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A MODEL OF NEAR-RATIONAL EXUBERANCE

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We study how the use of judgment or "add-factors" in forecasting may disturb the set of equilibrium outcomes when agents learn by using recursive methods. We isolate conditions under which new phenomena, which we call *exuberance equilibria*, can exist in a standard self-referential environment. Local indeterminacy is not a requirement for existence. We construct a simple asset-pricing example and find that exuberance equilibria, when they exist, can be extremely volatile relative to fundamental equilibria.

Keywords: Learning, Expectations, Excess Volatility, Bounded Rationality

1. INTRODUCTION

Judgment is a fact of life in macroeconomic forecasting. It is widely understood that even the most sophisticated econometric forecasts are adjusted before presentation. This adjustment is so pervasive that it is known as the use of "add-factors" —subjective changes to the forecast that depend on the forecaster's assessment of special circumstances that are not well summarized by the variables that are included in the econometric model.[1] Recently, some authors have argued that economic theory needs to take explicit account of the effects of judgment on the behavior of macroeconomic systems.[2]

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We wish to think of the news or add-factor that modifies the forecast as a qualitative, unique, commonly understood economy-wide variable. An example of a judgmental adjustment is suggested by Reifschneider, Stockton, and Wilcox (1997), when they discuss the "financial headwinds" that were thought to be inhibiting U.S. economic growth in the early to mid-1990s. As they discuss, the headwinds add-factor was used to adjust forecasts over a period of many quarters. It was communicated to the public prominently in speeches by Federal Reserve Chairman Alan Greenspan. It was thus widely understood throughout the economy and was highly serially correlated. This is the type of variable we have in mind, although by no means would we wish to restrict attention to this particular example. Other examples might include the Y2K millennium bug or the 9/11 terrorist attacks in the United States, as well as a host of more minor events thought to influence economic performance.

Conventional wisdom among economists suggests that judgment is all to the good in macroeconomic forecasting. Models are, of course, crude approximations of reality and must be supplemented with other information not contained in the model. Although we motivate our ideas in terms of macroeconomic forecasting, our framework applies more generally to economic environments where expectations and qualitative judgments about the effects of unique events play an important role.

Our focus in this paper is on how the add-factor or judgmental adjustment of forecasts may create more problems than it solves. In particular, we study ways in which judgmental adjustment may become self-fulfilling. We study macroeconomic models in which expectations play an important role. Following Evans and Honkapohja (2001), we replace the rational expectations (RE) assumption with one of recursive learning. This involves the assignment of a well-chosen perceived law of motion (PLM), an econometric forecasting model, to the agents. We supplement that model with a qualitative, judgmental adjustment variable and study the resulting dynamics.

The main contribution of the paper is to define the concept of an *exuberance equilibrium*. We impose three requirements on the judgmentally adjusted system to define this concept. The first is that any equilibrium reached is an RE equilibrium with limited information. For this we use the consistent expectations equilibrium (CEE) concept.[3] The second is a Nash equilibrium in the inclusion of judgment—given that all agents are using judgmentally adjusted forecasts, no agent wishes to discontinue using the judgmental adjustment. The last requirement is expectational stability or learnability of the equilibrium.

Our Nash equilibrium does not correspond exactly to an RE equilibrium. This is because the judgment variable is assumed to be unavailable in the statistical part of the forecasting. We think of this as reflecting the separation of the econometric forecasting unit from the actual decision makers. Decision makers treat the econometric forecast as an input to which they are free to add the judgment variable. The judgmentally adjusted forecasts are the basis for the decisions and actions of the agents, but the judgment variable cannot be extracted by the econometric

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forecasting unit and converted into a statistical time series that can be utilized in an econometric forecasting model. In a similar vein the decision makers face a dichotomy in their use of judgment: they either incorporate the variable as an add-factor or they ignore it and directly use the econometric forecast. This inability of the decision makers to transmit, in a quantitative way, the judgmental aspects behind their final economic decisions to the econometric forecasters is the source of the deviation from full RE and the reason for our use of the term "near-rationality."[4]

We isolate conditions under which exuberance equilibria exist in a standard dynamic framework in which the state of the system depends on expectations of future endogenous variables. We show that exuberance equilibria exhibit higher asymptotic variance of the endogenous variable than the standard RE equilibrium. (The standard RE equilibrium is also a solution in our framework.) As an application we study a simple univariate asset-pricing model. We interpret the exuberance equilibrium in the asset-pricing model as an example of "excess volatility."[5] In this paper we do not discuss policy applications. For a discussion of the implications for monetary policy, the reader is referred to related work in Bullard, Evans, and Honkapohja (2008).

Our results may lead one to view the possibility of exuberance equilibria as particularly worrisome, as exuberance equilibria may exist even in otherwise benign circumstances. In particular, we show that exuberance is a clear possibility even in the case in which the underlying RE equilibrium is unique (determinate). Thus an interesting and novel finding is the possibility of "sunspot-like" equilibria, but without requiring that the underlying RE equilibrium of the model be indeterminate.[6] In a sense, we find "sunspot-like" equilibria without indeterminacy.

2. ECONOMIES WITH JUDGMENTAL ADJUSTMENT

We study systems in which agents use recursive algorithms to learn equilibria. These systems can converge to an RE equilibrium provided that certain conditions are met. The conditions are outlined in some detail in Evans and Honkapohja (2001), and the key condition is known as expectational stability. In this paper, we alter the econometric model that the agents use to forecast in order to allow them to use judgmental adjustment.

We consider a scalar version of the model studied in Bullard et al. (2008):

$$y_t = \beta y_{t+1}^e + u_t + w_t, \tag{1}$$

in which β is a scalar parameter and y_t is the state of the system. Here u_t and w_t are stochastic shocks, assumed to be white noise. The second shock w_t is included to allow for partial correlation of the fundamental shocks with the judgmental adjustment to expectations, as discussed below. This extension to correlated judgment was not considered in Bullard et al. (2008).

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Equations of type (1) often describe economies linearized at a steady-state. We have normalized the steady-state component to zero, so there is no constant term. We use the notation y^e_{t+1} to represent the expectations of the agents in the model, which may initially be nonrational. These expectations are formed through the use of an econometric model, and the expectations from this model are denoted by $E^\star_t y_{t+1}$. We then add a judgmental adjustment variable to this econometric forecast, denoted by ξ_t. Thus expectations are given by

$$y^e_{t+1} = E^\star_t y_{t+1} + \xi_t. \tag{2}$$

Without the addition of judgment, this system would be the same as the systems analyzed in Evans and Honkapohja (2001). For those systems, convergence properties are well-established. Our goal in this paper is to understand how the addition of judgmental adjustment may influence convergence properties and lead to the existence of new equilibria in the economy.

2.1. The Nature of Judgmental Adjustment

The judgmental add-factor allows the agent to adjust the forecast in response to qualitative, unique events that may have an impact on the economy. Let us denote the qualitative "news" as η_t:

$$\eta_t = f w_t + \hat{\eta}_t. \tag{3}$$

We assume $f \geq 0$. In other words, the news consists of both information on the shock w_t as well as exogenous white noise $\hat{\eta}_t$. (We assume that w_t and $\hat{\eta}_t$ are independent.) The latter can be interpreted either as extraneous randomness or as measurement error of w_t. The news η_t is thought to have a persistent impact on y_{t+1} that is not captured by the econometric forecast $E^\star_t y_{t+1}$.

