J. Plasma Physics (1998), *vol.* 60, *part* 1, *pp.* 133–149. Printed in the United Kingdom
© 1998 Cambridge University Press

Magnetohydrodynamic equations for plasmas with finite-Larmor-radius effects

IGOR O. POGUTSE, ANDREI I. SMOLYAKOV and AKIRA HIROSE

Department of Physics and Engineering Physics, University of Saskatchewan, Saskatoon, Saskatoon, Canada S7N 5E2

(Received 10 July 1997 and in revised form 14 October 1997)

It is shown that the truncation of the infinite hierarchy of fluid equations obtained as moments of the Vlasov kinetic equation leads to a system of nonlinear equations that describe finite-Larmor-radius effects with good accuracy. Inertial terms in the momentum balance, viscosity and heat-flux evolution equations are crucial for a uniform description of the plasma response with an arbitrary Larmor radius. In the low-frequency ordering, the obtained equations are simplified by an expansion in the parameter $1/B$, where B is the equilibrium magnetic field. The results of the second-order $\mathcal{O}(1/B^2)$ and the fourth-order $\mathcal{O}(1/B^4)$ closures are compared. It is shown that the accuracy of the description improves for higher-order closures.

1. Introduction

The magnetohydrodynamic (MHD) description of a magnetized plasma has proved to be an effective tool in investigations of plasma dynamics in both laboratory and space conditions. Conventional MHD equations (Kulsrud 1983) are typically used for low-frequency perturbations ($\omega \ll \omega_i$, where ω_i is the ion cyclotron frequency) with the characteristic perpendicular length $k_\perp^{-1}$ much larger than the ion Larmor radius ρ: $k_\perp \rho \ll 1$. There are, however, many physical situations where the effects of a finite Larmor radius are important and plasma models should adequately include short-wavelength perturbations with $k_\perp \rho \gtrsim 1$. In general, such effects have to be described by the methods of kinetic theory. A full kinetic treatment is often prohibitively complicated, especially in the nonlinear case. This has stimulated many efforts to formulate simplified (reduced) schemes to describe the macroscopic behaviour of plasmas allowing for finite-Larmor-radius effects. In this paper we formulate nonlinear fluid equations that describe the effects of a finite Larmor radius with sufficient accuracy for arbitrary values of the $k_\perp \rho$ parameter. The equations obtained are asymptotically correct in both the limits of small $k_\perp \rho \ll 1$ and large $k_\perp \rho \gg 1$.

There exist several approaches to derive reduced MHD models. The approach of the gyrokinetic equation exploits the small parameter $\omega/\omega_{ci} \ll 1$ in the original Boltzmann kinetic equation to reduce particle dynamics to that of guiding centres. Further simplification is achieved for long-wavelength perturbations by an asymptotic expansion in the small parameter $k_\perp^2 \rho^2$. Reduced MHD equations (Hazeltine

https://doi.org/10.1017/S0022377898006552 Published online by Cambridge University Press

and Meiss 1985) can then be derived by taking moments of the drift-kinetic equation. Similar equations can be derived directly from the system of moments of the Vlasov equation (Macmahon 1965; Hsu *et al.* 1986). Such equations are asymptotically correct in the limit of small Larmor radius $k_\perp^2\rho^2 < 1$, and have been successfully applied to studies of nonlinear plasma dynamics in the low-frequency long-wavelength limit (Hazeltine and Meiss 1985). The effects of large Larmor radius with $k_\perp^2\rho^2 \gtrsim 1$, however, cannot be described with this approach.

In the gyrokinetic theory, the full plasma kinetic equation is averaged over the fast gyromotion to derive the gyrokinetic equation which desribes full Larmor-radius effects for arbitrary values of $k_\perp\rho$. By taking moments of the gyrokinetic equation, one can obtain an infinite series of gyrofluid equations (Brizard 1992). Truncation or closure of this series requires additional assumptions about the distribution function; the assumption of being close to the Maxwellian distribution is typically used (Bernstein 1985; Dorland and Hammet 1993). The gyrofluid equations are formulated in the guiding-centre space, and require a complicated nonlocal transformation back to the particle configuration space. Therefore simpler models are still desirable to model the evolution of plasmas with arbitrary $k_\perp^2\rho^2$.

It has previously been noted that the behaviour of a plasma with large Larmor radius (LLR), $k_\perp^2\rho^2 \gtrsim 1$, can be described by the single-fluid MHD equations modified by Hall inertia effects (Hassam and Huba 1988). These equations have been found useful in investigating the linear Kelvin–Helmholtz (Opp and Hassam 1991) and Rayleigh–Taylor (Velikovich 1991) instabilities in space and laboratory plasmas. Most recently, MHD equations with inertial terms have been used to analyse the dynamics of the $m = 1$ internal kink mode in a tokamak in the ion-kinetic regime, $k_\perp\rho \gtrsim 1$ (Aydemir 1991; Zakharov and Rogers 1992; Kuvshinov 1994). Kuvshinov (1994) has demonstrated that the MHD equations with the ion inertia retained recover the linear kinetic result with high accuracy for arbitrary values of $k_\perp\rho$. Moreover, he has also shown that in the linear limit the MHD ion response corresponds to the Padé approximant to the exact expression obtained from the linearized Vlasov equation. Thus, the MHD treatment can describe Larmor-radius effects even in the substantially kinetic region where the characteristic perturbation scale is smaller than the ion Larmor radius. The MHD model for the sub-Larmor region has been generalized recently by Kuvshinov and Mikhailovskii (1996) to include effects of plasma inhomogeneities.

In this paper, we derive the MHD equations, allowing for arbitrary-Larmor-radius effects by simply truncating the hierarchy of moment equations without invoking any subsidiary asymptotic expansion in the parameter $k_\perp\rho$. This truncation is equivalent to the assumption that the deviation of the distribution function from the Maxwellian is small, and that this deviation can be represented by a finite series of moments. As noted above, such an assumption is typically made in the gyrofluid theory, where it is used to truncate an infinite hierarchy of the moment equations generated by the full finite-Larmor-radius operator in the gyrokinetic equation (Brizard 1992; Dorland and Hammet 1993). We apply this assumption and truncate the chain of standard MHD equations generated directly from the Boltzmann kinetic equation. With this approach, we work only with fluid equations written in the real configuration space and avoid the complicated problem of nonlocal transformation from the guiding-centre variables.

In the low-frequency limit ($\omega/\omega_c \ll 1$), the truncated system can be reduced by an $1/B$ expansion. As shown below, the reduced fluid equations reasonably ap-

https://doi.org/10.1017/S0022377898006552 Published online by Cambridge University Press

proximate the plasma response for arbitrary values of the parameter $k_\perp^2\rho^2$. At second-order, $\mathcal{O}(1/B)^2$,† this expansion corresponds to inclusion of the leading-order inertial plasma velocity and inertial heat flux (Kuvshinov 1994; Kuvshinov and Mikhailovskii 1996). This approximation correctly recovers the leading-order terms in both asymptotic regimes for $k_\perp^2\rho^2 \ll 1$, and $k_\perp^2\rho^2 \gg 1$, thus providing the two-point Padé-type approximant for the perturbed plasma density and temperature. We demonstrate that in a homogeneous plasma the next order expansion in the parameter $1/B$ corresponds to the retention of the inertial collisionless gyroviscosity tensor that is determined from Grad-type equations (Grad 1949). We derive the reduced MHD equations to the next order, $\mathcal{O}(1/B^4)$, and show that these equations correctly recover Larmor-radius effects to $\mathcal{O}(k_\perp^4\rho^4)$ and the leading term of the asymptotic expansion for $k_\perp^2\rho^2 \gg 1$. This particular model uniformly describes kinetic Alfvén waves in the whole region from short ($k_\perp^2\rho^2 \gg 1$) to long ($k_\perp^2\rho^2 \ll 1$) wavelength scales, which is important, for instance, for localization properties of kinetic Alfvén waves in a plasma with a finite ion temperature. It is worth noting that these MHD equations are nonlinear and can be useful for numerical MHD simulations (Sovinec *et al.* 1996).

