, \dots, g_m \in G$.

We use the *second moment method*: we consider the set of all projections $a_k \in \mathcal{P}(D_{n_k})$, we compute the expected value of these traces, i.e. the average, and show that the variance, i.e. the deviation from the average, is sufficiently small.

Let us first exemplify in the case of one projection. For now, we fix $n \in \mathbb{N}$, dropping the n_k index. The cardinality of $\mathcal{P}(D_n)$ is 2^n, as each of the n diagonal entries can independently be 0 or 1. We identify $a \in \mathcal{P}(D_n)$ with this function $d_a : \{1, \dots, n\} \to \{0, 1\}$, representing the diagonal entries. Then $\mathrm{Tr}(a) = (1/n) \sum_{x\in\{1,\dots,n\}} d_a(x)$. Moreover, for each x, $d_a(x) = 1$ for exactly half of the matrices $a \in \mathcal{P}(D_n)$, i.e. $(1/2^n) \sum_{a\in\mathcal{P}(D_n)} d_a(x) = 1/2$. It follows that

$$\frac{1}{2^n} \sum_{a\in\mathcal{P}(D_n)} \mathrm{Tr}(a) = \frac{1}{2^n} \sum_{a\in\mathcal{P}(D_n)} \frac{1}{n} \sum_{x\in\{1,\dots,n\}} d_a(x) = \frac{1}{n2^n} \sum_x \sum_a d_a(x) = \frac{1}{n} \sum_x \frac{1}{2} = \frac{1}{2}.$$

This does not mean that we can find $a \in \mathcal{P}(D_n)$ such that $\mathrm{Tr}(a) = 1/2$ or close to this value. This is why we also compute the variance:

$$\begin{aligned}\frac{1}{2^n} \sum_{a\in\mathcal{P}(D_n)} \left(\mathrm{Tr}(a) - \frac{1}{2}\right)^2 &= \frac{1}{2^n} \sum_{a\in\mathcal{P}(D_n)} \mathrm{Tr}(a)^2 - \frac{1}{2^n} \sum_{a\in\mathcal{P}(D_n)} \mathrm{Tr}(a) + \frac{1}{4} \\ &= \frac{1}{2^n} \sum_{a\in\mathcal{P}(D_n)} \mathrm{Tr}(a)^2 - \frac{1}{4}.\end{aligned}$$

If this value is small enough, we can infer the existence of many projections with trace close to $1/2$. In order to compute $\sum \mathrm{Tr}(a)^2$, we notice that $\mathrm{Tr}(a)^2 = \mathrm{Tr}(a \otimes a)$. The function associated to $a \otimes a$ is $d_{a\otimes a} : \{1, \dots, n\} \times \{1, \dots, n\} \to \{0, 1\}$, defined as

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$d_{a\otimes a}(x, y) = d_a(x) \cdot d_a(y)$. Then $\mathrm{Tr}(a \otimes a) = (1/n^2) \sum_{x,y} d_{a\otimes a}(x, y)$. As before, we fix $(x, y) \in \{1, \ldots, n\}^2$ and estimate $(1/2^n) \sum_{a\in\mathcal{P}(D_n)} d_{a\otimes a}(x, y)$. If $x = y$, then this sum is $1/2$, as before. If $x \neq y$, then $d_{a\otimes a}(x, y) = 1$ if and only if $d_a(x) = d_a(y) = 1$. Only $1/4$ of the elements of $\mathcal{P}(D_n)$ satisfy this condition. All in all,

$$\frac{1}{2^n} \sum_{a\in\mathcal{P}(D_n)} \mathrm{Tr}(a)^2 = \frac{1}{n^2}\left(n \cdot \frac{1}{2} + (n^2 - n)\frac{1}{4}\right) = \frac{1}{4} + \frac{1}{4n}.$$

We proved that $(1/2^n) \sum(\mathrm{Tr}(a) - 1/2)^2 = 1/4n$. It means that, for any $\lambda > 0$, $(\mathrm{Tr}(a) - 1/2)^2 \geqslant \lambda/4n$ for at most $2^n/\lambda$ elements of $\mathcal{P}(D_n)$ (in this setting, this is Chebyshev's inequality). Let n be arbitrarily large such that we can choose a dimension where $\mathrm{Tr}(a)$ is close to $1/2$ for any proportion of projections we want.