Following Bullard et al. (2008) we assume that the amount of judgmental adjustment contained in y^e_{t+1} satisfies

$$(1 - \rho L)\, \xi_t = \eta_t, \tag{4}$$

where L is a lag operator, or $\xi_t = \rho \xi_{t-1} + \eta_t$. Thus the expected effect of past qualitative events, $\rho \xi_{t-1}$, plus the effect of today's news η_t, gives the complete judgmental adjustment at time t. A key property is persistence in the judgment, which we think is natural given the examples described above and because the endogenous variable has corresponding persistence in the equilibrium.

We stress that the assumption that ξ_t follows an AR(1) process has been adopted here for analytical convenience only. We caution the reader not to think of the judgmental adjustment as a quantifiable variable even though it is written in AR(1) form. Past judgments certainly have an impact on past forecasts, and in that sense they could be quantified. But an adjustment for the Korean War would not be comparable to adjustments for wage and price controls or for the Y2K

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millennium bug, or for a host of other, more minor events, and for this reason the past judgments do not provide useful time series information.

2.2. Econometric Forecasts

In the recursive learning literature, agents are assigned a PLM that summarizes how they use available economic data to form expectations about the future. The PLM is chosen to be consistent with the time-series properties of the equilibrium law of motion under RE. The actual evolution of system (1) will depend in part on how agents are forming expectations, including any judgmental adjustments they are making.

For the system with judgment, we show below that an ARMA(1,1) process,

$$y_t = by_{t-1} + v_t - av_{t-1}, \tag{5}$$

is appropriate because in equilibrium y_t will follow an ARMA(1,1) process. In this equation v_t is white noise and $|a| < 1$ and $|b| < 1$ are parameters. This can be written as

$$y_t = \theta(L)\, v_t, \tag{6}$$

where

$$\theta(L) = \frac{1 - aL}{1 - bL}.$$

Then

$$E_t^\star y_{t+1} = by_t - av_t = [b\theta(L) - a]\, v_t \tag{7}$$

is the best forecast in a mean square error (MSE) sense for the given PLM. We refer to (7) as the *econometric forecast*. This part of the agents' forecast depends on the econometric model alone.

2.3. Exuberance Equilibrium

We can now deduce the *actual law of motion* (ALM) for this system by combining the expectations (2) and the effect of these expectations on the state of the economy (1). This yields

$$\begin{aligned} y_t &= \beta y_{t+1}^e + u_t + w_t \\ &= \beta\left(\frac{b-a}{1-bL}\right) v_t + \frac{\beta}{1-\rho L}\eta_t + u_t + w_t \\ &= \beta\left(\frac{b-a}{1-bL}\right)\left(\frac{1-bL}{1-aL}\right) y_t + \frac{\beta}{1-\rho L}\eta_t + u_t + w_t. \end{aligned}$$

Solving for y_t and using (3), the ALM can be written as

$$y_t = \frac{1 - aL}{\beta(a-b) + 1 - aL}\left[(1-\rho L)^{-1}\beta(f w_t + \hat{\eta}_t) + (u_t + w_t)\right]. \tag{8}$$

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The judgment term η_t enters the ALM, because judgment is affecting the expectations of agents and through that channel is affecting actual macroeconomic outcomes y_t. Of course, the agents could decide not to use a judgmental adjustment, in which case the ALM would not involve this term, and this too would be an equilibrium. A critical aspect of the analysis will be to develop the conditions under which economic actors will use a judgmental adjustment, given that all other agents are making such an adjustment and causing the actual law of motion to be (8).

We proceed by defining the concept of an *exuberance equilibrium*. The general concept is that of an equilibrium in which the judgmental adjustment influences the actual evolution of the economy. The exuberance concept has three parts, and we discuss each in detail below. We first require that the PLM of the agents, their econometric forecast, is consistent with the actual data being generated by the economy. This is a type of nonfalsifiability assumption for the econometric model. We use the CEE concept to impose this condition.[7] A second requirement is that there is a Nash equilibrium in the use of judgment, so that given that all agents are using a judgmentally adjusted forecast, no agent wishes to discontinue using that forecast. Finally, because our agents are learning using recursive algorithms, we impose learnability as a requirement for an exuberance equilibrium.

3. ANALYSIS

3.1. Consistent Expectations

The main idea behind CEE is that econometricians in the model should be unable to reject their model of the economy. The econometric model should provide a good fit to the data produced by the economy itself. We impose this idea through a requirement that the autocovariance-generating function of the econometric model (the PLM) is equivalent to the autocovariance-generating function generated by the economy (the ALM).[8] We can analytically verify the existence of a solution to the equation implied by this statement for the univariate case.

The autocovariance generating function for the PLM is given by

$$G_{\text{PLM}}(z) = \sigma_v^2 \frac{(1 - az)\left(1 - az^{-1}\right)}{(1 - bz)\left(1 - bz^{-1}\right)}, \tag{9}$$

where σ_v^2 is the variance of v and z is a complex scalar.[9] For the ALM, the autocovariance-generating function is the sum of three such functions:

$$G_{\text{ALM}}(z) = G_{\hat{\eta}}(z) + G_u(z) + G_w(z)$$

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by the independence of $\hat{\eta}$, u, and w. The autocovariance-generating function for the ALM (8) is

$$G_{\text{ALM}}(z) = \frac{(1-az)(1-az^{-1})}{(1-\rho z)(1-\rho z^{-1})} \times \left\{ \frac{\beta^2\sigma_{\hat{\eta}}^2 + (1-\rho z)(1-\rho z^{-1})\sigma_u^2}{[\beta(a-b)+1-az]\left[\beta(a-b)+1-az^{-1}\right]} + \frac{\sigma_w^2(f\beta+1-\rho z)(f\beta+1-\rho z^{-1})}{[\beta(1-a)+1-az][\beta(1-a)+1-az^{-1}]} \right\}.$$

We use these functions to demonstrate the following result in Appendix A.

LEMMA 1. *There exists a CEE with* $b = \rho$ *and* $a \in [0, \rho]$.

Intuitively, $b = \rho$ in order to match the induced autoregressive process arising from judgment. Any other value of b would lead to a higher order process for y_t that would not be consistent with the PLM.

As shown in Appendix A, there are interesting limiting cases: when $\sigma_\eta^2 = \sigma_{\hat{\eta}}^2 + f^2\sigma_w^2 \to 0$, so that the relative variance of the judgment process is small, $a \to \rho$, whereas for $\sigma_u^2 + \sigma_w^2 \to 0$, so that the relative variance of the fundamental process is small, $a \to 0$. Because a solution $a \in [0, \rho]$ always exists,[10] the conditions for a CEE can always be met.

We now ask whether individual rationality holds with respect to inclusion of the judgment variable in making forecasts.

3.2. Incentives to Include Judgment

The agents can choose not to use the judgmentally adjusted forecast. But when all other agents are using the judgmental adjustment, they cause the ALM of the system to be given by equation (8). An individual agent faces the question of whether the outcomes generated by this ALM can be better forecast with a model that incorporates the judgmental adjustment or by one that ignores the judgmental adjustment. If each agent chooses the former course, then all agents will use the judgmentally adjusted forecast and the ALM will remain as given in equation (8). Of course, if each agent chooses the latter course, then no agent will use the judgmentally adjusted forecast, and the judgment variable will not affect equilibrium outcomes. The latter case leads to the standard RE equilibrium viewed here as a "no exuberance equilibrium."

