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A SIEVE PROBLEM AND ITS APPLICATION

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Abstract. Let θ be an arithmetic function and let $\mathcal{B}$ be the set of positive integers $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ which satisfy $p_{j+1} \leqslant \theta(p_1^{\alpha_1} \cdots p_j^{\alpha_j})$ for $0 \leqslant j < k$. We show that $\mathcal{B}$ has a natural density, provide a criterion to determine whether this density is positive, and give various estimates for the counting function of $\mathcal{B}$. When $\theta(n)/n$ is non-decreasing, the set $\mathcal{B}$ coincides with the set of integers n whose divisors $1 = d_1 < d_2 < \cdots < d_{\tau(n)} = n$ satisfy $d_{j+1} \leqslant \theta(d_j)$ for $1 \leqslant j < \tau(n)$.

§1. *Introduction.* Let θ be an arithmetic function, $\theta : \mathbb{N} \to \mathbb{R} \cup \{\infty\}$. We write $\mathcal{B}$ (or $\mathcal{B}_\theta$) to denote the set of positive integers containing $n = 1$ and all those $n \geqslant 2$ with prime factorization $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, $p_1 < p_2 < \cdots < p_k$, which satisfy

$$p_{j+1} \leqslant \theta(p_1^{\alpha_1} \cdots p_j^{\alpha_j}) \quad (0 \leqslant j < k), \tag{1}$$

where $p_1^{\alpha_1} \cdots p_j^{\alpha_j}$ is understood to be 1 when $j = 0$. Let $B(x)$ (or $B_\theta(x)$) be the number of positive integers $n \leqslant x$ in $\mathcal{B}$. The following list shows some examples of θ and its corresponding set $\mathcal{B}$.

- If $\theta(n) = 2n$, then $\mathcal{B}$ is the set of integers with 2-dense divisors, i.e. integers n which have a divisor in every interval $[y, 2y]$ for $1 \leqslant y \leqslant n$ (see [4, 5, 8]).
- If $\theta(n) = \sigma(n) + 1$, where $\sigma(n)$ is the sum-of-divisors function, then $\mathcal{B}$ is the set of practical numbers, i.e. integers n such that every $1 \leqslant m \leqslant n$ can be written as a sum of distinct positive divisors of n (see [4, 5, 8] and the references therein).
- If $\theta(n) = n + 1$, then $\mathcal{B}$ is the set of even φ-practical numbers, i.e. even integers n such that the polynomial $X^n - 1$ has a divisor in $\mathbb{Z}[X]$ of every degree from 1 to n (see [2, 6, 7]).

Building on earlier work by Tenenbaum [5] and Saias [4], we found [8, Theorem 1.2] that

$$B(x) \sim \frac{c_\theta x}{\log x} \quad (x \to \infty)$$

for some positive constant c_θ, provided $\theta(n)$ satisfies

$$\max(2, n) \leqslant \theta(n) \ll \frac{n \log 2n}{(\log\log 3n)^{1+\varepsilon}} \quad (n \geqslant 1)$$

for some $\varepsilon > 0$. This applies to each of the three examples listed above.

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In this note, our goal is to investigate the set $\mathcal{B}$ in general, without any restrictions on θ. We will show that $\mathcal{B}$ always has a natural density (Theorem 1) and provide a criterion to determine whether this natural density is positive or zero (Theorem 2). We give estimates for $B(x)$ with explicit error terms, first without any assumptions on θ (Theorem 1), and then under certain conditions on the size of $\theta(n)$ (Corollary 1 and Theorem 3).

As an application, we consider the following set, related to the distribution of divisors. Let $\mathcal{D}$ be the set of positive integers containing $n = 1$ and all those $n \geqslant 2$ whose divisors $1 = d_1 < d_2 < \cdots < d_{\tau(n)} = n$ satisfy

$$d_{j+1} \leqslant \theta(d_j) \quad (1 \leqslant j < \tau(n)). \tag{2}$$

We write $D(x)$ for the number of positive integers $n \leqslant x$ in $\mathcal{D}$. Theorem 4 shows that $\mathcal{B} = \mathcal{D}$ provided $\theta(n)/n$ is non-decreasing, so that all results concerning $B(x)$ also apply to $D(x)$ under this assumption.

§2. *Statement of results.* Let $P^+(n)$ (respectively $P^-(n)$) denote the largest (respectively smallest) prime factor of $n \geqslant 2$ and put $P^+(1) = 1$, $P^-(1) = \infty$.

Note that replacing $\theta(n)$ by $\max(\theta(n), P^+(n))$ in (1) leaves the set $\mathcal{B}$ unchanged, because of the assumption $p_1 < p_2 < \cdots < p_k$. Moreover, if $\theta(1) < 2$, then $\mathcal{B} = \{1\}$. Thus, we may assume from now on, without any loss of generality, that

$$\theta : \mathbb{N} \to \mathbb{R} \cup \{\infty\}, \qquad \theta(1) \geqslant 2, \qquad \theta(n) \geqslant P^+(n) \qquad (n \geqslant 2). \tag{3}$$

Let $\chi(n)$ be the characteristic function of the set $\mathcal{B}$. We shall see in Lemma 4 that the series

$$L = \sum_{n \geqslant 1} \frac{\chi(n)}{n} \prod_{p \leqslant \theta(n)} \left(1 - \frac{1}{p}\right)$$

converges to a value $0 \leqslant L \leqslant 1$. Theorem 1 shows that, for every choice of θ, the set $\mathcal{B}$ has a natural density, which is given by $1 - L$.

THEOREM 1. *Let θ satisfy (3). We have*

$$B(x) = (1 - L)x + o(x). \tag{4}$$

More precisely,

$$B(x) = (1 - L)x + O\left(1 + x \sum_{n \geqslant 1} \frac{\chi(n)}{n \log \theta(n)} \exp\left(-\frac{\max(0, \log(x/n))}{3 \log \theta(n)}\right)\right). \tag{5}$$

Theorem 2 provides a simple criterion to determine whether $L = 1$ and $B(x) = o(x)$, or $L < 1$ and $B(x) \sim (1 - L)x$.

THEOREM 2. *Let θ satisfy (3).*

(i) *Assume that $\theta(n) \leqslant f(n)$, where $n \log f(n)$ is increasing. If*

$$\sum_{n \geqslant 1} \frac{1}{n \log f(n)}$$

diverges, then $L = 1$ and $B(x) = o(x)$.

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(ii) *Assume that* $\theta(n) \geqslant f(n) \geqslant 2$, $f(n) \gg P^+(n)$, *and that for every* $t \geqslant 1$ *there exists an* $r \in \mathbb{N}$ *such that* $f(2^r n) \geqslant t f(n)$ *for all* $n \geqslant 1$. *If*

$$\sum_{n \geqslant 1} \frac{1}{n \log f(n)}$$

converges, then $L < 1$ *and* $B(x) \sim (1-L)x$.