The paper is organized as follows. In the next section, we give basic MHD moment equations of Grad type for the evolution of the plasma density, flow, pressure, viscosity tensor and heat flux. In Sec. 3, we discuss the basic reduction of the MHD equations in the parameter $1/B$. In Sec. 4, we derive the finite-Larmor-radius closures for homogeneous plasmas and compare them with the results of the linear kinetic analysis. Section 5 illustrates the derivation of an approximate dispersion relation for kinetic Alfvén waves. We summarize the results in Sec. 6.

2. Basic system of MHD moment equations

The general evolution equations for the plasma density, fluid velocity and plasma pressure are obtained as moments of the Vlasov kinetic equation

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla} f + \frac{e}{m}\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right)\cdot\frac{\partial f}{\partial \mathbf{v}} = 0. \tag{1}$$

Taking moments of this equation with weights $(1, \mathbf{v}, \frac{1}{2}m\mathbf{v}^2)$ we obtain the standard equations Braginskii (1963) for the plasma density, momentum and thermal energy:

$$\frac{\partial n}{\partial t} + \boldsymbol{\nabla}\cdot(n\mathbf{V}) = 0, \tag{2}$$

$$mn\frac{d\mathbf{V}}{dt} = en\left(\mathbf{E} + \frac{1}{c}\mathbf{V}\times\mathbf{B}\right) - \boldsymbol{\nabla} p - \boldsymbol{\nabla}\cdot\boldsymbol{\Pi}, \tag{3}$$

$$\frac{3}{2}\frac{\partial p}{\partial t} + \frac{5}{2}p\boldsymbol{\nabla}\cdot\mathbf{V} + \boldsymbol{\nabla}\cdot\mathbf{q} + \boldsymbol{\Pi} : \boldsymbol{\nabla}\mathbf{V} = 0. \tag{4}$$

The next order moment equations correspond to the Grad (1949) equations for the viscosity tensor $\boldsymbol{\Pi}$ and heat flux $\mathbf{q}$. We use the forms given by Mikhailovskii and

† Note that only even-order terms appear in the expressions for density and temperature: $(\widetilde{n}/n)_2 = \mathcal{O}(1/B^2)$, $(\widetilde{n}/n)_4 = \mathcal{O}(1/B^4), \ldots$; see Sec. 4.

https://doi.org/10.1017/S0022377898006552 Published online by Cambridge University Press

Tsypin (1984) for a uniform magnetic field:

$$\frac{d\mathbf{q}}{dt} + \frac{7}{5}\mathbf{q}\nabla\cdot\mathbf{V} + \frac{2}{5}(\nabla\mathbf{V})\cdot\mathbf{q} + \frac{7}{5}(\mathbf{q}\cdot\nabla)\mathbf{V} - \omega_c\mathbf{q}\times\mathbf{b} + \frac{5p}{2m}\nabla T + \frac{7}{2}\nabla\cdot\left(\frac{T}{m}\mathbf{\Pi}\right) - \frac{5T}{2m}\nabla\cdot\mathbf{\Pi} = 0, \tag{5}$$

$$\frac{d\mathbf{\Pi}}{dt} + \omega_c\,(\mathbf{b}\times\mathbf{\Pi} - \mathbf{\Pi}\times\mathbf{b}) + \mathbf{\Pi}\cdot\nabla\mathbf{V} + \{(\mathbf{\Pi}\cdot\nabla)\mathbf{V} + [(\mathbf{\Pi}\cdot\nabla)\mathbf{V}]^{\mathrm{T}} - \tfrac{2}{3}\mathbf{I}(\mathbf{\Pi}:\nabla\mathbf{V})\} + [p\nabla\mathbf{V} + p(\nabla\mathbf{V})^{\mathrm{T}} - \tfrac{2}{3}\mathbf{I}p\nabla\cdot\mathbf{V}] + \tfrac{2}{5}[\nabla\mathbf{q} + (\nabla\mathbf{q})^{\mathrm{T}} - \tfrac{2}{3}\mathbf{I}\nabla\cdot\mathbf{q}] = 0, \tag{6}$$

Here $\mathbf{b}$ is a unit vector along the equilibrium magnetic field, $\mathbf{I}$ is the unit tensor of the second rank, and superscript 'T' indicates a transposed tensor. While (2)–(4) are exact for a collisionless plasma, the equations for the heat flux and viscosity tensor are approximate since the higher-order moments were neglected in (5) and (6). As discussed below, the higher-order terms can be included to further improve the accuracy of the proposed closure scheme.

The general hierarchy of moment equations can be obtained from the expansion of the distribution function in terms of spherical functions (Shkarovsky *et al.* 1996; Balescu 1988):

$$f = \sum_{mnl}^{\infty} a_{mnl}Y_{lm}(\theta,\phi)L_n^{(l+1/2)}\left(\frac{mv^2}{2T}\right), \tag{7}$$

where $Y_{lm}(\theta,\phi)$ are spherical functions, $L_i^{(l+1/2)}(mv^2/2T)$ are Laguerre polynomials with argument $mv^2/2T$, v,θ and ϕ are the components of the velocity vector $\mathbf{v}$ in the spherical coordinate system, and $a_{mnl} = a_{mnl}(\mathbf{r},t)$ are the moment functions. It is convenient to rearrange (7) as a hierarchy of fluid moments (Callen 1989)

$$f = n_0\left(\frac{m}{2\pi T_0}\right)^{3/2}\exp\left(-\frac{mv^2}{2T_0}\right)\left[1 + \sum_{i=0}^{\infty}a_iL_i^{(1/2)} + \frac{2\mathbf{v}}{v_T^2}\cdot\sum_{i=0}^{\infty}\mathbf{u}_iL_i^{(3/2)} + \frac{12}{5mnv_T^4}(\mathbf{vv} - \tfrac{1}{3}v^2\mathbf{I}) : \sum_{i=0}^{\infty}\mathbf{P}_iL_i^{(5/2)} + \ldots + \right]. \tag{8}$$

Here a_i are the scalar moments, $a_0 = n/n_0$, $a_1 = T/T_0$; $\mathbf{u}_i$ are the vector moments, $\mathbf{u}_0 = \mathbf{V}$, $\mathbf{u}_1 = -2\mathbf{q}/5p$; and $\mathbf{P}_i$ are the second-rank tensor moments, $\mathbf{P}_0 = \mathbf{\Pi}$. Equations (2)–(6) are obtained by using the n, p, $\mathbf{V}$, $\mathbf{q}$ and $\mathbf{\Pi}$ sets of moments; all other higher-order moments are omitted.

3. Reduction of MHD equations

For the purpose of an analytical treatment, we simplify our basic moment equations by neglecting some nonlinear terms in (2)–(5)

$$\frac{3}{2}\frac{dp}{dt} + \frac{5}{2}p\nabla\cdot\mathbf{V} + \nabla\cdot\mathbf{q} = 0, \tag{9}$$

$$\frac{d\mathbf{\Pi}}{dt} + \omega_c\,(\mathbf{b}\times\mathbf{\Pi} - \mathbf{\Pi}\times\mathbf{b}) + [p\nabla\mathbf{V} + p(\nabla\mathbf{V})^{\mathrm{T}} - \tfrac{2}{3}\mathbf{I}p\nabla\cdot\mathbf{V}] + \tfrac{2}{5}[\nabla\mathbf{q} + (\nabla\mathbf{q})^{\mathrm{T}} - \tfrac{2}{3}\mathbf{I}\nabla\cdot\mathbf{q}] = 0, \tag{10}$$

https://doi.org/10.1017/S0022377898006552 Published online by Cambridge University Press

$$\frac{d\mathbf{q}}{dt} - \omega_c \mathbf{q} \times \mathbf{b} + \frac{5p}{2m}\boldsymbol{\nabla} T + \frac{T}{m}\boldsymbol{\nabla}\cdot\boldsymbol{\Pi} = 0. \tag{11}$$

The nomenclature $(\ldots)^{(\alpha)} = \mathcal{O}(1/B^{\alpha})$ is conveniently introduced to proceed with the $1/B$ expansions of (2), (3) and (9)–(11). Equations (3) and (11) are iterated as follows:

$$\mathbf{V}^{(n+1)} = \frac{1}{mn\omega_c}\mathbf{b} \times \left(mn\frac{d\mathbf{V}}{dt} - en\mathbf{E} + \boldsymbol{\nabla} p + \boldsymbol{\nabla}\cdot\boldsymbol{\Pi} \right)^{(n)}, \tag{12}$$

$$\mathbf{q}^{(n+1)} = \frac{1}{\omega_c}\mathbf{b} \times \left(\frac{d\mathbf{q}}{dt} + \frac{5p}{2m}\boldsymbol{\nabla} T + \frac{T}{m}\boldsymbol{\nabla}\cdot\boldsymbol{\Pi} \right)^{(n)}. \tag{13}$$