This individual rationality condition can be assessed by comparing the forecast error variance of the two models, (2) and (7), when all other agents are including judgment in their forecasts and hence causing the ALM to be given by (8). Using the condition from the consistent expectations calculation that $b = \rho$, we note

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that $v_t = \left(\frac{1-\rho L}{1-aL}\right) y_t$. The econometric forecast is therefore given by

$$E_t^{\star} y_{t+1} = \frac{\rho - a}{1 - \rho L} v_t = \frac{\rho - a}{1 - aL} y_t, \tag{10}$$

whereas the judgmentally adjusted forecast is given by

$$y_{t+1}^{e} = \frac{\rho - a}{1 - aL} y_t + \frac{1}{1 - \rho L} \eta_t. \tag{11}$$

Is it possible for the variance of the judgmentally adjusted forecast to be lower than the variance of the econometric forecast? It is. Consider the special case when $\sigma_{\hat{\eta}}^2, \sigma_w^2 \to 0$ so that the positive root $a \to \rho$. Then it is shown in Appendix B that, apart from additive terms in u_t that are identical for the two forecasts, the forecast errors are

$$\mathrm{FE}|_{u=0} = \frac{1 + \beta f - (\rho + kf)L}{1 - \rho L} w_{t+1} + \frac{\beta - kL}{1 - \rho L} \hat{\eta}_{t+1},$$

where $k = 1$ if judgment is included and $k = 0$ for the case of no judgment.

For the term involving $\hat{\eta}_{t+1}$ we have, with no judgment,

$$\mathrm{FE}_{NJ}^{\hat{\eta}} = \frac{\beta}{1 - \rho L} \hat{\eta}_{t+1},$$

whereas the forecast error with judgment is

$$\mathrm{FE}_{J}^{\hat{\eta}} = \frac{\beta(1 - \beta^{-1} L)}{1 - \rho L} \hat{\eta}_{t+1}.$$

Thus, as $\sigma_{\hat{\eta}}^2 \to 0$ the ratio between the variances of these terms is[11]

$$\frac{\mathrm{Var}[\mathrm{FE}_J^{\hat{\eta}}]}{\mathrm{Var}[\mathrm{FE}_{\mathrm{NJ}}^{\hat{\eta}}]} = 1 + \beta^{-2} - 2\beta^{-1}\rho.$$

This is less than one if and only if

$$\rho\beta > \tfrac{1}{2}. \tag{12}$$

For the term involving w_{t+1} we get for the relevant variances

$$\mathrm{Var}[\mathrm{FE}_{\mathrm{NJ}}^{w}] - \mathrm{Var}[\mathrm{FE}_{J}^{w}] \asymp \frac{f}{1 + \beta f} \left(2\rho - \frac{2\rho + f}{1 + \beta f}\right),$$

where $\asymp$ means "is positively proportional to." It is seen that the term in the brackets is nonnegative for all $f \geq 0$ when $\beta\rho > 1/2$.

By continuity, it follows that if $\beta > 1/2$ there are nontrivial judgment processes (with $\rho > 1/2\beta$ and $\sigma_{\hat{\eta}}^2, \sigma_w^2 > 0$ sufficiently small) for which the agents have

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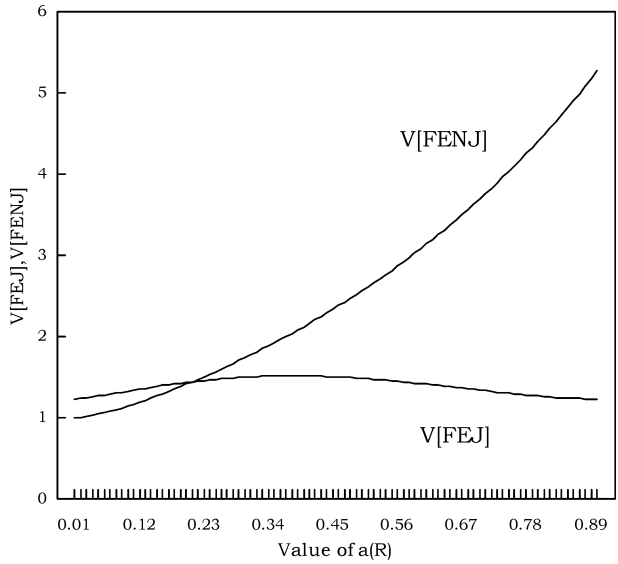


FIGURE 1. The variance of the forecast error, with (FEJ) and without (FENJ) judgment. The variance can be lower with judgment included, even for values of a far from ρ. Figure drawn for $\beta = 0.9$, $\rho = 0.9$.

incentives to include the process as an add-factor in their forecasts. The preceding argument considered the limiting case $a \to \rho$, but as we will numerically show below, it is not necessary for a to be close to ρ for our results to hold.

It is useful to consider further aspects of this comparison. For simplicity we now assume that $\sigma_w^2 = 0$ so that there is no correlation between fundamental shocks and judgment. Polar opposite to the case $\sigma_{\hat{\eta}}^2 \to 0$ is the special case $\sigma_u^2 \to 0$, in which case the positive root $a \to 0$. It can be shown that in this case

$$\text{Var}[\text{FE}_J] - \text{Var}[\text{FE}_{\text{NJ}}] = \left[\frac{(\beta^{-1} - \rho)^2}{1 - \rho^2}\right]\sigma_\eta^2.$$

This can never be less than zero, so that agents will not make the judgmental adjustment in this case.

By continuity, for the case of uncorrelated judgment, we deduce that for $R = \sigma_\eta^2/\sigma_u^2$ large it will not be individually rational to include judgment, but if $\rho\beta > \frac{1}{2}$ there are values of $R = \sigma_\eta^2/\sigma_u^2 \in (0, \infty)$ such that agents rationally choose to use a judgmentally adjusted forecast, given that all other agents are doing so.[12] The conclusion that it can be optimal to judgmentally adjust the econometric forecast is striking because this forecast already reflects the effects of judgment on the time series properties of the observable variables.

Figure 1 illustrates our points numerically for the case of uncorrelated judgment. For $a < \rho$, the forecast error variances involve the variance of an ARMA(2, 2)

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process, which is given in Appendix B. The figure is drawn for $\beta = 0.9$ and $\rho = 0.9$, which is a possible case. To draw Figure 1, we consider changes in R resulting from changes in σ_u^2 with σ_η^2 fixed. The variances of the forecast errors with and without judgment are plotted on the vertical axis, and the implied a is plotted on the horizontal axis. The special cases (i) $R \to \infty$, $a \to 0$ and (ii) $R \to 0$, $a \to \rho$, which were discussed above, are borne out in the figure. The figure also shows intermediate cases and indicates that a does not have to be particularly close to ρ for the individual rationality condition to be met. In fact, the two forecast error variances are equal at $a \approx 0.21$, which is far from the value of ρ in this example, which is 0.9. We conclude that the conditions for exuberance equilibria to exist are quite likely to be met for a wide range of relative variances R provided that both β and ρ are relatively close to one.[13]

3.3. Learnability

Because we have agents using recursive algorithms to learn, a plausibility condition for exuberance equilibrium is that any candidate equilibrium is learnable. We follow the literature on recursive learning to impose this requirement.[14]

In the current context, the CEE formulated above takes the form of an ARMA(1,1) process. Estimation of ARMA(1,1) processes is usually done using maximum-likelihood techniques, taking us beyond standard least squares estimation. Recursive maximum-likelihood (RML) algorithms are available, and they have formal similarities to recursive least-squares estimation.[15] Because this technical analysis is relatively unfamiliar, we confine the formal details to Appendix C. However, the results are easily summarized. Let a_t and b_t denote estimates at time t of the coefficients of the ARMA forecast function (7). Numerical computations using RML indicate convergence of (a_t, b_t) to (a, ρ), where $a > 0$ is the CEE value given in Lemma 1. Thus this CEE is indeed stable under learning. Moreover, in Section 3.4 we state a formal convergence result as part of our existence theorem.