(iii) $B(x) \sim x \Leftrightarrow L = 0 \Leftrightarrow \theta(n) = \infty$ *for all* $n \geqslant 1 \Leftrightarrow B(x) = [x]$.

Note that the three examples of θ listed in the introduction satisfy $L = 1$ and $B(x) = o(x)$. For an instance where $L < 1$, consider $\theta(n) = 2^n$, for which $L = 0.7734\ldots$ by numerical computation. (For $n \geqslant 30$, we used estimates for the Euler product with effective error bounds due to Rosser and Schoenfeld [**3**, Theorem 7].) Hence, (4) implies that $B(x) = cx(1+o(1))$ with $c = 1 - L = 0.2265\ldots$, while (5) shows that we have $B(x) = cx(1 + O(1/\log x))$ when $\theta(n) = 2^n$.

Corollary 1 generalizes this example to $\log\theta(n) \asymp n^a$, where a is a positive constant. We also consider the case $\log\theta(n) \asymp (\log 2n)^a$, where $a > 1$ is constant. (The notation $f(n) \asymp g(n)$ means that $f(n) \ll g(n)$ (i.e. $f(n) = O(g(n))$) and $g(n) \ll f(n)$.)

COROLLARY 1. *Let* θ *satisfy (3).*

(i) *If* $\log\theta(n) \asymp n^a$ *for some constant* $a > 0$, *then* $0 < L < 1$ *and*

$$B(x) = (1-L)x\left(1 + O\left(\frac{1}{\log x}\right)\right).$$

(ii) *If* $\log\theta(n) \asymp (\log 2n)^a$ *for some constant* $a > 1$, *then* $0 < L < 1$ *and*

$$B(x) = (1-L)x\left(1 + O\left(\frac{1}{(\log x)^{1-1/a}}\right)\right).$$

The error terms in Corollary 1 are easily derived from (5), using the trivial bound $\chi(n) \leqslant 1$. The claim that $0 < L < 1$ follows directly from Theorem 2, with (i) $f(n) = \max(2, \exp(cn^a))$ and (ii) $f(n) = \max(2, \exp(c(\log 2n)^a))$, where $c > 0$ is a suitable constant. Other examples of θ, for which $L < 1$, can be dealt with similarly.

When $B(x) = o(x)$, we need a different strategy for obtaining an asymptotic formula for $B(x)$, since the estimate (5) provides only an upper bound for $B(x)$ whenever $L = 1$. We will focus on the case $\theta(n) \asymp n^a$, where $a \geqslant 1$ is constant. Theorem 2 shows that $L = 1$ and $B(x) = o(x)$. Theorem 3 generalizes [**8**, Theorem 1.2], where the case $a = 1$ is established with $\lambda_1 = 1$.

THEOREM 3. *Let* θ *satisfy (3) and assume that* $\theta(n) \asymp n^a$ *for some constant* $a \geqslant 1$. *Then there are constants* $c_\theta > 0$ *and* $\lambda_a \in (0, 1]$ *such that*

$$B(x) = \frac{c_\theta x}{(\log x)^{\lambda_a}}\left(1 + O\left(\frac{1}{\log x}\right)\right). \tag{6}$$

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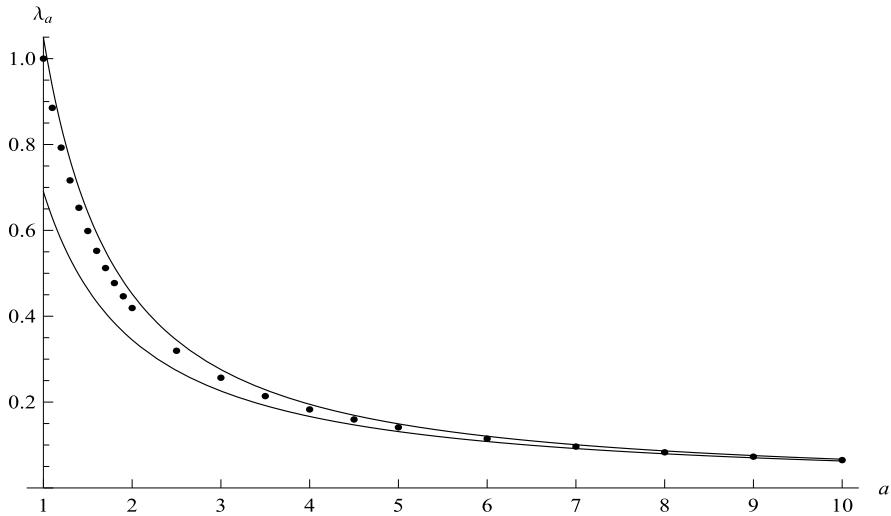


Figure 1: Values of λ_a together with the bounds (8) and (9).

Here $s = -\lambda_a$ is the unique solution in the interval $[-1, 0)$ of the equation

$$0 = s + \frac{e^{-\gamma}}{a(a+1)^s} + s\int_1^\infty (\omega(t) - e^{-\gamma})\frac{dt}{(at+1)^{s+1}}, \tag{7}$$

where $\omega(t)$ denotes Buchstab's function and γ is Euler's constant. For $a \geqslant 1$,

$$\lambda_a > \frac{e^{-\gamma}}{a}\left(1 + \frac{e^{-\gamma}\log(a+1)}{a} - \frac{0.16}{a}\right) \tag{8}$$

and

$$\lambda_a < \frac{e^{-\gamma}}{a}\left(1 + \frac{e^{-\gamma}\log(a+1)}{a} + \frac{\log^2(a+1)}{a^2}\right). \tag{9}$$

Figure 1 and Table 1 show several values of λ_a, obtained by solving equation (7) numerically.

As in [**8**, Theorem 1.2] with $a = 1$, one can consider a less restrictive condition on θ, such as $n^a(\log n)^{-b} \ll \theta(n) \ll n^a(\log n)^b$, where $0 \leqslant b < 1$, and establish the estimate (6) with the relative error term $O(1/\log x)$ replaced by $O(1/(\log x)^{1-b})$. However, we will not pursue this here.

We now turn to the distribution of divisors. Let $\mathcal{D}$ be the set defined in (2). When $\theta(n) = tn$, where t is constant, Tenenbaum [**5**, Lemma 2.2] showed that $\mathcal{D} = \mathcal{B}$. We want to generalize this result as much as possible. The example $\theta(n) = n + 1$, for which $\mathcal{D} = \{1, 2\}$ while $\mathcal{B}$ is infinite, illustrates that some condition on θ is required to ensure equality of these two sets. The condition we need is

$$\theta(n) \leqslant \theta(n+1), \qquad m\theta(n) \leqslant \theta(mn) \quad (n, m \geqslant 1, \gcd(n, m) = 1). \tag{10}$$

Note that (10) is satisfied if $\theta(n)/n$ is non-decreasing.

