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PROBLEMS AND SOLUTIONS

PROBLEM

03.2.1. Fixed Effects Estimation of the Population-Averaged Slopes in a Panel Data Random Coefficient Model

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Consider a linear panel data model with individual-specific slopes and also an individual-specific intercept. For a random draw i from the cross section, the model is

$$y_{it} = a_i + \mathbf{x}_{it}\mathbf{b}_i + u_{it}, \qquad t = 1, \ldots, T, \tag{1}$$

where $\mathbf{x}_{it}$ is $1 \times K$ and $\mathbf{b}_i$ is $K \times 1$. You are interested in estimating $\boldsymbol{\beta} \equiv E(\mathbf{b}_i)$, the population-averaged slope coefficients. In what follows, make the strict exogeneity assumption

$$E(u_{it}|\mathbf{x}_{i1}, \mathbf{x}_{i2}, \ldots, \mathbf{x}_{iT}, a_i, \mathbf{b}_i) = 0, \qquad t = 1, \ldots, T. \tag{2}$$

Let $\bar{\mathbf{x}}_i$ denote the $1 \times K$ vector of time averages and let $\ddot{\mathbf{x}}_i$ be the vector of time-demeaned variables. The $T \times K$ matrix $\ddot{\mathbf{X}}_i$ contains the $\ddot{\mathbf{x}}_i$ as its T rows. If $\mathbf{b}_i = \boldsymbol{\beta}$ then it is well known that the fixed effects, or within, estimator consistently estimates $\boldsymbol{\beta}$, as $N \to \infty$, holding T fixed, under assumption (2) and the rank condition

$$\text{rank } E(\ddot{\mathbf{X}}_i'\ddot{\mathbf{X}}_i) = K. \tag{3}$$

Essentially, condition (3) rules out time-constant regressors in addition to perfect collinearity. See, for example, Wooldridge (2002, Ch. 10). This problem asks you to study the properties of the fixed effects estimator in the more general model, (1).

a. Define $\mathbf{d}_i \equiv \mathbf{b}_i - \boldsymbol{\beta}$. Write equation (1) in terms of $\mathbf{d}_i$ and $\boldsymbol{\beta}$.
b. Let $\ddot{y}_{it}$ $t = 1, \ldots, T$ denote the time-demeaned quantities for the dependent variable and similarly for $\ddot{u}_{it}$. Write the time-demeaned equation in terms of $\ddot{y}_{it}$, $\ddot{\mathbf{x}}_i$, $\ddot{u}_{it}$, $\boldsymbol{\beta}$, and $\mathbf{d}_i$.
c. Show that under assumptions (2), (3), and

$$E(\mathbf{b}_i|\ddot{\mathbf{x}}_i) = E(\mathbf{b}_i), \qquad t = 1, \ldots, T, \tag{4}$$

the fixed effects estimator is consistent for $\boldsymbol{\beta}$. Explain why assumption (4) allows $\mathbf{b}_i$ to be correlated with the time average, $\bar{\mathbf{x}}_i$, and why allowing such correlation may be important in practice.
d. Can we relax (4) to the assumption that $\mathbf{b}_i$ is uncorrelated with each $\ddot{\mathbf{x}}_i$, $t = 1, \ldots, T$ and still maintain consistency? Explain.

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SOLUTION

01.2.1. ARMA Representation of Squared Markov Switching Heteroskedastic Models—Solution

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We can rewrite the equation expressing the dynamics of the volatility as

$$\sigma_t^2 = \mu_1 + (\mu_2 - \mu_1)\Delta_t. \quad \textbf{(1)}$$

The process $\{\Delta_t\}$ admits an AR(1) representation of the kind

$$\Delta_t = 1 - q + \lambda\Delta_{t-1} + v_t,$$

where $\lambda = -1 + p + q$ is the second eigenvalue of the transition matrix of the Markov chain (the other one being equal to 1) and v_t has the following probability structure (Hamilton 1989, p. 360):

$$\Pr[v_t = 1 - p | \Delta_{t-1} = 1] = p,$$

$$\Pr[v_t = -p | \Delta_{t-1} = 1] = 1 - p,$$

$$\Pr[v_t = -(1 - q) | \Delta_{t-1} = 0] = q,$$

$$\Pr[v_t = q | \Delta_{t-1} = 0] = 1 - q.$$

Rearranging equation (1) yields

$$\begin{aligned}\varepsilon_t^2 &= \mu_1 + (\mu_2 - \mu_1)\Delta_t + e_t \\ &= \mu_1 + (\mu_2 - \mu_1)(1 - q + \lambda\Delta_{t-1} + v_t) + e_t, \quad \textbf{(2)}\end{aligned}$$

where $e_t = \varepsilon_t^2 - \sigma_t^2$ is a mean-zero error. Applying the operator $(1 - \lambda L)$ to both sides of (2) and using the AR(1) structure of $\{\Delta_t\}$ yields

$$\varepsilon_t^2 = \mu_1(1 - \lambda) + (\mu_2 - \mu_1)(1 - q) + \lambda\varepsilon_{t-1}^2 + (\mu_2 - \mu_1)v_t + e_t - \lambda e_{t-1}. \quad \textbf{(3)}$$

The error term on the right-hand side of (3) admits a MA(1) representation

$$(\mu_2 - \mu_1)v_t + e_t - \lambda e_{t-1} = u_t - \beta u_{t-1},$$

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where β is the value less than one in absolute value that, together with σ_u^2, solves

$$(1+\beta^2)\sigma_u^2 = (1+\lambda^2)\sigma_e^2 + (\mu_2-\mu_1)^2\sigma_v^2$$
$$-\beta\sigma_u^2 = -\lambda\sigma_e^2. \quad \textbf{(4)}$$

In the case of interest

$$\sigma_e^2 = (\mathrm{E}(Z_t^4)-1)[(2\mu_1(\mu_2-\mu_1)+(\mu_2-\mu_1)^2)\mathrm{E}(\Delta_t)+\mu_1^2]$$

and

$$\sigma_v^2 = p(1-p)\mathrm{E}(\Delta_t) + q(1-q)(1-\mathrm{E}(\Delta_t)),$$

with $\mathrm{E}(\Delta_t) = (1-q)/(2-p-q)$.

Therefore we can write (3) as an ARMA(1,1)

$$\varepsilon_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + u_t - \beta u_{t-1},$$

where $\omega = \mu_1(1-\lambda) + (\mu_2-\mu_1)(1-q)$, $\alpha = -1+p+q$, and β is the solution to (4).

In the special case where $p+q=1$ we have that $\lambda = \alpha = \beta = 0$ and ε_t^2 is a white noise with mean $\omega = \mu_1 + (\mu_2-\mu_1)(1-q)$.

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