3.4. Existence and Properties of Equilibrium

We now collect the various results above. The following theorem gives the key results about the existence of an exuberance equilibrium and characterizes its asymptotic variance:

THEOREM 1. *Consider the model with judgment and suppose that* $\beta > 1/2$. *Then*

(i) *for appropriate AR(1) judgment processes there exists an exuberance equilibrium and*
(ii) *the exuberance equilibrium has a higher asymptotic variance than the RE equilibrium*

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Proof 1. (i) The preceding analysis has verified that the first two conditions for an exuberance equilibrium defined in Section 2.3, consistent expectations and incentives to include judgment, are met for all $\sigma_\eta^2 > 0$ sufficiently small. In Appendix C it is proved that the third condition also holds, that is, the CEE is stable under RML learning, when $\sigma_\eta^2 > 0$ is sufficiently small.

(ii) The RE equilibrium for the model is $y_t = u_t + w_t$ since $0 < \beta < 1$ and u_t and w_t are i.i.d. with mean zero. The exuberance equilibrium with $a > 0$ can be represented as the ARMA(1,1) process $y_t = \rho y_{t-1} + v_t - a v_{t-1}$, where a solves equation (A.3) given in Appendix A. From (A.2) of Appendix A it can be seen that $\sigma_v^2 a\,[\beta\,(a-\rho)+1] = \sigma_u^2\rho + \sigma_w^2(1+f\beta)\rho$. Thus

$$\sigma_v^2 = \frac{\rho}{a[\beta(a-\rho)+1]}\sigma_u^2 + \frac{(f\beta+1)\rho}{a[\beta(a-\rho)+1]}\sigma_w^2 > \sigma_u^2 + \sigma_w^2,$$

since $a < \rho$, $f \geq 0$ and $0 < \beta, \rho < 1$. Next, using the formula for the variance of an ARMA(1,1) process we have

$$\sigma_y^2 = \frac{1+a^2-2\rho a}{1-\rho^2}\sigma_v^2,$$

and since $\frac{1+a^2-2\rho a}{1-\rho^2} > 1$, the result follows. ■

The theorem states that in an exuberance equilibrium, the variance of the state variable y_t is larger than it would be in a fundamental RE equilibrium. This is because the REE has $y_t = u_t + w_t$, so that $\sigma_y^2 = \sigma_u^2 + \sigma_w^2$, but in an exuberance equilibrium $\sigma_y^2 > \sigma_u^2 + \sigma_w^2$.[16]

4. AN ASSET-PRICING EXAMPLE

A simple univariate example of the framework (1) is given by the standard present value model of asset pricing. One convenient way of obtaining this equation is the model of Brock and Hommes (1998). In their framework agents are myopic mean-variance maximizers who choose the quantity of riskless and risky assets in their portfolio to maximize the expected value of a quadratic utility function of end-of-period wealth.[17]

We modify their framework to allow for shocks to the supply of the risky asset.[18] For convenience we assume homogeneous expectations and constant known dividends. The temporary equilibrium is given by

$$p_{t+1}^e + d - R_f p_t = s_t,$$

where d is the dividend, p_t is the price of the asset, and $R_f > 1$ is the rate-of-return factor on the riskless asset. Here s_t is a linear function of the random supply of the risky asset per investor, assumed i.i.d. for simplicity.[19] Defining $y_t = p_t - \bar{p}$, where $\bar{s} = Es_t$ and $\bar{p} = (d - \bar{s})/(R_f - 1)$, we obtain (1) with $\beta = R_f^{-1}$,

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TABLE 1. Excess volatility

	σ_ξ/σ_u			
ρ	0.50	1.00	1.50	2.00
0.70	1.54	2.74	–	–
0.80	1.85	3.62	5.58	–
0.90	2.70	5.82	9.11	12.43
0.95	3.99	8.75	13.64	18.56

Note: Exuberance equilibria in the asset-pricing model. A dash indicates that exuberance equilibrium does not exist. Entries give the ratio of the standard deviation of y to the standard deviation of u.

$u_t = -R_f^{-1}(s_t - \bar{s})$, and $w_t \equiv 0$. We assume that $0 < \beta < 1$, so that the model is determinate, that is, under RE there is a unique nonexplosive solution, given by the "fundamentals" solution $y_t = u_t$. In particular, under RE, sunspot solutions do not exist.

Theorem 1 shows that the basic asset-pricing model is consistent with excess volatility. If investors incorporate judgmental factors that are strongly serially correlated, they will find that this improves their forecasts, but in an exuberance equilibrium this will also generate significant stationary asset price movements in excess of those associated with fundamental factors. The stationarity of our exuberance movements is in marked contrast to the literature on rational asset price bubbles. Because the latter are explosive, the literature on rational bubbles has been punctuated by controversy and complicated by the need to construct valid tests for nonstationary bubbles. Exuberance equilibria offer an alternative approach to modeling bubbles within a stationary time series framework.

A natural question is whether the excess volatility associated with an exuberance equilibrium is economically meaningful, or whether the exuberance conditions outlined in Theorem 1 are met only for situations in which the variance σ_y^2 is just trivially larger than the fundamental variance. In particular, it is of interest to know whether the excess volatility effect isolated in the theorem is large enough to be comparable to empirical estimates of the degree of excess volatility in financial data. One famous calculation due to Shiller (1981) puts the ratio of the standard deviation of U.S. stock prices to the standard deviation of prices based on fundamentals alone at between 5 and 13.[20]

Table 1 provides illustrative calculations of exuberance equilibria for representative parameter values. In the REE, $y_t = u_t$, the standard deviation of prices is σ_u The table gives relative volatility results in exuberance equilibria for different ratios σ_ξ/σ_u, ranging from 0.5 to 2.0. Since $\sigma_\xi^2 = \sigma_\eta^2/(1-\rho^2)$, the innovation variance associated with judgment is quite modest for high values of ρ. We examine an empirically plausible case in which the discount factor $\beta = 0.95$ and the degree of serial correlation ρ is relatively high. A dash in the table indicates that an exuberance equilibrium does not exist for the indicated parameter values. The

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entries in the table are a measure of excess volatility corresponding to Shiller's (1981) concept, namely, σ_y/σ_u. The results indicate that these measures are often in the range of 5–13 estimated by Shiller.

We conclude on the basis of this illustrative calculation that the model can generate substantial excess volatility without difficulty. We remark that if we push the discount factor β closer to unity, the degree of excess volatility can rise to very high levels for high degrees of serial correlation, with σ_y many hundreds of times larger than σ_u. In this sense, the model can generate arbitrarily large amounts of excess volatility. We think that these results are striking because they indicate the possibility of near-rational exuberance in a model that is usually regarded as well-behaved. In addition to excess volatility, there are, of course, other stylized facts and puzzles concerning asset prices, involving the related issues of Sharpe ratios, predictability of excess returns, and the equity premium puzzle.[21] Because our exuberance equilibria generate both excess volatility and positively serially correlated prices, we think that our enlarged class of equilibria may be able to shed light on these phenomena. However, investigation of these issues would require a serious calibration of the model, and we reserve this topic for future research.

5. RATIONALITY TESTS OF JUDGMENT

Our exuberance equilibrium is near-rational but not fully rational. There are two ways in which we have imposed assumptions that deviate from full rationality. First, the judgment process ξ_t is assumed to be not directly available to (or usable by) econometric forecasters, who rely purely on the observables y_t. This seems realistic because ξ_t is the adjustment the judgmental forecasters believe is appropriate due to "unique" qualitative events. This procedure thus reflects a natural division of labor in which the econometricians produce the best statistical forecast based on the observable variables of interest, and the judgmental forecasters modify these forecasts as they think is appropriate. Although $\xi_t = y^e_{t+1} - E_t y^\star_{t+1}$ may possibly be obtainable by the econometricians (at least with a lag), we would expect the judgmental forecasters to resist the incorporation of ξ_t into the econometric model.