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Table 1: Truncated values of λ_a.

a	λ_a	a	λ_a
1	1	2.5	0.3195...
1.1	0.8854...	3	0.2567...
1.2	0.7927...	3.5	0.2139...
1.3	0.7164...	4	0.1829...
1.4	0.6526...	4.5	0.1595...
1.5	0.5985...	5	0.1412...
1.6	0.5522...	6	0.1147...
1.7	0.5122...	7	0.09639...
1.8	0.4772...	8	0.08301...
1.9	0.4464...	9	0.07283...
2	0.4191...	10	0.06484...

THEOREM 4. *Let θ satisfy (3).*

(i) *We have $\mathcal{D} \subseteq \mathcal{B}$ and hence $D(x) \leqslant B(x)$.*

(ii) *If θ satisfies (10), then $\mathcal{D} = \mathcal{B}$ and hence $D(x) = B(x)$.*

As an example, consider $\theta(n) = n^2 + 1$. Theorems 3 and 4 show that the number of integers $n \leqslant x$ whose divisors $1 = d_1 < d_2 < \cdots < d_{\tau(n)} = n$ satisfy $d_{j+1} \leqslant d_j^2 + 1$ for $1 \leqslant j < \tau(n)$ is given by

$$D(x) = \frac{cx}{(\log x)^{\lambda_2}}\left(1 + O\left(\frac{1}{\log x}\right)\right),$$

where c is a positive constant and $\lambda_2 = 0.4191\ldots$.

With $\theta(n) = 2^n$, Corollary 1 and Theorem 4 show that the number of integers $n \leqslant x$ whose divisors $1 = d_1 < d_2 < \cdots < d_{\tau(n)} = n$ satisfy $d_{j+1} \leqslant 2^{d_j}$ for $1 \leqslant j < \tau(n)$ is given by

$$D(x) = (1 - L)x\left(1 + O\left(\frac{1}{\log x}\right)\right),$$

where $1 - L = 0.2265\ldots$.

The proof of Theorem 1 is based on the functional equation in Lemma 3 and an estimate for the number of integers without small prime factors in Lemma 2. Theorem 2 is established with the help of Theorem 1. The proof of Theorem 3, which requires the most amount of work, is modeled after [**8**, Theorem 1.2], with the added difficulty that the poles of the Laplace transform (i.e. the solutions of equation (7)) now depend on the parameter a (see Lemma 9). The proof of Theorem 4 generalizes that of Tenenbaum [**5**, Lemma 2.2].

§3. *Preliminaries.* Let

$$\Phi(x, y) = \#\{1 \leqslant n \leqslant x : P^-(n) > y\}$$

and define $\Phi(x, \infty) = 1$ for $x \geqslant 1$. For $u \geqslant 1$, Buchstab's function $\omega(u)$ is defined as the unique continuous solution to the equation

$$(u\omega(u))' = \omega(u - 1) \quad (u > 2)$$

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with initial condition $u\omega(u) = 1$ for $1 \leqslant u \leqslant 2$. Let $\omega(u) = 0$ for $u < 1$ and define ω at 1 and ω' at 1 and 2 by right-continuity. Let $\Gamma(u)$ denote the usual gamma function.

The calculation of the values of λ_a in Table 1 and the approximation of several integrals in the proof of Lemma 9 require estimates for integrals involving $\omega(t) - \mathrm{e}^{-\gamma}$. For that purpose, we used exact formulas for $\omega(t)$ for $0 \leqslant t \leqslant 5$, derived with the help of Mathematica. To estimate the contribution from $t > 5$, we used a table of zeros and relative extrema of $\omega(t) - \mathrm{e}^{-\gamma}$ on the interval [5, 10.3355] due to Cheer and Goldston [**1**], and the estimate (ii) from Lemma 1 for $t \geqslant 10.3355$.

LEMMA 1. *We have:*

(i) $|\omega'(u)| \leqslant 1/\Gamma(u+1) \quad (u \geqslant 0)$;

(ii) $|\omega(u) - \mathrm{e}^{-\gamma}| \leqslant 1/\Gamma(u+1) \quad (u \geqslant 0)$.

Proof. See [**8**, Lemma 2.1]. □

LEMMA 2. *Let* $u = \log x/\log y$.

(i) *For* $x \geqslant 1$, $y \geqslant 2$, *we have*

$$\Phi(x, y) = \mathrm{e}^{\gamma} x\omega(u) \prod_{p \leqslant y} \left(1 - \frac{1}{p}\right) + O\left(\frac{y}{\log y} + \frac{x\mathrm{e}^{-u/3}}{(\log y)^2}\right).$$

(ii) *For* $x \geqslant y \geqslant 2$, *we have*

$$\Phi(x, y) = x \prod_{p \leqslant y} \left(1 - \frac{1}{p}\right) + O\left(\frac{x\mathrm{e}^{-u/3}}{\log y}\right).$$

Proof. (i) See [**8**, Lemma 2.2]. Part (ii) follows from (i) and Lemma 1. □

LEMMA 3. *Let* θ *satisfy (3). For* $x \geqslant 0$, *we have*

$$[x] = \sum_{n \leqslant x} \chi(n)\, \Phi(x/n, \theta(n)).$$

Proof. In [**8**, Lemma 2.3], we proved this result assuming that $\theta(n) < \infty$. The inclusion of ∞ in the possible values of θ has no effect on the proof. The basic idea is that every integer $1 \leqslant m \leqslant x$ factors uniquely as $m = nr$, where $n \in \mathcal{B}$ and r is counted in $\Phi(x/n, \theta(n))$. □

LEMMA 4. *Let* θ *satisfy (3). The series*

$$L = \sum_{n \geqslant 1} \frac{\chi(n)}{n} \prod_{p \leqslant \theta(n)} \left(1 - \frac{1}{p}\right)$$

converges and $0 \leqslant L \leqslant 1$.