There are a number of important differences between the diagonal and off-diagonal components of the viscosity tensor. In particular, the successive iterations are different for diagonal and off-diagonal components. The explicit equations for the components of the viscosity tensor in a Cartesian coordinate system (x, y, z), with the magnetic field along the z axis, are

$$\frac{d\Pi_{xy}}{dt} + \omega_c\left(\Pi_{xx} - \Pi_{yy}\right) + p\left(\frac{\partial V_x}{\partial y} + \frac{\partial V_y}{\partial x}\right) + \frac{2}{5}\left(\frac{\partial q_x}{\partial y} + \frac{\partial q_y}{\partial x}\right) = 0, \tag{14}$$

$$\frac{d}{dt}\Pi_{xx} - 2\omega_c\Pi_{xy} + p\left(2\frac{\partial V_x}{\partial x} - \frac{2}{3}\boldsymbol{\nabla}\cdot\mathbf{V}\right) + \frac{2}{5}\left(2\frac{\partial q_x}{\partial x} - \frac{2}{3}\boldsymbol{\nabla}\cdot\mathbf{q}\right) = 0, \tag{15}$$

$$\frac{d}{dt}\Pi_{yy} + 2\omega_c\Pi_{xy} + p\left(2\frac{\partial V_y}{\partial y} - \frac{2}{3}\boldsymbol{\nabla}\cdot\mathbf{V}\right) + \frac{2}{5}\left(2\frac{\partial q_y}{\partial y} - \frac{2}{3}\boldsymbol{\nabla}\cdot\mathbf{q}\right) = 0. \tag{16}$$

For calculations of the diagonal components it is convenient to introduce the following combinations of the components Π_{xx} and Π_{yy}:

$$\frac{d}{dt}\left(\Pi_{xx} - \Pi_{yy}\right) - 4\omega_c\Pi_{xy} + 2p\left(\frac{\partial V_x}{\partial x} - \frac{\partial V_y}{\partial y}\right) + \frac{4}{5}\left(\frac{\partial q_x}{\partial x} - \frac{\partial q_y}{\partial y}\right) = 0, \tag{17}$$

$$\frac{d}{dt}\left(\Pi_{xx} + \Pi_{yy}\right) + \tfrac{2}{3}\left(p\boldsymbol{\nabla}\cdot\mathbf{V} + \tfrac{2}{5}\boldsymbol{\nabla}\cdot\mathbf{q}\right) = 0. \tag{18}$$

Equation (18) demonstrates an interesting property of the diagonal components of the viscosity tensor. It follows that, in general, $\Pi_{xx} + \Pi_{yy}$ is not zero, but rather is determined by the total compressibility of the plasma velocity and heat flux. For the standard gyroviscosity tensor Braginskii (1963), we have $\Pi_{xx} = -\Pi_{yy}$. In (18), $\Pi_{xx} + \Pi_{yy} \neq 0$ because of the finite compressibility of the plasma velocity and heat flux. As shown below, the compressibility of the plasma flow in the uniform magnetic field is due to plasma inertia, and therefore is an effect of first-order in ω/ω_c. Such effects were not considered in Braginskii (1963). The viscosity tensor $\boldsymbol{\Pi}$ has zero trace, $\Pi_{xx} + \Pi_{yy} + \Pi_{zz} = 0$; hence, because of finite plasma compressibility, the component Π_{zz} of the viscosity tensor is finite even for two-dimensional perturbations. For the purely two-dimensional case, the two-dimensional plasma pressure $p_\perp = p + \frac{1}{2}(\Pi_{xx} + \Pi_{yy})$ can be introduced so that the residual two-dimensional gyroviscosity tensor $\boldsymbol{\Pi}'$ (Kogan *et al.* 1965) restores the property $\Pi'_{xx} = -\Pi'_{yy}$. Note that the compressibility term (the term in large parentheses in (18)) vanishes identically owing to diamagnetic cancellation (Tsai *et al.* 1970) even in a plasma with inhomogeneous density and temperature. This term is finite for an inhomogeneous magnetic field, where it gives rise the Chew–Goldberger–Law type of evolution equation for $\Pi_{xx} + \Pi_{yy}$ (Chew *et al.* 1956; Ilgisonis 1992, 1995).

https://doi.org/10.1017/S0022377898006552 Published online by Cambridge University Press

Equation (17) gives the following expansion for the off-diagonal component of the viscosity tensor:

$$\frac{d}{dt}\left(\Pi_{xx}-\Pi_{yy}\right)^{(n)}-4\omega_c\Pi_{xy}^{(n+1)}+2p\left(\frac{\partial V_x}{\partial x}-\frac{\partial V_y}{\partial y}\right)^{(n)}+\frac{4}{5}\left(\frac{\partial q_x}{\partial x}-\frac{\partial q_y}{\partial y}\right)^{(n)}=0. \quad (19)$$

The antisymmetric part of the diagonal components is determined from the equation

$$\frac{d\Pi_{xy}^{(n)}}{dt}+\omega_c\left(\Pi_{xx}-\Pi_{yy}\right)^{(n+1)}+p\left(\frac{\partial V_x}{\partial y}+\frac{\partial V_y}{\partial x}\right)^{(n)}+\frac{2}{5}\left(\frac{\partial q_x}{\partial y}+\frac{\partial q_y}{\partial x}\right)^{(n)}=0. \quad (20)$$

Similarly to the plasma continuity and pressure equations, the equation for the symmetric part of the diagonal components has no large $1/B$ parameter. Its iterations are found from the equation

$$\frac{d}{dt}\left(\Pi_{xx}+\Pi_{yy}\right)^{(n)}+\tfrac{2}{3}\left(p\boldsymbol{\nabla}\cdot\mathbf{V}+\tfrac{2}{5}\boldsymbol{\nabla}\cdot\mathbf{q}\right)^{(n)}=0. \quad (21)$$

Using (12)–(20), one obtains the following expressions at the respective orders.

3.1. First order

The first-order plasma flow is obtained from (3) by neglecting inertial and viscous terms:

$$\mathbf{V}^{(1)}=\frac{c}{B}\mathbf{b}\times\boldsymbol{\nabla}\phi+\frac{1}{mn\omega_c}\mathbf{b}\times\boldsymbol{\nabla}p. \quad (22)$$

Note that we order the $\mathbf{E}\times\mathbf{B}$ and diamagnetic (the second term on the right-hand side of (22)) drifts to be of the same order. This assumption is essential to obtain the correct plasma response in the large-Larmor-radius limit, $k_\perp^2\rho^2\gg 1$. In the limit of $B\to 0$, the plasma is unmagnetized, and the expression (22) for $\mathbf{V}^{(1)}$ is formally divergent. A finite value of $\mathbf{V}^{(1)}$, is maintained, however, by the requirement $\widetilde{n}/n_0=-e\phi/T_0$ corresponding to Boltzmann equilibrium. (Note that in the limit $k_\perp^2\rho^2\gg 1$, perturbations of the plasma temperature are negligible.)

