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A note on extensions of p-adic representations of $\mathrm{GL}_2(\mathbb{Q}_p)$

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Abstract. We compute extension groups in the category of duals of p-adic Banach space representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. Focusing on representations arising from the p-adic local Langlands correspondence for generic Galois representations, we classify these extensions completely. These results are then applied to prove the vanishing of extensions between the dual p-adic Banach space representations attached to reducible Galois representations and supercuspidal Galois isotypic components of the p-adic étale cohomology of the finite-level Drinfeld spaces.

This paper is dedicated to the fond memory of the first author's mother, the late Mrs. Snigdha Banerjee.

1 Introduction

Fix a prime p and let $\mathrm{G}_{\mathbb{Q}_p} := \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ denote the absolute Galois group of $\mathbb{Q}_p$. The study of ℓ-adic representations of $\mathrm{G}_{\mathbb{Q}_p}$, in the case where $\ell = p$, was pioneered by Fontaine, who laid the foundations of p-adic Hodge theory, and was further developed by Berger [1] and Colmez [6]. The p-adic Langlands program, initiated by Breuil and further developed by Berger, Colmez, Dospinescu, Emerton, Kisin, and Paškūnas, associates a p-adic admissible unitary Banach space representation $\Pi(V)$ of the group $\mathrm{G} := \mathrm{GL}_2(\mathbb{Q}_p)$ to a representation V of $\mathrm{G}_{\mathbb{Q}_p}$ of dimension ≤ 2.

While the bijection between the isomorphism classes of the above categories of representations is well-understood, a deeper understanding of the correspondence requires analyzing the derived category of representations; specifically, the extension groups between them. These extensions are central to the compatibility between the deformation theory of Galois representations and the deformation theory of G-representations [21].

Significant progress has been made in computing extension groups in the context of smooth mod p representations of G. Emerton [14] calculated the $\mathrm{Ext}^\bullet$ groups in the category of locally admissible mod p representations of G as an application of his computation of the right derived functors of the ordinary parts. Breuil and Paškūnas

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[4] investigated similar extensions for $\mathrm{GL}_2(\mathcal{O}_F)$ (where $\mathcal{O}_F$ is the ring of integers of some finite extension $F/\mathbb{Q}_p$), while Colmez [6] also performed foundational calculations for G in characteristic p. In [20], Paškūnas completed the computation of $\mathrm{Ext}^1(V_1, V_2)$ for absolutely irreducible mod p representations V_1, V_2 (treating the previously open cases where V_1 is supersingular and V_2 is supersingular or a twist of Steinberg).

However, in characteristic 0, the literature is sparse. Hauseux [18, 19] has determined the extensions between unitary continuous p-adic and smooth mod p principal series representations of $G'(F)$ in the generic case for split connected reductive groups G'. In [22], Paškūnas has calculated Ext^1 between two supercuspidal representations of G in the p-adic setting.

The novelty of this short note lies in the computation of extensions in the category of duals of finite length admissible p-adic Banach space representations, specifically when one of the representations is dual of a non-ordinary representation. We say that a p-adic Banach space representation is non-ordinary if it is not a subquotient of a principal series representation. This dual category arises naturally in Colmez's construction of the p-adic local Langlands correspondence [6]. If D denotes the (φ, Γ)-module attached to a p-adic Galois representation V via Fontaine's equivalence [16], then for $\delta := \varepsilon^{-1} \det V$ (where ε is the p-adic cyclotomic character), the global sections of Colmez's sheaf $D \boxtimes_\delta \mathbb{P}^1$ fit into the fundamental exact sequence [6]:

$$0 \to \Pi(V)^* \otimes \delta \to D \boxtimes_\delta \mathbb{P}^1 \to \Pi(V) \to 0. \tag{1.1}$$

Here, $\Pi(V)^*$ denotes the continuous dual equipped with the compact-open topology. The functor $\mathbf{V}$ (Colmez's Montréal functor) and Π satisfy the relation $\mathbf{V}(\Pi(V)) \cong V$.

Since the category of admissible representations with a central character on p-adic Banach spaces does not have enough injectives, we utilize Yoneda's construction to compute the extension groups in $\mathrm{Ban}^{\mathrm{adm}}_{\mathrm{G},\zeta}(E)$ (cf. Section 2 for the definition), denoted by $\mathrm{Ext}^i_{\mathrm{G},\zeta}$. Note that in the presence of enough injectives, Yoneda's definition is equivalent to the derived functor construction.

Our main result provides a complete classification of these extensions under a suitable genericity condition. Let E be a finite extension of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}$, uniformizer ϖ, and residue field $\kappa := \mathcal{O}/(\varpi)$. We denote by ω the mod p cyclotomic character.

Let V be an absolutely irreducible two-dimensional E-representation of $\mathrm{G}_{\mathbb{Q}_p}$, and let $\overline{V}$ denote the semisimplification of its reduction modulo p. We say that $\overline{V}$ is *generic* (in the sense of Paškūnas [21]) if $\overline{V}$ is irreducible or $\overline{V} \cong \psi_1 \oplus \psi_2$ is split with characters ψ_1, ψ_2 satisfying $\psi_1\psi_2^{-1} \notin \{1, \omega^{\pm 1}\}$.

Theorem 1.1 *Let V be an absolutely irreducible two-dimensional representation of $\mathrm{G}_{\mathbb{Q}_p}$ over E such that its reduction $\overline{V}$ is generic. Let π be an absolutely irreducible admissible unitary E-Banach space representation of* G. *If $\mathbf{V}(\pi) \neq V$, then*

$$\mathrm{Ext}^i_{\mathrm{G},\zeta}(\pi^*, \Pi(V)^*) = 0 \quad \forall\, i \geq 0,$$

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where $\mathbf{V}$ *denotes Colmez's Montréal functor. Suppose that* $\mathbf{V}(\pi) = V$. *Then for every integer* $i \geq 0$,

$$\mathrm{Ext}^i_{\mathrm{G},\zeta}\left(\pi^*, \Pi(V)^*\right) \cong \begin{cases} E^{\binom{3}{i}}, & 0 \leq i \leq 3, \\ 0, & i \geq 4. \end{cases}$$