Furthermore, older ξ_{t-j} represent different unique events, unrelated to the current judgmental variable. Econometric models sometimes incorporate dummy variables (or other proxies) to capture the quantitative effects of qualitative events, but as the events become more distant such variables tend to get dropped and rolled into the unobserved random shocks in order to preserve degrees of freedom. The impact of recent qualitative events could be estimated by incorporating dummy variables into the econometric model, but for forecasting purposes this would be unhelpful and would still leave the problem of forecasting the future impact of qualitative factors to the judgmental forecasters.

The second way in which our exuberance equilibrium is not fully rational is that the incentive condition is assumed to be dichotomous. This also seems realistic, because its inclusion is determined by the judgmental forecaster. Furthermore, econometric tests of whether "all" of ξ_t should have been included would often

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TABLE 2. Test rejection rates

	ρ					
n	0.7	0.8	0.85	0.9	0.95	0.99
120	47.6	5.1	0.1	0	0.4	2.0
240	86.8	15.9	1.4	0	0.4	1.7
480	99.7	48.0	3.5	0	0.4	1.0

Note: Exuberance equilibria in the asset-pricing model. Percent of test rejections at 5% level of the null hypothesis that including judgment is fully rational, that is, Ho: $k = 1$. Results given are based on 1,000 replications.

have low power. Suppose that we allowed for just a proportion $k \in [0, 1]$ of the judgment to be included in the forecast. It can be shown that the minimum MSE occurs at $k = \beta\rho$. For $\beta\rho$ near one, rationality tests using

$$y_{t+1} - y^e_{t+1} = (1 - k)\,\xi_t + \zeta_{t+1} \tag{13}$$

of the null hypothesis $H_0 : k = 1$ have low power, and considerable data would be required to detect that not all of ξ_t should be optimally included.

We illustrate this point in Table 2, which takes into account both aspects of bounded rationality discussed above. Suppose that econometricians do have access ex post to the judgmentally adjusted forecasts y^e_{t+1}, and therefore to $\xi_t = y^e_{t+1} - E^\star_t y_{t+1}$, and that they estimate (13) and test the null hypothesis $H_0 : k = 1$ that the inclusion of judgment is fully rational. For the purposes of this test we set the discount factor at $\beta = 1 - (0.05/12) = 0.9958$ in line with a real monthly risk-free rate of return of $0.05/12$.[22] We also set $\sigma^2_\xi/\sigma^2_u = 1.0$. The three sample sizes shown correspond to 10, 20, and 40 years of monthly data, and the nominal significance level of the test is set at 5%. When ρ is below 0.8, one would expect to eventually detect a deviation from full rationality. However, it can be seen that for ρ at or above 0.85, rejection of the null is unlikely even with 40 years of data. In particular, for $\rho = 0.9$ or 0.95 any deviation of the judgmental forecasts from full rationality would be virtually undetectable except with enormous sample sizes. Furthermore, these cases correspond to large, empirically plausible values of excess volatility: for the parameter settings of Table 2 we have excess volatility measures of $\sigma_y/\sigma_u = 8.40$ for $\rho = 0.9$ and $\sigma_y/\sigma_u = 16.39$ for $\rho = 0.95$.

From Table 2 we see that for an exuberance equilibrium with ρ values above 0.85, decision makers are likely to conclude that the functional division of labor between econometricians, who supply forecasts based on the observable variable y_t, and judgmental forecasters, who adjust these forecasts to take account of perceived qualitative events omitted from the econometric model, is entirely appropriate. Because an exuberance equilibrium is a CEE, the econometricians are fully taking into account the predictable serial correlation properties of the variable being forecast. At the same time, the mean square forecast error is smaller for the

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judgmental forecasts than for the pure econometric forecast, and thus there is a clear gain to forecast performance in making use of the judgmental adjustment.

The uniqueness of qualitative events is also relevant to the issue at hand. Suppose, for example, that $\rho = 0.8$ and that rationality tests eventually indicate a statistically significant deviation from full rationality, with an estimated value near $k = 0.8$. It does not really seem plausible that forecasters would decide to downweight current judgmental adjustments, based on the finding that such adjustments over the past 20 years or so have been about 20% too high, because past judgmental adjustments mainly concerned different qualitative events, and because the adjustments may have been made by different judgmental forecasters. Furthermore, even if on this basis current judgment is downweighted, and even if this eventually results in the role of judgment being gradually extinguished over time, a new qualitative event will at some point suggest the need once again for judgment, with the judgment process again becoming persistent. In this sense, an economy in which exuberance equilibria exist always remains "subject to judgment."

6. CONCLUSIONS AND POSSIBLE EXTENSIONS

We have studied how a new phenomenon, *exuberance equilibria*, may arise in standard macroeconomic environments. We assume that agents are learning in the sense that they are employing and updating econometric models used to forecast the future values of variables they care about. Unhindered, this learning process would converge to an RE equilibrium in the economies we study. We investigated the idea that decision makers may be tempted to include judgmental adjustments to their forecasts if all others in the economy are similarly judgmentally adjusting their forecasts. The judgmental adjustment, or add factor, is a pervasive and widely acknowledged feature of actual macroeconometric forecasting in industrialized economies. We obtain conditions under which such add-factoring can become self-fulfilling, altering the actual dynamics of the economy significantly but in a way that remains consistent with the econometric model of the agents.

To develop our central points, we have made some strong simplifying assumptions. We have assumed that the exuberance or judgment variables take a simple autoregressive form, but this assumption is mainly made for convenience. Although we do believe that judgmental adjustments exhibit a strong positive serial correlation, a more complicated stationary stochastic process could instead be used. The assumption of identical judgments of different (representative) agents is correspondingly restrictive. Allowing for differences in judgments by individual agents would probably make the conditions for exuberance equilibrium more difficult to achieve. On the other hand, this could create new phenomena, such as momentum effects arising when a large fraction of agents begin to agree in their judgments.

The incorporation of judgment into decisions, in the form of adjustments to econometric forecasts, can have a self-fulfilling feature in the sense that decision makers would believe ex post that their judgment had improved their forecasts.

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This result is similar in spirit to the self-fulfilling nature of sunspot equilibria, but with the novel feature that it can arise in determinate models in which there is a unique RE equilibrium that depends only on fundamentals. This widens the set of models in which self-fulfilling fluctuations might plausibly emerge. In particular, we have shown that exuberance equilibria can arise in the standard asset-pricing model, generating substantial excess volatility.

NOTES

1. See Reifschneider et al. (1997) for a discussion of the extent of judgmental adjustment in macroeconomic forecasting at the Federal Reserve.

2. See Svensson (2003, 2005). Related work includes Svensson and Tetlow (2005) and Jansson and Vredin (2001).

3. See Sargent (1991), Marcet and Sargent (1995), and Hommes and Sorger (1998).

4. The term "near-rationality" has been used elsewhere in the literature, often to mean less-than-full maximization of utility. See, for example, Akerlof and Yellen (1985) and Caballero (1995). Ball (2000) analyzes a model in which the agents use a forecasting model that does not encompass the equilibrium law of motion—a "restricted perception." Our concept is based on full optimization but subject to the restriction that some information is not quantifiable—"judgment." Our concept of near-rationality is discussed further in Section 5.

5. "Exuberance" (which in our equilibria leads to both positive and negative deviations from the fundamentals solution) has a long informal tradition as a potential explanation of asset price "bubbles." For its possible role in "financial fragility" see Lagunoff and Schreft (1999).