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Proof. Lemma 2(ii) implies that $\lim_{x\to\infty} \Phi(x, y)/x = \prod_{p\leqslant y}(1 - 1/p)$. From Lemma 3, we have

$$\sum_{1\leqslant n\leqslant N} \frac{\chi(n)}{n} \prod_{p\leqslant\theta(n)} \left(1 - \frac{1}{p}\right) \leqslant 1$$

for every $N \geqslant 1$. The result now follows since the terms of the series are $\geqslant 0$. □

§4. *Proof of Theorems 1 and 2.*

Proof of Theorem 1. Since $\Phi(x/n, \theta(n)) = 1$ when $n \leqslant x < n\theta(n)$, Lemma 3 yields

$$\begin{aligned}
[x] &= B(x) + \sum_{n\leqslant x} \chi(n)\,(\Phi(x/n, \theta(n)) - 1) \\
&= B(x) + \sum_{n\theta(n)\leqslant x} \chi(n)\,(\Phi(x/n, \theta(n)) - 1) \\
&= B(x) + x \sum_{n\theta(n)\leqslant x} \frac{\chi(n)}{n} \prod_{p\leqslant\theta(n)} \left(1 - \frac{1}{p}\right) + O(E_1(x)),
\end{aligned}$$

where

$$E_1(x) = x \sum_{n\theta(n)\leqslant x} \frac{\chi(n)}{n \log\theta(n)} \exp\left(-\frac{\log(x/n)}{3\log\theta(n)}\right),$$

by Lemma 2(ii). Thus,

$$[x] = B(x) + Lx + O(E_1(x) + E_2(x)),$$

where

$$E_2(x) = x \sum_{n\theta(n)>x} \frac{\chi(n)}{n \log\theta(n)}.$$

This completes the proof of (5). The estimate (4) follows, since $E_1(x) + E_2(x) = o(x)$, by Lemma 4. □

Proof of Theorem 2. (i) Assume that $L < 1$, so that $B(n) \geqslant cn$ for some $c > 0$ and all $n \geqslant 1$. Let $g(n) = \log f(n)$ and assume that $ng(n)$ is increasing. Partial summation applied twice yields

$$\begin{aligned}
\sum_{n\leqslant N} \frac{\chi(n)}{ng(n)} &= \frac{B(N)}{Ng(N)} + \sum_{n\leqslant N-1} B(n)\left(\frac{1}{ng(n)} - \frac{1}{(n+1)g(n+1)}\right) \\
&\geqslant \frac{cN}{Ng(N)} + \sum_{n\leqslant N-1} cn\left(\frac{1}{ng(n)} - \frac{1}{(n+1)g(n+1)}\right) \\
&= \sum_{n\leqslant N} \frac{c}{ng(n)},
\end{aligned}$$

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which grows unbounded as N increases. But this is impossible, since

$$\sum_{n\leqslant N}\frac{\chi(n)}{ng(n)}\leqslant\sum_{n\leqslant N}\frac{\chi(n)}{n\log\theta(n)}\ll\sum_{n\leqslant N}\frac{\chi(n)}{n}\prod_{p\leqslant\theta(n)}\Big(1-\frac{1}{p}\Big)\leqslant L\leqslant 1,$$

by Lemma 4. Thus, $L=1$ and $B(x)=o(x)$.

(ii) Assume that $\theta(n)\geqslant f(n)\geqslant 2$, where $f(n)\gg P^+(n)$, and that

$$\sum_{n\geqslant 1}\frac{1}{n\log f(n)}$$

converges. Then there exists a $t\geqslant 1$ such that $tf(n)\geqslant P^+(n)$ and

$$L_{tf}\leqslant\sum_{n\geqslant 1}\frac{1}{n}\prod_{p\leqslant tf(n)}\Big(1-\frac{1}{p}\Big)<1.$$

Theorem 1 implies that $B_{tf}(x)\gg x$. According to the hypothesis, there is an $r\in\mathbb{N}$ such that $f(2^rn)\geqslant tf(n)$ for all $n\geqslant 1$. Thus, if n is counted in $B_{tf}(x/2^r)$, then 2^rn is counted in $B_f(x)$. This yields

$$B_\theta(x)\geqslant B_f(x)\geqslant B_{tf}(x/2^r)\gg_r x.$$

Part (iii) is an immediate consequence of Theorem 1. □

§5. *Proof of Theorem 3.* Throughout this section, we assume that θ satisfies (3) and that

$$B(x)\ll\frac{x}{(\log x)^\nu}\quad(x\geqslant 2)\tag{11}$$

for some suitable $\nu\in[0,1]$, to be determined later. Clearly, $\nu=0$ is admissible. All constants implied by $\ll$ and the big O notation may depend on θ, and therefore on a, but are otherwise absolute. Lemmas 5 through 8 correspond to Lemmas 5.3 through 5.7 of [**8**]. The main difference is the assumption on the size of $B(x)$ for the purpose of estimating the error terms, for which we use (11) here.

LEMMA 5. *For $x\geqslant e$, we have*

$$B(x)=x-x\sum_{n\theta(n)\leqslant x}\frac{\chi(n)}{n}\,\mathrm{e}^\gamma\omega\Big(\frac{\log x/n}{\log\theta(n)}\Big)\prod_{p\leqslant\theta(n)}\Big(1-\frac{1}{p}\Big)+O\Big(\frac{x}{(\log x)^{1+\nu}}\Big).$$

Proof. As in the proof of Theorem 1, we have

$$B(x)=[x]-\sum_{n\theta(n)\leqslant x}\chi(n)\,(\Phi(x/n,\theta(n))-1).\tag{12}$$

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We apply Lemma 2(i) to estimate each occurrence of $\Phi(x/n, \theta(n))$ in (12). The contribution from the error term $O(y/\log y)$ is

$$\ll \sum_{n\theta(n)\leqslant x} \chi(n)\frac{\theta(n)}{\log\theta(n)} \ll \sum_{n^{1+a}\ll x} \chi(n)\frac{n^a}{\log 2n}$$
$$\ll \frac{x^{a/(1+a)}}{\log x} \sum_{n\ll x^{1/(1+a)}} \chi(n) \ll \frac{x}{(\log x)^{\nu+1}},$$

by (11). For the contribution from the error term $O(xe^{-u/3}/(\log y)^2)$, we split up the range of summation by powers of 2 and use (11) to get

$$\ll \sum_{n\theta(n)\leqslant x} \chi(n)\frac{x}{n(\log\theta(n))^2}\exp\left(-\frac{\log x/n}{3\log\theta(n)}\right)$$
$$\ll \sum_{n\ll x^{1/(1+a)}} \frac{x}{n(\log 2n)^{2+\nu}}\exp\left(-\frac{\log 2x}{A\log 2n}\right) \ll \frac{x}{(\log x)^{1+\nu}},$$

where A is some suitable positive constant. □

LEMMA 6. *For $x \geqslant e$, we have*

$$B(x) = x\sum_{n\geqslant 1}\frac{\chi(n)}{n\log\theta(n)}\left(e^{-\gamma} - \omega\left(\frac{\log x/n}{\log\theta(n)}\right)\right) + O\left(\frac{x}{(\log x)^{1+\nu}}\right).$$

Proof. We have $L = 1$ by Theorem 2. Since $\omega(u) = 0$ for $u < 1$, Lemma 5 shows that

$$B(x) = x\sum_{n\geqslant 1}\frac{\chi(n)}{n}\prod_{p\leqslant\theta(n)}\left(1-\frac{1}{p}\right)\left(1 - e^{\gamma}\omega\left(\frac{\log x/n}{\log\theta(n)}\right)\right) + O\left(\frac{x}{(\log x)^{1+\nu}}\right).$$