This specific category of duals is also of significant arithmetic interest due to its geometric realization. Recall that Drinfeld spaces form a tower of rigid analytic covers of the p-adic upper half-plane $\Omega_{Dr,p} := \mathbb{P}^1_{\mathbb{Q}_p} - \mathbb{P}^1(\mathbb{Q}_p)$. This tower is defined over $\breve{\mathbb{Q}}_p$, the completion of the maximal unramified extension of $\mathbb{Q}_p$, and is equipped with actions of $\mathrm{GL}_2(\mathbb{Q}_p)$, the units of the nonsplit quaternion algebra over $\mathbb{Q}_p$, and the Weil group $W_{\mathbb{Q}_p}$. Specifically, when V is a two-dimensional supercuspidal representation of $\mathrm{G}_{\mathbb{Q}_p}$ (i.e., the associated Weil–Deligne representation is irreducible), we derive from the work of Colmez et al. [7, 8] that the V-isotypic part of the p-adic étale cohomology of the finite-level Drinfeld spaces (see Section 4 for details) is isomorphic to a direct sum of copies of the dual $\Pi(V)^*$ (cf. isomorphism (4.1)). Consequently, we determine that there is no extension of the V-isotypic part of the p-adic étale cohomology of these finite-level Drinfeld spaces by the dual of the p-adic local Langlands associated with a reducible local Galois representation. Here, $\mathcal{M}_\infty$ denotes the (non-complete) projective limit of the finite-level Drinfeld spaces over $\mathbb{C}_p := \widehat{\overline{\mathbb{Q}}_p}$ (defined precisely in Section 4).

Corollary 1.2 *Let* $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(E)$ *be a global Galois representation with the corresponding local representation* $\rho_p \cong \begin{pmatrix} \eta_1 & \star \\ 0 & \eta_2 \end{pmatrix} \otimes \eta$, *where* $\eta_1, \eta_2 : \mathbb{Q}_p^\times \to \mathcal{O}_E^\times$ *are continuous integral characters and* $\eta : \mathrm{G}_{\mathbb{Q}_p} \to E^\times$. *We also assume that* $\eta_1 \cdot \eta_2^{-1} \notin \{\varepsilon^{\pm 1}\}$. *Suppose* ρ_p *is potentially crystalline with distinct Hodge–Tate weights between* $(0, k-1)$ *for* $k \geq 2$. *Then, for any absolutely irreducible supercuspidal representation* V *of* $\mathrm{G}_{\mathbb{Q}_p}$ *of dimension* ≥ 2 *such that its mod* p *reduction* $\overline{V}$ *is generic, we have*

$$\mathrm{Ext}^i_{\mathrm{G},\zeta}(\Pi(\rho_p)^*, \mathrm{H}^1_{\acute{e}t}(\mathcal{M}_\infty, E(1))[V]) = 0 \quad \textit{for all } i \geq 0.$$

2 *p*-adic and mod *p* local Langlands for $\mathrm{GL}_2(\mathbb{Q}_p)$

2.1 The Montréal functor and the trianguline correspondence

Following [9, 21], we recall some basic facts about the p-adic and mod p local Langlands correspondence. As before, we fix a finite extension E of $\mathbb{Q}_p$ with a ring of integers $\mathcal{O}$, a uniformizer ϖ, and a residue field κ.

Fix a continuous character $\zeta : Z \to \mathcal{O}^\times$ of the center Z of G. We denote by $\mathrm{Mod}^{\mathrm{sm}}_{\mathrm{G}}(\mathcal{O})$ the category of smooth $\mathcal{O}$-torsion G-modules. Within this, let $\mathrm{Mod}^{\mathrm{sm}}_{\mathrm{G},\zeta}(\mathcal{O})$ be the full subcategory consisting of representations on which Z acts via ζ. Furthermore, we define $\mathrm{Mod}^{\mathrm{fin}}_{\mathrm{G},\zeta}(\mathcal{O})$ and $\mathrm{Mod}^{\mathrm{lfin}}_{\mathrm{G},\zeta}(\mathcal{O})$ as the full subcategories of representations that are, respectively, of finite length and locally of finite length as $\mathcal{O}[\mathrm{G}]$-modules. A detailed study of these categories can be found in [13, Section 2]. In his Montréal lectures, Colmez introduced an exact covariant $\mathcal{O}$-linear functor

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$\mathbf{V} : \mathrm{Mod}^{\mathrm{fin}}_{\mathrm{G},\zeta}(\mathcal{O}) \to \mathrm{Mod}_{\mathrm{G}_{\mathbb{Q}_p}}(\mathcal{O})$, where $\mathrm{Mod}_{\mathrm{G}_{\mathbb{Q}_p}}(\mathcal{O})$ is the category of $\mathcal{O}$-modules with a continuous $\mathrm{G}_{\mathbb{Q}_p}$-action. Let $\mathrm{Ban}^{\mathrm{adm}}_{\mathrm{G}}(E)$ be the category of unitary admissible E-Banach space representations of G and let $\mathrm{Ban}^{\mathrm{adm.fl}}_{\mathrm{G},\zeta}(E)$ be the full subcategory consisting of finite length objects with central character ζ. For any $\Pi \in \mathrm{Ban}^{\mathrm{adm}}_{\mathrm{G}}(E)$, one may choose an open bounded G-invariant lattice Θ in Π. Then $\Theta/\varpi^n\Theta$ is a smooth admissible representation of G for all $n \geq 1$, implying it is locally finite. This allows one to define $\mathbf{V}(\Pi) := E \otimes \varprojlim \mathbf{V}(\Theta/\varpi^n\Theta)$. Due to the commensurability of all open bounded lattices in Π, the definition is independent of the specific lattice Θ chosen. The functor $\mathbf{V}$ satisfies the following properties:

(1) It is exact and covariant from $\mathrm{Ban}^{\mathrm{adm.fl}}_{\mathrm{G},\zeta}(E)$ to finite-dimensional continuous E-representations of $\mathrm{G}_{\mathbb{Q}_p}$.

(2) The functor $\Pi \mapsto \mathbf{V}(\Pi)$ induces a bijection between the isomorphism classes of absolutely irreducible non-ordinary $\Pi \in \mathrm{Ban}^{\mathrm{adm}}_{\mathrm{G}}(E)$ and two-dimensional absolutely irreducible continuous E-representations of $\mathrm{G}_{\mathbb{Q}_p}$.

