proof of this. We have

$$f(\theta) = \frac{\sin\theta - \theta\cos\theta}{\theta - \sin\theta} = \frac{\int_0^\theta x\sin x\,dx}{\int_0^\theta 1 - \cos x\,dx} \tag{2}$$

But $\dfrac{1-\cos x}{x\sin x} = \dfrac{\tan\frac{1}{2}x}{x}$ so, since $\dfrac{\tan x}{x}$ is increasing, $\dfrac{1-\cos x}{x\sin x} < \dfrac{1-\cos\theta}{\theta\sin\theta}$ for $0 < x < \theta$ so that $\theta\sin\theta\int_0^\theta 1-\cos x\,dx < (1-\cos\theta)\int_0^\theta x\sin x\,dx$. By the quotient rule on (2), this establishes that $f'(\theta) < 0$ for $0 < \theta < \frac{1}{2}\pi$ (indeed, for $0 < \theta < \pi$).

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Trigonometric identities from regular polygons

Introduction

This piece concerns trigonometric identities derived from regular n-gons by projecting their sides onto an inradius, length r, or circumradius, length R, to which the sum of the projections is equal. We find we can generalise such equations according to the form of n. This yields (at least) five useful cases and either one or two identities for each. Where there are two, one has the inradius as subject, the other, the circumradius. As shown in Figure 1,

$$r = \frac{1}{2}\cot\left(\frac{\pi}{n}\right) \qquad \text{and} \qquad R = \frac{1}{2}\operatorname{cosec}\left(\frac{\pi}{n}\right),$$

and the two are related by the equation $4R^2 = 4r^2 + 1$.

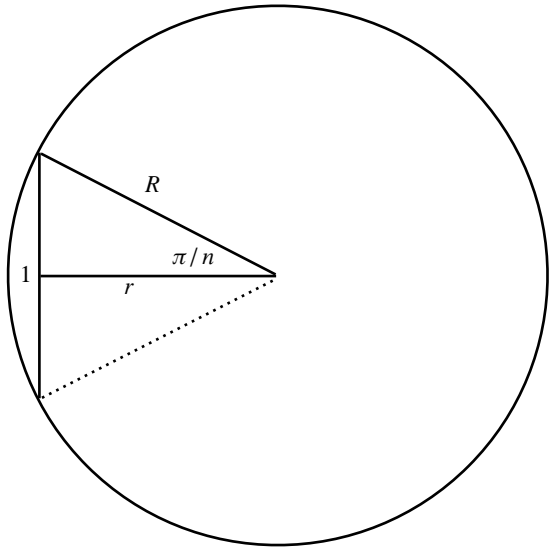


FIGURE 1

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Figure 2 shows the 10-gon with unit side used in this way.

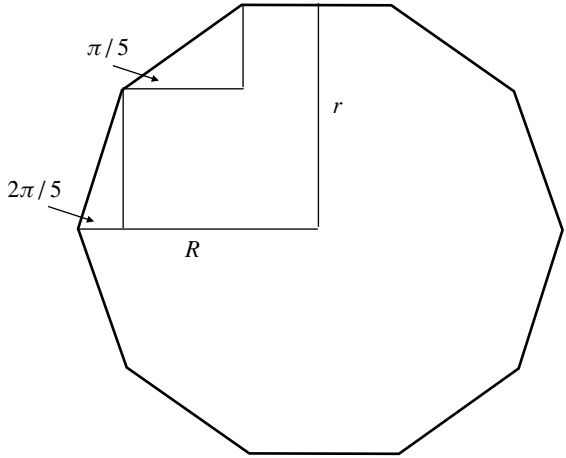


FIGURE 2

We have:

$$R = \frac{1}{2}\operatorname{cosec}\left(\frac{\pi}{10}\right) = \cos\left(\frac{2\pi}{5}\right) + \cos\left(\frac{\pi}{5}\right) + \frac{1}{2},$$
$$r = \frac{1}{2}\cot\left(\frac{\pi}{10}\right) = \sin\left(\frac{2\pi}{5}\right) + \sin\left(\frac{\pi}{5}\right),$$

which combine to give

$$\cos\left(\frac{4\pi}{5}\right) + 2\cos\left(\frac{2\pi}{5}\right) + \cos\left(\frac{\pi}{5}\right) = 2\left[\sin\left(\frac{2\pi}{5}\right)\sin\left(\frac{\pi}{5}\right) - \cos\left(\frac{2\pi}{5}\right)\cos\left(\frac{\pi}{5}\right)\right].$$

The symmetry of the regular n-gons produces sums of terms in sine or cosine whose arguments run in arithmetic progression, as we shall shortly show.