The proof of the theorem is not more difficult. Choose $\theta_k : G \to P_{n_k}$ for $k \in \mathbb{N}$ such that these maps represent θ, i.e. $\theta(g) = \Pi_{k\to\omega}\theta_k(g)$. In order to construct the required embedding $\overline{\theta}$, fix $\varepsilon > 0$ and $F \subset G$ a finite subset with m its cardinality. We want to find a sufficiently large n_k and $a \in D_{n_k}$ such that not only $|\mathrm{Tr}(a) - 1/2| < \varepsilon$, but also $|\mathrm{Tr}(p_1 a^{s_1} p_1^* \cdot \cdots \cdot p_m a^{s_m} p_m^*) - 1/2^m| < \varepsilon$, where $p_1, \ldots, p_m$ is an enumeration of the set $\{\theta_k(g) : g \in F\}$ and $s_j \in \{0, 1\}$ with $a^1 = a$ and $a^0 = 1 - a$. Denote by N the number of these inequalities ($N = 2^m$).

We choose $\lambda = N + 1$ and show, using the above method, that each of these conditions fails for at most $2^{n_k}/\lambda$ projections $a \in D_{n_k}$. This implies the existence of a projection that simultaneously satisfies all those inequalities. We identify a permutation matrix $p \in P_{n_k}$ with an element of $\mathrm{Sym}(n_k)$, meaning a function $\{1, \ldots, n_k\} \to \{1, \ldots, n_k\}$. In order to greatly simplify the writing, we assume that for all entries $x \in \{1, \ldots, n_k\}$, the permutations $p_1, \ldots, p_m$ take different values, i.e. the set $\{p_1(x), \ldots, p_m(x)\}$ has cardinality m for each x. This is true for most entries in a sofic representation. The complete proof will separate the set $\{1, \ldots, n_k\}$ into these 'good' points and 'bad' points, the error that the definition of sofic groups allows. With this assumption, we have

$$\frac{1}{2^{n_k}} \sum_{a\in\mathcal{P}(D_{n_k})} \mathrm{Tr}(p_1 a^{s_1} p_1^* \cdot \cdots \cdot p_m a^{s_m} p_m^*) = \frac{1}{2^m}.$$

As before, this computation is done by fixing an entry $x \in \{1, \ldots, n_k\}$ and counting how often $b = p_1 a^{s_1} p_1^* \cdot \cdots \cdot p_m a^{s_m} p_m^*$ is 1 on this position. Note that $d_b(x) = 1$ if and only if $d_{p_j a^{s_j} p_j^*}(x) = 1$ for each $j = 1, \ldots, m$. Also, $d_{p_j a^{s_j} p_j^*}(x) = d_{a^{s_j}}(p_j(x))$. In the end, $d_b(x) = 1$ if and only if $d_a(p_j(x)) = s_j$ for all $j = 1, \ldots, m$. Here we use the fact that the numbers $\{p_j(x)\}_j$ are distinct, so that indeed $(1/2^{n_k}) \sum_a d_{p_1 a^{s_1} p_1^* \cdots p_m a^{s_m} p_m^*}(x) = (1/2^m)$.

The more difficult part is to compute the variance:

$$\frac{1}{2^{n_k}} \sum_{a\in\mathcal{P}(D_{n_k})} \left(\mathrm{Tr}(p_1 a^{s_1} p_1^* \cdot \cdots \cdot p_m a^{s_m} p_m^*) - \frac{1}{2^m}\right)^2$$
$$= \frac{1}{2^{n_k}} \sum_{a\in\mathcal{P}(D_{n_k})} \mathrm{Tr}(p_1 a^{s_1} p_1^* \cdot \cdots \cdot p_m a^{s_m} p_m^*)^2 - \frac{1}{4^m}.$$

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Using the same notation $b = p_1 a^{s_1} p_1^* \cdot \cdots \cdot p_m a^{s_m} p_m^*$, we have that $\mathrm{Tr}(b)^2 = \mathrm{Tr}(b \otimes b)$. Then $d_{b\otimes b}(x, y) = 1$ if and only if $d_a(p_j(x)) = d_a(p_j(y)) = s_j$ for all $j = 1, \ldots, m$. If $\{p_j(x) : j = 1, \ldots, m\}$ and $\{p_j(y) : j = 1, \ldots, m\}$ are disjoint sets, then these conditions hold for exactly 2^{n_k-2m} projections $a \in \mathcal{P}(D_{n_k})$. If those two sets intersect, we are not interested in computing the number of good projections. It can be zero, or 2^{n_k-m} if $x = y$. Thus, we need to count the number of pairs (x, y) such that $\{p_j(x) : j = 1, \ldots, m\}$ and $\{p_j(y) : j = 1, \ldots, m\}$ are disjoint. Assume that they are not. Then there are $j_1, j_2 \in \{1, \ldots, m\}$ such that $p_{j_1}(x) = p_{j_2}(y)$ or $y = p_{j_2}^{-1} p_{j_1}(x)$. In total we get m^2 forbidden values of y for any fixed x. Hence, the number of good (x, y) pairs is $n_k(n_k - m^2)$, which is a generalization of our previous $n(n-1)$ (actually the number of good pairs is slightly less, as we have to exclude also those for which $\{p_j(x)\}_j$ or $\{p_j(y)\}_j$ are not collections of distinct numbers). We can now provide an estimate