6. Indeterminacy and sunspot equilibria are distinct concepts, as discussed in Benhabib and Farmer (1999). We consider only linear models, for which the existence of stationary sunspot equilibria requires indeterminacy—see, for example, Propositions 2 and 3 of Chiappori and Guesnerie (1991).

7. CEE and RE equilibrium with limited information are equivalent in our linear settings.

8. See, e.g., Hommes and Sorger (1998) and Branch and McGough (2005).

9. See Brockwell and Davis (1991, pp. 417–420) or Hamilton (1994, pp. 266–268).

10. Appendix A also makes it clear that there is a second, negative value of a that equates the two autocovariance-generating functions. We found that the other conditions for exuberance equilibrium are not met at this value of a, and we refer to it only in passing in the remainder of the paper.

11. See, for instance, Harvey (1981, p. 40). The variance of $x_t = [(1 + mL)/(1 - \ell L)]\epsilon_t$ is $[(1 + m^2 + 2\ell m)/(1 - \ell^2)]\sigma_\epsilon^2$.

12. The case with $a \approx \rho$ is a near-common factor representation of the time series, but the required variances remain continuous in the parameters, as can be seen from the formulas in Appendix B.

13. One suspects that the individual rationality constraint is binding at values $\rho\beta < 1/2$. For example, if $\rho = 0.7$ and $\beta = 0.7$, an analogous numerical exercise shows that the judgmentally adjusted forecast is never preferable to the econometric forecast.

14. Evans and Honkapohja (2001) give a systematic treatment of adaptive learning and its implications in macroeconomics. Evans and Honkapohja (1999), Marimon (1997), and Sargent (1993, 1999) provide surveys of the field.

15. They are also called recursive prediction error (RPE) algorithms—see Evans and Honkapohja (1994) and Marcet and Sargent (1995) for other uses of RPE methods in learning.

16. To examine the amount of correlation between the fundamental $u_t + w_t$ and the judgment innovation η_t, we also considered the limit $\sigma_{\hat{\eta}}^2 \to 0$ and computed the correlation for different values of σ_u^2, σ_w^2, and f under the constraints of learning convergence and that inclusion of judgment is a CEE. For example, if $\beta = \rho = 0.95$, $f = 1$ and $\sigma_{\hat{\eta}}^2$ very small, the correlation can be pushed beyond 0.9 before the conditions start to fail.

17. There are, of course, alternative ways to derive the same equation.

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18. The role of share supply has recently been stressed in the finance literature, e.g., see Ofek and Richardson (2003) and Hong, Scheinkman, and Xiong (2006).

19. Using the notation of Brock and Hommes (1998), $s_t = a\sigma^2 z_{st}$, where σ^2 is the conditional variance of excess returns (assumed constant), a is a parameter of the utility function, and z_{st} is the (random) asset supply.

20. Shiller (1981) actually compared the variance of equity prices to the variance of their ex post price (the present value of actual future dividends), but the latter must exceed the variance of the fundamentals price under RE.

21. See, for example, Lettau and Uhlig (2002).

22. In the Brock and Hommes (1998) setup, β is the inverse of the risk-free real rate-of-return factor. The value chosen here corresponds to an annual discount rate of 5% per annum (p.a.), but the results of Table 2 are quite similar if 3% p.a. (or 7% p.a.) is used.

23. For further details on the algorithm see Section 2.2.3 of Ljung and Soderstrom (1983). The algorithm is often called an RPE algorithm.

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APPENDIX A: CONDITIONS FOR CEE

It can be seen from the form of $G_{\mathrm{ALM}}(z)$ that, for arbitrary a and b, the ALM is an ARMA(2,2) process. As we will now show, there are choices of a and b that yield $G_{\mathrm{PLM}}(z) = G_{\mathrm{ALM}}(z)$. These choices of a and b also have the property that the corresponding ALM takes an ARMA(1,1) form that matches the PLM. This is possible if a and b are chosen so that there is a common factor in the numerator and denominator of the expression on the right-hand side of $G_{\mathrm{ALM}}(z)$.

We now set $G_{\mathrm{PLM}}(z) = G_{\mathrm{ALM}}(z)$, under the condition $b = \rho$, so that the poles of the autocovariance generating functions agree. This yields

$$\sigma_v^2 = \left\{ \frac{\beta^2\sigma_{\tilde{\eta}}^2 + (1-\rho z)\,(1-\rho z^{-1})\sigma_u^2}{[\beta\,(a-b)+1-az][\beta(a-b)+1-az^{-1}]} \right.$$

$$\left. + \frac{\sigma_w^2(f\beta+1-\rho z)(f\beta+1-\rho z^{-1})}{[\beta(1-a)+1-az][\beta(1-a)+1-az^{-1}]} \right\}$$

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or

$$\sigma_v^2 [\beta (a-\rho) + 1 - az][\beta (a-\rho) + 1 - az^{-1}]$$
$$= \beta^2 \sigma_{\hat{\eta}}^2 + (1-\rho z)(1-\rho z^{-1})\sigma_u^2 + \sigma_w^2 (f\beta + 1 - \rho z)(f\beta + 1 - \rho z^{-1}).$$

This equation can be written as

$$\sigma_v^2 \{[1 + \beta (a-\rho)]^2 + a^2\} - \sigma_v^2 a [\beta (a-\rho) + 1](z + z^{-1}) = \beta^2 \sigma_{\hat{\eta}}^2 + \sigma_u^2 (1+\rho^2)$$
$$- \sigma_u^2 \rho (z + z^{-1}) + \sigma_w^2 [(f\beta + 1)^2 + \rho^2 - (f\beta + 1)\rho (z + z^{-1})].$$

For the autocovariances of the PLM and ALM to be equal, the coefficients on the powers of z in this equation must be equal. Equating these, we obtain the two equations

$$\sigma_v^2 \{[1 + \beta (a-\rho)]^2 + a^2\} = \beta^2 \sigma_{\hat{\eta}}^2 + \sigma_u^2 (1+\rho^2) + \sigma_w^2 [(f\beta + 1)^2 + \rho^2] \qquad \textbf{(A.1)}$$

and

$$\sigma_v^2 a [\beta (a-\rho) + 1] = \sigma_u^2 \rho + \sigma_w^2 (f\beta + 1)\rho. \qquad \textbf{(A.2)}$$

We wish to solve for a value of a such that $|a| < 1$. Solving equation (A.2) for σ_v^2 and substituting the result into equation (A.1), at a CEE a solves

$$\hat{F}(a) \equiv \hat{c}_2 a^2 + \hat{c}_1 a + \hat{c}_0 = 0 \qquad \textbf{(A.3)}$$

with

$$\hat{c}_2 = -\hat{t}(1+\beta^2) + \hat{s}\beta,$$
$$\hat{c}_1 = -2\hat{t}\beta(1-\rho\beta) + \hat{s}(1-\rho\beta),$$
$$\hat{c}_0 = -\hat{t}(1-\rho\beta)^2,$$

where $\hat{s} = \beta^2 \sigma_{\hat{\eta}}^2 + (1+\rho^2)\sigma_u^2 + \sigma_w^2 [(1+f\beta)^2 + \rho^2]$ and $\hat{t} = \sigma_u^2 \rho + \sigma_w^2 \rho (1 + f\beta)$. Clearly, $\hat{F}(0) < 0$ when $f \geq 0$. Next we consider $\hat{F}(\rho)$. Using Mathematica, it is verified that

$$\hat{F}(\rho) = \beta\rho [\beta \sigma_{\hat{\eta}}^2 + \sigma_w^2 f(\beta f + 1 - \rho^2)].$$

Clearly, $\hat{F}(\rho) > 0$ for $f \geq 0$. It follows that there exists a CEE with $0 < a < \rho$. Furthermore, by inspecting $\hat{F}(\rho)$ it is seen that $a \to \rho$ when $\sigma_\eta^2 \to 0$. Also $a \to 0$ when $\sigma_u^2 + \sigma_w^2 \to 0$ since $\hat{t} \to 0$.

