Canad. Math. Bull. 2026, pp. 1–23
http://dx.doi.org/10.4153/S0008439526101830
© The Author(s), 2026. Published by Cambridge University Press on behalf of Canadian Mathematical Society. This is an Open Access article, distributed under the terms of the Creative Commons Attribution licence (https://creativecommons.org/licenses/by/4.0), which permits unrestricted re-use, distribution and reproduction, provided the original article is properly cited.

The Kromatic symmetric function: A K-theoretic analog of X_G

Logan Crew, Oliver Pechenik, and Sophie Spirkl

Abstract. Schur functions are a basis of the symmetric function ring that represent Schubert cohomology classes for Grassmannians. Replacing the cohomology ring with K-theory yields a rich combinatorial theory of inhomogeneous deformations, where Schur functions are replaced by their K-analogs, the *symmetric Grothendieck functions*. We initiate a theory of the *Kromatic symmetric function* $\overline{X}_G$, a K-theoretic analog of the chromatic symmetric function X_G of a graph G. The Kromatic symmetric function is a generating series for graph colorings in which vertices receive any nonempty set of colors such that neighboring color sets are disjoint. Our main result lifts a theorem of Gasharov (1996), showing that when G is a claw-free incomparability graph, $\overline{X}_G$ is a positive sum of symmetric Grothendieck functions. This suggests a topological interpretation of Gasharov's theorem. Kromatic symmetric functions of path graphs are not positive in any of several K-analogs of the e-basis, demonstrating that the Stanley–Stembridge conjecture (1993) does not have such a lift to K-theory and so is unlikely to be amenable to a topological perspective. We define a vertex-weighted extension of $\overline{X}_G$ which admits a deletion–contraction relation. Finally, we give a K-analog for $\overline{X}_G$ of the monomial-basis expansion of X_G.

1 Introduction

The *chromatic symmetric function* X_G of a graph G was introduced by Stanley [39] as a generalization of Birkhoff's *chromatic polynomial* [5]. While the chromatic polynomial enumerates proper graph colorings by the number of colors used, X_G also records how many times each color is used. A recent boom of research regarding X_G has focused on the *Stanley–Stembridge conjecture* [42], which proposes (in a reformulation by Guay-Paquet [17]) that unit interval graphs have chromatic symmetric functions that expand positively in the e-basis of the ring Sym of symmetric functions. A proof of this conjecture was recently announced by Hikita [19], by interpreting the coefficients as probabilities.

Received by the editors November 25, 2025; accepted February 2, 2026.
Published online on Cambridge Core February 16, 2026.
This work was supported by the National Natural Science Foundation of China (Grant No. 12271328). We acknowledge the support of the Natural Sciences and Engineering Research Council of Canada (NSERC), [funding reference numbers RGPIN-2020-03912, RGPIN-2021-00010, and RGPIN-2022-03093]. Cette recherche a été financée par le Conseil de recherches en sciences naturelles et en génie du Canada (CRSNG), [numéros de référence RGPIN-2020-03912, RGPIN-2021-00010, et RGPIN-2022-03093]. This project was funded in part by the Government of Ontario.
AMS subject classification: **05C15**, **05C31**, **05E05**.
Keywords: Chromatic symmetric function, Grothendieck polynomial, K-theory, deletion–contraction relation, Stanley–Stembridge conjecture.

https://doi.org/10.4153/S0008439526101830 Published online by Cambridge University Press

To the best of our knowledge, Hikita's result [19] does not apply to various generalizations of the chromatic symmetric function and corresponding lifts of the Stanley–Stembridge conjecture, which had been considered both as possible avenues for proving the conjecture and to gain a better understanding of it. Examples of this latter approach include the *chromatic quasisymmetric function* and *Shareshian–Wachs conjecture* of [38] (further studied in [1, 2, 7, 8]), the *chromatic nonsymmetric functions* of Haglund–Wilson [18] (further studied in [43]), and Gebhard–Sagan's [16] chromatic symmetric function in noncommuting variables combined with notions of *(e)-positivity* and *appendable (e)-positivity* (further studied in [3, 11, 13]). Our work provides a novel generalization of X_G in the same vein.

An important appearance of the ring of symmetric functions Sym is as the cohomology of complex *Grassmannians* (parameter spaces for linear subspaces of a vector space) or more precisely for the classifying space BU. Here, the *Schubert classes* derived from a natural cell decomposition of BU are represented by the *Schur function* basis s_λ of Sym. A richer perspective into the topology of BU is obtained by replacing cohomology with a generalized cohomology theory. In particular, there has been much focus on studying the associated combinatorics of the *K-theory* ring (see [6, 28, 32, 44]). In this context, many of the classical objects of symmetric function theory are seen to have interesting K-analogs, often resembling "superpositions" of classical objects. For example, classical *semistandard Young tableaux* are replaced by *set-valued tableaux* (allowing multiple labels per cell), while Schur functions are replaced by *Grothendieck polynomials* $\overline{s}_\lambda$ (inhomogeneous deformations of s_λ).

Our work introduces a K-analog of the chromatic symmetric function X_G, enumerating colorings of the graph G that assign a nonempty set of distinct colors to each vertex such that adjacent vertices receive disjoint sets. While our *Kromatic symmetric function* $\overline{X}_G$ is new, similar functions have been previously considered. The first such functions were originally discussed by Stanley [40] in the context of graph analogs of symmetric functions, with connections to the real-rootedness of polynomials, and by Gasharov [15] in the context of Schur expansions. Recently, as part of his effort to refine Schur-positivity results and the Stanley–Stembridge conjecture, Hwang [20] studied a similar quasisymmetric function for graphs endowed with a fixed map $\alpha : V(G) \to \mathbb{N}$ that dictates the size of the set of colors each vertex receives. To connect chromatic quasisymmetric functions of vertex-weighted graphs to *horizontal-strip LLT polynomials*, Tom [45] has considered a variant for fixed α with repeated colors allowed. Our work appears to be the first to connect these ideas to the combinatorics of K-theoretic Schubert calculus. (However, [29] (see also, [37]) is similar in spirit to our work, developing a K-theoretic analog of the *Postnikov–Shapiro algebra* [36], an apparently unrelated invariant of graphs). After this manuscript appeared in preprint form[1], Marberg [27] gave an interesting Hopf-algebra-theoretic interpretation of $\overline{X}_G$, as well as a quasisymmetric analog from the same perspective.

In this article, having introduced the *Kromatic symmetric function*, we begin to develop its combinatorial theory. We show that the Kromatic symmetric function $\overline{X}_G$ for any graph G expands positively in a K-theoretic analog (that we also introduce) of the *monomial basis* of Sym. In this expansion, the coefficients enumerate coverings

[1] An extended abstract of this work also appeared in the proceedings of FPSAC 2023 [9].

https://doi.org/10.4153/S0008439526101830 Published online by Cambridge University Press

of the graph by (possibly overlapping) *stable sets*. We further extend the definition of $\overline{X}_G$ to a vertex-weighted setting, where we give a deletion–contraction relation analogous to that developed by the first and last authors [10] for the vertex-weighted version of X_G.

Our main result is that the Kromatic symmetric function of a *claw-free incomparability graph* expands positively in the symmetric Grothendieck basis $\bar{s}_\lambda$ of Sym, lifting to K-theory a celebrated result of Gasharov [15] that such graphs have Schur-positive chromatic symmetric functions. While all known proofs of Gasharov's theorem are representation-theoretic or purely combinatorial, the existence of our K-theoretic analog suggests that both results likely also have an interpretation in terms of the topology of Grassmannians. Precisely, for each claw-free incomparability graph G, there should be a subvariety of the Grassmannian whose cohomology class is represented by X_G and whose K-theoretic structure sheaf class is represented by $\overline{X}_G$. It would be very interesting to have an explicit construction of such subvarieties.

On the other hand, we show that the Kromatic symmetric functions $\overline{X}_{P_n}$ of path graphs P_n generally do not expand positively in either of two K-theoretic deformations we propose for the e-basis of Sym. This fact suggests that the Stanley–Stembridge conjecture is not naturally interpreted in terms of the cohomology of Grassmannians and is unlikely to be amenable to such topological tools from Schubert calculus. (Note that Hikita's proof [19] appears to be derived from representation-theoretic considerations, although these are hidden in the actual writeup.) We hope these observations can play a similar role to [12] in limiting the range of potential generalizations of the Stanley–Stembridge conjecture.

This article is organized as follows. In Section 2, we provide an overview of the background and notation used from symmetric function theory (Section 2.1), K-theoretic Schubert calculus (Section 2.2), and graph theory (Section 2.3). In Section 3, we formally introduce the Kromatic symmetric function $\overline{X}_G$ and give its basic properties, including a formula for the expansion in a new K-analog of the monomial basis of Sym and a deletion–contraction relation for a vertex-weighted generalization. We also give our main theorem that the Kromatic symmetric functions of claw-free incomparability graphs expand positively in symmetric Grothendieck functions, lifting the main result of [15]. In Section 4, we introduce two different K-theoretic analogs of the e-basis of Sym and show that the Kromatic symmetric function $\overline{X}_{P_3}$ of a 3-vertex path graph P_3 is not positive in either analog, casting doubt on hopes for a Schubert calculus-based generalization of the Stanley–Stembridge conjecture.

2 Background

Throughout this work, $\mathbb{N}$ denotes the set of (strictly) positive integers. We write $[n]$ for the set of positive integers $\{1, 2, \ldots, n\}$. If S is any set, 2^S denotes the power set of all subsets of S.