$$\sum_{a\in\mathcal{P}(D_{n_k})} \mathrm{Tr}(p_1 a^{s_1} p_1^* \cdot \cdots \cdot p_m a^{s_m} p_m^*)^2 \leqslant \frac{1}{n_k^2}\left(n_k(n_k - m^2)\frac{1}{4^m} + n_k m^2 \frac{1}{2^m}\right)$$
$$= \frac{1}{4^m} + \frac{c_m}{n_k},$$

where c_m is a constant on m. Now we know that $(\mathrm{Tr}(p_1 a^{s_1} p_1^* \cdot \cdots \cdot p_m a^{s_m} p_m^*) - 1/2^m)^2 < \lambda c_m/n_k$ for all but at most $2^{n_k}/\lambda$ elements in $\mathcal{P}(D_{n_k})$. We choose a sufficiently large n_k such that $\lambda c_m/n_k < \varepsilon^2$ and we are done. □

Before proving the result of this section, we cite the following proposition.

PROPOSITION 4.2. [**Pău11**, Proposition 3.3] *Let θ_1, θ_2 be two embeddings of $L^\infty(X, \mu)$ in $\Pi_{k\to\omega} D_{n_k}$. Then there exists $p \in \Pi_{k\to\omega} P_{n_k}$ such that $\theta_2 = p\theta_1 p^*$.*

THEOREM 4.3. *Let $\theta : G \to \Pi_{k\to\omega} \mathrm{Sym}(n_k)$ be a sofic representation. Then there exists $p \in \Pi_{k\to\omega} \mathrm{Sym}(n_k)$ such that $p\theta p^* \subset \mathcal{GA}$.*

Proof. By Theorem 4.1, we have an extension $\overline{\theta} : L^\infty(Y) \rtimes_\beta G \to \Pi_{k\to\omega} M_{n_k}$. By the above proposition, there is $p \in \Pi_{k\to\omega} P_{n_k}$ such that $\mathrm{St}^*(L^\infty(X)) = p\overline{\theta}(L^\infty(Y))p^{-1}$. For any $g \in G$, $p\theta(g)p^*$ is acting on this abelian subalgebra. By Theorem 3.8, $p\theta(g)p^*$ is in $\mathcal{GA}$. □

This theorem shows that, when investigating sofic groups, we can restrict our study from the universal sofic group to the subgroup of generalized automorphisms.

5. *The Coxeter semi-length*

In this section we investigate the connection between generalized maps and the order relation on X_ω. This study leads us to the Coxeter length, which we now define.

Definition 5.1. For $p \in \mathrm{Sym}(n)$, the *Coxeter length* is defined as

$$\ell_C(p) = \frac{2}{n(n-1)} \mathrm{Card}\{i < j : p(i) > p(j)\}.$$

If $p(i) = n + 1 - i$, then $\ell_C(p) = 1$, independently of n. The factor $2/n(n-1)$ from the definition plays the role of normalizing the length.

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PROPOSITION 5.2. *For $p \in \mathrm{Sym}(n)$, $\ell_C(p) \leqslant 2 \cdot \ell_H(p)$.*

Proof. Let us assume that $\ell_H(p) = k/n$ with $k \geq 0$. Since p has $n-k$ fixed points, the number of pairs (i, j) such that $i < j$ and $p(i) < p(j)$ is at least $\binom{n}{n-k} = \frac{1}{2}(n-k)(n-k-1)$. Therefore,

$$\ell_C(p) \leq 1 - \frac{(n-k)(n-k-1)}{n(n-1)} = \frac{2kn - k(k+1)}{n(n-1)} \leq \frac{2kn-2k}{n(n-1)} = \frac{2k}{n} = 2 \cdot \ell_H(p). \quad \square$$

Notice that ℓ_C is a *length* on $\mathrm{Sym}(n)$, that is, $0 \leq \ell_C(p) = \ell_C(p^{-1})$, $\ell_C(p_1 p_2) \leq \ell_C(p_1) + \ell_C(p_2)$ for every $p, p_1, p_2 \in \mathrm{Sym}(n)$, and it is equal to zero only at the identity. We call the semi-length a function that satisfies these properties except possibly the last one.