The first-order diamagnetic heat flux is obtained from (11).

$$\mathbf{q}^{(1)}=\frac{5cp_0}{2eB}\mathbf{b}\times\boldsymbol{\nabla}T. \quad (23)$$

Note that the first-order fluxes are incompressible for a homogeneous plasma:

$$\boldsymbol{\nabla}\cdot\mathbf{V}^{(1)}=\boldsymbol{\nabla}\cdot\mathbf{q}^{(1)}=0. \quad (24)$$

3.2. Second order

At second order, the plasma inertia (polarization) drift and inertial heat flux are obtained from (12) and (13) by using (22) and (23):

$$\mathbf{V}^{(2)}=\frac{1}{\omega_B}\mathbf{b}\times\frac{d}{dt}\mathbf{V}^{(1)}, \quad (25)$$

$$\mathbf{q}^{(2)}=\frac{1}{\omega_B}\mathbf{b}\times\frac{d}{dt}\mathbf{q}^{(1)}. \quad (26)$$

The plasma flow velocity and heat flux are compressible at this order. Their contributions to the continuity and energy-balance equations are described by the expressions

$$\boldsymbol{\nabla}\cdot\mathbf{V}^{(2)}=-\frac{c}{B\omega_c}\frac{d}{dt}\nabla_\perp^2\left(\phi+\frac{p}{en_0}\right), \quad (27)$$

https://doi.org/10.1017/S0022377898006552 Published online by Cambridge University Press

$$\boldsymbol{\nabla}\cdot\mathbf{q}^{(2)} = -\frac{5cp_0}{2eB\omega_c}\frac{d}{dt}\nabla_\perp^2 T. \tag{28}$$

The components of the viscosity tensor at second order can be determined from the equations

$$\Pi_{xy}^{(2)} = \frac{p}{2\omega_c}\left(\frac{\partial V_x^{(1)}}{\partial x} - \frac{\partial V_y^{(1)}}{\partial y}\right) + \frac{1}{5\omega_c}\left(\frac{\partial q_x^{(1)}}{\partial x} - \frac{\partial q_y^{(1)}}{\partial y}\right), \tag{29}$$

$$\Pi_{xx}^{(2)} - \Pi_{yy}^{(2)} = -\frac{p}{\omega_c}\left(\frac{\partial V_x^{(1)}}{\partial y} + \frac{\partial V_y^{(1)}}{\partial x}\right) - \frac{2}{5\omega_c}\left(\frac{\partial q_x^{(1)}}{\partial y} + \frac{\partial q_y^{(1)}}{\partial x}\right). \tag{30}$$

These equations are derived from (14) and (17) by neglecting the inertial terms. The sum of the second-order components is obtained from (18) by substituting the second-order expressions for the plasma velocity and the heat flux:

$$\frac{d}{dt}\left(\Pi_{xx}^{(2)} + \Pi_{yy}^{(2)}\right) + \tfrac{2}{3}\left(p\boldsymbol{\nabla}\cdot\mathbf{V}^{(2)} + \tfrac{2}{5}\boldsymbol{\nabla}\cdot\mathbf{q}^{(2)}\right) = 0. \tag{31}$$

Using (25) and (26), we find from (29)–(31)

$$\Pi_{xy}^{(2)} = -\frac{cp}{B\omega_c}\frac{\partial^2}{\partial x\,\partial y}\Phi - \frac{cp}{eB\omega_c}\frac{\partial^2}{\partial x\,\partial y}T, \tag{32}$$

$$\Pi_{xx}^{(2)} - \Pi_{yy}^{(2)} = -\frac{cp}{B\omega_c}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)\Phi - \frac{cp}{eB\omega_c}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)T, \tag{33}$$

$$\Pi_{xx}^{(2)} + \Pi_{yy}^{(2)} = \frac{2cp}{3B\omega_c}\nabla_\perp^2\Phi + \frac{2cp}{3eB\omega_c}\nabla_\perp^2 T. \tag{34}$$

Here the function $\Phi = \phi + p/en_0$ has been introduced.

The plasma compressibility shown in (31) and (34) is associated with inertial effects. In the limit of negligible inertia ($\omega/\omega_c \to 0$), the last term in (31) disappears and we recover the usual relation $\Pi_{xx} = -\Pi_{yy}$. In this approximation, the plasma viscosity is similar to that given by Braginskii with additional terms due to heat-flux gradients.

3.3. Third order

The third-order plasma flow and heat flux are determined by the viscosity effects, and are obtained by further iterations of (12) and (11):

$$\mathbf{V}^{(3)} = \frac{1}{mn\omega_c}\mathbf{b}\times\boldsymbol{\nabla}\cdot\boldsymbol{\Pi}^{(2)}, \tag{35}$$

$$\mathbf{q}^{(3)} = \frac{T}{m\omega_c}\mathbf{b}\times\boldsymbol{\nabla}\cdot\boldsymbol{\Pi}^{(2)}. \tag{36}$$

According to our expansion procedure, the inertial terms of the type

$$\mathbf{V}_p^{(n+1)} = \frac{1}{\omega_B}\mathbf{b}\times\frac{d}{dt}\mathbf{V}^{(n)}, \tag{37}$$

$$\mathbf{q}_p^{(n+1)} = \frac{1}{\omega_B}\mathbf{b}\times\frac{d}{dt}\mathbf{q}^{(n)}, \tag{38}$$

are neglected at all orders higher than the second. Such terms correspond to higher-order inertial effects, $\mathcal{O}((\omega/\omega_c)^n)$, $n > 1$, which are omitted in our low-frequency ordering.

https://doi.org/10.1017/S0022377898006552 Published online by Cambridge University Press

By using (32)–(34), we find that the plasma velocity and the heat flux are incompressible at this order:

$$\boldsymbol{\nabla}\cdot\mathbf{V}^{(3)}=\boldsymbol{\nabla}\cdot\mathbf{q}^{(3)}=0. \tag{39}$$

It follows from (39) and (18) that

$$\frac{d}{dt}\left(\Pi_{xx}^{(3)}+\Pi_{yy}^{(3)}\right)=0. \tag{40}$$

Then, the inertial gyroviscosity tensor is defined by the expressions

$$\Pi_{xy}^{(3)}=\frac{p}{2\omega_c}\left(\frac{\partial V_x^{(2)}}{\partial x}-\frac{\partial V_y^{(2)}}{\partial y}\right)+\frac{1}{5\omega_c}\left(\frac{\partial q_x^{(2)}}{\partial x}-\frac{\partial q_y^{(2)}}{\partial y}\right)+\frac{1}{4\omega_c}\frac{d}{dt}\left(\Pi_{xx}^{(2)}-\Pi_{yy}^{(2)}\right), \tag{41}$$

$$\Pi_{xx}^{(3)}=-\Pi_{yy}^{(3)}=-\frac{p}{2\omega_c}\left(\frac{\partial V_x^{(2)}}{\partial y}+\frac{\partial V_y^{(2)}}{\partial x}\right)-\frac{1}{5\omega_c}\left(\frac{\partial q_x^{(2)}}{\partial y}+\frac{\partial q_y^{(2)}}{\partial x}\right)-\frac{1}{\omega_c}\frac{d}{dt}\Pi_{xy}^{(2)}. \tag{42}$$

Note that for a homogeneous plasma, the gyroviscous plasma velocity and heat flux do not contribute to the continuity and energy-balance equations at this order. Therefore the standard gyroviscous cancellation, which is typical for a plasma with inhomogeneous density and temperature (Hazeltine and Meiss 1985), does not appear in the present formulation.

3.4. Fourth order

The fourth-order plasma velocity and heat flux are obtained from (12) and (13):

$$\mathbf{V}^{(4)}=\frac{1}{mn\omega_c}\mathbf{b}\times\boldsymbol{\nabla}\cdot\boldsymbol{\Pi}^{(3)}, \tag{43}$$

$$\mathbf{q}^{(3)}=\frac{T}{m\omega_c}\mathbf{b}\times\boldsymbol{\nabla}\cdot\boldsymbol{\Pi}^{(3)}. \tag{44}$$

By using (41) and (42), we obtain

$$\boldsymbol{\nabla}\cdot\mathbf{V}^{(4)}=\frac{11cT}{12m\omega_c^3B}\frac{d}{dt}\nabla_\perp^4\left(\phi+\frac{p}{en_0}\right)+\frac{11cT}{12em\omega_c^3B}\frac{d}{dt}\nabla_\perp^4T, \tag{45}$$

$$\boldsymbol{\nabla}\cdot\mathbf{q}^{(4)}=\frac{11cpT}{12m\omega_c^3B}\frac{d}{dt}\nabla_\perp^4\left(\phi+\frac{p}{en_0}\right)+\frac{11cpT}{12em\omega_c^3B}\frac{d}{dt}\nabla_\perp^4T. \tag{46}$$

4. Finite-Larmor-radius closures for a homogeneous plasma

In this section, we formulate a closed system of nonlinear MHD equations for the evolution of the plasma density and plasma pressure. These equations are obtained from the continuity and energy-balance equations with closures for the plasma velocity and the heat flux. The closure relations are written in terms of the plasma density and pressure and the electric potential. We consider two different closure schemes. In the second-order closure, the plasma velocity and heat flux are calculated to the second order in $1/B$, $\mathbf{V}^{(2)}\approx\mathbf{q}^{(2)}=\mathcal{O}(1/B^2)$. The fourth-order closure, in turn, uses the fourth-order expressions of the plasma velocity and heat flux, $\mathbf{V}^{(4)}\approx\mathbf{q}^{(4)}=\mathcal{O}(1/B^4)$. Only the first-order inertial terms, $\mathcal{O}(\omega/\omega_c)$, are taken into account.