2.1 Partitions and symmetric functions

In this section, we give a brief overview of the necessary background material. Further details can be found in the textbooks of Stanley [41], Manivel [26], and Macdonald [25].

https://doi.org/10.4153/S0008439526101830 Published online by Cambridge University Press

An ***integer partition*** $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k)$ is a finite nonincreasing sequence of positive integers. We define $\ell(\lambda)$ to be the length of the sequence λ (so above, $\ell(\lambda) = k$). We define $r_i(\lambda)$ to be the number of occurrences of i as a part of λ (so, e.g., $r_1(2,1,1,1) = 3$). If

$$\sum_{i=1}^{\ell(\lambda)} \lambda_i = n,$$

we say that λ is a partition of n, and we write $\lambda \vdash n$. The ***Young diagram of shape*** λ is a set of squares called ***cells***, left- and top-justified (i.e., in "English notation"), such that the ith row from the top contains λ_i cells. For example, the Young diagram of shape $(2,2,1)$ is . Let $C(\lambda)$ denote the set of cells of the Young diagram of shape λ. If $\mathsf{c} \in C(\lambda)$ is a cell of the Young diagram of shape λ, we write $\mathsf{c}^{\uparrow}$ for the cell immediately above c (assuming it exists), $\mathsf{c}^{\rightarrow}$ for the cell immediately right of c, and so on. We write λ^{T} for the ***transpose*** of λ, the integer partition whose Young diagram is obtained from that of λ by exchanging rows and columns.

Let $S_{\mathbb{N}}$ denote the set of all permutations of the set $\mathbb{N}$ fixing all but finitely-many elements. A ***symmetric function*** $f \in \mathbb{C}[\![x_1, x_2, \ldots,]\!]$ is a power series of bounded degree such that for each permutation $\sigma \in S_{\mathbb{N}}$, we have $f(x_1, x_2, \ldots) = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots)$. The set $\mathrm{Sym} \subset \mathbb{C}[\![x_1, x_2, \ldots]\!]$ of symmetric functions forms a $\mathbb{C}$-vector space. Furthermore, if Λ^d denotes the set of symmetric functions that are homogeneous of degree d, then each Sym^d is a vector space, and

$$\mathrm{Sym} = \bigoplus_{d=0}^{\infty} \mathrm{Sym}^d$$

as graded vector spaces.

The dimension of Sym^d as a $\mathbb{C}$-vector space is equal to the number of integer partitions of d, and many bases of symmetric functions are conveniently indexed by integer partitions. Below, we provide some commonly used bases that will be used in this article.

Definition 2.1 The following are bases of Sym:

- the ***monomial symmetric functions*** $\{m_\lambda\}$, defined as

$$m_\lambda = \sum x_{i_1}^{\lambda_1} \ldots x_{i_{\ell(\lambda)}}^{\lambda_{\ell(\lambda)}},$$

  where the sum ranges over all distinct monomials formed by choosing distinct positive integers $i_1, \ldots, i_{\ell(\lambda)}$;
- the ***augmented monomial symmetric functions*** $\{\widetilde{m}_\lambda\}$, defined as

$$\widetilde{m}_\lambda = \left(\prod_{i=1}^{\infty} r_i(\lambda)!\right) m_\lambda;$$

- the ***elementary symmetric functions*** $\{e_\lambda\}$, defined by

$$e_n = \prod_{i_1 < \cdots < i_n} x_{i_1} \ldots x_{i_n}; \quad e_\lambda = e_{\lambda_1} \ldots e_{\lambda_{\ell(\lambda)}};$$

https://doi.org/10.4153/S0008439526101830 Published online by Cambridge University Press

- and the ***complete homogeneous symmetric functions*** $\{h_\lambda\}$, defined by

$$h_n = \prod_{i_1 \le \cdots \le i_n} x_{i_1} \dots x_{i_n}; \quad h_\lambda = h_{\lambda_1} \dots h_{\lambda_{\ell(\lambda)}}.$$

The space of symmetric functions is equipped with a natural inner product $\langle \cdot, \cdot \rangle$; it may be defined by

$$\langle h_\lambda, m_\mu \rangle = \delta_{\lambda,\mu},$$

where $\delta_{\bullet,\bullet}$ denotes the *Kronecker delta function*.

We will also need the basis of Schur functions. A ***Young tableau*** of shape λ is a function $T : C(\lambda) \to \mathbb{N}$, typically visualized by writing the value $T(\mathsf{c})$ in the cell c. A Young tableau T of shape λ is ***semistandard*** if for each cell $\mathsf{c} \in C(\lambda)$, we have $T(\mathsf{c}) \le T(\mathsf{c}^{\rightarrow})$ and $T(\mathsf{c}) < T(\mathsf{c}^{\downarrow})$ whenever the cells in question exist. We write $\mathrm{SSYT}(\lambda)$ for the set of all semistandard Young tableaux of shape λ. The ***Schur functions*** s_λ is defined by

$$s_\lambda = \sum_{T \in \mathrm{SSYT}(\lambda)} x^T, \quad \text{where} \quad x^T = \prod_{\mathsf{c} \in C(\lambda)} x_{T(\mathsf{c})}.$$

As λ ranges over integer partitions, the Schur functions are another basis of Sym. The inner product on Sym also satisfies

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu}.$$

When $f \in \mathrm{Sym}$ is a symmetric function and $\{b_\lambda\}$ is a basis of symmetric functions indexed by integer partitions λ, the notation $[b_\mu]f$ denotes the coefficient of b_μ when f is expanded in the b-basis. A symmetric function $f \in \mathrm{Sym}$ is said to be ***b-positive*** if $[b_\mu]f$ is nonnegative for every integer partition μ.

2.2 *K*-theoretic Schubert calculus

The ***Grassmannian*** $\Gamma_k = \mathrm{Gr}_k(\mathbb{C}^\infty)$ is the parameter space of k-dimensional vector subspaces of the space of all eventually-zero sequences of complex numbers. The space Γ_k can be given the structure of a projective Ind-variety and has a cell decomposition into cells Γ_λ indexed by partitions with at most k parts. Each Γ_λ induces a cohomology class $\sigma_\lambda \in H^\star(\Gamma_k)$ and classically we have $H^\star(\Gamma_k) \cong \mathrm{Sym}_k = \mathrm{Sym} \cap \mathbb{C}[x_1, \dots, x_k]$ with the isomorphism taking the class of the cell σ_λ to the Schur polynomial $s_\lambda(x_1, \dots, x_k)$.

Each cell-closure in Γ_k also has a structure sheaf, inducing a class in the representable ***K-theory*** ring $K^0(\Gamma_k)$. These K-theoretic classes are represented by inhomogeneous symmetric polynomials called *Grothendieck polynomials* $\bar{s}_\lambda(x_1, \dots, x_k)$.

A ***set-valued tableau*** of shape λ is a filling T of each cell of $C(\lambda)$ with a nonempty *set* of positive integers. The set-valued tableau T is ***semistandard*** if for each cell $\mathsf{c} \in C(\lambda)$, we have $\max T(\mathsf{c}) \le \min T(\mathsf{c}^{\rightarrow})$ and $\max T(\mathsf{c}) < \min T(\mathsf{c}^{\downarrow})$ whenever the cells in question exist. In other words, T is semistandard if every Young tableau formed by choosing one number from the set of each cell is semistandard. For example,

https://doi.org/10.4153/S0008439526101830 Published online by Cambridge University Press

1	1, 2	2, 3, 9
2, 6	8	

is a set-valued tableau of shape $\lambda = (3, 2)$. Let $\mathrm{SV}(\lambda)$ denote the set of all semistandard set-valued tableaux of shape λ. The ***symmetric Grothendieck function*** $\bar{s}_\lambda$ is

$$\bar{s}_\lambda = \sum_{T \in \mathrm{SV}(\lambda)} (-1)^{|T|-\ell(\lambda)} x^T,$$

where $|T| = \sum_{\mathsf{c} \in C(\lambda)} |T(\mathsf{c})|$ and $x^T = \prod_{\mathsf{c} \in C(\lambda)} \prod_{i \in T(\mathsf{c})} x_i$. Note that $\bar{s}_\lambda$ contains terms of degree greater than or equal to $|\lambda|$, and that the sum of all of its lowest-degree terms is equal to s_λ. This tableau formula for $\bar{s}_\lambda$ is due to Buch [6]. For further background on K-theoretic Schubert calculus and symmetric Grothendieck functions, see [28, 32].

We will also need the ***dual symmetric Grothendieck function*** $\underline{s}_\lambda$ defined by

$$\langle \bar{s}_\lambda, \underline{s}_\mu \rangle = \delta_{\lambda,\mu}.$$

Dual symmetric Grothendieck functions were first introduced explicitly in [22] in relation to the *K-homology* of Γ_k; however, they are also implicit in the earlier work [6]. Each $\underline{s}_\lambda$ contains terms of degree less than or equal to $|\lambda|$; moreover, the sum of all of its highest-degree terms is equal to s_λ. Although an attractive tableau formula for $\underline{s}_\lambda$ was given in [22], we do not recall it here, as we will not need it.

2.3 Graphs and coloring

Here, we recall basic notions, terminology, and notations from graph theory. For further details, see the textbooks [14, 47].

A ***graph*** G consists of a set V of ***vertices***, and a set E of unordered pairs of distinct vertices called ***edges***. All graphs in this article are *simple*, so there are no loops and no multi-edges. When $\{v_1, v_2\} \in E(G)$, we will typically denote this edge by v_1v_2 and say v_1 and v_2 are ***adjacent***. Two graphs G, G' are ***isomorphic*** if there is a bijection $\phi : V(G) \to V(G')$ such that, for all vertices $v, w \in V(G)$, we have $vw \in E(G)$ if and only if $\phi(v)\phi(w) \in E(G')$. In this article, we consider graphs up to isomorphism.