History

From the simplicity of the constructions one might have expected sums of sines or cosines with arguments in arithmetic progression to appear in the work of the Ancients, but the author has turned up nothing which can be construed as such before the sixteenth century, when, allowing for differences in notation, François Viète (1540-1603) worked on expressions of this kind*. Two such identities are associated with Joseph-Louis Lagrange (1736-1813). Peter Dirichlet (1805-1859) recast one as a function of a continuous variable, a vital contribution to the analysis of wave forms we now associate with the name of Joseph Fourier (1768-1830). Paul Dirac (1902-1984) took a limiting case as the basis of a particular 'hyperfunction' which could describe the distribution of point masses and charges as conceived in the emerging theory of quantum mechanics. Going back two millennia, one can speculate that, whereas the Ancients studied geometry for its own sake, trigonometry was

* The author would like to thank Bob Burn for that information.

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simply an aid with which to make the calculations that astronomy and land surveying demanded. It seems that, primarily at any rate, they were interested only in the relation between the length of a chord and the angle it subtended at the centre of the unit circle. This is indeed the central modern concern with trigonometry except that we centre the circle at the origin of a coordinate system or Argand diagram and set the chord in motion.

The 'arithmetic sequence' property and a sample derivation

The identities are listed in Table 3. As in any other case, the students can demonstrate the results by induction. But they can also *construct* a proof, given the regular n-gon as a template.

We give the sides unit length and take single quadrants. The five cases are distinguished by the arrangement of sides within each as shown in Table 1. Below the figures we show the number of sides allowed in the gap to complete them. (It may well be that cases beyond these five are of interest, though we assume that odd values of n would not prove useful.)

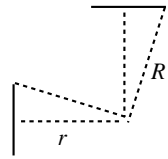
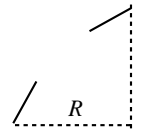
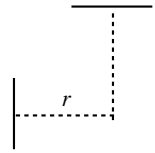
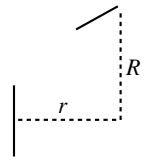
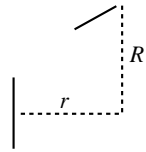
	$n\,(p)\,,\ p \geqslant 1$:		Gap:
1.	$4p$		$p - 1$
2.	$8p$		$2\,(p - 1)$
3.	$4\,(2p + 1)$		$2p$
4.	$2\,(2p + 1)$		$p - 1$
5.	$2\,(4p + 1)$		$2p - 1$

TABLE 1

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Figure 3 shows a representative vertex and the projections of the adjacent sides. We see that the arithmetic sequence of arguments arises because ϕ_r and ϕ_{r-1} differ by a constant, the exterior angle of the n-gon.

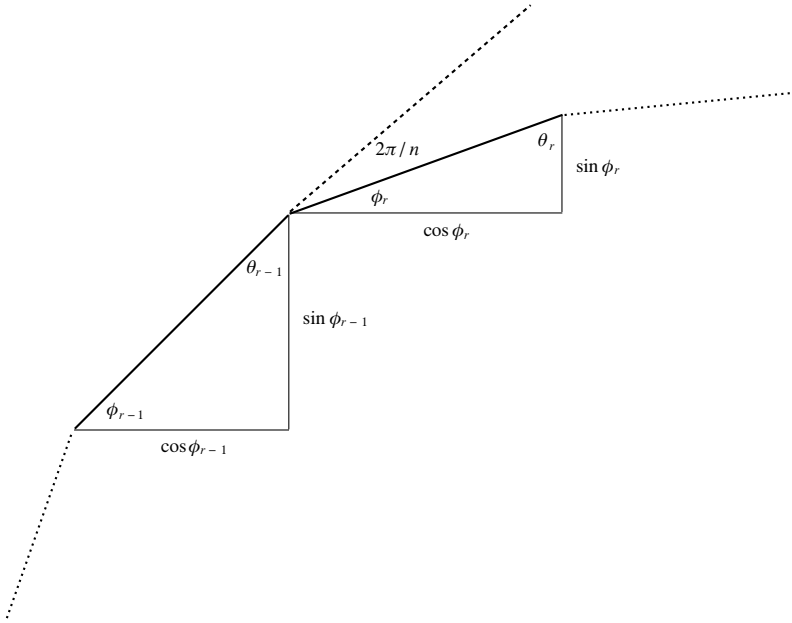


FIGURE 3

This accounts for what happens in the 'gaps'. Table 1 shows that the geometry at the quadrant boundaries depends on the form of n. Since in any particular case this dependency is of little mathematical interest, we shall derive just one pair of results: 1(a) and (b).