https://doi.org/10.1017/S0022377898006552 Published online by Cambridge University Press

4.1. Second-order closure

The first-order fluxes are incompressible in a homogenous plasma, (24), and do not contribute to the plasma dynamics. The second-order reduced continuity and energy-balance equations are written as

$$\frac{\partial n}{\partial t} + \boldsymbol{\nabla} \cdot \left(n \mathbf{V}^{(2)} \right) = 0, \tag{47}$$

$$\frac{3}{2}\frac{dp}{dt} + \frac{5}{2} p \boldsymbol{\nabla} \cdot \mathbf{V}^{(2)} + \boldsymbol{\nabla} \cdot \mathbf{q}^{(2)} = 0. \tag{48}$$

By using the closure relations (27) and (28), we find the following equations:

$$\frac{\partial n}{\partial t} + \boldsymbol{\nabla} \cdot \left[\frac{n}{\omega_B} \mathbf{b} \times \frac{d}{dt} \left(\frac{c}{B} \mathbf{b} \times \boldsymbol{\nabla} \phi + \frac{1}{mn\omega_c} \mathbf{b} \times \boldsymbol{\nabla} p \right) \right] = 0, \tag{49}$$

$$\begin{aligned} \frac{3}{2}\frac{dp}{dt} + \frac{5}{2} p \boldsymbol{\nabla} \cdot \left[\frac{1}{\omega_B} \mathbf{b} \times \frac{d}{dt} \left(\frac{c}{B} \mathbf{b} \times \boldsymbol{\nabla} \phi + \frac{1}{mn\omega_c} \mathbf{b} \times \boldsymbol{\nabla} p \right) \right] \\ + \boldsymbol{\nabla} \cdot \left[\frac{1}{\omega_B} \mathbf{b} \times \frac{d}{dt} \left(\frac{5cp}{2eB} \mathbf{b} \times \boldsymbol{\nabla} T \right) \right] = 0. \end{aligned} \tag{50}$$

We note that by neglecting the temperature perturbations in (49), Kuvshinov (1994) has derived a nonlinear equation for the ion response,

$$\frac{\partial n}{\partial t} + \boldsymbol{\nabla} \cdot \left[\frac{n}{\omega_B} \mathbf{b} \times \frac{d}{dt} \left(\frac{c}{B} \mathbf{b} \times \boldsymbol{\nabla} \phi + \frac{T}{mn\omega_c} \mathbf{b} \times \boldsymbol{\nabla} n \right) \right] = 0, \tag{51}$$

which can apparently be cast into Hamiltonian form by an appropriate variable transformation (Kuvshinov *et al.* 1994; Schep *et al.* 1994).

Equations (49) and (50) describe the nonlinear evolution of the plasma density and temperature. In the linear limit, these equations reduce to the system

$$(1+z)\frac{\widetilde{n}}{n_0} + z\frac{\widetilde{T}}{T_0} = -z\frac{e\phi}{T_0}, \tag{52}$$

$$z\frac{\widetilde{n}}{n_0} + (\tfrac{3}{2} + \tfrac{7}{2}z)\frac{\widetilde{T}}{T_0} = -z\frac{e\phi}{T_0}, \tag{53}$$

where $z \equiv k^2\rho^2$. Solving these equations, we obtain the second-order expressions for the perturbed density and temperature:

$$\left(\frac{\widetilde{n}}{n_0}\right)_2 = -\frac{e\phi}{T_0}\frac{z(1+\frac{5}{3}z)}{1+\frac{10}{3}z+\frac{5}{3}z^2}, \tag{54}$$

$$\left(\frac{\widetilde{T}}{T_0}\right)_2 = -\frac{2e\phi}{3T_0}\frac{z}{1+\frac{10}{3}z+\frac{5}{3}z^2}. \tag{55}$$

In the limit of small length scales, $k^2\rho^2 \gg 1$, we obtain the asymptotic expressions

$$\left(\frac{\widetilde{n}}{n_0}\right)_2 \to -\frac{e\phi}{T_0}, \tag{56}$$

$$\left(\frac{\widetilde{T}}{T_0}\right)_2 \to 0. \tag{57}$$

https://doi.org/10.1017/S0022377898006552 Published online by Cambridge University Press

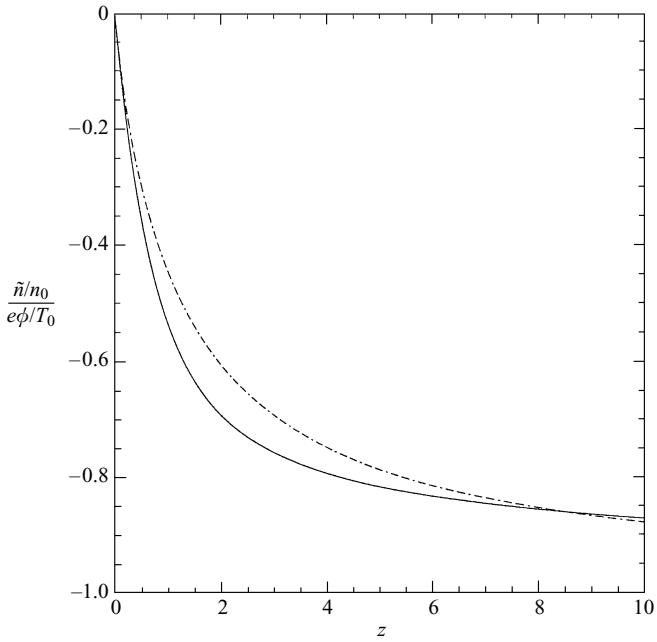


Figure 1. Normalized perturbation of the plasma density $(\widetilde{n}/n_0)/(e\phi/T_0)$ as a function of the parameter $z = k^2\rho^2$. The solid line is the exact kinetic expression (60), and the dot-dashed line is the second-order approximation (54).

In the long-wavelength limit, $k^2\rho^2 \ll 1$, the leading-order asymptotic forms are

$$\left(\frac{\widetilde{n}}{n_0}\right)_2 \to -\frac{e\phi}{T_0}z, \tag{58}$$

$$\left(\frac{\widetilde{T}}{T_0}\right)_2 \to -\frac{2e\phi}{3T_0}z. \tag{59}$$

The expressions (56)–(59) correspond to the asymptotics of the exact linear kinetic expressions (Smolyakov *et al.* 1995)

$$\left(\frac{\widetilde{n}}{n_0}\right)_K = -\frac{e\phi}{T_0}\left[1 - I_0(z)\exp(-z)\right], \tag{60}$$

$$\left(\frac{\widetilde{T}}{T_0}\right)_K = -\frac{2e\phi}{3T_0}z\exp(-z)\left[I_0(z) - I_1(z)\right]. \tag{61}$$

The approximate expressions (54) and (55) show reasonably good agreement with the exact results for the whole region of values of z. They are plotted together with the kinetic expressions (60) and (61) in Figs 1 and 2.