(3) If $\psi : \mathbb{Q}_p^\times \to \mathcal{O}^\times$ is a continuous character, then we may also consider it as a continuous character $\psi : \mathrm{G}_{\mathbb{Q}_p} \to \mathcal{O}^\times$ via the local Artin map, and for all $\Pi \in \mathrm{Ban}^{\mathrm{adm}}_{\mathrm{G}}(E)$, we have the isomorphism:

$$\mathbf{V}(\Pi \otimes \psi \circ \det) \cong \mathbf{V}(\Pi) \otimes \psi.$$

(4) The Montréal functor maps ordinary Banach space representations of G to one-dimensional representations of $\mathrm{G}_{\mathbb{Q}_p}$. More precisely, as a functor from $\mathrm{Mod}^{\mathrm{fin}}_{\mathrm{G},\zeta}(\mathcal{O})$, $\mathbf{V}$ satisfies the following:

$$\mathbf{V}(\delta \circ \det) = 0,\ \mathbf{V}(St \otimes \delta \circ \det) = \varepsilon\delta,\ \mathbf{V}(\mathrm{Ind}^{\mathrm{G}}_{P}(\chi_1 \otimes \chi_2\varepsilon^{-1})) \cong \chi_2,$$

where P is the Borel subgroup of upper-triangular matrices in G, $\mathrm{Ind}^{\mathrm{G}}_{P}$ denotes the (smooth) parabolic induction, χ_1, χ_2, and δ are continuous characters of $\mathbb{Q}_p^\times$, and St is the Steinberg representation.

Let $H \subset \mathrm{G}$ be a compact open subgroup, and let $\mathcal{O}[[H]]$ represent its completed group algebra. We define $\mathrm{Mod}^{\mathrm{pro\,aug}}_{\mathrm{G}}(\mathcal{O})$ as the category of profinite, linearly topological $\mathcal{O}[[H]]$-modules M equipped with a continuous G-action, such that the action of $\mathcal{O}[H]$ induced by $\mathcal{O}[[H]]$ aligns with the restriction of the G-action. Morphisms in this category are continuous G-equivariant homomorphisms of topological $\mathcal{O}[[H]]$-modules. This definition is well-defined regardless of the choice of H, as any two compact open subgroups of G are commensurable. As noted in [13, Equation 2.2.8], taking Pontryagin dual yields an anti-equivalence between $\mathrm{Mod}^{\mathrm{sm}}_{\mathrm{G}}(\mathcal{O})$ and $\mathrm{Mod}^{\mathrm{pro\,aug}}_{\mathrm{G}}(\mathcal{O})$. The Pontryagin dual of an $\mathcal{O}$-module M is defined as

$$M^\vee := \mathrm{Hom}^{cont}_{\mathcal{O}}(M, E/\mathcal{O}),$$

where $E/\mathcal{O}$ carries the discrete topology and $M^\vee$ the compact-open topology. There exists a canonical isomorphism $M^{\vee\vee} \cong M$.

Let $\mathfrak{C}(\mathcal{O})$ be the full subcategory of $\mathrm{Mod}^{\mathrm{pro\,aug}}_{\mathrm{G}}(\mathcal{O})$ whose objects consist of all M isomorphic to $\pi^\vee$ for some $\pi \in \mathrm{Mod}^{\mathrm{lfin}}_{\mathrm{G},\zeta}(\mathcal{O})$. This category $\mathfrak{C}(\mathcal{O})$ is abelian and has exact projective limits and projective envelopes. Let $\mathrm{Rep}_{\mathrm{G}_{\mathbb{Q}_p}}(\mathcal{O})$ denote the category

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of continuous $G_{\mathbb{Q}_p}$-representations on compact $\mathcal{O}$-modules. Paškūnas [21] introduced a functor $\check{\mathbf{V}} : \mathfrak{C}(\mathcal{O}) \to \mathrm{Rep}_{G_{\mathbb{Q}_p}}(\mathcal{O})$ defined as follows: for an object $M \in \mathfrak{C}(\mathcal{O})$ of finite length, set $\check{\mathbf{V}}(M) := \mathbf{V}(M^\vee)^\vee(\varepsilon\zeta)$, where $\vee$ denotes the Pontryagin dual and ζ is treated as a character of $G_{\mathbb{Q}_p}$ via local class field theory. More generally, by writing $M \cong \varprojlim M_i$ as a limit over finite-length quotients in $\mathfrak{C}(\mathcal{O})$, one defines $\check{\mathbf{V}}(M) := \varprojlim \check{\mathbf{V}}(M_i)$. By construction, this functor is also covariant and exact. Under this normalization of $\check{\mathbf{V}}$, the following relations hold:

$$\check{\mathbf{V}}(\pi^\vee) \cong \mathbf{V}(\pi), \quad \check{\mathbf{V}}((\mathrm{Ind}_P^G \chi_1 \otimes \chi_2\omega^{-1})^\vee) = \chi_1, \quad \check{\mathbf{V}}((St \otimes \eta \circ \det)^\vee) = \eta, \tag{2.1}$$

where π represents a supersingular representation and η is a continuous character of $\mathbb{Q}_p^\times$.

Consider $\Pi \in \mathrm{Ban}_{G,\zeta}^{\mathrm{adm}}(E)$ with an open, bounded G-invariant lattice Θ. Its Schikhof dual is defined as $\Theta^d := \mathrm{Hom}_{\mathcal{O}}(\Theta, \mathcal{O})$, endowed with the topology of pointwise convergence. It is the unit ball for the dual representation Π^*. The relationship between the Schikhof and Pontryagin duals is given by

$$\Theta^d \otimes_{\mathcal{O}} \mathcal{O}/\varpi^n\mathcal{O} \cong (\Theta/\varpi^n\Theta)^\vee.$$

According to [21, Lemma 4.11], Θ^d belongs to $\mathfrak{C}(\mathcal{O})$. As Θ^d is $\mathcal{O}$-torsion free and $\check{\mathbf{V}}$ preserves exactness and covariance, $\check{\mathbf{V}}(\Theta^d)$ remains $\mathcal{O}$-torsion free. In this context, we define $\check{\mathbf{V}}(\Pi)$ as $\check{\mathbf{V}}(\Theta^d) \otimes_{\mathcal{O}} E$. The commensurability of different open lattices in Π ensures that $\check{\mathbf{V}}(\Pi)$ is independent of the choice of Θ.

Lemma 2.1 *Assume Π and Θ are defined as above. Then $\check{\mathbf{V}}(\Pi) \cong \mathbf{V}(\Pi)^*(\varepsilon\zeta)$, where $*$ denotes the E-linear dual. Furthermore, if $\check{\mathbf{V}}(\Pi)$ is two-dimensional with determinant $\varepsilon\zeta$, then $\check{\mathbf{V}}(\Pi) \cong \mathbf{V}(\Pi)$.*

Proof This is proved in [21, Lemma 5.49]. ∎

The surjectivity of the functor $\mathbf{V}$ is established in [6] for odd p (and in [10] for $p = 2$) via the construction of a representation $\Pi(V)$ of G satisfying $\mathbf{V}(\Pi(V)) = V$ for any two-dimensional representation V of $G_{\mathbb{Q}_p}$. The construction goes through Fontaine's equivalence [16] relating representations of $G_{\mathbb{Q}_p}$ and (φ, Γ)-modules, similar to the construction of $\Pi \mapsto \mathbf{V}(\Pi)$. The resulting p-adic local Langlands correspondence $V \mapsto \Pi(V)$ exhibits several notable features:

- compatibility with reduction modulo p;
- consistent behavior within families;
- compatibility with the classical local Langlands correspondence.