The contribution from n with $\log n \leqslant \sqrt{\log x}$ is $\ll x\exp(-\sqrt{\log x})$. For those n for which $\log n > \sqrt{\log x}$, we use the estimate

$$\prod_{p\leqslant\theta(n)}\left(1-\frac{1}{p}\right) = \frac{e^{-\gamma}}{\log\theta(n)}\left(1 + O\left(\frac{1}{(\log n)^4}\right)\right).$$

The contribution from the error term is

$$\ll x\sum_{\log n > \sqrt{\log x}}\frac{1}{n(\log n)^5} \ll \frac{x}{(\log x)^2} \ll \frac{x}{(\log x)^{1+\nu}}.$$ □

LEMMA 7. *For $x \geqslant e$, we have*

$$B(x) = x\sum_{n\geqslant 2}\frac{\chi(n)}{an\log n}\left(e^{-\gamma} - \omega\left(\frac{\log x/n}{a\log n}\right)\right) + O\left(\frac{x}{(\log x)^{1+\nu}}\right).$$

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Proof. Since $\theta(n) \asymp n^a$, we have $\log\theta(n) = a\log n + O(1)$. Inserting this estimate for each instance of $\log\theta(n)$ in Lemma 6 yields the desired result. For more details of the calculations, see [**8**, Lemma 5.6], where the case $a = 1$ is treated. □

LEMMA 8. *For $x \geqslant e$, we have*

$$B(x) = x\int_{\mathrm{e}}^{\infty} \frac{B(y)}{ay^2\log y}\left(\mathrm{e}^{-\gamma} - \omega\left(\frac{\log x/y}{a\log y}\right)\right)dy + O\left(\frac{x}{(\log x)^{1+\nu}}\right).$$

Proof. This follows from applying partial summation to the sum in Lemma 7. □

From Lemma 8, we have, for $x \geqslant e$,

$$B(x) = x\,\alpha - x\int_{\mathrm{e}}^{\infty} \frac{B(y)}{ay^2\log y}\,\omega\left(\frac{\log x/y}{a\log y}\right)dy + O\left(\frac{x}{(\log x)^{1+\nu}}\right), \tag{13}$$

where

$$\alpha := \mathrm{e}^{-\gamma}\int_{\mathrm{e}}^{\infty} \frac{B(y)}{ay^2\log y}\,dy.$$

For $x \geqslant e$, let $z \geqslant 0$ be given by

$$z = \log\log(x)$$

and let

$$G_\theta(z) := \frac{B(\exp(\mathrm{e}^z))}{\exp(\mathrm{e}^z)} = \frac{B(x)}{x}.$$

Dividing (13) by x and changing variables in the integral via $u = \log\log y$, we get, for $z \geqslant 0$,

$$\begin{aligned} G_\theta(z) &= \alpha - \frac{1}{a}\int_0^z G_\theta(u)\,\omega((\mathrm{e}^{z-u}-1)/a)\,du + E_\theta(z) \\ &= \alpha - \frac{1}{a}\int_0^z G_\theta(u)\,\Omega_a(z-u)\,du + E_\theta(z), \end{aligned} \tag{14}$$

where

$$E_\theta(z) \ll \mathrm{e}^{-(1+\nu)z} \tag{15}$$

and

$$\Omega_a(u) := \omega((\mathrm{e}^u - 1)/a).$$

Equation (14) leads to the equation of Laplace transforms

$$\widehat{G}_\theta(s) = \frac{\alpha}{s} - \frac{1}{a}\widehat{G}_\theta(s)\,\widehat{\Omega}_a(s) + \widehat{E}_\theta(s) \quad (\mathrm{Re}(s) > 0),$$

which we solve for $\widehat{G}_\theta(s)$ to get

$$\widehat{G}_\theta(s) = \frac{\alpha}{s(1+(1/a)\widehat{\Omega}_a(s))} + \frac{\widehat{E}_\theta(s)}{1+(1/a)\widehat{\Omega}_a(s)} \quad (\mathrm{Re}(s) > 0).$$

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Let $F_a(z)$ be given by

$$F_a(z) = 1 - \frac{1}{a}\int_0^z F_a(u)\,\Omega_a(z-u)\,du. \tag{16}$$

Equation (16) is an error-free, rescaled version of equation (14), which depends on a, but does not involve θ. Note that the upper limit of the integral could be replaced by $z - \log(a+1)$, since $\omega(t) = 0$ for $t < 1$. Thus, $F_a(z) = 1$ for $0 \leqslant z \leqslant \log(a+1)$ and, for $z > \log(a+1)$, $F_a(z)$ is determined by the values of $F_a(u)$ for $u \in [0, z - \log(a+1)]$. Hence, (16) defines the continuous function $F_a(z)$ for $z \geqslant 0$. As with $G_\theta(z)$, we find that the Laplace transform of $F_a(z)$ is given by

$$\widehat{F}_a(s) = \frac{1}{s(1 + (1/a)\widehat{\Omega}_a(s))} \quad (\mathrm{Re}(s) > 0)$$

and, therefore,

$$\widehat{G}_\theta(s) = \alpha\widehat{F}_a(s) + s\widehat{F}_a(s)\widehat{E}_\theta(s). \tag{17}$$

The Laplace transform of $\Omega_a(u)$, defined for $\mathrm{Re}(s) > 0$, is given by

$$\begin{aligned}\widehat{\Omega}_a(s) &= \int_0^\infty \omega((\mathrm{e}^u - 1)/a)\mathrm{e}^{-us}\,du = a\int_0^\infty \omega(t)\frac{dt}{(at+1)^{s+1}} \\ &= \frac{\mathrm{e}^{-\gamma}}{s(a+1)^s} + a\int_1^\infty (\omega(t) - \mathrm{e}^{-\gamma})\frac{dt}{(at+1)^{s+1}}.\end{aligned}$$

The last equation extends $\widehat{\Omega}_a(s)$ to a meromorphic function on $\mathbb{C}$ with a simple pole at $s = 0$. We will need to investigate the location of zeros of the entire function $g_a(s)$, defined by

$$\begin{aligned}g_a(s) = 1/\widehat{F}_a(s) &= s\left(1 + \frac{1}{a}\widehat{\Omega}_a(s)\right) \\ &= s + \frac{\mathrm{e}^{-\gamma}}{a(a+1)^s} + s\int_1^\infty (\omega(t) - \mathrm{e}^{-\gamma})\frac{dt}{(at+1)^{s+1}}.\end{aligned}$$

LEMMA 9. *For $a \geqslant 1$, $g_a(s)$ has a simple real zero at $s = -\lambda_a$, where $\lambda_1 = 1$, $\lambda_a \in (0,1)$ for $a > 1$,*

$$\lambda_a > \frac{\mathrm{e}^{-\gamma}}{a}\left(1 + \frac{\mathrm{e}^{-\gamma}\log(a+1)}{a} - \frac{0.16}{a}\right) =: l_a$$

and

$$\lambda_a < \frac{\mathrm{e}^{-\gamma}}{a}\left(1 + \frac{\mathrm{e}^{-\gamma}\log(a+1)}{a} + \frac{\log^2(a+1)}{a^2}\right) =: u_a.$$

For $a \geqslant 1$, $g_a(s)$ has no other zero with $\mathrm{Re}(s) \geqslant -1 - u_a$.