4.2. Fourth-order closure

Higher accuracy is achieved with the fourth-order closure in which the higher-order terms (in $1/B$) are taken into account:

$$\frac{\partial n}{\partial t} + \boldsymbol{\nabla}\cdot\left(n\mathbf{V}^{(2)} + n\mathbf{V}^{(4)}\right) = 0, \tag{62}$$

https://doi.org/10.1017/S0022377898006552 Published online by Cambridge University Press

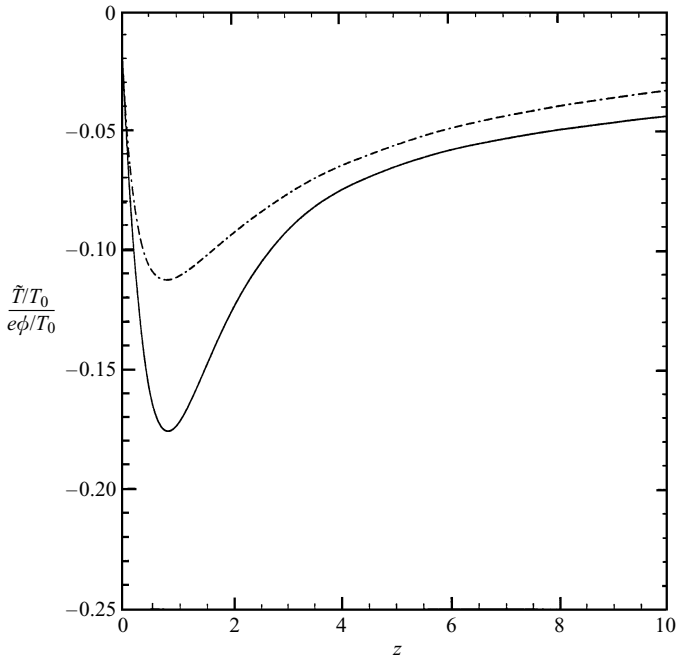


Figure 2. Normalized perturbation of the plasma temperature $(\widetilde{T}/T_0)/(e\phi/T_0)$ as a function of the parameter $z = k^2\rho^2$. The solid line is the exact kinetic expression (61) and the dot-dashed line is the second-order approximation (55).

$$\frac{3}{2}\frac{\partial p}{\partial t} + \frac{5}{2}p\boldsymbol{\nabla}\cdot(\mathbf{V}^{(2)} + \mathbf{V}^{(4)}) + \boldsymbol{\nabla}\cdot\mathbf{q}^{(4)} = 0. \tag{63}$$

Invoking the fourth-order closure relations for the plasma velocity and the heat flux, we obtain

$$\begin{aligned}\frac{\partial n}{\partial t} &+ \boldsymbol{\nabla}\cdot\left[\frac{n}{\omega_B}\mathbf{b}\times\frac{d}{dt}\left(\frac{c}{B}\mathbf{b}\times\boldsymbol{\nabla}\phi + \frac{1}{mn\omega_c}\mathbf{b}\times\boldsymbol{\nabla}p\right)\right] \\ &+ \frac{11cp}{12m\omega_c^3 B}\frac{d}{dt}\nabla_\perp^4\left(\phi + \frac{p}{en_0}\right) + \frac{11cp}{12em\omega_c^3 B}\frac{d}{dt}\nabla_\perp^4 T = 0,\end{aligned} \tag{64}$$

$$\begin{aligned}\frac{3}{2}\frac{dp}{dt} &+ \frac{5}{2}p\boldsymbol{\nabla}\cdot\left[\frac{1}{\omega_B}\mathbf{b}\times\frac{d}{dt}\left(\frac{c}{B}\mathbf{b}\times\boldsymbol{\nabla}\phi + \frac{1}{mn\omega_c}\mathbf{b}\times\boldsymbol{\nabla}p\right)\right] \\ &+ \boldsymbol{\nabla}\cdot\left[\frac{1}{\omega_B}\mathbf{b}\times\frac{d}{dt}\left(\frac{5cp}{2eB}\mathbf{b}\times\boldsymbol{\nabla}T\right)\right] \\ &+ \frac{11cpT}{6m\omega_c^3 B}\frac{d}{dt}\nabla_\perp^4\left(\phi + \frac{p}{en_0}\right) + \frac{11cpT}{6em\omega_c^3 B}\frac{d}{dt}\nabla_\perp^4 T = 0.\end{aligned} \tag{65}$$

In the linear limit, (64) and (65) give the following equations for the perturbations of plasma density and temperature:

$$\left(1 + z + \tfrac{11}{12}z^2\right)\frac{\widetilde{n}}{n_0} + \left(z + \tfrac{11}{6}z^2\right)\frac{\widetilde{T}}{T_0} = -\left(z + \tfrac{11}{12}z^2\right)\frac{e\phi}{T_0}, \tag{66}$$

$$\left(z + \tfrac{11}{6}z^2\right)\frac{\widetilde{n}}{n_0} + \left(\tfrac{3}{2} + \tfrac{7}{2}z + \tfrac{11}{3}z^2\right)\frac{\widetilde{T}}{T_0} = -\left(z + \tfrac{11}{6}z^2\right)\frac{e\phi}{T_0}. \tag{67}$$

https://doi.org/10.1017/S0022377898006552 Published online by Cambridge University Press

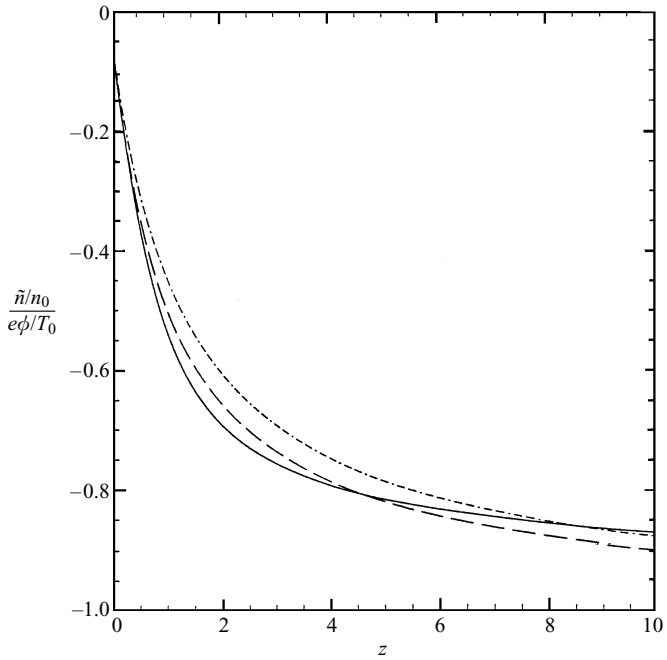


Figure 3. Normalized perturbation of the plasma density $(\widetilde{n}/n_0)/(e\phi/T_0)$ as a function of the parameter $z = k^2\rho^2$. The solid line is the exact kinetic expression (60), the dot-dashed line is the second-order approximation (54) and the dashed line is the fourth-order approximation (54).

We then obtain the following expressions for the fourth-order response:

$$\left(\frac{\widetilde{n}}{n_0}\right)_4 = -z\frac{e\phi}{T_0}\frac{1+\frac{31}{12}z+\frac{77}{36}z^2}{1+\frac{10}{3}z+\frac{181}{36}z^2+\frac{77}{36}z^3}, \tag{68}$$

$$\left(\frac{\widetilde{T}}{T_0}\right)_4 = -\frac{2e\phi}{3T_0}\frac{z(1+\frac{11}{6}z)}{1+\frac{10}{3}z+\frac{181}{36}z^2+\frac{77}{36}z^3}. \tag{69}$$

Comparisons between the second- and the fourth-order approximations are given in Figs 3 and 4. One can see that the density perturbations are reasonably approximated even by the second-order closure. However, the second-order closure is not as good in describing the temperature perturbations. The fourth-order closure is noticeably better in matching the kinetic response for temperature perturbations, especially for intermediate values of $z \approx 1$.

Although (54), (55), (68) and (69) do provide two-point approximations to the exact expressions at the points $z = 0$ and $z = \infty$, they are not the two-point Padé appoximants in a rigorous sense. The standard Padé approximation for the ion kinetic response (Pegoraro and Schep 1986), given by

$$\left(\frac{\widetilde{n}}{n_0}\right)_P = -\frac{e\phi}{T_0}\frac{z}{1+z}, \tag{70}$$

correctly recovers the two terms in the corresponding asymptotic series for the exact kinetic expression: $\widetilde{n}/n \to -ze\phi/T_0$ for $z \ll 1$, and $\widetilde{n}/n \to 0$ for $z \to \infty$. This approximation is the ratio of two first-order polynomials, $(\widetilde{n}/n)_P \propto P_1^1$, where

https://doi.org/10.1017/S0022377898006552 Published online by Cambridge University Press