The ***complete graph*** K_d with d vertices is the graph such that $V(K_d) = [d]$, and

$$E(K_d) = \{vw : v, w \in [d], v \neq w\}.$$

The n-vertex ***path*** P_n has vertex set $V(P_n) = [n]$ and edge set $E(P_n) = \{uv : u, v \in [n], v - u = 1\}$. The ***claw*** $K_{1,3}$ has vertex set $V(K_{1,3}) = [4]$ and edge set $E(K_{1,3}) = \{\{1, 2\}, \{1, 3\}, \{1, 4\}\}$.

An ***induced subgraph*** of a graph G is a graph H such that $V(H) \subseteq V(G)$ and

$$E(H) = \{vw \in E(G) : v, w \in V(H)\}.$$

We say the graph G is ***H-free*** if no induced subgraph of G is isomorphic to H. We will be especially interested in claw-free graphs.

https://doi.org/10.4153/S0008439526101830 Published online by Cambridge University Press

A ***stable set*** (or ***independent set***) of a graph G is a set $S \subseteq V(G)$ of vertices such that for each $v, w \in S$, $vw \notin E(G)$. A ***clique*** of a graph G is a set $S \subseteq V(G)$ of vertices such that for each $v \neq w \in S$, $vw \in E(G)$.

For $\alpha : V(G) \to \mathbb{N}$ a vertex weight function of the graph G, the ***α-clan graph of G*** is the graph $C_\alpha(G)$ obtained by blowing up each vertex v into a clique of $\alpha(v)$ vertices. More formally, $C_\alpha(G)$ has vertex set $V(C_\alpha(G)) = \{(v, i) : v \in V(G), i \in [\alpha(v)]\}$. In $C_\alpha(G)$, the vertices (v, i) and (w, j) are adjacent either if $vw \in E(G)$ or if both $v = w$ and $i \neq j$.

Given a vertex $v \in V(G)$, its ***open neighborhood*** $N(v)$ is defined by $N(v) = \{w : vw \in E(G)\}$. Given $S \subseteq V(G)$ and $v \in V(G)$ with $v \notin S$, we let $vS \subseteq E(G)$ denote the set of edges $\{vs : s \in S\}$. The ***contraction*** of a graph G by a pair of distinct vertices $v, w \in V(G)$, denoted G/vw, is the graph with vertex set

$$V(G/vw) = (V(G)\backslash\{v, w\}) \cup \{z_{vw}\},$$

where z_{vw} is a new vertex, and edge set

$$E(G/vw) = \big(E(G)\backslash\big(vN(v) \cup wN(w)\big)\big) \cup \big(z_{vw}N(v) \cup z_{vw}N(w)\big).$$

A ***coloring*** of a graph G is a function $\kappa : V(G) \to \mathbb{N}$. A coloring κ of G is ***proper*** if $\kappa(a) \neq \kappa(b)$ whenever $ab \in E(G)$.

The ***chromatic symmetric function*** [39] of a graph G is the power series

$$X_G = \sum_{\kappa} \prod_{v \in V(G)} x_{\kappa(v)}$$

where the first sum ranges over all proper colorings κ of G. Note that, for every graph G, $X_G \in \mathrm{Sym}$.

2.4 Posets and their incomparability graphs

A ***poset*** (partially-ordered set) $(P, \leq)$ is a set P together with a binary relation $\leq$ that is ***transitive*** ($a \leq b$ and $b \leq c$ implies $a \leq c$), ***reflexive*** ($a \leq a$), and ***weakly antisymmetric*** ($a \leq b$ and $b \leq a$ implies $a = b$). For $a, b \in P$, we write $a < b$ if $a \leq b$ and $a \neq b$. We often write P as shorthand for $(P, \leq)$ and decorate the relation as $\leq_P$ for clarity as needed. For more background on posets than is provided here, see [47].

When $a, b \in P$ are such that $a \not\leq b$ and $b \not\leq a$, we say a and b are ***incomparable***. We write $\mathbf{n}$ for the unique totally ordered n-element poset and call such a poset a ***chain***. The ***sum*** $P + Q$ of posets $(P, \leq_P), (Q, \leq_Q)$ is the disjoint union of sets $P \sqcup Q$ with the relation $a \leq_{P+Q} b$ if and only if either $a, b \in P$ with $a \leq_P b$ or $a, b \in Q$ with $a \leq_Q b$.

We say $(Q, \leq_Q)$ is a ***subposet*** of $(P, \leq_P)$ if Q is a subset of P and, for all $a, b \in Q$, we have $a \leq_Q b$ if and only if $a \leq_P b$. Two posets $(P, \leq_P), (Q, \leq_Q)$ are ***isomorphic*** if there is a bijection $\phi : P \to Q$ such that, for all $a, b \in P$, we have $a \leq_P b$ if and only if $\phi(a) \leq_Q \phi(b)$. If $(P, \leq_P), (Q, \leq_Q)$ are any two posets, we say that $(P, \leq_P)$ is ***$(\mathbf{Q}, \leq_\mathbf{Q})$-free*** if no subposet of P is isomorphic to Q. We will be mostly interested in posets that are $(\mathbf{3} + \mathbf{1})$-free.

Associated with any poset P is its ***incomparability graph*** $I(P)$. This is the graph whose vertex set is $V(I(P)) = P$ and whose edge set is $E(I(P)) = \{ab : a, b \in P, a \not\leq b, b \not\leq a\}$. That is to say, edges connect incomparable elements of the poset. It is

https://doi.org/10.4153/S0008439526101830 Published online by Cambridge University Press

straightforward to see that the poset P is $(\mathbf{3}+\mathbf{1})$-free if and only if its incomparability graph is claw-free; however, many claw-free graphs are not incomparability graphs of posets.

3 The Kromatic symmetric function

3.1 Main definition

A ***vertex-weighted graph*** (G, α) consists of a graph G together with a function $\alpha: V(G) \to \mathbb{N}$; we call α the ***weight function*** on the vertices of G. A ***proper $\boldsymbol{\alpha}$-coloring*** of G is a function $\kappa : V(G) \to 2^{\mathbb{N}} \setminus \{\varnothing\}$ assigning to each $v \in V(G)$ a set of $\alpha(v)$ distinct colors in $\mathbb{N}$, subject to the constraint that when $uv \in E(G)$, we have $\kappa(u) \cap \kappa(v) = \varnothing$. Note that these conditions are equivalent to saying that every choice of a single element from each $\kappa(v)$ yields a proper coloring of G. A ***proper set coloring*** of G is a proper α-coloring for some weight function on the vertices of G.

The ***set chromatic symmetric function*** of the vertex-weighted graph (G, α) is

$$X_G^\alpha = \sum_\kappa \prod_{v \in V(G)} \prod_{i \in \kappa(v)} x_i,$$

where the first sum runs over all proper α-colorings of G. Note that up to a scalar factor depending only on α, the set chromatic symmetric function X_G^α equals the chromatic symmetric function $X_{C_\alpha(G)}$ of the α-clan graph of G.

Definition 3.1 The ***Kromatic symmetric function*** of a graph G is the symmetric power series

$$\overline{X}_G = \sum_\alpha X_G^\alpha,$$

where α ranges over all weight functions of the vertex set $V(G)$.

In other words, $\overline{X}_G$ enumerates all colorings of G by nonempty sets of colors, such that adjacent vertices receive disjoint sets of colors. Note that $\overline{X}_G$ is not a homogeneous symmetric function, but rather consists of X_G plus terms of degree higher than $|V(G)|$.

Remark 3.2 Stanley [40] considered a function Y_G related to $\overline{X}_G$, although with two differences. Firstly, Y_G uses the rescaled power series $X_{C_\alpha(G)}$ in place of X_G^α. Secondly, Y_G allows $\alpha(v) = 0$, whereas the Kromatic symmetric function $\overline{X}_G$ only considers strictly positive vertex weights. We are unaware of any further study of the functions Y_G since their introduction in [40].

Remark 3.3 It is easy to observe that the Kromatic symmetric function $\overline{X}_G$ of any graph G is m-positive. Moreover, one may also check that $\overline{X}_G$ is positive in the basis $\{\omega(p_\lambda)\}_\lambda$, where p_λ denotes the *power sum symmetric function* and ω is the standard involution on symmetric functions.

https://doi.org/10.4153/S0008439526101830 Published online by Cambridge University Press

Although weight functions are used in the definition of $\overline{X}_G$, the function $\overline{X}_G$ is independent of any particular one. We will find it useful to also consider a vertex-weighted analog of $\overline{X}_G$. Let α and ω be independent vertex weight functions on G. Define

$$X^{\alpha}_{(G,\omega)} = \sum_{\kappa} \prod_{v \in V(G)} \left(\prod_{i \in \kappa(v)} x_i \right)^{\omega(v)},$$

where again the first sum runs over all proper α-colorings of G. Finally, we define the ***vertex-weighted Kromatic symmetric function*** of the vertex-weighted graph (G, ω) to be

$$\overline{X}_{(G,\omega)} = \sum_{\alpha} X^{\alpha}_{(G,\omega)},$$

where the sum is over all weight functions α. In this way, $\overline{X}_{(G,\omega)}$ is a generating function for proper set colorings of G.

3.2 A *K*-theoretic monomial expansion

For λ an integer partition, let K_λ denote the vertex-weighted complete graph $(K_{\ell(\lambda)}, \omega)$, where $\omega(i) = \lambda_i$ for each i. It is straightforward to see that $X_{K_\lambda} = \widetilde{m}_\lambda$, the augmented monomial symmetric function. Thus, by analogy, we define

$$\overline{\widetilde{m}}_\lambda \coloneqq \overline{X}_{K_\lambda}.$$

We call $\overline{\widetilde{m}}_\lambda$ the ***K-theoretic augmented monomial symmetric function***. To justify this definition, we show that the Kromatic symmetric function of every graph (even every vertex-weighted graph) is a positive sum of K-theoretic augmented monomial symmetric functions.