COROLLARY 5.3. *The function $\ell_C : \Pi_{k\to\omega} \mathrm{Sym}(n_k) \to [0, 1]$, defined as $\ell_C(\Pi_{k\to\omega} p_k) = \lim_{k\to\omega} \ell_C(p_k)$, is a well-defined semi-length on the universal sofic group.*

The following proposition provides a nice characterization of elements of the universal sofic group that act trivially on (X, μ).

PROPOSITION 5.4. *Let $p = \Pi_{k\to\omega} p_k \in \Pi_{k\to\omega} \mathrm{Sym}(n_k)$. Then $p|_X = \mathrm{Id}$ if and only if $\ell_C(p) = 0$.*

Proof. We define $\mathrm{Inv}(p) = \{(x, y) \in X_\omega^2 : x \leqslant y,\ p(x) > p(y)\} \cup \{(x, y) \in X_\omega^2 : x > y,\ p(x) \leq p(y)\}$. This set can be described as $\mathrm{Inv}(p) = \{(x, y) \in X_\omega^2 : x \in I_y \Delta p^{-1} I_{p(y)}\}$. By Fubini's theorem, $\mu_\omega \times \mu_\omega(\mathrm{Inv}(p)) = \int_{X_\omega} \mu_\omega(I_y \Delta p^{-1} I_{p(y)})\, d\mu_\omega(y)$. Moreover,

$$\begin{aligned}\mu_\omega \times \mu_\omega(\mathrm{Inv}(p)) &= \lim_\omega \frac{2\,\mathrm{Card}(\{(x_k, y_k) \in \{1, \ldots, n_k\}^2 : x_k \leqslant y_k,\ p_k(x_k) > p_k(y_k)\})}{{n_k}^2} \\ &= \ell_C(p).\end{aligned}$$

It follows that $\ell_C(p) = 0$ if and only if $\int_{y\in X_\omega} \mu_\omega(I_y \Delta p^{-1} I_{p(y)})\, d\mu_\omega = 0$ if and only if $I_y = p^{-1} I_{p(y)}$ for μ_ω-almost all $y \in X_\omega$.

If $I_y = p^{-1} I_{p(y)}$, then $\mu_\omega(I_y) = \mu_\omega(I_{p(y)})$, so $\mathrm{St}(y) = \mathrm{St}(p(y))$ almost everywhere. This is the definition of $p|_X = \mathrm{Id}$.

Assume now that $p|_X = \mathrm{Id}$. By Proposition 2.7, $I_y = I_{p(y)}$ and, by Proposition 3.7, $I_{p(y)} = p^{-1} I_{p(y)}$. Hence, $I_y = p^{-1} I_{p(y)}$ and thus $\ell_C(p) = 0$. $\square$

Definition 5.5. Denote $\mathcal{GI} = \{p \in \Pi_{k\to\omega} \mathrm{Sym}(n_k) : \ell_C(p) = 0\}$.

OBSERVATION 5.6. *Using Theorem 3.6, for any countable group G, we can construct a map $\theta : G \to \Pi_{k\to\omega} P_{n_k}$ such that $\theta(g)\theta(h)\theta(gh)^{-1} \in \mathcal{GI}$ for all $g, h \in G$. By the previous proposition, it follows that every group is metric approximable by symmetric groups endowed with the Coxeter length. Unfortunately, the Coxeter length is not invariant (constant on conjugacy classes), otherwise, by Glebsky's result* [**Gel17**, Proposition 9]*, every group would have been sofic. The reader can consult the reference* [**HR17**, Definition 1.1] *for the notion of a metric approximable group by an invariant length. Also,* [**Pes08**, §2.3] *shows why it is important to have invariant lengths.*

In the next section we shall see that the group of generalized automorphisms is an extension of $\mathcal{GI}$ by $\mathrm{Aut}(X, \mu)$.

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By using a result of Diaconis and Graham [**DG77**], we can link the group of $\mathcal{GI}$ with the notion of *total displacement*, with no additional effort.

Definition 5.7. For $p \in \mathrm{Sym}(n)$, the *normalized total displacement* is defined as

$$T(p) := \frac{2}{n(n-1)} \sum_{i=1}^{n} |p(i) - i|.$$

The name *total displacement* comes from [**Knu98**]; however, Diaconis and Graham [**DG77**] already showed the relation between Coxeter distance and total displacement.

PROPOSITION 5.8. [**DG77**, Theorem 2] *For any $p \in \mathrm{Sym}(n)$, we have $\ell_C(p) \leq T(p) \leq 2\ell_C(p)$.*

It follows that the normalized total displacement can be defined also for elements of the universal sofic group, as an ultralimit.