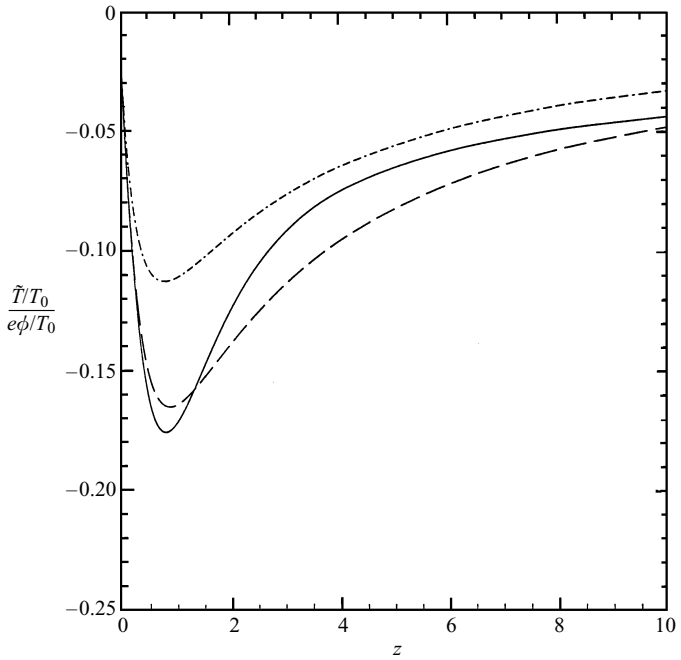


Figure 4. Normalized perturbation of the plasma temperature $(\widetilde{T}/T_0)/(e\phi/T_0)$ as a function of the parameter $z = k^2\rho^2$. The solid line is the exact kinetic expression (61), the dot-dashed line is the second-order approximation (55), and the dashed line is the fourth-order approximation (69).

P_l^m is the ratio of two polynomials of degrees l and m respectively. The second-order approximation (54) recovers the same two terms in the asymptotic series, but is the ratio of two second-order polynomials, $(\widetilde{n}/n)_2 \propto P_2^2$. The fourth-order approximation (68) correctly matches the three asymptotic terms, $(\widetilde{n}/n)_4 \to (-z + 3z^2/4)e\phi/T_0$ for $z \ll 1$, and $(\widetilde{n}/n)_4 \to 0$ for $z \to \infty$, but is the ratio of two third-order polynomials, $(\widetilde{n}/n)_4 \propto P_3^3$.

5. Kinetic Alfvén waves

The dynamic equations derived in Sec. 4 can be used to describe the Alfvén perturbations for arbitrary values of the Larmor radius (cf. Marchenko *et al.* 1996). The electromagnetic field of the Alfvén wave can be described with the electrostatic potential ϕ and the parallel component of the vector potential A:

$$j_\parallel = \frac{c}{4\pi}\,|\boldsymbol{\nabla} \times \mathbf{B}_\perp| = -\frac{c}{4\pi}\nabla_\perp^2 A, \tag{71}$$

$$E_\parallel = -\nabla_\parallel \phi - \frac{1}{c}\frac{\partial A}{\partial t}. \tag{72}$$

The electron continuity equation

$$\frac{d_0 n}{dt} - \frac{1}{e}\nabla_\parallel j_\parallel = 0 \tag{73}$$

and the generalized Ohm's law

$$0 = -T_e \nabla_\parallel n - e n_0 E_\parallel, \tag{74}$$

https://doi.org/10.1017/S0022377898006552 Published online by Cambridge University Press

can be combined with (71) and (72) to give the following system describing the electron response:

$$\frac{dn}{dt} + \frac{c}{4\pi e}\nabla_\parallel \nabla_\perp^2 A = 0, \tag{75}$$

$$\frac{\partial}{\partial t}A + c\nabla_\parallel \left(\phi - \frac{T_e}{e}\frac{\widetilde{n_e}}{n_0}\right) = 0. \tag{76}$$

The ion component responds only to the perturbations in the electrostatic potential and can be described by (68). Combining it with (75) and (76), we obtain the dispersion equations for kinetic Alfvén waves for arbitrary ion temperature

$$\omega^2 = \left(\frac{T_e}{T_i}z + \frac{1 + \frac{10}{3}z + \frac{181}{36}z^2 + \frac{77}{36}z^3}{1 + \frac{31}{12}z + \frac{77}{36}z^2}\right) c_A^2 k_\parallel^2. \tag{77}$$

In the long-wavelength limit, $z \ll 1$, we recover the familiar form for the kinetic Alfvén waves (Hasegawa and Chen 1976):

$$\omega^2 = c_A^2 k_\parallel^2 (1 + k_\perp^2 \rho_s^2 + \tfrac{3}{4} k_\perp^2 \rho_i^2), \tag{78}$$

where $\rho_s^2 = T_e/m_i\omega_i^2$ and $\rho_i^2 = T_i/m_i\omega_i^2$. At small scales, $z \gg 1$, we obtain

$$\omega^2 = \left(1 + \frac{T_e}{T_i}\right) c_A^2 k_\parallel^2 k_\perp^2 \rho_i^2. \tag{79}$$

A comparison between the approximate dispersion equation (77) and the exact kinetic result (Mikhailovskii 1967; Hasegawa and Chen 1976) for $T_i/T_e = 1$,

$$\omega^2 = z\left[1 + \frac{1}{1 - I_0(z)\exp(-z)}\right] c_A^2 k_\parallel^2, \tag{80}$$

is shown in Fig. 5.

6. Conclusions

We have analysed the structure of the MHD equations obtained by truncation of the Grad-type hierarchy. We have demonstrated that the finite-dimensional representation (given by (8)) can describe perturbations in plasmas with arbitrary Larmor radius. As noted previously (Kuvshinov 1994; Kuvshinov and Mikhailovskii 1996; Hassam and Huba 1988), the inertial terms in the momentum-balance as well as the viscosity and heat-flux evolution equations are crucial for a uniform description of the plasma response for arbitrary Larmor radius (the importance of the inertial heat flux for the dispersion properties of compressional Alfvén waves has also been noted by Mikhailovskii and Smolyakov 1985). Our analysis seems to indicate that a 'structural' expansion of the infinite hierarchy of MHD equations given by (8) is equivalent to the series of the two-point approximation (of Padé type). We cannot prove that this expansion is an asymptotically rigorous procedure. However, our calculations show that the accuracy of the plasma response increases with the number of terms retained. We have compared the results of expansions of the second order $\mathcal{O}(1/B^2)$ (n, p, $\mathbf{V}$ and $\mathbf{q}$ moments are retained in the truncated series) and fourth order $\mathcal{O}(1/B^4)$ (n, p, $\mathbf{V}$, $\mathbf{q}$ and $\mathbf{\Pi}$ moments are retained). We have shown that the fourth-order closure reproduces the exact kinetic plasma response to the accuracy of fourth-order terms, $\mathcal{O}(k_\perp^4\rho^4)$. The fourth-order closure gives a better

https://doi.org/10.1017/S0022377898006552 Published online by Cambridge University Press

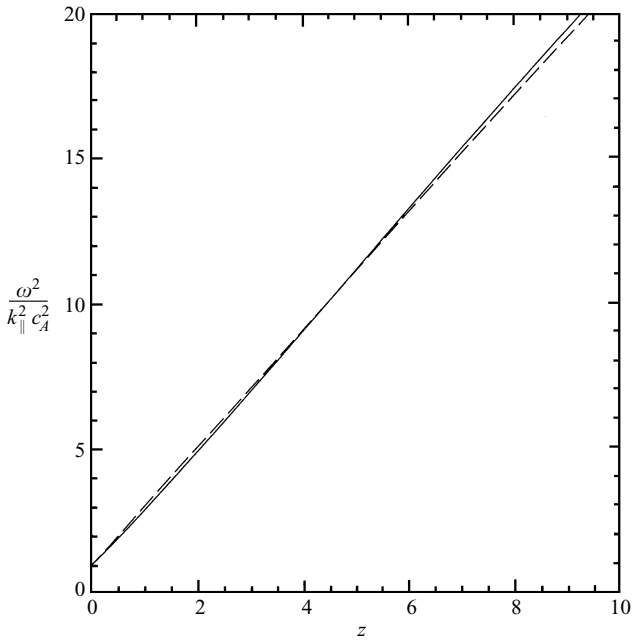


Figure 5. Normalized frequency $\omega^2/k_\parallel^2 c_A^2$ of kinetic Alfvén waves as a function of the parameter $z = k^2\rho^2$ for $T_e/T_i = 1$. The solid line is the exact kinetic expression (80) and the dashed line is the fourth-order approximation (77).

match for arbitrary $k_\perp\rho$. The improvement in accuracy is especially pronounced for temperature perturbations in the intermediate regime, $k_\perp\rho \approx 1$.