1(a): Looking at the top figure in Table 1, we see that we can start our sequence down on the left with $\phi_0 = \frac{1}{2}\pi$. This gives

$$\phi_1 = \left(\frac{n-4}{2n}\right)\pi = \left(\frac{p-1}{2p}\right)\pi.$$

This is the first argument of the $(p-1)$ 'gap' terms, which proceed with common difference $\dfrac{-2\pi}{n} = -1\left(\dfrac{1}{2p}\right)\pi$. Adding the half-side at the top of the Figure, we have

$$r = \frac{1}{2}\cot\left(\frac{\pi}{4p}\right) = \cos\left(\left(\frac{p-1}{2p}\right)\pi\right) + \cos\left(\left(\frac{p-2}{2p}\right)\pi\right) + \ldots + \cos\left(\left(\frac{1}{2p}\right)\pi\right) + \frac{1}{2}.$$

Exploiting the geometric symmetry of the figure, (we can project either horizontally or vertically), we can rewrite the whole identity in terms of sine, as in the Table 3 entry. To put it another way, for every angle, its complement occurs.

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1(b): We imagine rotating the Table 1 figure to bring a circumradius vertical. Our subject is now R. We take ϕ_1 as half the interior angle which is $\left(\frac{n-2}{2n}\right)\pi = \left(\frac{2p-1}{4p}\right)\pi$ and add p terms with the general form

$$\left[\left(\frac{2p-1}{4p}\right) - \left(\frac{1}{2p}\right)s\right]\pi = \left(\frac{2p-1-2s}{4p}\right)\pi,$$

taking the sum from $s = 0$ to $s = p - 1$. We obtain

$$R = \frac{1}{2}\operatorname{cosec}\left(\frac{\pi}{4p}\right) = \cos\left(\left(\frac{2p-1}{4p}\right)\pi\right) + \cos\left(\left(\frac{2p-3}{4p}\right)\pi\right) + \ldots + \cos\left(\frac{\pi}{4p}\right).$$

Again, we have the sine/cosine symmetry.

Using the identities alone and in combination to reveal particular relations

Because the identities have just two subjects and the value ranges of the five cases overlap, we can equate corresponding expressions to obtain relations between particular sines and cosines, and sometimes particular values expressed in surds. In Table 2 are some examples. Most can be checked by the use of simpler, standard identities.

Case	p value	Case	p value	Expression
5(a)	1			$4\cos\left(\frac{\pi}{5}\right)\sin\left(\frac{\pi}{10}\right) = 1$
5(b)	1			$\sin\left(\frac{\pi}{10}\right) + 2\sin^2\left(\frac{\pi}{5}\right) = 1$
1(a)	3	3	1	$\cos\left(\frac{\pi}{12}\right) = \frac{\sqrt{2}(\sqrt{3}-1)}{4}$
1(a)	5	3	2	$\cos\left(\frac{8\pi}{20}\right) + \cos\left(\frac{6\pi}{20}\right) + \cos\left(\frac{4\pi}{20}\right) + \cos\left(\frac{2\pi}{20}\right)$ $= \sqrt{2}\left(\cos\left(\frac{3\pi}{20}\right) + \cos\left(\frac{\pi}{20}\right)\right)$
1(b)	2	2	1	$(\sqrt{2}-1)\cos\left(\frac{\pi}{8}\right) = \cos\left(\frac{3\pi}{8}\right)$
4(a)	2	5(a)	1	$\cos\left(\frac{3\pi}{10}\right) = \cos\left(\frac{\pi}{10}\right)\left(4\cos^2\left(\frac{\pi}{10}\right) - 3\right)$
4(b)	4	5(a)	2	$\cos\left(\frac{7\pi}{18}\right) + \cos\left(\frac{5\pi}{18}\right)$ $= \left(\cos\left(\frac{3\pi}{18}\right) + \cos\left(\frac{\pi}{18}\right)\right)\left(2\cos\left(\frac{4\pi}{18}\right) - 1\right)$