An explicit construction of $\Pi(V)$ for trianguline V is given in [12, Section 6]. We recall it here. Note that Emerton's description in the Banach space setting is the same as the characteristic zero translation of Colmez's mod p results [6, Théorème 0.11] via the extension of the Montréal functor to the Banach space setting described above. Throughout this construction, the superscript $\mathcal{C}^0$ indicates that the induction is taken in the category of continuous functions.

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(1) (Absolutely reducible) Let $V \cong \begin{pmatrix} \eta_1 & 0 \\ 0 & \eta_2 \end{pmatrix} \otimes \eta$ with η_1, η_2 continuous *integral* characters and $\eta : G_{\mathbb{Q}_p} \to E^\times$ be a continuous character. In this case,

- If $\eta_1\eta_2^{-1} \neq \varepsilon^{\pm 1}$, then
$$\Pi(V) \cong \mathrm{Ind}_P^G(\eta_1 \otimes \eta_2\varepsilon^{-1})^{\mathcal{C}^0} \otimes \eta \bigoplus \mathrm{Ind}_P^G(\eta_2 \otimes \eta_1\varepsilon^{-1})^{\mathcal{C}^0} \otimes \eta.$$
- If $\eta_1\eta_2^{-1} = \varepsilon$, then
$$\Pi(V) \cong \eta_1 \circ \det \otimes B(2, \infty) \oplus \mathrm{Ind}_P^G(\eta_1\varepsilon^{-1} \otimes \eta_1\varepsilon)^{\mathcal{C}^0} \otimes \eta,$$
where $B(2, \infty)$ denotes the universal unitary completion of the smooth induction $\mathrm{Ind}_P^G(|\,|_p^{-1} \otimes |\,|_p)^{sm}$.

(2) (Reducible non-split, case I)Let $V \cong \begin{pmatrix} \eta_1 & \star \\ 0 & \eta_2 \end{pmatrix} \otimes \eta$ with η_1, η_2 as above, $\star \neq 0$ and $\eta_1 \cdot \eta_2^{-1} \neq \varepsilon^{\pm 1}$, then $\Pi(V)$ sits inside the following exact sequence in the category $\mathrm{Ban}_G^{\mathrm{adm}}(E)$:
$$0 \to \pi_1 \otimes \eta \to \Pi(V) \to \pi_2 \otimes \eta \to 0,$$
where $\pi_1 := \mathrm{Ind}_P^G(\eta_2 \otimes \eta_1)^{\mathcal{C}^0}$ and $\pi_2 := \mathrm{Ind}_P^G(\eta_1\varepsilon \otimes \eta_2\varepsilon^{-1})^{\mathcal{C}^0}$.

(3) (Reducible non-split, case II) Suppose $V \cong \begin{pmatrix} \eta & \star \\ 0 & \eta\varepsilon^{-1} \end{pmatrix}$ or $V \cong \begin{pmatrix} \eta\varepsilon^{-1} & \star \\ 0 & \eta \end{pmatrix}$ with η as above and $\star \neq 0$. Then the corresponding G-representation $\Pi(V)$ has a Jordan–Hölder filtration with Jordan–Hölder factors $\widehat{St}$, $\underline{1}$, and $\mathrm{Ind}_P^G(\eta\varepsilon^{-1} \otimes \eta\varepsilon)^{\mathcal{C}^0}$, where $\widehat{St}$ denotes the universal unitary completion of the Steinberg representation St.

2.2 Blocks and semisimple mod *p* LLC

In [21], Paškūnas defined an equivalence relation on the set of (isomorphism classes of) irreducible objects of $\mathrm{Mod}_{G,\zeta}^{\mathrm{lfin}}(\mathcal{O})$, where $\tau \sim \pi$ if and only if there exists a sequence of irreducible representations $\tau = \tau_0, \tau_1, \ldots, \tau_n = \pi$ such that for each i, one of the following holds: 1) $\tau_i = \tau_{i+1}$; 2) $\mathrm{Ext}_G^1(\tau_i, \tau_{i+1}) \neq 0$; and 3) $\mathrm{Ext}_G^1(\tau_{i+1}, \tau_i) \neq 0$. An equivalence class is called a block. The category $\mathrm{Mod}_{G,\zeta}^{\mathrm{lfin}}(\mathcal{O})$ is locally finite and, by general results of Gabriel [17], it decomposes as a product
$$\mathrm{Mod}_{G,\zeta}^{\mathrm{lfin}}(\mathcal{O}) \cong \prod_{\mathfrak{B}} \mathrm{Mod}_{G,\zeta}^{\mathrm{lfin}}(\mathcal{O})^{\mathfrak{B}}$$
of indecomposable subcategories, where the product runs over the set of all blocks $\mathfrak{B}$ of $\mathrm{Mod}_{G,\zeta}^{\mathrm{lfin}}(\mathcal{O})$. Here, $\mathrm{Mod}_{G,\zeta}^{\mathrm{lfin}}(\mathcal{O})^{\mathfrak{B}}$ denotes the full subcategory of $\mathrm{Mod}_{G,\zeta}^{\mathrm{lfin}}(\mathcal{O})$ consisting of those representations such that all their irreducible subquotients lie in $\mathfrak{B}$. Dually, we obtain a decomposition:
$$\mathfrak{C}(\mathcal{O}) \cong \prod_{\mathfrak{B}} \mathfrak{C}(\mathcal{O})^{\mathfrak{B}},$$
where $M \in \mathfrak{C}(\mathcal{O})$ lies in $\mathfrak{C}(\mathcal{O})^{\mathfrak{B}}$ if and only if $M^\vee$ lies in $\mathrm{Mod}_{G,\zeta}^{\mathrm{lfin}}(\mathcal{O})^{\mathfrak{B}}$.