The results of the present work indicate that one can obtain a useful system of MHD equations by an appropriate truncation of the infinite hierarchy of moments of the Vlasov kinetic equation. To allow for the effects of arbitrary Larmor radius, such a system should include the evolution equations for the heat flux and the collisionless viscosity tensor. It is worth noting that the truncation of the infinite hierarchy of moment equations is based on the same assumption that is used in the gyrofluid theory, namely that the deviation of the distribution function from a Maxwellian is small and can be represented by a finite number of terms (Dorland and Hammet 1993). However, our approach does not employ the cumbersome gyrokinetic formalism and avoids a number of problems inherent to it: for example, our equations are written in the real fluid configuration space rather than in the guiding-centre space as in the gyrofluid theory.

The minimal useful model for nonlinear simulations, like those contemplated in Sovinec *et al.* (1996), should consist of the continuity, momentum and energy-balance equations and equations for the evolution of the heat flux. Equations (2)–(5) can form a basis for such a nonlinear plasma model. Inclusion of the equation for the evolution of the viscosity tensor will substantially improve the accuracy of the plasma response. For numerical simulations it is not necessary to expand the above equations in the small parameter ω/ω_c, as has been done in this work. The complete equations (2)–(5) can be solved directly by advancing them in time. Then this system will describe not only the effects of arbitrary Larmor radius, but will also be able to reproduce the gyrotropic phenomena associated with the finiteness

https://doi.org/10.1017/S0022377898006552 Published online by Cambridge University Press

of ω/ω_c. In particular, it is expected that a description of the Bernstein waves may be possible within the MHD model (Gould 1996).

Acknowledgements

This work was supported by the Natural Sciences and Engineering Research Council of Canada. We thank the referees for suggestions that improved the manuscript.

References

Aydemir, A. Y. 1991 Nonlinear studies of $m = 1$ modes in high temperature plasmas. *Phys. Fluids* **B3**, 3649–3472.

Balescu, R. 1988 *Transport Processes in Plasmas.* Vol. 1: *Classical Transport Theory*, p. 317. North-Holland, Amsterdam.

Bernstein, I. B. 1985 Generalized gyrokinetics. *Phys. Fluids* **28**, 1342–1353.

Braginskii, S. I. 1963 Transport processes in a plasma. *Reviews of Plasma Physics*, Vol. 1 (ed. M. A. Leontovich), pp. 205–311. Consultants Bureau, New York.

Brizard A. 1992 Nonlinear gyrofluid description of turbulent magnetized plasmas. *Phys. Fluids* **B4**, 1213–1228.

Callen J. D. A. 1989 Models of plasma confinement and heating in tokamaks. *The College of Engineering Lecture Series, University of Wisconsin Plasma Report* UWPR 89-2.

Chew, G. F., Golberger, M. L. and Low, F. E. 1956 The Boltzmann equation and the one-fluid hydromagnetic equations in the absence of particle collisions. *Proc. R. Soc. Lond.* **A236**, 112–118.

Dorland, W. and Hammet, G. W. 1993 Gyrofluid turbulence with kinetic effects. *Phys. Fluids* **B5**, 812–835.

Gould, R. W. 1996 Fluid description of Bernstein waves. *Bull. Am. Phys. Soc.* **41**, 1522.

Grad, H. 1949 On the kinetic theory of rarefied gases. *Commun. Pure Appl. Maths* **2**, 331–407.

Hasegawa, A. and Chen, L. 1976 Kinetic processes plasma heating by resonant mode conversion of Alfvén waves. *Phys. Fluids* **19**, 1924–1935.

Hassam, A. B. and Huba, J. D. 1988 Magnetohydrodynamics equations for systems with large Larmor radius. *Phys. Fluids* **31**, 318–325.

Hazeltine, R. D. and Meiss, J. D. 1985 Shear–Alfvén wave dynamics in toroidally confined plasma. *Phys. Rep.* **121**, 1–164.

Hsu, C. T., Hazeltine, R. D. and Morrison, P. J. 1986 A generalized reduced fluid model with finite ion-gyroradius effects. *Phys. Fluids* **29**, 1480–1487.

Ilgisonis, V. I. 1992 Integrals of the drift motion of particles with finite Larmor radius. *Soviet Phys. JETP* **74**, 31–35.

Ilgisonis, V. I. 1995 Guiding-center theory for three-dimensional collisionless finite Larmor radius plasmas. *Phys. Plasmas* **2**, 2387–2397.

Kogan, E. Ya, Moiseev, S. S. and Oraevkii, V. N. 1965 *J. Appl. Mech. Tech. Phys.* **1**, 25–28.

Kulsrud, R. M. 1983 MHD description of plasma. *Handbook of Plasma Physics*, Vol. 1 (ed. M. N. Rozenbluth and R. Z. Sagdeev), pp. 115–146. North-Holland, Amsterdam.

Kuvshinov, B. N. 1994 Magnetohydrodynamic model for plasma instabilities in the ion-kinetic regime. *Plasma Phys. Contr. Fusion*, **36**, 867–877.

Kuvshinov, B. N. and Mikhailovskii, A. B. 1996 MHD model including small-scale perturbations in plasma with temperature variations. *Plasma Rep.* **22**, 529–534.

Kuvshinov, B. N., Pegoraro, F. and Schep, T. J. 1994 Hamiltonian formulation of low-frequency, nonlinear plasma dynamics. *Phys Lett.* **191A**, 296–300.

Macmahon, A. 1965 Finite gyro-radius corrections to the hydromagnetic equations for a Vlasov plasma. *Phys. Fluids* **8**, 1840–1845.

Marchenko, V., Denton, R. E. and Hudson, M. K. 1996 A magnetohydrodynamic model of kinetic Alfvén waves with finite ion gyroradius. *Phys. Plasmas* **3**, 3861–3863.

Mikhailovskii, A. B. 1967 Oscillations of an inhomogeneous plasma. *Reviews of Plasma Physics*, Vol. 3 (ed. M. A. Leontovich) pp. 159–227. Consultants Bureau, New York.

https://doi.org/10.1017/S0022377898006552 Published online by Cambridge University Press

Mikhailovskii, A. B. and Smolyakov, A. I. 1985 Theory of low-frequency solitons. *Soviet Phys. JETP* **61**, 109-117.

Mikhailovskii, A. B. and Tsypin, V. S. 1984 Transport equations in a curvilinear magnetic field. *Beitr. Plasmaphys.* **24**, 335–354.

Opp, E. N. and Hassam, A. B. 1991 Kelvin–Helmholtz instability in systems with large effective Larmor radius. *Phys. Fluids* **B3**, 885–892.

Pegoraro, F. and Schep, T. J. 1986 Theory of ballooning modes in the ballooning representation. *Plasma Phys. Contr. Fusion* **28**, 647–667.

Schep, T. J., Pegoraro, F. and Kuvshinov, B. N. 1994 Generalized two-fluid theory of nonlinear magnetic structures. *Phys. Plasmas* **1**, 2843–2852.

Shkarovsky, I. P., Johnston, T. W. and Bachynski, M. P. 1966 *The Particle Kinetics of Plasmas*. Addison-Wesley, Reading, MA.

Smolyakov, A. I., Pogutse, I. O. and Hirose, A. 1995 Fluid model of collisionless plasma with finite Larmor radius effects. *Phys. Plasmas.* **2**, 4451–4454.

Sovinec, C. R., Barnes, D. C., Nebel, R. A. and Gianakon, T. A. 1996 The physical model of the NIMROD code. *Bull. Am. Phys. Soc.* **41**, 1361.

Tsai, S.-T., Perkins, F. W. and Stix, T. H. 1970 Thermal conductivity and low frequency waves in collisional plasma. *Phys. Fluids* **13**, 2108–2114.

Velikovich, A. L. 1991 Rayleigh–Taylor instability of a plasma–vacuum boundary in the limit of a large Larmor radius. *Phys. Fluids* **B3**, 492–494.

Zakharov, L. E. and Rogers, B. 1992 Two-fluid magnetohydrodynamic descriprion of the internal kink mode in tokamaks. *Phys. Fluids* **B4**, 3285–3301.

https://doi.org/10.1017/S0022377898006552 Published online by Cambridge University Press