TABLE 2

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Form	Case	Identity
$n = 4p$	1(a)	$r = \frac{1}{2}\cot\left(\frac{\pi}{4p}\right) = \frac{1}{2} + \sum_{s=1}^{p-1} \cos\left(\left(\frac{p-s}{2p}\right)\pi\right)$
		$= \frac{1}{2} + \sum_{s=1}^{p-1} \sin\left(\left(\frac{p-s}{2p}\right)\pi\right)$
	1(b)	$R = \frac{1}{2}\operatorname{cosec}\left(\frac{\pi}{4p}\right) = \sum_{s=0}^{p-1} \cos\left(\left(\frac{2p-1-2s}{4p}\right)\pi\right)$
		$= \sum_{s=0}^{p-1} \sin\left(\left(\frac{2p-1-2s}{4p}\right)\pi\right)$
$n = 8p$	2	$R = \frac{1}{2}\operatorname{cosec}\left(\frac{\pi}{8p}\right) = \sqrt{2}\sum_{s=1}^{p} \cos\left(\left(\frac{2p+1-2s}{8p}\right)\pi\right)$
$n = 4(2p+1)$	3	$r = \frac{1}{2}\cot\left(\frac{\pi}{4(2p+1)}\right) = \frac{1}{2} + \sqrt{2}\sum_{s=1}^{p} \cos\left(\left(\frac{2p+1-2s}{4(2p+1)}\right)\pi\right)$
$n = 2(2p+1)$	4(a)	$r = \frac{1}{2}\cot\left(\frac{\pi}{2(2p+1)}\right) = \sum_{s=1}^{p} \cos\left(\left(\frac{2p+1-2s}{2(2p+1)}\right)\pi\right)$
	4(b)	$R = \frac{1}{2}\operatorname{cosec}\left(\frac{\pi}{2(2p+1)}\right) = \frac{1}{2} + \sum_{s=1}^{p} \sin\left(\left(\frac{2p+1-2s}{2(2p+1)}\right)\pi\right)$
$n = 2(4p+1)$	5(a)	$r = \frac{1}{2}\cot\left(\frac{\pi}{2(4p+1)}\right)$
		$= 2\cos\left(\frac{p}{4p+1}\right)\pi \sum_{s=1}^{p} \cos\left(\left(\frac{2p+1-2s}{2(4p+1)}\right)\pi\right)$
	5(b)	$R = \frac{1}{2}\operatorname{cosec}\left(\frac{\pi}{2(4p+1)}\right)$
		$= \frac{1}{2} + 2\sin\left(\frac{p}{4p+1}\right)\pi \sum_{s=1}^{p} \cos\left(\left(\frac{2p+1-2s}{2(4p+1)}\right)\pi\right)$

TABLE 3

Two standard identities for comparison

In Case 1 we noted the complementarity of sine and cosine. This is apparent in the following standard identities for sums of sines and cosines whose arguments run in arithmetic progression:

$$\sum_{s=0}^{s=n-1} \sin(a + sd) = \frac{\sin\left(\frac{1}{2}nd\right)}{\sin\left(\frac{1}{2}d\right)} \sin\left(a + (n-1)\frac{d}{2}\right),$$

$$\sum_{s=0}^{s=n-1} \cos(a + sd) = \frac{\sin\left(\frac{1}{2}nd\right)}{\sin\left(\frac{1}{2}d\right)} \cos\left(a + (n-1)\frac{d}{2}\right).$$

The students may attempt constructive proofs of these. To take the first as an example, we multiply the left-hand sequence by the right, and rewrite

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each product so formed as a difference of cosines. We then have the following equation stack, where, on the right, the second term of the kth equation cancels the first term of the $(k + 1)$th:

$$\sin(a)\sin\left(\frac{d}{2}\right) = \frac{1}{2}\left[\cos\left(a - \frac{d}{2}\right) - \cos\left(a + \frac{d}{2}\right)\right].$$

$$\sin(a + d)\sin\left(\frac{d}{2}\right) = \frac{1}{2}\left[\cos\left(a + \frac{d}{2}\right) - \cos\left(a + \frac{3}{2}d\right)\right].$$

$$\cdots$$

$$\sin(a + (n-2)d)\sin\left(\frac{d}{2}\right) = \frac{1}{2}\left[\cos\left(a + (2n-5)\frac{d}{2}\right) - \cos\left(a + (2n-3)\frac{d}{2}\right)\right].$$

$$\sin(a + (n-1)d)\sin\left(\frac{d}{2}\right) = \frac{1}{2}\left[\cos\left(a - (2n-3)\frac{d}{2}\right) - \cos\left(a + (2n-1)\frac{d}{2}\right)\right].$$