There can be a second, negative root. However, our numerical results indicate that the CEE corresponding to the negative root is not learnable.

APPENDIX B: IMPACT OF JUDGMENT

The induced ALM, as depicted in equation (8), is

$$y_t = \frac{1 - aL}{\beta (a-b) + 1 - aL} [(1 - \rho L)^{-1} \beta (f w_t + \hat{\eta}_t) + (u_t + w_t)]. \qquad \textbf{(B.1)}$$

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By substituting equation (B.1) into both (10) and (11), we can write the two types of forecasts in terms of the shocks u_t and η_t. These expressions become

$$E_t^\star y_{t+1} = \frac{\rho - a}{\beta\,(a-\rho) + 1 - aL}\left(\frac{\beta}{1-\rho L}\eta_t + u_t + w_t\right)$$

in the case of no judgment and

$$y_{t+1}^e = \frac{\rho - a}{\beta\,(a-\rho) + 1 - aL}\left(\frac{\beta}{1-\rho L}\eta_t + u_t + w_t\right) + \frac{1}{1-\rho L}\eta_t$$

in the case of the judgmentally adjusted forecast. The actual state of the economy at time $t+1$ is, from equation (B.1),

$$y_{t+1} = \frac{1 - aL}{\beta\,(a-\rho) + 1 - aL}\left(\frac{\beta}{1-\rho L}\eta_{t+1} + u_{t+1} + w_{t+1}\right). \tag{B.2}$$

We can therefore compute forecast errors in each of the two cases. We get

$$\begin{aligned} y_{t+1} - y_{t+1}^e = {} & \frac{1}{1+\beta(a-\rho) - aL} \\ & \times \left\{\left[\frac{\beta f(1-aL) - (\rho - a)\beta f L}{1-\rho L} + (1-aL)\right] w_{t+1}\right. \\ & \left. + \frac{\beta(1-aL) - (\rho-a)\beta L}{1-\rho L}\hat{\eta}_{t+1}\right\} - \frac{kfL}{1-\rho L}w_{t+1} - \frac{kL}{1-\rho L}\hat{\eta}_{t+1} \\ & + \text{term in } u_{t+1}, \end{aligned} \tag{B.3}$$

where $k = 1$ if judgment is included and zero otherwise. These equations simplify to those given in the text for the case $a \to \rho$.

When $\sigma_{\hat{\eta}}^2 \to 0$, $\sigma_w^2 \to 0$, the relevant terms in the forecast error for assessing judgment are

$$\frac{1 + \beta f - (\rho + kf)L}{1-\rho L}w_{t+1} + \frac{\beta - kL}{1-\rho L}\hat{\eta}_{t+1}.$$

For the second term the comparison is in the main text. For the term involving w_{t+1} we get for the relevant variances

$$\text{Var}\,|_{k=0} - \text{Var}\,|_{k=1} \asymp \frac{f}{1+\beta f}\left(2\rho - \frac{2\rho + f}{1+\beta f}\right),$$

where $\asymp$ means "is positively proportional to." It is seen that the term in the brackets is nonnegative for all $f \geq 0$ when $\beta\rho > 1/2$. This implies that adding a small positive correlation between judgment and unobserved fundamentals does not alter the incentive condition for inclusion of judgment. In other words, if $\beta\rho > 1/2$ an individual agent will make the judgmental adjustment to the forecast for sufficiently small values of $\sigma_{\hat{\eta}}^2$ and σ_w^2.

Next, we give the formula to compute the variance of the ARMA(2,2) process needed for the numerics behind Figure 1. Apart from the lead coefficient $\beta/[1+\beta(a-\rho)]$, each forecast error process is in the generic class

$$x_t = \frac{1 + m_1 L + m_2 L^2}{1 - \ell_1 L - \ell_2 L^2}\epsilon_t.$$

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Using the procedure described on pp. 26–27 of Granger and Newbold (1986), one can solve simultaneously for $\text{Var}(x_t)$, $\text{Cov}(x_t, x_{t-1})$, and $\text{Cov}(x_t, x_{t-2})$ to obtain

$$\text{Var}\,(x_t) = \frac{x_{\text{num}}}{x_{\text{den}}}\sigma_\epsilon^2, \tag{B.4}$$

where

$$x_{\text{num}} = \frac{(1+\ell_2)\,\ell_1\,(m_1 + m_2\ell_1 + m_2 m_1)}{1-\ell_2} + (m_1 + m_2\ell_1)\,(\ell_1 + m_1) + \left(1 + 2m_2\ell_2 + m_2^2\right)$$

and

$$x_{\text{den}} = 1 - \frac{\ell_1^2}{1-\ell_2} - \frac{\ell_2\ell_1^2}{1-\ell_2} - \ell_2^2.$$

Considering the forecast error in the case without judgment included, equation (B.3) with $k = 0$, we set $m_1 = m_2 = \ell_2 = 0$ and $\ell_1 = a/\,[1 + \beta\,(a-\rho)]$ in equation (B.4). For the case with judgment, we set

$$\begin{aligned} m_1 &= -\,(1+a\beta)\,\beta^{-1}, \\ m_2 &= a\beta^{-1}, \\ \ell_1 &= \frac{a + \rho\,[1+\beta\,(a-\rho)]}{1+\beta\,(a-\rho)}, \\ \ell_2 &= \frac{-a\rho}{1+\beta\,(a-\rho)}, \end{aligned}$$

and a is determined by β, ρ, and σ_η^2/σ_u^2.

APPENDIX C: RECURSIVE MAXIMUM LIKELIHOOD

We now consider recursive estimation when the PLM is an ARMA(1,1) process, that is,

$$y_t = by_{t-1} + v_t + cv_{t-1},$$

where y_t is observed but the white noise process v_t is not observed. Let b_t and c_t denote the estimates of b and c using data through time $t-1$. The econometricians are assumed to use a RML algorithm, which we now describe.[23]

Let $\phi_t' = (b_t, c_t)$. To implement the algorithm an estimate ε_t of v_t is required. Let $\varepsilon_t = y_t - x_{t-1}'\phi_{t-1}$, where $x_{t-1}' = (y_{t-1}, \varepsilon_{t-1})$. y_t is given by $y_t = \beta[E_t^\star y_{t+1} + \xi_t] + u_t + w_t$, where $E_t^\star y_{t+1} = b_{t-1}y_t + c_{t-1}\varepsilon_t$. The RML algorithm is as follows:

$$\begin{aligned} \psi_t &= -c_{t-1}\psi_{t-1} + x_t, \\ \phi_t &= \phi_{t-1} + t^{-1}R_{t-1}^{-1}\psi_{t-1}\varepsilon_t, \\ R_t &= R_{t-1} + t^{-1}(\psi_{t-1}\psi_{t-1}' - R_{t-1}). \end{aligned}$$

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The question of interest is whether ϕ_t converges to a CEE. Convergence can be studied using the associated ordinary differential equation

$$\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = R^{-1} E\psi_t(\phi)\varepsilon_t(\phi), \tag{C.1}$$

$$\frac{\mathrm{d}R}{\mathrm{d}\tau} = E\psi_t(\phi)\psi_t(\phi)' - R. \tag{C.2}$$

Here $y_t(\phi)$, $\psi_t(\phi)$, and $\varepsilon_t(\phi)$ denote the stationary processes for y_t, ψ_t, and ε_t with ϕ_t set at a constant value ϕ. Using the stochastic approximation tools discussed in Marcet and Sargent (1989), Evans and Honkapohja (1998) and Chapter 6 of Evans and Honkapohja (2001), it can be shown that the RML algorithm locally converges provided that the associated ordinary differential equation is locally asymptotically stable (analogous instability results are also available). Numerically, convergence of (C.1) and (C.2) can be verified using a discrete time version of the differential equation. A first-order state space form is convenient for computing the expectations $E\psi_t(\phi)\varepsilon_t(\phi)$ and $E\psi_t(\phi)\psi_t(\phi)'$, and this procedure was used for the numerical illustrations given in the main text.