First, we need some additional definitions. We define a ***stable set cover*** C of a graph G to be a collection of (distinct) stable sets of G such that every vertex of $V(G)$ is in at least one element of C. In symbols, this means that

$$\bigcup_{S \in C} S = V(G);$$

note that this union is not required to be disjoint. We write $\mathsf{SSC}(G)$ for the family of all stable set covers of G. For $C \in \mathsf{SSC}(G)$, if G is endowed with a vertex weight function ω, let $\lambda(C)$ be the partition of length $|C|$ whose parts are $\sum_{v \in S} \omega(v)$ for $S \in C$. Finally, let the ***color class*** of the color i in a proper set coloring κ be

$$\{v \in V(G) : i \in \kappa(v)\},$$

the set of vertices of G that receive color i (possibly among other colors) under κ.

Proposition 3.4 *For any vertex-weighted graph (G, ω), we have*

$$\overline{X}_{(G,\omega)} = \sum_{C \in \mathsf{SSC}(G)} \overline{\widetilde{m}}_{\lambda(C)}.$$

https://doi.org/10.4153/S0008439526101830 Published online by Cambridge University Press

Proof The monomials of $\overline{X}_{(G,\omega)}$ correspond to proper set colorings κ of G. For each such κ, note that the set of its color classes is a stable set cover C_κ of G.

For each $C \in \mathsf{SSC}(G)$, the monomials of $\overline{\widetilde{m}}_{\lambda(C)}$ enumerate all proper set colorings κ of G such that:

- each $S \in C$ is the color class of at least one color i under κ, and
- for each nonempty $T \subseteq V(G)$ with $T \notin C$, there is no color j such that T is the color class of j under κ.

In other words, the monomials of $\overline{\widetilde{m}}_{\lambda(C)}$ correspond to all proper set colorings κ of G such that $C_\kappa = C$.

Since this correspondence between the monomials of $\overline{X}_{(G,\omega)}$ and those of $\sum_{C\in\mathsf{SSC}(G)} \overline{\widetilde{m}}_{\lambda(C)}$ is a weight-preserving bijection, the two power series are equal. ■

We define the ***K-theoretic power sum symmetric function*** $\overline{p}_\lambda$ to be the Kromatic symmetric function of a disjoint union of vertices with weights $\lambda_1, \ldots, \lambda_{\ell(\lambda)}$. Observe that the ordinary chromatic symmetric function of this graph yields the classical power sum symmetric function p_λ. For some small graphs G, the expansions of $\overline{X}_G$ in the $\overline{\widetilde{m}}_\lambda$-basis and the $\overline{p}$-basis are collected in Table 1.

Remark 3.5 It is natural to ask if the classical p-basis expansions of X_G lift to $\overline{p}$-basis expansions of $\overline{X}_G$. However, it is unclear what this expansion would look like. Attempting to naively modify Stanley's inclusion–exclusion proof [39, Theorem 2.5] of this expansion for unweighted graphs fails because it uses the fact that if we take a connected graph G and evaluate $\sum_\kappa \prod_{v\in V(G)} x_{\kappa(v)}$ over all κ such that adjacent vertices receive the same color, this yields $p_{|V(G)|}$, since all vertices must have the same color. But in the Kromatic case, the corresponding statement is that color sets of adjacent vertices have nonempty intersection, which yields many possibilities for the corresponding sum over all such colorings. In particular, the result depends on more than just $|V(G)|$, making analysis more difficult. It would be interesting to modify this expansion in a way that explains, for example, the data of Table 1. (Since this manuscript first appeared as a preprint, Laura Pierson [34] has given a complicated formula for the $\overline{p}$-basis expansion of $\overline{X}_G$, which in particular establishes its integrality.)

3.3 A deletion–contraction relation

The Kromatic symmetric function for vertex-weighted graphs also admits a deletion–contraction relation, analogous to that of [10] for the chromatic symmetric function, although somewhat more complicated. We first need to set up some additional notation.

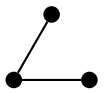
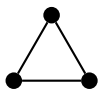
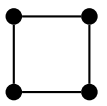
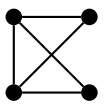
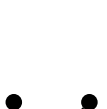
Recall that, given $S \subseteq V(G)$ and $v \in V(G)$ with $v \notin S$, vS denotes the set of edges $\{vs : s \in S\} \subseteq E(G)$. Let (G, ω) be a vertex-weighted graph, and let v, w be distinct vertices such that $e = vw \notin E(G)$. The graph $G^\star$ has vertex set

$$V(G^\star) = V(G) \cup \{z^\star\},$$

https://doi.org/10.4153/S0008439526101830 Published online by Cambridge University Press

https://doi.org/10.4153/S0008439526101830 Published online by Cambridge University Press

Table 1: Kromatic symmetric functions of some small graphs as determined by implementing the deletion–contraction relation of Proposition 3.7 in Python, expressed in the K-theoretic $\overline{\widetilde{m}}$-basis, as well as in the $\overline{p}$-basis. Since the latter expansion is infinite, we write explicitly only the $\overline{p}_\lambda$ with $|\lambda| \leq |V(G)| + 1$, suppressing higher order terms ("h.o.t.").

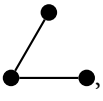
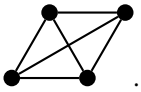
Graph G	$\overline{X}_G$ in the $\overline{\widetilde{m}}$-basis	$\overline{X}_G$ in the $\overline{p}$-basis
	$\overline{\widetilde{m}}_{(1^3)} + \overline{\widetilde{m}}_{(2,1)} + 2\overline{\widetilde{m}}_{(2,1^2)} + \overline{\widetilde{m}}_{(2,1^3)}$	$\overline{p}_{(1^3)} - 2\overline{p}_{(2,1)} + \overline{p}_{(3)} - 4\overline{p}_{(2,1^2)} + \overline{p}_{(2^2)} + 7\overline{p}_{(3,1)} - 4\overline{p}_{(4)} + \text{h.o.t.}$
	$\overline{\widetilde{m}}_{(1^3)}$	$\overline{p}_{(1^3)} - 3\overline{p}_{(2,1)} + 2\overline{p}_{(3)} - 6\overline{p}_{(2,1^2)} + 3\overline{p}_{(2^2)} + 12\overline{p}_{(3,1)} - 9\overline{p}_{(4)} + \text{h.o.t.}$
	$\overline{\widetilde{m}}_{(1^4)} + 2\overline{\widetilde{m}}_{(2,1^2)} + \overline{\widetilde{m}}_{(2^2)} + 4\overline{\widetilde{m}}_{(2,1^3)} + 4\overline{\widetilde{m}}_{(2^2,1)} + 2\overline{\widetilde{m}}_{(2,1^4)} + 6\overline{\widetilde{m}}_{(2^2,1^2)} + 4\overline{\widetilde{m}}_{(2^2,1^3)} + \overline{\widetilde{m}}_{(2^2,1^4)}$	$\overline{p}_{(1^4)} - 4\overline{p}_{(2,1^2)} + 2\overline{p}_{(2^2)} + 4\overline{p}_{(3,1)} - 3\overline{p}_{(4)} - 8\overline{p}_{(2,1^3)} + 12\overline{p}_{(2^2,1)} + 20\overline{p}_{(3,1^2)} - 16\overline{p}_{(3,2)} - 28\overline{p}_{(4,1)} + 20\overline{p}_{(5)} + \text{h.o.t.}$
	$\overline{\widetilde{m}}_{(1^4)} + \overline{\widetilde{m}}_{(2,1^2)} + 2\overline{\widetilde{m}}_{(2,1^3)} + \overline{\widetilde{m}}_{(2,1^4)}$	$\overline{p}_{(1^4)} - 5\overline{p}_{(2,1^2)} + 2\overline{p}_{(2^2)} + 6\overline{p}_{(3,1)} - 4\overline{p}_{(4)} - 10\overline{p}_{(2,1^3)} + 28\overline{p}_{(3,1^2)} + 16\overline{p}_{(2^2,1)} - 22\overline{p}_{(3,2)} - 42\overline{p}_{(4,1)} + 30\overline{p}_{(5)} + \text{h.o.t.}$
	$\overline{\widetilde{m}}_{(1^4)} + 3\overline{\widetilde{m}}_{(2,1^2)} + \overline{\widetilde{m}}_{(3,1)} + 6\overline{\widetilde{m}}_{(2,1^3)} + 3\overline{\widetilde{m}}_{(2^2,1)} + 3\overline{\widetilde{m}}_{(3,1^2)} + 3\overline{\widetilde{m}}_{(2,1^4)} + 9\overline{\widetilde{m}}_{(2^2,1^2)} + 3\overline{\widetilde{m}}_{(3,1^3)} + 3\overline{\widetilde{m}}_{(3,2,1)} + 9\overline{\widetilde{m}}_{(2^2,1^3)} + \overline{\widetilde{m}}_{(2^3,1)} + 9\overline{\widetilde{m}}_{(3,2,1^2)} + \overline{\widetilde{m}}_{(3,1^4)} + 3\overline{\widetilde{m}}_{(2^2,1^4)} + 3\overline{\widetilde{m}}_{(2^3,1^2)} + 9\overline{\widetilde{m}}_{(3,2,1^3)} + 3\overline{\widetilde{m}}_{(3,2^2,1)} + 9\overline{\widetilde{m}}_{(3,2^2,1^2)} + 3\overline{\widetilde{m}}_{(3,2,1^4)} + 3\overline{\widetilde{m}}_{(2^3,1^3)} + 9\overline{\widetilde{m}}_{(3,2^2,1^3)} + \overline{\widetilde{m}}_{(3,2^3,1)} + \overline{\widetilde{m}}_{(2^3,1^4)} + 3\overline{\widetilde{m}}_{(3,2^2,1^4)} + 3\overline{\widetilde{m}}_{(3,2^3,1^2)} + 3\overline{\widetilde{m}}_{(3,2^3,1^3)} + \overline{\widetilde{m}}_{(3,2^3,1^4)}$	$\overline{p}_{(1^4)} - 3\overline{p}_{(2,1^2)} + 3\overline{p}_{(3,1)} - \overline{p}_{(4)} - 6\overline{p}_{(2,1^3)} + 3\overline{p}_{(2^2,1)} + 15\overline{p}_{(3,1^2)} - 3\overline{p}_{(3,2)} - 16\overline{p}_{(4,1)} + 7\overline{p}_{(5)} + \text{h.o.t.}$

where $z^\star$ is a new vertex, and edge set

$$E(G^\star) = E(G) \cup \{vw, vz^\star, wz^\star\} \cup z^\star N(v) \cup z^\star N(w).$$