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For each fixed block $\mathrm{Mod}^{\mathrm{lfin}}_{\mathrm{G},\zeta}(\mathcal{O})^{\mathfrak{B}}$, there is a finite extension E' of E with ring of integers $\mathcal{O}'$, such that $\mathrm{Mod}^{\mathrm{lfin}}_{\mathrm{G},\zeta}(\mathcal{O})^{\mathfrak{B}} \otimes_{\mathcal{O}} \mathcal{O}'$ decomposes into a finite product of indecomposable subcategories, each of which remains indecomposable after further extension of scalars. Such absolutely indecomposable blocks have been classified in [23], which are described as follows:

(1) $\mathfrak{B} = \{\pi\}$, with π supersingular,
(2) $\mathfrak{B} = \{\mathrm{Ind}_P^{\mathrm{G}} \chi_1 \otimes \chi_2\omega^{-1}, \mathrm{Ind}_P^{\mathrm{G}} \chi_2 \otimes \chi_1\omega^{-1}\}$, with $\chi_1\chi_2^{-1} \notin \{1, \omega^{\pm 1}\}$,
(3) $\mathfrak{B} = \{\mathrm{Ind}_P^{\mathrm{G}} \chi \otimes \chi\omega^{-1}\}$, with $p \geq 3$,
(4) $\mathfrak{B} = \{1, St\} \otimes \chi \circ \det$, with $p = 2$,
(5) $\mathfrak{B} = \{1, St, \mathrm{Ind}_P^{\mathrm{G}} \omega \otimes \omega^{-1}\} \otimes \chi \circ \det$, with $p \geq 5$,
(6) $\mathfrak{B} = \{1, St, \omega \circ \det, St \otimes \omega \circ \det\} \otimes \chi \circ \det$, with $p = 3$,

where χ_1, χ_2, and χ are smooth characters of $\mathbb{Q}_p^\times$ and ω is the mod p cyclotomic character given by $x \mapsto x|x|(\mathrm{mod}\ p)$.

To each block above, one may attach a semi-simple two-dimensional κ-representation $\overline{\rho_{\mathfrak{B}}}$ of $\mathrm{G}_{\mathbb{Q}_p}$ using the semi-simple mod p correspondence of Breuil [2, 3]: in case (1), $\overline{\rho_{\mathfrak{B}}}$ is absolutely irreducible, and Colmez's functor $\mathbf{V}$ maps π to $\overline{\rho_{\mathfrak{B}}}$, in case (2), $\overline{\rho_{\mathfrak{B}}} = \chi_1 \oplus \chi_2$, in cases (3) and (4), $\overline{\rho_{\mathfrak{B}}} = \chi \oplus \chi$, and in cases (5) and (6), $\overline{\rho_{\mathfrak{B}}} = \chi \oplus \chi\omega$, where we consider characters of $\mathrm{G}_{\mathbb{Q}_p}$ as characters of $\mathbb{Q}_p^\times$ via local class field theory, normalized so that uniformizers correspond to geometric Frobenii. We note that the determinant of $\overline{\rho_{\mathfrak{B}}}$ is equal to $\zeta\varepsilon$ modulo $\bar{\omega}$, where ε is the p-adic cyclotomic character.

3 Vanishing results for $\mathrm{Ext}^\bullet$ groups

We keep the notation from the previous sections: E is a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$, uniformizer $\bar{\omega}$, and residue field κ.

3.1 Two-dimensional pseudocharacters

We begin by reviewing the definition and some properties of two-dimensional pseudocharacters. Since these facts are standard, we quote much of this and the following paragraph directly from [21, Appendix A]. Let $\mathcal{G}$ denote a profinite group, and let $(A, \mathfrak{m}_A)$ be a local artinian $\mathcal{O}$-algebra. Throughout, we assume that $p > 2$. A continuous map $T : \mathcal{G} \to A$ is defined as a two-dimensional A-valued pseudocharacter if it adheres to the following conditions:

(1) $T(1) = 2$;
(2) $T(gh) = T(hg)$ for every $g, h \in \mathcal{G}$;
(3) the relation $T(g)T(h)T(k) - T(g)T(hk) - T(h)T(gk) - T(k)T(gh) + T(ghk) + T(gkh) = 0$ holds for all $g, h, k \in \mathcal{G}$.

It is a well-known result that for any continuous representation $\rho : \mathcal{G} \to \mathrm{GL}_2(A)$, the associated trace function $\mathrm{tr}\rho$ constitutes a two-dimensional pseudocharacter. Given a two-dimensional pseudocharacter $T : \mathcal{G} \to A$, one can define a continuous group homomorphism $\mathcal{G} \to A^\times$ via the formula $D(g) := \frac{T(g)^2 - T(g^2)}{2}$ (see [5, Proof of Proposition 7.23]). As shown in [5, Lemma 7.7 and Proposition 7.23], the mapping

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$T \mapsto (T, D)$ establishes a bijection between the set of two-dimensional pseudocharacters and the set of pairs (T, D), where $D : \mathcal{G} \to A^\times$ is a continuous group homomorphism and $T : \mathcal{G} \to A$ is a continuous function which satisfies: $T(1) = 2$, $T(gh) = T(hg)$, and the identity $D(g)T(g^{-1}h) - T(g)T(h) + T(gh) = 0$ for all $g, h \in \mathcal{G}$.

Consider a continuous representation $\rho : \mathcal{G} \to \mathrm{GL}_2(\kappa)$. Let $D^{\mathrm{ps}}_{\mathrm{tr}\rho}$ denote the functor from local artinian augmented $\mathcal{O}$-algebras with residue field κ to the category of sets, where $D^{\mathrm{ps}}_{\mathrm{tr}\rho}(A)$ consists of all two-dimensional A-valued pseudocharacters T lifting $\mathrm{tr}\rho$, that is, satisfying the condition $T \equiv \mathrm{tr}\rho \pmod{\mathfrak{m}_A}$. Provided that $\mathrm{Hom}^{cont}(\mathcal{H}, \mathbb{F}_p)$ is a finite-dimensional $\mathbb{F}_p$-vector space for every open subgroup $\mathcal{H}$ of $\mathcal{G}$, this functor $D^{\mathrm{ps}}_{\mathrm{tr}\rho}$ is pro-represented by a complete local Noetherian $\mathcal{O}$-algebra. Notably, this finiteness criterion is met when $\mathcal{G}$ is the absolute Galois group of a local field.

We usually work with a modified version where the determinant is fixed. Let $\psi : \mathcal{G} \to \mathcal{O}^\times$ be a continuous character that lifts $\det\rho$. We define $D^{\mathrm{ps},\psi}_{\mathrm{tr}\rho}$ as the subfunctor of $D^{\mathrm{ps}}_{\mathrm{tr}\rho}$, where $T \in D^{\mathrm{ps},\psi}_{\mathrm{tr}\rho}(A)$ if and only if $\frac{T(g)^2 - T(g^2)}{2}$ coincides with (the image of) $\psi(g)$ for all $g \in \mathcal{G}$. We refer to $D^{\mathrm{ps},\psi}_{\mathrm{tr}\rho}$ as a deformation problem with a fixed determinant. If $R^{\mathrm{ps}}_{\mathrm{tr}\rho}$ pro-represents $D^{\mathrm{ps}}_{\mathrm{tr}\rho}$, it follows that $D^{\mathrm{ps},\psi}_{\mathrm{tr}\rho}$ is pro-represented by a quotient of this ring, which we denote by $R^{\mathrm{ps},\psi}_{\mathrm{tr}\rho}$.