Summing terms, there remain on the right the left-hand part of the first term and the right-hand part of the last:

$$\frac{1}{2}\left[\cos\left(a - \frac{d}{2}\right) - \cos\left(a + (2n-1)\frac{d}{2}\right)\right].$$

We now reverse the first process and write the resulting difference of cosines as a product of sines. The required sum is then the quotient of this product and $\sin\left(\frac{1}{2}d\right)$. Students may apply a similar method to establish the Lagrange/Dirichlet equation mentioned in the section on its history

$$1 + 2\sum_{s=1}^{n}\cos(sd) = \frac{\sin\left(\frac{1}{2}(2n+1)d\right)}{\sin\left(\frac{1}{2}d\right)}.$$

For our immediate purpose, we can use the identities to check the results we have derived by projection. Note that, since in every case n is even and a is either $\left(\frac{n-2}{2n}\right)\pi$ or $\left(\frac{n-4}{2n}\right)\pi$, and $d = -\frac{2\pi}{n}$ in all cases, every argument is a multiple of $\frac{\pi}{n}$.

Conclusion

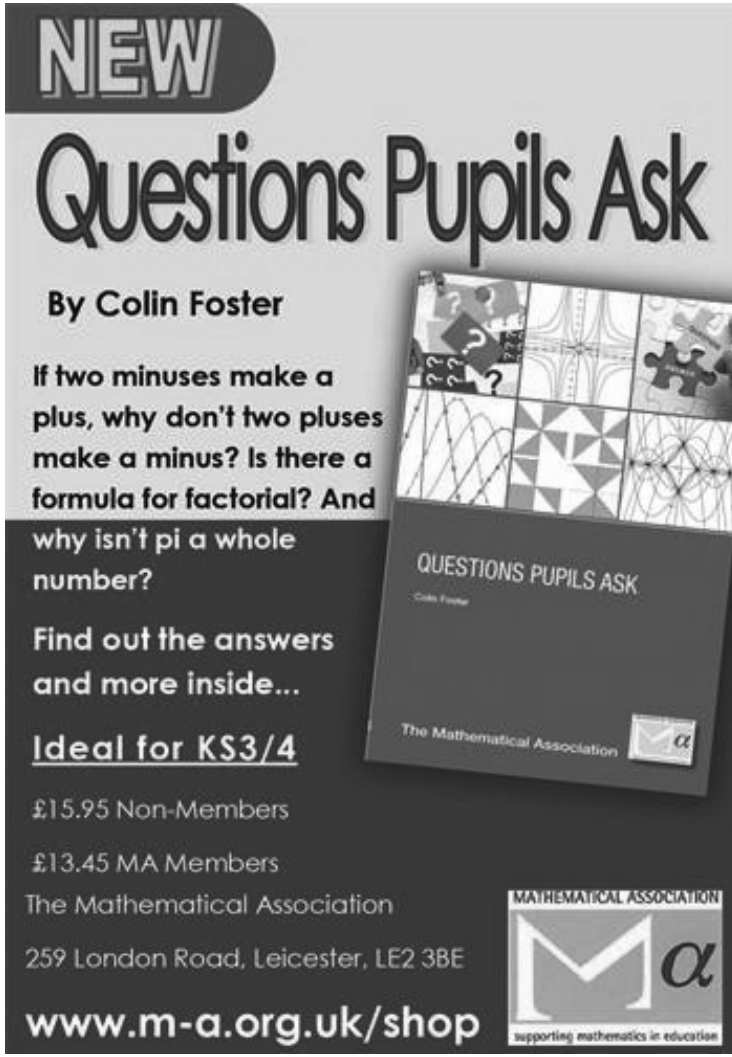
In the third and fifth sections we tried to provide students with cases where a constructive proof provides an obvious alternative to an inductive one. In the fourth section we showed how our set of identities could be mined for particular results. Despite this possibility, our cursory historical survey in the second section yielded no evidence that regular polygons had been used in the way described. In recent centuries such scaffolding would in any case have been removed from the final mathematical structure. Furthermore, beyond a small set, trigonometric identities are arguably losing their importance. The chief use of trigonometric identities in modern mathematics is to permit integration by substitution but the contemporary mathematician is more likely to settle for a numerical approximation than

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spend time searching for the right identity. The author would however be interested to hear from historians who have found diagrams like Figure 2 among earlier sources or, alternatively, found identities like (1) to (5) and wondered about their origin.

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