If G has a vertex weight function ω, we define an induced vertex weight function $\omega^\star$ on $G^\star$ by

$$\omega^\star(u) = \begin{cases} \omega(v) + \omega(w), & \text{if } u = z^\star; \\ \omega(u), & \text{if } u \in V(G). \end{cases}$$

Example 3.6 If G is the tree , with the leaves being v and w, then $G^\star$ is the complete graph .

We also define graphs G^1, G^2 with vertex sets

$$V(G^i) = V(G)$$

and edge sets

$$E(G^1) = E(G) \cup e \cup vN(w) \quad \text{and} \quad E(G^2) = E(G) \cup e \cup wN(v).$$

When G has a vertex weight function ω, there are induced vertex weight functions ω^i on G^i given by

$$\omega^1(u) = \begin{cases} \omega(v) + \omega(w), & \text{if } u = v; \\ \omega(u), & \text{otherwise;} \end{cases}$$

and

$$\omega^2(u) = \begin{cases} \omega(v) + \omega(w), & \text{if } u = w; \\ \omega(u), & \text{otherwise.} \end{cases}$$

In the contracted graph G/e, we give it the weight function ω/e defined by

$$(\omega/e)(u) = \begin{cases} \omega(v) + \omega(w), & \text{if } u = z_{vw}; \\ \omega(u), & \text{otherwise.} \end{cases}$$

Finally, let $G \cup e$ be the graph $(V(G), E(G) \cup \{e\})$.

Proposition 3.7 *Let (G, ω) be a vertex-weighted graph, and let v and w be distinct vertices such that $e = vw \notin E(G)$. Then*

$$\overline{X}_{(G,\omega)} = \overline{X}_{(G/e,\omega/e)} + \overline{X}_{(G\cup e,\omega)} + \overline{X}_{(G^1,\omega^1)} + \overline{X}_{(G^2,\omega^2)} + \overline{X}_{(G^\star,\omega^\star)}. \tag{3.1}$$

Proof The proof is a direct bijection between the proper set colorings contributing to the left and right sides of Equation (3.1) as indicated below. In each case, it is

https://doi.org/10.4153/S0008439526101830 Published online by Cambridge University Press

straightforward to verify that the given correspondence is reversible, and that the monomials produced by the corresponding colorings are identical.

- Proper set colorings κ of (G, ω) such that $\kappa(v) = \kappa(w)$ correspond to all proper set colorings κ/e of $(G/e, \omega/e)$ by
$$(\kappa/e)(u) = \begin{cases} \kappa(v), & \text{if } u = z_{vw}; \\ \kappa(u), & \text{otherwise.} \end{cases}$$
- Proper set colorings κ of (G, ω) such that $\kappa(v) \cap \kappa(w) = \varnothing$ are in exact correspondence with all the proper set colorings of $(G \cup e, \omega)$.
- Proper set colorings κ of (G, ω) such that $\kappa(v) \subsetneq \kappa(w)$ correspond to all proper set colorings κ^1 of (G^1, ω^1) by
$$\kappa^1(u) = \begin{cases} \kappa(v), & \text{if } u = v; \\ \kappa(w) \backslash \kappa(v), & \text{if } u = w; \\ \kappa(u), & \text{otherwise.} \end{cases}$$
- Proper set colorings κ of (G, ω) such that $\kappa(w) \subsetneq \kappa(v)$ correspond to all proper set colorings κ^2 of (G^2, ω^2) by
$$\kappa^2(u) = \begin{cases} \kappa(v) \backslash \kappa(w), & \text{if } u = v; \\ \kappa(w), & \text{if } u = w; \\ \kappa(u), & \text{otherwise.} \end{cases}$$
- Proper set colorings κ of (G, ω) that fit into none of the previous categories (i.e., those such that each of the sets
$$\kappa(v) \cap \kappa(w), \kappa(v) \backslash \kappa(w), \kappa(w) \backslash \kappa(v)$$
are nonempty) correspond to all the proper set colorings $\kappa^\star$ of $(G^\star, \omega^\star)$ by
$$\kappa^\star(u) = \begin{cases} \kappa(v) \cap \kappa(w), & \text{if } u = z^\star; \\ \kappa(v) \backslash \kappa(w), & \text{if } u = v; \\ \kappa(w) \backslash \kappa(v), & \text{if } u = w; \\ \kappa(u), & \text{otherwise.} \end{cases}$$

This completes the proof of the deletion–contraction relation. ∎

The deletion–contraction relation of Proposition 3.7 can be used to yield algorithmically the $\widetilde{\widetilde{m}}_\lambda$-expansion of a Kromatic symmetric function $\overline{X}_{(G,\omega)}$ in an alternative fashion to Proposition 3.4. Define the ***total stability*** of a graph G to be $\mathtt{ts}(G) = |\mathsf{SS}(G)| - |V(G)|$, where $\mathsf{SS}(G)$ denotes the collection of all stable sets of G. Since any single vertex of a graph is a stable set, we may view the total stability as the number of nontrivial stable sets. Thus, note that $\mathtt{ts}(G) \geq 0$ and that equality holds if and only if G is a complete graph.

Corollary 3.8 *Recursively applying Proposition 3.7 to a vertex-weighted graph (G, ω) (iteratively applying it to an arbitrary nonedge of each non-complete graph formed) terminates in a sum of Kromatic symmetric functions of vertex-weighted complete graphs, yielding the $\widetilde{\widetilde{m}}_\lambda$ expansion of $\overline{X}_{(G,\omega)}$.*

https://doi.org/10.4153/S0008439526101830 Published online by Cambridge University Press

Proof We proceed by induction on the total stability $\mathsf{ts}(G)$. If $\mathsf{ts}(G) = 0$, then G is a complete graph and the result is trivial. Otherwise, it is sufficient to show that after applying Proposition 3.7 to (G, ω), each of the resulting five graphs

$$G/e, G \cup e, G^1, G^2, G^\star$$

has strictly smaller total stability than G does. We consider each of these five graphs in turn.

- $(\underline{G/e})$: We have $|V(G/e)| = |V(G)| - 1$. On the other hand, the stable sets of G/e not containing z_{vw} are in obvious bijection with the stable sets of G containing neither v nor w. Moreover, there is a bijection between the stable sets of G/e containing z_{vw} and $\{S \in \mathsf{SS}(G) : v, w \in S\}$. Together, this gives a bijection between stable sets of G/e and those stable sets of G which contain either both or neither of v and w. Since $\{v\}$ and $\{w\}$ are stable sets of G, we have $|\mathsf{SS}(G/e)| \leq |\mathsf{SS}(G)| - 2$, and so

  $$\mathsf{ts}(G/e) = |\mathsf{SS}(G/e)| - |V(G/e)| \leq \big(|\mathsf{SS}(G)| - 2\big) - \big(|V(G)| - 1\big) = \mathsf{ts}(G) - 1,$$

  as needed.
- $(\underline{G \cup e})$: We have $|V(G \cup e)| = |V(G)|$. Clearly, $\mathsf{SS}(G \cup e) \subseteq \mathsf{SS}(G)$. However, this inclusion is strict since $\{v, w\} \in \mathsf{SS}(G) \backslash \mathsf{SS}(G \cup e)$. Thus, $\mathsf{ts}(G \cup e) < \mathsf{ts}(G)$.
- $(\underline{G^1})$: We have $|V(G^1)| = |V(G)|$. Again, it is clear that $\mathsf{SS}(G^1) \subseteq \mathsf{SS}(G)$ and the inclusion is strict since $\{v, w\} \in \mathsf{SS}(G) \backslash \mathsf{SS}(G^1)$.
- $(\underline{G^2})$: The analysis is the same as for G^1.
- $(\underline{G^\star})$: We have $|V(G^\star)| = |V(G)| + 1$. Let $X = \{S \in \mathsf{SS}(G) : v, w \in S\}$ and let $Y = \mathsf{SS}(G) \backslash X$. There is an obvious injection of $\{S : \mathsf{SS}(G^\star) : z^\star \notin S\}$ into Y, since $vw \in E(G^\star)$. We may also biject $\{S : \mathsf{SS}(G^\star) : z^\star \in S\}$ with X by mapping $S \mapsto (S \backslash \{z^\star\}) \cup \{v, w\}$. This latter map is well-defined since such an S does not include v, w, or any vertex in $N(v)$ or $N(w)$. Combining these bijections yields a bijection of $\mathsf{SS}(G^\star)$ with $\mathsf{SS}(G)$, so $|\mathsf{SS}(G^\star)| = |\mathsf{SS}(G)|$. We conclude that

  $$\mathsf{ts}(G^\star) = |\mathsf{SS}(G^\star)| - |V(G^\star)| \leq |\mathsf{SS}(G)| - (|V(G)| + 1) < |\mathsf{SS}(G)| - |V(G)| = \mathsf{ts}(G),$$

  as needed.