Recall that for the identity functor $I_{\mathcal{C}}$ of any category $\mathcal{C}$, the natural transformations $\eta : I_{\mathcal{C}} \to I_{\mathcal{C}}$ form a commutative monoid, called the center of the category $\mathcal{C}$. The following theorem characterizes the center of the category $\mathrm{Mod}^{\mathrm{lfin}}_{\mathrm{G},\psi}(\mathcal{O})^{\mathfrak{B}}$, as established in [22, Theorem 1.3] (refer also to [21]).

Theorem 3.1 *Suppose the block $\mathfrak{B}$ is given by (1) or (2) above. There exists a natural isomorphism between the center of the category $\mathrm{Mod}^{\mathrm{lfin}}_{\mathrm{G},\psi}(\mathcal{O})^{\mathfrak{B}}$ and the ring $R^{\mathrm{ps},\psi}_{\mathrm{tr}\overline{\rho}_{\mathfrak{B}}}$.*

3.2 Block decomposition in the Banach setting

In the context of Banach space representation, for a block $\mathfrak{B}$, let $\mathrm{Ban}^{\mathrm{adm}}_{\mathrm{G},\psi}(E)^{\mathfrak{B}}$ be the full subcategory of $\mathrm{Ban}^{\mathrm{adm}}_{\mathrm{G},\psi}(E)$ consisting of those Π, such that for some (equivalently any) open bounded G-invariant lattice Θ in Π, all the irreducible subquotients of $\Theta \otimes_{\mathcal{O}} \kappa$, which is an object of $\mathrm{Mod}^{\mathrm{lfin}}_{\mathrm{G},\psi}(\mathcal{O})$, lie in $\mathfrak{B}$. It is shown in [21, Proposition 5.36] that $\mathrm{Ban}^{\mathrm{adm}}_{\mathrm{G},\psi}(E)$ decomposes into a direct sum of subcategories:

$$\mathrm{Ban}^{\mathrm{adm}}_{\mathrm{G},\psi}(E) \cong \bigoplus_{\mathfrak{B}} \mathrm{Ban}^{\mathrm{adm}}_{\mathrm{G},\psi}(E)^{\mathfrak{B}}, \tag{3.1}$$

where the sum runs over the set of all blocks $\mathfrak{B}$ of $\mathrm{Mod}^{\mathrm{lfin}}_{\mathrm{G},\psi}(\mathcal{O})$.

To understand the finer structure of these blocks, we first make explicit the action of the generic fiber of the relevant pseudodeformation ring. For any $\Pi \in \mathrm{Ban}^{\mathrm{adm}}_{\mathrm{G},\psi}(E)^{\mathfrak{B}}$ equipped with an open bounded G-invariant lattice Θ, the quotients Θ/ϖ^n are objects of $\mathrm{Mod}^{\mathrm{lfin}}_{\mathrm{G},\psi}(\mathcal{O})^{\mathfrak{B}}$ for all $n \geq 1$. Theorem 3.1 gives a natural action of $R^{\mathrm{ps},\psi}_{\mathrm{tr}\overline{\rho}_{\mathfrak{B}}}$ on Θ/ϖ^n. Upon taking the inverse limit and inverting p, we obtain a natural homomorphism $R^{\mathrm{ps},\psi}_{\mathrm{tr}\overline{\rho}_{\mathfrak{B}}}[1/p] \to \mathrm{End}^{\mathrm{cont}}_{\mathrm{G}}(\Pi)$.

This action allows us to decompose the finite length objects. The Chinese remainder theorem, see [21, Theorem 4.36], yields an equivalence of categories:

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$$\mathrm{Ban}_{G,\psi}^{\mathrm{adm.fl}}(E)^{\mathfrak{B}} \cong \bigoplus_{\mathfrak{n}\in \mathrm{maxSpec} R^{\mathrm{ps},\psi}_{\mathrm{tr}\overline{\rho}_{\mathfrak{B}}}[1/p]} \mathrm{Ban}_{G,\psi}^{\mathrm{adm.fl}}(E)^{\mathfrak{B}}_{\mathfrak{n}}, \tag{3.2}$$

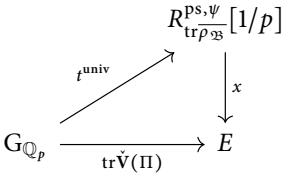
where, for a maximal ideal $\mathfrak{n}$ of $R^{\mathrm{ps},\psi}_{\mathrm{tr}\overline{\rho}_{\mathfrak{B}}}[1/p]$, $\mathrm{Ban}_{G,\psi}^{\mathrm{adm.fl}}(E)^{\mathfrak{B}}_{\mathfrak{n}}$ consists of those finite length representations that are killed by a power of $\mathfrak{n}$.

3.3 Proof of main theorem 1.1

Consider an absolutely irreducible representation $\Pi \in \mathrm{Ban}_{G,\psi}^{\mathrm{adm}}(E)^{\mathfrak{B}}$. As established in the previous section, we have a natural homomorphism $x : R^{\mathrm{ps},\psi}_{\mathrm{tr}\overline{\rho}_{\mathfrak{B}}}[1/p] \to \mathrm{End}_G^{\mathrm{cont}}(\Pi)$. Because Π is absolutely irreducible, Schur's Lemma [21, Corollary 4.42] implies the isomorphism $\mathrm{End}_G^{\mathrm{cont}}(\Pi) \cong E$.

Finally, assuming the block $\mathfrak{B}$ corresponds to types (1) or (2) as defined in the preceding section, [22, Corollary 2.25] provides the subsequent commutative diagram:

(3.3)



where t^{univ} is the universal pseudocharacter of $G_{\mathbb{Q}_p}$. Let $\mathfrak{n}$ be the maximal ideal of $R^{\mathrm{ps},\psi}_{\mathrm{tr}\overline{\rho}_{\mathfrak{B}}}[1/p]$ corresponding to Π in the above diagram. Then, by the construction of the specialization map x, we have $\mathfrak{n} \subseteq \mathrm{Ann}_{R^{\mathrm{ps},\psi}_{\mathrm{tr}\overline{\rho}_{\mathfrak{B}}}[1/p]}(\Pi)$. Therefore, Π is an element of $\mathrm{Ban}_{G,\psi}^{\mathrm{adm.fl}}(E)^{\mathfrak{B}}_{\mathfrak{n}}$.