We now prove convergence analytically for all $0 < \beta, \rho < 1$ with $\sigma_{\hat{\eta}}^2$ and σ_w^2 sufficiently small. This completes the proof of part (i) in Theorem 1. We rewrite the system (C.1) and (C.2) in the form

$$\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = (R)^{-1} g(\phi),$$

$$\frac{\mathrm{d}R}{\mathrm{d}\tau} = M_\psi(\phi) - R,$$

where we have introduced the simplifying notation $g(\phi) = E\psi_t(\phi)\varepsilon_t(\phi)$ and $M_\psi(\phi) = E\psi_t(\phi)\psi_t(\phi)'$. An equilibrium $\bar{\phi}$, $\bar{R}$ of the system is defined by $g(\bar{\phi}) = 0$ and $\bar{R} = M_\psi(\bar{\phi})$. As mentioned in Appendix A, there can be two equilibrium values $\bar{\phi}' = (\rho, -a)$ determined by the solutions to the quadratic (A.3), but we here focus on the solution with $0 < a < 1$. Recall that for this solution $a \to \rho$ as $\sigma_\eta^2 \to 0$.

Linearizing the system at the equilibrium point, it can be seen that the linearized system has a block diagonal structure, in which one block has the eigenvalues equal to -1 (with multiplicity four) and the eigenvalues of the other block are equal to those of the "small" differential equation

$$\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = (\bar{R})^{-1} J(\bar{\phi})(\phi - \bar{\phi}), \tag{C.3}$$

where $J(\phi)$ is the Jacobian matrix of $g(\phi)$. The system (C.1) and (C.2) is therefore locally asymptotically stable if the coefficient matrix $(\bar{R})^{-1} J(\bar{\phi})$ of the two-dimensional linear system (C.3) has a negative trace and a positive determinant. Since $(\bar{R})^{-1} = [\det(\bar{R})]^{-1}\mathrm{adj}(\bar{R})$, we have

$$\mathrm{Tr}[(\bar{R})^{-1} J(\bar{\phi})] = [\det(\bar{R})]^{-1}\mathrm{Tr}[\mathrm{adj}(\bar{R}) J(\bar{\phi})],$$

$$\det[(\bar{R})^{-1} J(\bar{\phi})] = \det[(\bar{R})^{-1}]\det[J(\bar{\phi})].$$

Now $\det(\bar{R}) > 0$ as $\bar{R}$ is a matrix of second moments and thus positive definite for $\sigma_\eta^2 > 0$. It thus remains to prove that $\mathrm{Tr}[\mathrm{adj}(\bar{R}) J(\bar{\phi})] < 0$ and $\det[J(\bar{\phi})] > 0$ when $\sigma_\eta^2 > 0$ is sufficiently small.

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We consider the values of $\mathrm{Tr}[\mathrm{adj}(\bar{R})J(\bar{\phi})]$ and $\det[J(\bar{\phi})]$ when $\sigma_\eta^2 \to 0$. Using the definition of ε_t, the explicit form of $g(\phi)$ is

$$g(\phi) = E\psi_{t-1}(\phi)x_{t-1}' \left[(1-\beta b-\beta c)^{-1}\beta \begin{pmatrix} -bc \\ -c^2 \end{pmatrix} - \begin{pmatrix} b \\ c \end{pmatrix} \right] + (1-\beta b-\beta c)^{-1}\beta\rho E\psi_{t-1}(\phi)\xi_{t-1},$$

where the moment matrices $E\psi_{t-1}(\phi)x_{t-1}'$ and $E\psi_{t-1}(\phi)\xi_{t-1}$ can be computed from the state space form

$$AX_t = CX_{t-1} + H\begin{bmatrix} u_t \\ \hat{\eta}_t \\ w_t \end{bmatrix}, \quad \text{with } X_t = \begin{pmatrix} y_t \\ \varepsilon_t \\ \xi_t \\ \psi_t \\ \psi_{t-1} \end{pmatrix},$$

$$A = \begin{pmatrix} 1 & -(1-\beta b)^{-1}\beta c & -(1-\beta b)^{-1}\beta & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -b & -c & 0 & 0 & 0 \\ 0 & 0 & \rho & 0 & 0 \\ 0 & 0 & 0 & -c & 0 \\ 0 & 0 & 0 & 0 & -c \end{pmatrix}, \quad H = \begin{pmatrix} (1-\beta b)^{-1} & 0 & (1-\beta b)^{-1} \\ 0 & 0 & 0 \\ 0 & 1 & f \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For the limit $\sigma_{\hat{\eta}}^2 \to 0$ and $\sigma_w^2 \to 0$ we first set $\sigma_w^2 = \lambda\sigma_{\hat{\eta}}^2$, where $\lambda > 0$ is arbitrary. It can be computed using Mathematica 4 (routine available on request) that $\mathrm{Tr}[\mathrm{adj}(\bar{R})J(\bar{\phi})]$ and $\det[J(\bar{\phi})]$ have the following properties as functions of (using temporary notation) $\omega \equiv \sigma_{\hat{\eta}}^2$:

$$\lim_{\omega\to 0}\mathrm{Tr}[\mathrm{adj}(\bar{R})J(\bar{\phi})] = \lim_{\omega\to 0}\frac{\mathrm{d}}{\mathrm{d}\omega}\mathrm{Tr}[\mathrm{adj}(\bar{R})J(\bar{\phi})] = 0,$$

$$\lim_{\omega\to 0}\det[J(\bar{\phi})] = \lim_{\omega\to 0}\frac{\mathrm{d}}{\mathrm{d}\omega}\det[J(\bar{\phi})] = 0,$$

$$\lim_{\omega\to 0}\frac{d^2}{d\omega^2}\mathrm{Tr}[\mathrm{adj}(\bar{R})J(\bar{\phi})] = -\frac{4\beta^2\rho^2[\beta + f\lambda(1+f\beta-\rho^2)]^2}{(1-\beta\rho)(\rho^2-1)^6} < 0,$$

$$\lim_{\omega\to 0}\frac{\mathrm{d}^2}{\mathrm{d}\omega^2}\det[J(\bar{\phi})] = \frac{2\beta^2\rho^2[\beta + f\lambda(1+f\beta-\rho^2)]^2}{(\rho^2-1)^6} > 0.$$

The two latter derivatives are evidently locally smooth functions in view of the form of $g(\phi)$. Expressing $\mathrm{Tr}[\mathrm{adj}(\bar{R})J(\bar{\phi})]$ and $\det[J(\bar{\phi})]$ in terms of Taylor series, these results show that

$$\mathrm{Tr}[\mathrm{adj}(\bar{R})J(\bar{\phi})] < 0 \text{ and } \det[J(\bar{\phi})] > 0$$

for $\sigma_\eta^2 > 0$ sufficiently small.

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