Therefore, the corollary follows by induction on ts. ∎

3.4 Grothendieck positivity

In 1996, Gasharov [15] proved that X_G is Schur-positive for G a claw-free incomparability graph of a poset P by showing that $[s_\lambda]X_G$ enumerates objects he called *P-tableaux*. We now define a generalization of these objects that is enumerated by $[\bar{s}_\lambda]\overline{X}_G$. Informally, a *Grothendieck P-tableau of shape* λ consists of a P-tableau (as defined in [15]) of shape μ for some $\mu \subseteq \lambda$ layered with a semistandard Young tableau of shape λ/μ that also satisfies "flagging" restrictions on each row. (Similar flagging conditions appear in [24] with relation to Schur-basis expansions of Grothendieck symmetric functions; we do not know a direct relation between our Grothendieck P-tableaux and [24], nor with the *flagged tableaux* of [46].)

https://doi.org/10.4153/S0008439526101830 Published online by Cambridge University Press

Definition 3.9 Let P be a poset and λ an integer partition. A ***Grothendieck P-tableau of shape $\boldsymbol{\lambda}$*** is a filling T of the cells of the Young diagram of λ with elements of $P \sqcup \mathbb{N}$ such that:

- the cells filled with elements of P form the Young diagram of some partition $\mu \subseteq \lambda$ (and so the cells filled with positive integers form the Young diagram of the skew shape λ/μ);
- for each $p \in P$, there exists at least one cell c with $T(c) = p$;
- for each cell c with $T(\mathsf{c}) \in P$,
  - we have $T(\mathsf{c}) <_P T(\mathsf{c}^{\rightarrow})$, if $T(\mathsf{c}^{\rightarrow}) \in P$, and
  - we have $T(\mathsf{c}) \not>_P T(\mathsf{c}^{\downarrow})$, if $T(\mathsf{c}^{\downarrow}) \in P$;
- for each cell c with $T(\mathsf{c}) \in \mathbb{N}$,
  - we have $T(\mathsf{c}) \leq T(\mathsf{c}^{\rightarrow})$, if $T(\mathsf{c}^{\rightarrow}) \in \mathbb{N}$,
  - we have $T(\mathsf{c}) < T(\mathsf{c}^{\downarrow})$, if $T(\mathsf{c}^{\downarrow}) \in \mathbb{N}$, and
  - we have $T(\mathsf{c}) \leq i - 1$, if c is in row i (in particular, the first row contains no positive integers).

Example 3.10 Let $\lambda = (3,3,2,1)$. Consider the poset $P = \mathbf{3} + \mathbf{1}$ with four elements a, b, c, d satisfying $a < b < c$ and no other inequalities. Then

a	b	c
d	1	1
b	2	
1		

and

a	b	c
a	c	1
d	1	
d		

are Grothendieck P-tableaux of shape λ.

Theorem 3.11 *If G is a claw-free incomparability graph, then the Kromatic symmetric function $\overline{X}_G$ is Grothendieck-positive. Moreover, the coefficient $[\bar{s}_\lambda]\overline{X}_G$ counts the number of Grothendieck P-tableaux of shape λ.*

Proof The basic structure of our proof is as follows. We use a generalized Jacobi–Trudi formula [23, Equation (4)] to write a dual symmetric Grothendieck function as a sum of products of complete homogeneous symmetric functions. Then, for any graph G, the inner product of this expression with $\overline{X}_G$ yields a formula for the coefficient of $\bar{s}_\lambda$ in the Grothendieck expansion of $\overline{X}_G$ in terms of its monomial expansion. In the case that G is a claw-free incomparability graph, we then extend Gasharov's [15] theory of *P-arrays* to collect terms in this expansion and extend his sign-reversing involution to cancel all terms except those corresponding to Grothendieck P-tableaux.

Now, we give the details of this argument. Let G be a claw-free incomparability graph and let P be a poset such that $G = I(P)$. The graph G being claw-free is equivalent to the poset P being $(\mathbf{3} + \mathbf{1})$-free.

Fix a positive integer $n \in \mathbb{N}$ and a partition $\lambda \vdash n$. Let $k = \ell(\lambda)$. Let $N \geq 2n$ be fixed, and let S_N denote the symmetric group of permutations of $[N]$, with identity element id_{S_N}. For $\pi \in S_N$, the ***sign*** of π is

$$\operatorname{sgn} \pi = \begin{cases} +1, & \text{if } \pi \text{ is in the } \textit{alternating group } A_N \subset S_N; \\ -1, & \text{otherwise.} \end{cases}$$

https://doi.org/10.4153/S0008439526101830 Published online by Cambridge University Press

We will write

$$\left(\!\!\binom{n}{k}\!\!\right) \coloneqq \binom{n+k-1}{k}$$

as a shorthand for the number of k-element *multisets* with elements of n types.

We will use the notation that for f a symmetric function, $f[x_N] \coloneqq f(x_1, x_2, \dots, x_N, 0, 0, \dots)$ is a restriction to finitely many variables, and $f[x_N + r] \coloneqq f(x_1, \dots, x_N, 1, 1, \dots, 1, 0, 0, \dots)$ is a restriction to $N + r$ variables with r of them set equal to 1. Recall the dual symmetric Grothendieck function $\underline{s}_\lambda$ from Section 2.2. Lascoux and Naruse [23, Equation (4)] show that $\underline{s}_\lambda$ may be expanded as

$$\begin{aligned}\underline{s}_\lambda[x_N] &= \det(s_{\lambda_i - i + j}[x_N + i - 1]) \\ &= \sum_{\pi \in S_N} \operatorname{sgn}(\pi) \prod_i s_{\lambda_i - i + \pi(i)}[x_N + i - 1],\end{aligned}$$

a K-theoretic analog of the Jacobi–Trudi formula.

We apply here the simplification from just before [23, Equation (4)],

$$s_m[x_N + r] = \sum_{j=0}^{m} \binom{r+j-1}{j} s_{m-j}[x_N] = \sum_{j=0}^{m} \left(\!\!\binom{r}{j}\!\!\right) s_{m-j}[x_N],$$

to simplify the above to

$$\begin{aligned}\underline{s}_\lambda[x_N] &= \sum_{\pi \in S_N} \operatorname{sgn}(\pi) \prod_{i=1}^{\check{\mathrm{R}}(\lambda)} \left(\sum_{l=0}^{\lambda_i - i + \pi(i)} \binom{i+l-2}{l} s_{\lambda_i - i + \pi(i) - l}[x_N] \right) \\ &= \sum_{\pi \in S_N} \operatorname{sgn}(\pi) \sum_{\substack{(l_1, \dots, l_{l(\lambda)}) \\ l_i \le \lambda_i - i + \pi(i)}} \left(\prod_i \left(\!\!\binom{i-1}{l_i}\!\!\right) h_{\lambda_i - i + \pi(i) - l_i}[x_N] \right)\end{aligned} \tag{3.2}$$

From here, we can pass to general symmetric functions for convenience since equality on a sufficiently large finite number of variables implies equality on infinitely many variables. Thus, we find that

$$\begin{aligned}\underline{s}_\lambda &= \sum_{\pi \in S_N} \operatorname{sgn}(\pi) \sum_{\substack{(l_1, \dots, l_{l(\lambda)}) \\ l_i \le \lambda_i - i + \pi(i)}} \left(\prod_i \left(\!\!\binom{i-1}{l_i}\!\!\right) h_{\lambda_i - i + \pi(i) - l_i} \right) \\ &= \sum_{\pi \in S_N} \operatorname{sgn}(\pi) \sum_{\substack{(l_1, \dots, l_{l(\lambda)}) \\ l_{\pi(i)} \le \lambda_{\pi(i)} - \pi(i) + i}} \left(\prod_{\pi(i)} \left(\!\!\binom{\pi(i)-1}{l_{\pi(i)}}\!\!\right) h_{\lambda_{\pi(i)} - \pi(i) + i - l_{\pi(i)}} \right),\end{aligned} \tag{3.3}$$

where the last equality follows by replacing π by π^{-1}. We also note that an essentially equivalent formula for $\underline{s}_\lambda$ appears in the work of Iwao [21].

Taking the inner product of both sides of Equation (3.3) with $\overline{X}_G$, we find that

$$[\overline{s}_\lambda]\overline{X}_G = \langle \underline{s}_\lambda, \overline{X}_G \rangle = \sum_{\pi \in S_N} \operatorname{sgn}(\pi) \sum_{\substack{(l_1, \dots, l_{l(\lambda)}) \\ l_{\pi(i)} \le \lambda_{\pi(i)} - \pi(i) + i}} \left(\prod_{\pi(i)=1}^{\ell(\lambda)} \left(\!\!\binom{\pi(i)-1}{l_{\pi(i)}}\!\!\right) \right) [m_{\lambda(\pi, l_1, \dots, l_{l(\lambda)})}]\overline{X}_G, \tag{3.4}$$

https://doi.org/10.4153/S0008439526101830 Published online by Cambridge University Press

where $\lambda(\pi, l_1, \ldots, l_{l(\lambda)})$ is the partition whose multiset of parts is $\{\lambda_{\pi(i)} - \pi(i) + i - l_{\pi(i)}\}_{i=1}^{l(\lambda)}$, and $[m_{\lambda(\pi,l_1,\ldots,l_{l(\lambda)})}]\overline{X}_G$ denotes the coefficient of the corresponding monomial symmetric function in the expansion of $\overline{X}_G$.