Proof Since π and $\Pi(V)$ are absolutely irreducible, their continuous duals π^* and $\Pi(V)^*$ are also absolutely irreducible. Therefore, by [9, Theorem 1.4 and Corollary 1.6], they are of finite length. Now if π^* and $\Pi(V)^*$ are in $\mathrm{Ban}_{G,\zeta}^{\mathrm{adm}}(E)^{\mathfrak{B}}$ for different $\mathfrak{B}$, then by the block decomposition (3.1), we are done. Assume they lie in $\mathrm{Ban}_{G,\zeta}^{\mathrm{adm}}(E)^{\mathfrak{B}}$ for the same block $\mathfrak{B}$. By Lemma 2.1, we have $\check{\mathbf{V}}(\Pi(V)^*) = V$, which is an absolutely irreducible representation of dimension 2. Set $r := \check{\mathbf{V}}(\pi^*)$, which is also an absolutely irreducible E-representation of $G_{\mathbb{Q}_p}$. Let $\mathfrak{n}$ be the maximal ideal of $R^{ps,\zeta\varepsilon}_{\mathrm{tr}\overline{\rho}_{\mathfrak{B}}}[1/p]$ corresponding to $\mathrm{tr}\, V$, and let $\mathfrak{m}$ be the maximal ideal of $R^{ps,\zeta\varepsilon}_{\mathrm{tr}\overline{\rho}_{\mathfrak{B}}}[1/p]$ corresponding to $\mathrm{tr}\, r$. Since $\overline{V}$ is generic, the above discussion implies $\Pi(V)^* \in \mathrm{Ban}_{G,\zeta}^{\mathrm{adm}}(E)^{\mathfrak{B}}_{\mathfrak{n}}$ and $\pi^* \in \mathrm{Ban}_{G,\zeta}^{\mathrm{adm}}(E)^{\mathfrak{B}}_{\mathfrak{m}}$. By the Brauer–Nesbitt theorem, the trace character is determined by the semisimplification, so $\mathrm{tr}\, V$ and $\mathrm{tr}\, r$ depend only on $V^{ss} \cong V$ and $r^{ss} \cong r$, respectively. Thus, $\mathfrak{n} = \mathfrak{m}$ can occur only when $V \cong r$.

If π is ordinary, then by the isomorphisms in (2.1), r is one-dimensional, so r can never be isomorphic to V. When π is non-ordinary, the first isomorphism in (2.1) tells us $r = \mathbf{V}(\pi)$. Therefore, if $\mathbf{V}(\pi) \neq V$, then [24, Corollary 6.8] yields $\mathrm{Ext}^i_{G,\zeta}(\pi^*, \Pi(V)^*) = 0$ for all $i \geq 0$.

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For the case $\mathbf{V}(\pi) = V = \mathbf{V}(\Pi(V))$, we have the equality $\mathfrak{n} = \mathfrak{m}$ between the maximal ideals of $R^{ps,\zeta_\varepsilon}_{\mathrm{tr}\,\overline{\rho}_{\mathfrak{B}}}[1/p]$. Following the proof of [22, Corollary 2.26], we get that $\mathrm{Ban}^{\mathrm{adm.fl}}_{\mathrm{G},\zeta}(E)^{\mathfrak{B}}_{\mathfrak{n}}$ is anti-equivalent to the category of A-modules of finite length, where A denotes the completion of $R^{ps,\zeta_\varepsilon}_{\mathrm{tr}\,\overline{\rho}_{\mathfrak{B}}}[1/p]$ at $\mathfrak{n}$. Via this anti-equivalence, the objects π^* and $\Pi(V)^*$ are both identified with the residue field $\kappa(\mathfrak{n})$ of A. Consequently, we obtain the identification:

$$\mathrm{Ext}^i_{\mathrm{G},\zeta}(\pi^*, \Pi(V)^*) \cong \mathrm{Ext}^i_A(\kappa(\mathfrak{n}), \kappa(\mathfrak{n})).$$

Moreover, in the proof of [22, Corollary 2.26], it is shown that A is a complete regular local ring of dimension 3. Using the Koszul complex, we compute $\mathrm{Ext}^i_A(\kappa(\mathfrak{n}), \kappa(\mathfrak{n}))$, and by [26, Exercise 4.5.6], this group is a $\binom{3}{i}$-dimensional $\kappa(\mathfrak{n})$-vector space. As the residue field $\kappa(\mathfrak{n})$ is identified with E via the specialization map x, we have the required dimension results. ∎

Remark 3.2 In the case that V is residually irreducible with π ordinary, the vanishing of $\mathrm{Ext}^1_{\mathrm{G},\zeta}$ can be proved by purely automorphic methods using the results of [21, Section 4] and [22, Lemma 2.16].

4 Rational cohomology of Drinfeld space

We recall the construction of the Drinfeld spaces as described in [7, Section 0.1]. For $\ell \neq p$, works of Faltings, Fargues, Harris, and Taylor establish that the étale cohomology of the Drinfeld spaces encodes both the classical local Langlands and the classical Jacquet–Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$. It is expected that the p-adic étale cohomology groups similarly encode the hypothetical p-adic local Langlands.

Let $\mathrm{G} := \mathrm{GL}_2(\mathbb{Q}_p)$, D denote the nonsplit quaternion algebra with center $\mathbb{Q}_p$, and $\check{\mathrm{G}}$ be the group of invertible elements of D. We write $\mathcal{O}_D$ for the maximal order of D, and ϖ_D is a uniformizer of $\mathcal{O}_D$. The level structure is given by the sequence of subgroups

$$\check{\mathrm{G}}_n := \begin{cases} \mathcal{O}_D^\times & \text{if } n = 0, \\ 1 + \varpi_D^n \mathcal{O}_D & \text{if } n \geq 1. \end{cases}$$