COROLLARY 5.9. *Let $p = \Pi_{k\to\omega} p_k \in \Pi_{k\to\omega} \mathrm{Sym}(n_k)$. Then $p \in \mathcal{GI} \Leftrightarrow \lim_{k\to\omega} T(p_k) = 0$.*

6. *A short exact sequence*

THEOREM 6.1. *The following is a short exact sequence:* $1 \to \mathcal{GI} \to \mathcal{GA} \to \mathrm{Aut}(X, \mu) \to 1$.

Proof. Define $\Psi : \mathcal{GA} \to \mathrm{Aut}(X, \mu)$ by $\Psi(p) = \varphi$, where $p|_X = \varphi$. Then Ψ is a morphism and, by Proposition 3.6, it is surjective. By definition, $Ker(\Psi) = \mathcal{GI}$. This proves the statement. □

Settling the type of this extension seems challenging. We can at least prove that it is not trivial, i.e. $\mathcal{GA}$ is not obtained as a direct product via the maps contained in the short exact sequence.

PROPOSITION 6.2. *The commutant of $\mathcal{GI}$ in the universal sofic group is trivial.*

Proof. Let $p = \Pi_{k\to\omega} p_k \in \Pi_{k\to\omega} \mathrm{Sym}(n_k)$ be an element commuting with $\mathcal{GI}$. Let $s_k^1, s_k^2 \in \mathrm{Sym}(n_k)$ be defined as follows: $s_k^1(i) = i + 1$ with the exception of $s_k^1(n_k) = 1$ and $s_k^2(2i) = 2i$, $s_k^2(2i + 1) = 2i + 3$, again with the exception of the largest odd number smaller than n_k, for which s_k^2 is equal to 1. It is easy to see that $s_1 = \Pi_{k\to\omega} s_k^1$ and $s_2 = \Pi_{k\to\omega} s_k^2$ are elements of $\mathcal{GI}$. It follows that $ps_1 = s_1 p$ and $ps_2 = s_2 p$.

Let A_k be the set of points $i \in \{1, \ldots, n_k\}$ such that $p_k s_k^1(i) = s_k^1 p_k(i)$ and $p_k s_k^2(i) = s_k^2 p_k(i)$. We know that $\mathrm{Card}(A_k)/n_k \to_{k\to\omega} 1$. Then, for $i \in A_k$, the first condition implies that $p_k(i + 1) = p_k(i) + 1$ and the second one amounts to i and $p_k(i)$ having the same parity.

Let B_k be the set of non-fixed even points of p_k, i.e. $B_k = \{2i : p_k(2i) \neq 2i\}$. Let C_k be a maximal subset of B_k with the property that $p_k(C_k) \cap C_k = \emptyset$. Then $\mathrm{Card}(C_k) \geqslant \mathrm{Card}(B_k)/3$ (if $x \in B_k \backslash C_k$ cannot be added to C_k, it means that either $p_k(x) \in C_k$ or $x \in p_k(C_k)$).

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Construct s_k^3 as follows: $s_k^3(2i) = 2i + 1$ and $s_k^3(2i + 1) = 2i$ for any i such that $2i \in C_k$, and $s_k^3(i) = i$ otherwise. It is easy to see that $s_3 = \Pi_{k\to\omega} s_k^3$ is in $\mathcal{GI}$, so $ps_3 = s_3 p$. However, for $2i \in A_k \cap C_k$, we have

$$p_k s_k^3(2i) = p_k(2i + 1) = p_k(2i) + 1,$$
$$s_k^3 p_k(2i) = p_k(2i) \text{ because } p_k(2i) \notin C_k \text{ and it is even.}$$

It follows that $d_H(p_k s_k^3, s_k^3 p_k) \geqslant \text{Card}(A_k \cap C_k)/n_k$. As $ps_3 = s_3 p$, we get that $\text{Card}(A_k \cap C_k)/n_k$ converges in the ultralimit to 0. As $\text{Card}(A_k)/n_k \to_{k\to\omega} 1$, it must be that $\text{Card}(C_k)/n_k \to_{k\to\omega} 0$. Then $\text{Card}(B_k)/n_k \to_{k\to\omega} 0$, so almost all even points in $\{1, \ldots, n_k\}$ are fixed points for p_k. A similar argument can be performed for odd points. □

COROLLARY 6.3. *The extension* $1 \to \mathcal{GI} \to \mathcal{GA} \to \text{Aut}(X, \mu) \to 1$ *is not trivial.*

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