Proof. We have $g_a(0) = \mathrm{e}^{-\gamma}/a > 0$ and

$$g_a(-1) = -1 + \mathrm{e}^{-\gamma}(1 + 1/a) - (2\mathrm{e}^{-\gamma} - 1) = \mathrm{e}^{-\gamma}(1/a - 1).$$

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Thus, $g_1(-1) = 0$ and $\lambda_1 = 1$. Assume that $a > 1$. We have $g_a(-1) < 0$, so that $g_a(s)$ has a zero in the interval $(-1, 0)$, since $g_a(s)$ is real if s is real. For $s \in [-1, 0]$, we have

$$I_a(s) := \int_1^\infty (\omega(t) - \mathrm{e}^{-\gamma}) \frac{dt}{(at+1)^{s+1}}$$
$$\leqslant \frac{1}{(a+1)^{s+1}} \int_1^\infty |\omega(t) - \mathrm{e}^{-\gamma}|\, dt < \frac{0.16}{(a+1)^{s+1}},$$

by numerical computation of the last integral, and

$$I_a(s) = \int_1^{\mathrm{e}^\gamma} (t^{-1} - \mathrm{e}^{-\gamma}) \frac{dt}{(at+1)^{s+1}} + \int_{\mathrm{e}^\gamma}^\infty (\omega(t) - \mathrm{e}^{-\gamma}) \frac{dt}{(at+1)^{s+1}}$$
$$\geqslant \frac{1}{(a\mathrm{e}^\gamma + 1)^{s+1}} (\gamma - \mathrm{e}^{-\gamma}(\mathrm{e}^\gamma - 1)) - \frac{1}{(a\mathrm{e}^\gamma + 1)^{s+1}} \int_{\mathrm{e}^\gamma}^\infty |\omega(t) - \mathrm{e}^{-\gamma}|\, dt$$
$$\geqslant \frac{\gamma - 1 + \mathrm{e}^{-\gamma} - 0.021}{(a\mathrm{e}^\gamma + 1)^{s+1}} > \frac{0.11}{(a\mathrm{e}^\gamma + 1)^{s+1}}.$$

Since $s \leqslant 0$,

$$g_a(s) < s + \frac{\mathrm{e}^{-\gamma}}{a(a+1)^s} + s \frac{0.11}{(a\mathrm{e}^\gamma + 1)^{s+1}} =: g_a^+(s)$$

and

$$g_a(s) > s + \frac{\mathrm{e}^{-\gamma}}{a(a+1)^s} + s \frac{0.16}{(a+1)^{s+1}} =: g_a^-(s).$$

Hence, $g_a(-l_a) > g_a^-(-l_a)$ and $g_a(-u_a) < g_a^+(-u_a)$ if $u_a \leqslant 1$. A calculus exercise, made easier with the help of a computer, shows that $g_a^-(-l_a) > 0$ and $g_a^+(-u_a) < 0$ for $a \geqslant 1$. Hence, $g_a(s)$ has a zero in $(-1, 0) \cap (-u_a, -l_a)$.

It remains to show that, for $a \geqslant 1$, $g_a(s)$ has no other zero with $\mathrm{Re}(s) \geqslant -1 - u_a$. We write

$$h_a(s) = g_a(s) - s, \qquad \mu_a = u_a + 1.$$

For $\mathrm{Re}(s) \geqslant -\mu_a$,

$$|h_a(s)| \leqslant \frac{\mathrm{e}^{-\gamma}}{a(a+1)^{-\mu_a}} + |s| \int_1^\infty |\omega(t) - \mathrm{e}^{-\gamma}| \frac{dt}{(at+1)^{-\mu_a+1}}.$$

Since $\mu_a > 1$ and $at + 1 < (a+1)t$ for $t \geqslant 1$, the last integral is

$$\leqslant (a+1)^{\mu_a - 1} \int_1^\infty |\omega(t) - \mathrm{e}^{-\gamma}| t^{\mu_a - 1}\, dt.$$

First assume that $a \geqslant 10$. We have

$$\mu_a - 1 \leqslant \mu_{10} - 1 < 0.1,$$
$$(a+1)^{\mu_a - 1} \leqslant (10+1)^{\mu_{10} - 1} < 1.2$$

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and

$$\frac{\mathrm{e}^{-\gamma}}{a(a+1)^{-\mu_a}} \leqslant \frac{\mathrm{e}^{-\gamma}}{10(10+1)^{-\mu_{10}}} < 0.73.$$

Thus, for $a \geqslant 10$,

$$|h_a(s)| \leqslant 0.73 + 1.2|s| \int_1^\infty |\omega(t) - \mathrm{e}^{-\gamma}| t^{0.1}\, dt < 0.73 + 0.21|s|,$$

by numerical computation of the integral. Hence, we have $|h_a(s)| < |s|$ provided $|s| > 0.93$. On the boundary of the rectangle R with corners $-\mu_a \pm \mathrm{i}T, \alpha \pm \mathrm{i}T$, where $\alpha, T \geqslant 5$, we have $|s| > 0.93$, since $\mu_a > 1$. Rouché's theorem implies that $g_a(s)$ and s have the same number of zeros, i.e. exactly one zero, with $\mathrm{Re}(s) \geqslant -\mu_a$.