Let $\Omega_{Dr,p} := \mathbb{P}^1_{\mathbb{Q}_p} - \mathbb{P}^1(\mathbb{Q}_p)$ denote Drinfeld's p-adic upper half-plane. In [11], Drinfeld introduced certain covers $\check{\mathcal{M}}_n$ of $\Omega_{Dr,p}$. This covering is defined over $\check{\mathbb{Q}}_p := \widehat{\mathbb{Q}_p^{nr}}$ and the action of the Weil group $W_{\mathbb{Q}_p}$ is compatible with the natural action of $\check{\mathbb{Q}}_p$. There is a natural covering map $\check{\mathcal{M}}_{n+1} \to \check{\mathcal{M}}_n \to \Omega_{Dr,p}$ compatible with the action of G and $\check{\mathrm{G}}$. The zeroth level of this tower is given by $\check{\mathcal{M}}_0 = \mathbb{Z} \times \Omega_{\mathrm{Dr},p}$, while for $n \geq 1$, the space $\check{\mathcal{M}}_n$ is a Galois cover of $\check{\mathcal{M}}_0$, with the Galois group $\check{\mathrm{G}}_0/\check{\mathrm{G}}_n = \mathcal{O}_D^\times/(1 + \varpi_D^n \mathcal{O}_D)$. Define $\mathcal{M}_{n,\mathbb{C}_p} := \mathbb{C}_p \times_{\check{\mathbb{Q}}_p} \check{\mathcal{M}}_n$ and let $\mathcal{M}_\infty$ denote the (non-complete) projective limit of $\mathcal{M}_{n,\mathbb{C}_p}$. We now study the structure of the p-adic étale cohomology of the Drinfeld spaces as a G-representation.

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As before, let E be a finite extension of $\mathbb{Q}_p$ with the residue field κ. Fix a de Rham representation V of $\mathrm{G}_{\mathbb{Q}_p} := \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ of dimension 2 with Hodge–Tate weights in $(0,1)$. One can associate to V the filtered $E-(\varphi, N, \mathrm{G}_{\mathbb{Q}_p})$-module $\mathbf{D}_{\mathrm{pst}}(V)$, a rank-two module over $E \otimes_{\mathbb{Q}_p} \mathbb{Q}_p^{\mathrm{nr}}$. By Fontaine's construction [15], we have the associated Weil–Deligne representation $\mathrm{WD}(V) := \mathrm{WD}(\mathbf{D}_{\mathrm{pst}}(V))$ of dimension 2 over E. We write $\mathrm{LL}(V) := \mathrm{LL}(\mathrm{WD}(V))$ for the irreducible smooth E-representation of G arising via the classical local Langlands correspondence. It is an infinite-dimensional G-representation and may be recovered as the space of G-smooth vectors inside the p-adic local Langlands correspondence $\Pi(V)$. We say that V is supercuspidal if $\mathrm{WD}(V)$ is irreducible (in which case $\mathrm{LL}(V)$ is supercuspidal). This implies, in particular, that $N = 0$ on $\mathbf{D}_{\mathrm{pst}}(V)$ (i.e., V is potentially crystalline). Likewise, we denote by $\mathrm{JL}(V) := \mathrm{JL}(\mathrm{LL}(V))$ the smooth irreducible E-representation of $\check{\mathrm{G}}$, obtained via the Jacquet–Langlands correspondence. The representations $\mathrm{WD}(V)$, $\mathrm{LL}(V)$, and $\mathrm{JL}(V)$ depend only on the $E-(\varphi, N, \mathrm{G}_{\mathbb{Q}_p})$-module $\mathbf{D}_{\mathrm{pst}}(V)$, whereas $\Pi(V)$ genuinely depends on V itself. Unlike $\mathrm{LL}(V)$, the representation $\mathrm{JL}(V)$ is finite-dimensional over E.

The following fundamental result of Colmez–Dospinescu–Nizioł([7, Theorem 0.2]; see also [8, Theorem 0.14]) describes the V-isotypic component of the first p-adic étale cohomology of the finite-level Drinfeld spaces as a $\mathrm{G} \times \check{\mathrm{G}}$-representation.

Theorem 4.1 *Let V be an absolutely irreducible E-representation of $\mathrm{G}_{\mathbb{Q}_p}$ of dimension ≥ 2.*

(i) *If V is supercuspidal of dimension* 2 *and of Hodge–Tate weights* 0 *and* 1*, then*

$$\mathrm{Hom}_{\mathrm{W}_{\mathbb{Q}_p}}\left(V,\, E \otimes_{\mathbb{Q}_p} \mathrm{H}^1_{\acute{e}t}(\mathcal{M}_\infty, \mathbb{Q}_p(1))\right) = \mathrm{JL}(V) \otimes_E \Pi(V)^*.$$

(ii) *In all other cases,*

$$\mathrm{Hom}_{\mathrm{W}_{\mathbb{Q}_p}}\left(V,\, E \otimes_{\mathbb{Q}_p} \mathrm{H}^1_{\acute{e}t}(\mathcal{M}_\infty, \mathbb{Q}_p(1))\right) = 0.$$

Proof of Corollary 1.2: As $\mathrm{JL}(V)$ is a finite-dimensional E-vector space, from the above Theorem 4.1, we obtain the following isomorphism of G-representations:

$$\mathrm{H}^1_{\mathrm{et}}(\mathcal{M}_\infty, E(1))[V] \cong (\Pi(V)^*)^{\oplus r}, \tag{4.1}$$

where V is an absolutely irreducible E-representation of $\mathrm{G}_{\mathbb{Q}_p}$ and

$$r = \begin{cases} \dim_E(\mathrm{JL}(V)) & \text{if } V \text{ is supercuspidal of dimension 2 and H-T weights 0 and 1,} \\ 0 & \text{otherwise.} \end{cases}$$

By the isomorphism (4.1), it is enough to prove that $\mathrm{Ext}^i_{\mathrm{G},\zeta}(\Pi(\rho_p)^*, \Pi(V)^*) = 0$ for all $i \geq 0$. From Emerton's description of $\Pi(\rho_p)$ for reducible ρ_p in Section 2.1, we get the exact sequence of duals:

$$0 \to \pi_2^* \to \Pi(\rho_p)^* \to \pi_1^* \to 0, \tag{4.2}$$

where π_1 and π_2 are (twists of) principal series representations on a p-adic Banach space. Applying $\mathrm{Hom}_{\mathrm{G}}(_, \Pi(V)^*)$ to the above short exact sequence, we get the

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following long exact sequence:

$$0 \to \cdots \to \mathrm{Ext}^i_{\mathrm{G},\zeta}(\pi_1^*, \Pi(V)^*) \to \mathrm{Ext}^i_{\mathrm{G},\zeta}(\Pi(\rho_p)^*, \Pi(V)^*) \to \mathrm{Ext}^i_{\mathrm{G},\zeta}(\pi_2^*, \Pi(V)^*) \to \dots$$

Using Theorem 1.1 on this exact sequence, we get the result.

Remark 4.2 Recently, Vanhaecke [25, Théorème 1.1] extended Theorem 4.1 to arbitrary weights by working with the p-adic étale cohomology with coefficients in the symmetric powers of the universal local system on the Drinfeld tower. In this broader setting, one readily verifies that Corollary 1.2 continues to hold.

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