We now proceed to extend the proof of Gasharov [15] to the Kromatic symmetric function. Let P be a poset. Taking an isomorphic copy of P if necessary, we can assume that no positive integers are elements of P and that $\varnothing$ is not an element of P. A ***Grothendieck P-array of type*** $\boldsymbol{\lambda}$ is a pair (π, A), where $\pi \in S_N$ and A is a map $A : \mathbb{N} \times \mathbb{N} \to P \sqcup \mathbb{N} \sqcup \{\varnothing\}$ satisfying the following properties (we write a_{ij} as shorthand for the element $A((i,j))$):

- a_{ij} must equal $\varnothing$ unless $i \leq \ell(\lambda)$ and $j \leq \lambda_{\pi(i)} - \pi(i) + i$;
- for each $p \in P$, there is some $(i,j) \in \mathbb{N} \times \mathbb{N}$ such that $a_{ij} = p$;
- if $a_{ij} \in P$ for some $(i,j) \in \mathbb{N} \times \mathbb{N}$ with $j > 1$, then $a_{i(j-1)} \in P$ and $a_{i(j-1)} <_P a_{ij}$;
- if $a_{ij} \in \mathbb{N}$, then $a_{ij} \leq \pi(i) - 1$;
- if $a_{ij} \in \mathbb{N}$ for some $(i,j) \in \mathbb{N} \times \mathbb{N}$ with $j > 1$, then either $a_{i(j-1)} \in P$, or $a_{i(j-1)} \in \mathbb{N}$ and $a_{i(j-1)} \leq a_{ij}$.

We generally think of A as a partial filling of an infinite matrix by elements of P, where coordinates (i,j) with $a_{ij} = \varnothing$ are thought of as unfilled. Under this interpretation, the bullet points state that in addition to the restriction on which integers can appear in which row, the entries in each row of A are left-justified and consist of an increasing chain in P followed by a weakly increasing string of positive integers.

From the definitions, it is now straightforward to verify that the sum in Equation (3.4) is equal to

$$[\bar{s}_\lambda]\overline{X}_G = \sum_{(\pi,A)} \operatorname{sgn}(\pi), \tag{3.5}$$

where the sum ranges over all Grothendieck P-arrays of type λ. The choice of π determines the shape of the array, the choice of l_i gives the number of cells in row i that contain positive integers, the m-coefficient covers all choices of poset elements filling the appropriate shape (note that each stable set in G corresponds uniquely to a chain in P), and the product of multiset coefficients covers all possible choices of weakly increasing sequences of positive integers for the rows in the remaining cells.

We claim that the sum in Equation (3.5) evaluates to the number of Grothendieck P-tableaux of shape λ. This will follow by exhibiting a sign-reversing involution Ψ of the set of all pairs (π, A) of Grothendieck P-arrays that are not Grothendieck P-tableaux.

First, note that if π is a non-identity permutation, then for i such that $\pi(i) > \pi(i+1)$ we have $\lambda_{\pi(i)} - \pi(i) + i < \lambda_{\pi(i+1)} - \pi(i+1) + i + 1$, so any Grothendieck P-array with π not equal to the identity permutation is not a Grothendieck P-tableau, as it does not have partition shape.

The sign-reversing involution Ψ that we need is a mild extension of that given by Gasharov in his original proof [15, Proof of Theorem 3]. We call a position (i,j) of a Grothendieck P-array with $i \geq 2$ a ***flaw***

https://doi.org/10.4153/S0008439526101830 Published online by Cambridge University Press

- if $a_{ij} \in P$, and either $a_{(i-1)j} \notin P$ or $a_{ij} <_P a_{(i-1)j}$; or
- if $a_{ij} \in \mathbb{Z}$, and either $a_{(i-1)j} = \varnothing$ or $a_{(i-1)j} \in \mathbb{Z}$ with $a_{ij} \leq a_{(i-1)j}$.

That is, (i, j) is a flaw if it and the cell above it violate the conditions for the array to be a Grothendieck P-tableau. In particular, a Grothendieck P-array is a Grothendieck P-tableau if and only if it has no flaws.

The involution Ψ is as follows. Given (π, A) a non-tableau Grothendieck P-array, let c be the leftmost column in which a flaw occurs. Let r be the bottom-most row in which column c has a flaw. We define $\Psi(\pi, A) = (\pi', A')$, where

- π' is formed by applying the transposition $(r-1 \;\; r)$ to π;
- A' is formed by swapping each $a_{(r-1)j}$ with $a_{r(j+1)}$ for every $j \geq c$ (i.e., swapping the elements of row $r-1$ that are weakly right of column c with those elements of row r that are strictly right of column c).

We write a'_{ij} for the entry in position (i, j) of the array A'.

Clearly, the map Ψ is sign-reversing, so it suffices to show that Ψ takes non-tableau Grothendieck P-arrays to non-tableau Grothendieck P-arrays and that Ψ is an involution. To establish these properties, it is enough to show that Ψ is well-defined and preserves the flaw used (since it is straightforward to observe that no flaw is created to the left of or below the used flaw).

We split into cases based on whether or not $a_{rc} \in P$. If $a_{rc} \in P$ and $c > 1$, note that $a_{r(c-1)} \in P$ as well, so necessarily $a_{(r-1)(c-1)} \in P$, as the opposite would contradict our choice of (r, c) as a leftmost flaw.

Given this observation, it is simple to verify that Ψ is a flaw-preserving involution when $a_{rc} \in P$, by using the same argument as in [15] (the additional cases where some cells contain positive integers are straightforward). The crux of this part of Gasharov's argument is that, if in the newly formed array A', we have that $a'_{(r-1)(c-1)}$ and $a'_{(r-1)c}$ are both in P, then they must satisfy $a'_{(r-1)(c-1)} <_P a'_{(r-1)c}$. This follows from the fact that $a_{(r-1)(c-1)} <_P a_{r(c+1)}$ in A whenever both are in P, which in turn follows from P being a $(\mathbf{3}+\mathbf{1})$-free poset (otherwise consider $a_{(r-1)(c-1)}$ and $a_{r(c-1)} <_P a_{rc} <_P a_{r(c+1)}$). Additionally, the integers in rows r and $r+1$ of A' satisfy the required upper bounds, since these bounds changed correspondingly as π changed to π'.

Suppose instead that $a_{rc} \in \mathbb{N}$. A key point is that if $a_{(r-1)(c-1)}$ exists, then we cannot have $a_{(r-1)(c-1)} = \varnothing$ or $a_{(r-1)(c-1)} \in \mathbb{N}$ with $a_{(r-1)(c-1)} \geq a_{rc}$, as otherwise it is straightforward to see that $(r, c-1)$ is a flaw strictly further left than (r, c). Thus, if it exists, either $a_{(r-1)(c-1)} \in P$ or $a_{(r-1)(c-1)} \in \mathbb{N}$ with $a_{(r-1)(c-1)} < a_{rc}$.

Either way, we may verify that Ψ produces rows consisting of a chain in P followed by a sequence of weakly increasing integers, since before the swap, if $a_{(r-1)(c-1)} \in \mathbb{N}$, then we have $a_{(r-1)(c-1)} < a_{rc} \leq a_{r(c+1)}$. The only extra detail to check is that all integers that remain in their original rows $r-1$ and r are less than or equal to $\pi'(r-1)-1$ and $\pi'(r)-1$, respectively. For row $r-1$, this is immediate, since $a_{(r-1)(c-1)} < a_{rc}$ in A. It is also clear for row r, provided that $a_{(r-1)c} \in \mathbb{N}$, since then $a_{rc} \leq a_{(r-1)c}$ in A.

Thus, we need only consider row r in A' in the case that $a_{(r-1)c} = \varnothing$. In this case, the length of row r of A is strictly larger than the length of row $r-1$, so we have

$$\lambda_r - \pi(r) + r > \lambda_{r-1} - \pi(r-1) + r - 1,$$

https://doi.org/10.4153/S0008439526101830 Published online by Cambridge University Press

which implies that

$$\pi(r-1) > \pi(r) + (\lambda_{r-1} - \lambda_r) - 1 \geq \pi(r) - 1.$$

Therefore, since $\pi(r-1) > \pi(r) - 1$, clearly $\pi'(r) = \pi(r-1) \geq \pi(r)$, so $\pi'(r) - 1 \geq \pi(r) - 1$, and thus the integers that remain in row r after applying Ψ satisfy the appropriate row bound.

In conclusion, Ψ is a sign-reversing involution on non-tableau Grothendieck P-arrays, so all such terms cancel in Equation (3.5), yielding that the coefficient $[\overline{s}_\lambda]\overline{X}_G$ equals the number of Grothendieck P-tableaux of shape λ, as desired. ∎

It is highly suggestive that Theorem 3.11 (and Gasharov's Schur-analog) should have an interpretation and proof via the topology of Grassmannians. We would be very interested in a solution to the following.

Problem 3.12 For each claw-free incomparability graph G, find a corresponding subvariety V_G of the Grassmannian such that the cohomology class of V_G is represented in Sym by X_G and the structure sheaf class of V_G is represented by $\overline{X}_G$.

4 Conjectures

4.1 Analogs of the Stanley–Stembridge conjecture

Section 3.4 shows that Schur-positivity of X_G when G is a claw-free incomparability graph lifts to an analog for $\overline{X}_G$. It is natural to ask if it is similarly possible to lift the ***Stanley-Stembridge conjecture*** – claiming that such X_G are e-positive – to the context of the Kromatic symmetric function. However, it appears that the answer is "no."

We propose two definitions for a lift of the e-basis to the K-theoretic setting. On the one hand, e-basis elements in usual symmetric function theory may be defined in terms of fillings of single-column Young diagrams, so we may lift this formula.