Next, assume that $1 \leqslant a \leqslant 10$. We have

$$\mu_a - 1 \leqslant \mu_1 - 1 < 1.1,$$
$$(a+1)^{\mu_a - 1} \leqslant (1+1)^{\mu_1 - 1} < 2.1$$

and

$$\frac{\mathrm{e}^{-\gamma}}{a(a+1)^{-\mu_a}} \leqslant \frac{\mathrm{e}^{-\gamma}}{(1+1)^{-\mu_1}} < 2.4.$$

Thus, for $1 \leqslant a \leqslant 10$,

$$|h_a(s)| \leqslant 2.4 + 2.1|s| \int_1^\infty |\omega(t) - \mathrm{e}^{-\gamma}| t^{1.1}\, dt < 2.4 + 0.5|s|.$$

Hence, $|h_a(s)| < |s|$ provided $|s| > 4.8$. On the boundary of the rectangle R, we have $|s| \geqslant 5$, with the possible exception of the segment with end points $-\mu_a \pm 5\mathrm{i}$. For s on this segment, and $1 \leqslant a \leqslant 10$, we evaluate $|h_a(s)/s|$ numerically to get

$$\frac{|h_a(s)|}{|s|} = \frac{|h_a(-\mu_a + \mathrm{i}\tau)|}{|-\mu_a + \mathrm{i}\tau|} < 0.98 \quad (1 \leqslant a \leqslant 10, -5 \leqslant \tau \leqslant 5).$$

Hence, $|h_a(s)| < |s|$ on the boundary of R, and Rouché's theorem implies that $g_a(s)$ has exactly one zero with $\mathrm{Re}(s) \geqslant -\mu_a$. □

LEMMA 10. *We write* $s = \sigma + \mathrm{i}\tau$ *and*

$$H_a(\sigma) := \frac{1}{a(a+1)^\sigma} + \frac{1}{a} \int_1^\infty |\omega'(t)| \frac{dt}{(at+1)^\sigma}.$$

If $g_a(s) = 0$, *then* $|\tau| \leqslant H_a(\sigma)$. *For* $|\tau| \geqslant 2H_a(\sigma)$, *we have*

$$\frac{1}{g_a(s)} = \frac{1}{s} + O\left(\frac{H_a(\sigma)}{\sigma^2 + \tau^2}\right).$$

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Proof. Integration by parts shows that

$$g_a(s) = s + \frac{1}{a(a+1)^s} + \frac{1}{a}\int_1^\infty \omega'(t)\frac{dt}{(at+1)^s} = s + H_a(\sigma)\xi_a(s),$$

where $\xi_a(s) \in \mathbb{C}$ with $|\xi_a(s)| \leqslant 1$. Thus, any zero of $g_a(s)$ must satisfy $|\tau| \leqslant |s| \leqslant H_a(\sigma)$. We have

$$g_a(s) = s\left(1 + \frac{H_a(\sigma)\xi_a(s)}{s}\right),$$

from which the estimate for $1/g_a(s)$ follows. □

LEMMA 11. *The function $F_a(z)$ defined by (16) satisfies*

$$F_a(z) = c_a \mathrm{e}^{-\lambda_a z} + O_a(\mathrm{e}^{-\mu_a z}) \quad (z \geqslant 0) \tag{18}$$

and

$$F_a'(z) = \tilde{c}_a \mathrm{e}^{-\lambda_a z} + O_a(\mathrm{e}^{-\mu_a z}) \quad (z \geqslant 0) \tag{19}$$

for constants c_a, $\tilde{c}_a = -\lambda_a c_a$, where $\mu_a > \lambda_a + 1$. Here $F_a'(z)$ denotes the right derivative of $F_a(z)$.

Proof. We evaluate the inverse Laplace integral

$$F_a(z) = \frac{1}{2\pi \mathrm{i}} \int_{1-\mathrm{i}\infty}^{1+\mathrm{i}\infty} \widehat{F}_a(s)\mathrm{e}^{zs}\, ds.$$

Let $\mu = \mu_a = u_a + 1$ from Lemma 9 and put $T = \exp(z(\mu+1))$. Since the result is trivial for bounded z, we may assume that z is sufficiently large such that $T > 2H(-\mu)$. We have

$$\int_{1+\mathrm{i}T}^{1+\mathrm{i}\infty} \widehat{F}_a(s)\mathrm{e}^{zs}\, ds = \int_{1+\mathrm{i}T}^{1+\mathrm{i}\infty} \frac{1}{s}\mathrm{e}^{zs}\, ds + O(\mathrm{e}^z/T) = O(\mathrm{e}^{-\mu z}),$$

by Lemma 10 and integration by parts applied to the last integral. We apply the residue theorem to the rectangle with vertices $-\mu \pm \mathrm{i}T$, $1 \pm \mathrm{i}T$. The contribution from the horizontal segments can be estimated by Lemma 10 as

$$\int_{-\mu+\mathrm{i}T}^{1+\mathrm{i}T} \widehat{F}_a(s)\mathrm{e}^{zs}\, ds \ll \frac{\mathrm{e}^z}{T} = O(\mathrm{e}^{-\mu z}).$$

For the vertical segment with $\mathrm{Re}(s) = -\mu$, we have

$$\left|\int_{-\mu-\mathrm{i}2H_a(-\mu)}^{-\mu+\mathrm{i}2H_a(-\mu)} \widehat{F}_a(s)\mathrm{e}^{zs}\, ds\right| \leqslant 4H_a(-\mu) \max_{|\tau|\leqslant 2H_a(-\mu)} |\widehat{F}_a(-\mu+\mathrm{i}\tau)|\mathrm{e}^{-\mu z} = O(\mathrm{e}^{-\mu z})$$

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and

$$\int_{-\mu+\mathrm{i}2H_a(-\mu)}^{-\mu+\mathrm{i}T} \widehat{F}_a(s)\mathrm{e}^{zs}\,ds = \int_{-\mu+\mathrm{i}2H_a(-\mu)}^{-\mu+\mathrm{i}T} \left(\frac{1}{s}+O(\tau^{-2})\right)\mathrm{e}^{zs}\,ds = O(\mathrm{e}^{-\mu z}).$$

The residue theorem now yields

$$F_a(z) = \mathrm{Res}(\widehat{F}_a(s)\mathrm{e}^{zs}; -\lambda_a) + O(\mathrm{e}^{-\mu z}) = c_a\mathrm{e}^{-\lambda_a z} + O(\mathrm{e}^{-\mu z}),$$

which completes the proof of (18).

If $0 \leqslant z < \log(a+1)$, we have $F_a(z) = 1$ and hence $F_a'(z) = 0$. If $z \geqslant \log(a+1)$, (16) implies that

$$F_a'(z) = -\frac{1}{a}\int_0^{z-\log(a+1)} F_a(u)\,\omega'\left(\frac{\mathrm{e}^{z-u}-1}{a}\right)\frac{\mathrm{e}^{z-u}}{a}\,du - \frac{1}{a}F_a(z-\log(a+1)). \tag{20}$$

The estimate (19) follows from applying (18) to each occurrence of F_a on the right-hand side of (20), and the fact that $\omega'(t) \ll_A t^{-A}$ for any constant A, by Lemma 1. □

We are now ready to finish the proof of Theorem 3. Since $\widehat{F_a'}(s) = s\widehat{F}_a(s) - F_a(0)$ and $F_a(0) = 1$, (17) yields

$$\widehat{G}_\theta(s) = \alpha\widehat{F}_a(s) + \widehat{E}_\theta(s) + \widehat{F_a'}(s)\widehat{E}_\theta(s)$$

and thus

$$G_\theta(z) = \alpha F_a(z) + E_\theta(z) + \int_0^z F_a'(z-u)E_\theta(u)\,du. \tag{21}$$

We estimate $F_a(z)$ and $F_a'(z-u)$ using Lemma 11. Since $E_\theta(u) \ll \mathrm{e}^{-(\nu+1)u}$ by (15), $\nu \geqslant 0$ and $\lambda_a \leqslant 1$, (21) shows that $G_\theta(z) \ll z\mathrm{e}^{-\lambda_a z}$. Thus, $\nu = \lambda_a - \varepsilon$ is acceptable in (11) for every $\varepsilon > 0$. With this choice of ν, (21) shows that $G_\theta(z) \ll \mathrm{e}^{-\lambda_a z}$, which means that $\nu = \lambda_a$ is acceptable in (11). Hence, we assume that $\nu = \lambda_a$ and $E_\theta(u) \ll \mathrm{e}^{-(\lambda_a+1)u}$ for the remainder of this proof. The contribution to the integral in (21) from the main term in Lemma 11 is

$$\begin{aligned}
&\int_0^z \tilde{c}_a\mathrm{e}^{-\lambda_a(z-u)}E_\theta(u)\,du \\
&\quad= \tilde{c}_a\mathrm{e}^{-\lambda_a z}\left(\int_0^\infty \mathrm{e}^{\lambda_a u}E_\theta(u)\,du + O\left(\int_z^\infty \mathrm{e}^{\lambda_a u}\mathrm{e}^{-(\lambda_a+1)u}\,du\right)\right) \\
&\quad= \tilde{c}_a\mathrm{e}^{-\lambda_a z}(\beta + O(\mathrm{e}^{-z})),
\end{aligned}$$

say. The contribution from the error term in Lemma 11 to the integral in (21) is

$$\ll \int_0^z \mathrm{e}^{-\mu_a(z-u)}\mathrm{e}^{-(\lambda_a+1)u}\,du \ll \mathrm{e}^{-(\lambda_a+1)z},$$

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since $\mu_a > \lambda_a + 1$. Hence, (21) implies that

$$G_\theta(z) = (\alpha c_a + \beta \tilde{c}_a)\mathrm{e}^{-\lambda_a z} + O(\mathrm{e}^{-(\lambda_a+1)z}).$$

With $c_\theta := \alpha c_a + \beta \tilde{c}_a$, we get

$$G_\theta(z) = B(x)/x = c_\theta (\log x)^{-\lambda_a} + O((\log x)^{-(\lambda_a+1)}).$$

It remains to show that $c_\theta > 0$ if $a \geqslant 1$. Since $\theta(n) \gg n^a \geqslant n$, there exists an integer k such that $\theta(2^k n) \geqslant 2n$ for all $n \geqslant 1$. With $\theta_0(n) = 2n$, we have that $n \in \mathcal{B}_{\theta_0}$ implies that $2^k n \in \mathcal{B}_\theta$. Hence, $B_\theta(x) \geqslant B_{\theta_0}(x/2^k) \gg x/\log x$, by [**8**, Theorem 1.2]. Since $\lambda_a + 1 > 1$, we must have $c_\theta > 0$.

§6. *Proof of Theorem 4.* (i) Assume that $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k} \in \mathcal{D}$, where $p_1 < p_2 < \cdots < p_k$. Let $0 \leqslant j < k$ and write $d_i = p_1^{\alpha_1} \cdots p_j^{\alpha_j}$. The next larger divisor, d_{i+1}, satisfies $d_{i+1} \geqslant p_{j+1}$, since d_{i+1} must be divisible by at least one of the primes $p_{j+1}, \ldots, p_k$. Thus,

$$p_{j+1} \leqslant d_{i+1} \leqslant \theta(d_i) = \theta(p_1^{\alpha_1} \cdots p_j^{\alpha_j}),$$

which means that $n \in \mathcal{B}$.

(ii) Assume that θ satisfies (10). To show that $n \in \mathcal{B}$ implies that $n \in \mathcal{D}$ for all $n \geqslant 2$, we proceed by induction on k, the number of distinct prime factors of n. When $k = 1$, $n = p^\alpha \in \mathcal{B}$ implies that $p \leqslant \theta(1)$. For $1 \leqslant j \leqslant \alpha$, we have

$$d_{j+1} = p^j = p^{j-1} p \leqslant p^{j-1}\theta(1) \leqslant \theta(p^{j-1}) = \theta(d_j),$$

which means that $n \in \mathcal{D}$.

Now assume that, for some $k \geqslant 1$ and each $m = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, if $m \in \mathcal{B}$, then $m \in \mathcal{D}$. Let $n = mp^\alpha \in \mathcal{B}$, where $p > p_k$ and $m = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$. By the definition of $\mathcal{B}$, we have $m \in \mathcal{B}$ and hence $m \in \mathcal{D}$. Assume that $1 = t_1 < t_2 < \cdots < t_r = m$ are the divisors of m. Then the divisors of n, say $1 = d_1 < d_2 < \cdots < d_l = n$, are of the form $d_i = p^\beta t_j$, where $0 \leqslant \beta \leqslant \alpha$, $1 \leqslant j \leqslant r$. If $j < r$, we have

$$d_{i+1} \leqslant p^\beta t_{j+1} \leqslant p^\beta \theta(t_j) \leqslant \theta(p^\beta t_j) = \theta(d_i),$$

as desired. If $j = r$, then $d_i = p^\beta m$ and we may assume that $0 \leqslant \beta < \alpha$, that is, $d_i < n$. If $p > m$, then $p^{\beta+1} > p^\beta m = d_i$. Also, $n \in \mathcal{B}$ implies that $p \leqslant \theta(m)$, so

$$d_{i+1} \leqslant p^{\beta+1} \leqslant p^\beta \theta(m) \leqslant \theta(p^\beta m) = \theta(d_i).$$

If $p \leqslant m$, then $t_s \leqslant m/p < t_{s+1}$ for some s with $1 \leqslant s < r$. Now $d_i = p^\beta m < p^{\beta+1} t_{s+1}$, so

$$d_{i+1} \leqslant p^{\beta+1} t_{s+1} \leqslant p^{\beta+1}\theta(t_s) \leqslant \theta(p^{\beta+1} t_s).$$

Since $t_s \leqslant m/p$, we have $p^{\beta+1} t_s \leqslant mp^\beta = d_i$ and hence $\theta(p^{\beta+1} t_s) \leqslant \theta(d_i)$, as $\theta(n)$ is non-decreasing. Thus, $d_{i+1} \leqslant \theta(d_i)$ also holds in this case, which means that $n \in \mathcal{D}$.

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