Definition 4.1 The ***tableau K-elementary symmetric function*** $\overline{e}_\lambda$ is given by

$$\overline{e}_n = \overline{s}_{1^n} \quad \text{and} \quad \overline{e}_\lambda = \overline{e}_{\lambda_1} \dots \overline{e}_{\lambda_{\ell(\lambda)}}.$$

On the other hand, we may also define $e_n = \frac{1}{n!} X_{K_n}$, and lift this characterization.

Definition 4.2 The ***graph K-elementary symmetric function*** is given by

$$\overline{e}'_n = \frac{1}{n!}\overline{X}_{K_n} \quad \text{and} \quad \overline{e}'_\lambda = \overline{e}'_{\lambda_1} \dots \overline{e}'_{\lambda_{\ell(\lambda)}}.$$

It is reasonable to hope (for extending the Stanley–Stembridge conjecture) that $\overline{X}_G$ is positive in one of these K-theoretic e-bases, whenever G is a claw-free incomparability graph, or even just when G is a unit interval graph. However, one can compute that $\overline{X}_{P_3}$ is not positive in either K-theoretic e-basis $\{\overline{e}_\lambda\}$ or $\{\overline{e}'_\lambda\}$, dashing any such hopes. (See Table 1 for the $\overline{\widetilde{m}}$-basis expansion of $\overline{X}_{P_3}$.)

https://doi.org/10.4153/S0008439526101830 Published online by Cambridge University Press

The terms of $\overline{X}_{P_3}$ that are homogeneous of degree 3 must come from tableau or graph K-elementary symmetric functions of degree 3, and have coefficients corresponding to e-expansion of X_{P_3}. Since $X_{P_3} = 3e_3 + e_{21}$, one sees that the terms of $\overline{X}_{P_3}$ for $|\lambda| = 3$ in the $\overline{e}$-basis are $3\overline{e}_3 + \overline{e}_{21}$, and in the $\overline{e}'$-basis are $3\overline{e}'_3 + \overline{e}'_{21}$. However, we now encounter problems with the $|\lambda| = 4$ terms. In particular, both $\overline{e}_{21}$ and $\overline{e}'_{21}$ are supported on the monomial $x_1^2x_2^2$, with two distinct variables each of degree 2. However, it is easy to check that there is no proper set coloring of P_3 using exactly 1 twice and 2 twice; thus, these monomials must be cancelled by $\overline{e}_\mu$ or $\overline{e}'_\mu$ terms with strictly negative coefficients.

This breakdown is so fundamental suggests that it may not be possible to reasonably generalize e-positivity to the Kromatic symmetric function, in stark contrast with the generalization of Schur-positivity given in Theorem 3.11. This suggests that the Stanley–Stembridge is not amenable to a topological interpretation along the lines of Problem 3.12.

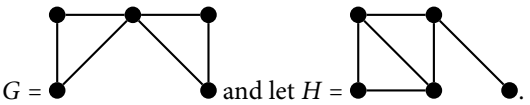
4.2 Distinguishing graphs by $\overline{X}_G$

It is widely believed that the chromatic symmetric function is a complete invariant for ***trees*** (i.e., connected graphs without cycles). We propose the following weakening of this statement as a stepping stone that may be easier to prove.

Conjecture 4.3 *Let T, T' be trees such that $\overline{X}_T = \overline{X}_{T'}$. Then $T \cong T'$.*

As evidence for Conjecture 4.3 being potentially easier than the corresponding statement for X_T, we observe that the Kromatic symmetric function distinguishes some graphs with equal chromatic symmetric function. Indeed, we are not currently aware of any graphs $G \not\cong G'$ with $\overline{X}_G = \overline{X}_{G'}$. (In a follow-up to this article, Laura Pierson [33] conjectures that the Kromatic symmetric function is a complete invariant for all graphs and provides some further evidence toward this conjecture.) [Note added in proof: This conjecture has now been disproved by Pierson and Soham Samanta in [35].]

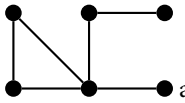
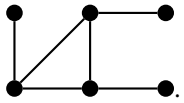
Example 4.4 cf. [39, p. 170] Let



It is straightforward to compute that $X_G = X_H$. By Proposition 3.4, $\overline{X}_H$ has a nonzero coefficient of $\overline{\widetilde{m}}_{(2^3)}$, as one can easily find a covering of H by three stable sets, each of size 2. On the other hand, G has a vertex that is connected to every other vertex, so any stable set containing this vertex must have size 1. Hence, by Proposition 3.4, the $\overline{\widetilde{m}}$-expansion of $\overline{X}_G$ only involves $\overline{\widetilde{m}}_\lambda$ such that λ contains a part of size 1. In particular, $\overline{X}_G \neq \overline{X}_H$.

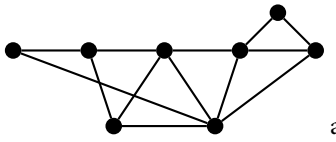
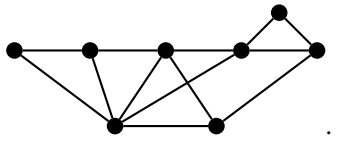
https://doi.org/10.4153/S0008439526101830 Published online by Cambridge University Press

Example 4.5 cf. [31, Figure 9] Let

$G =$  and let $H =$ .

Then $X_G = X_H$. Note that G has a vertex v adjacent to all but one other vertex. Hence, no stable set containing v can have size greater than 2. Therefore, by Proposition 3.4, the $\widetilde{\overline{m}}$-expansion of $\overline{X}_G$ only involves $\widetilde{\overline{m}}_\lambda$ such that λ contains a part of size at most 2. On the other hand, H can be covered in a unique fashion by three stable sets of size 3. Hence, by Proposition 3.4, $\widetilde{\overline{m}}_{(3^3)}$ appears with coefficient 1 in the $\widetilde{\overline{m}}$-expansion of $\overline{X}_H$. Hence, $\overline{X}_G \neq \overline{X}_H$.

Example 4.6 cf. [4, Figure 5] Let

$G =$  and let $H =$ .

Then $X_G = X_H$, and in fact the stronger condition holds that these graphs have equal Tutte symmetric function [4], or equivalently equal U-polynomial [30]. Note that there is a vertex v of H with only two non-neighbors, and that these non-neighbors are adjacent. Hence, no stable set containing v can have size greater than 2. Therefore, by Proposition 3.4, the $\widetilde{\overline{m}}$-expansion of $\overline{X}_H$ only involves $\widetilde{\overline{m}}_\lambda$ such that λ contains a part of size at most 2. On the other hand, G can be covered by stable sets of size 3. Hence, by Proposition 3.4, some $\widetilde{\overline{m}}_\lambda$ where all parts of λ are at least 3 appears with positive coefficient in the $\widetilde{\overline{m}}$-expansion of $\overline{X}_G$. Hence, $\overline{X}_G \neq \overline{X}_H$.

In each of these examples, it is easy to distinguish the graphs' Kromatic symmetric functions because the graphs disagree on $\min_v \max|I_v|$, where the min ranges across all vertices v, and I_v is a stable set containing v. It would be interesting to investigate more generally the extent to which the multiset of numbers $\{\max|I_v| : v \text{ a vertex}\}$ distinguishes Kromatic symmetric functions.

Acknowledgements We are grateful to Per Alexandersson, Eric Marberg, and Laura Pierson for helpful comments on earlier drafts of this manuscript and Boris Shapiro for alerting us to [37].

References

[1] A. Abreu and A. Nigro, *Chromatic symmetric functions from the modular law*. J. Combin. Theory, Ser. A **180**(2021), Article no. 105407, 30 pp.
[2] P. Alexandersson and R. Sulzgruber, *A combinatorial expansion of vertical-strip LLT polynomials in the basis of elementary symmetric functions*. Adv. Math. **400**(2022), Article no. 108256, 58 pp.
[3] F. Aliniaeifard, V. Wang, and S. van Willigenburg, *The chromatic symmetric function of a graph centred at a vertex*. Electron. J. Combin. **31**(2024), no. 4, Article no. 4.22, 34 pp.

https://doi.org/10.4153/S0008439526101830 Published online by Cambridge University Press

[4] J. Aliste-Prieto, L. Crew, S. Spirkl, and J. Zamora, *A vertex-weighted Tutte symmetric function, and constructing graphs with equal chromatic symmetric function*. Electron. J. Comb. **28**(2021), no. 2, P2–1.
[5] G. D. Birkhoff, *A determinant formula for the number of ways of coloring a map*. Ann. Math. **14**(1912), nos. 1–4, 42–46.
[6] A. S. Buch, *A Littlewood-Richardson rule for the K-theory of Grassmannians*. Acta Math. **189**(2002), no. 1, 37–78.
[7] S. Cho and J. Hong, *Positivity of chromatic symmetric functions associated with Hessenberg functions of bounce number 3*. Electron. J. Comb. **29**(2022), no. 2, Article no. 2.19, 37 pp.
[8] L. Colmenarejo, A. H. Morales, and G. Panova, *Chromatic symmetric functions of Dyck paths and q-rook theory*. Eur. J. Comb. **107**(2023), Article no. 103595, 36 pp.
[9] L. Crew, O. Pechenik, and S. Spirkl, *The Kromatic symmetric function: A K-theoretic analogue of X_G*. Sém. Lothar. Combin. **89B**(2023), Article no. 77, 12 pp.
[10] L. Crew and S. Spirkl, *A deletion–contraction relation for the chromatic symmetric function*. Eur. J. Comb. **89**(2020), Article no. 103143, 20 pp.
[11] S. Dahlberg, *A new formula for Stanley's chromatic symmetric function for unit interval graphs and e-positivity for triangular ladder graphs*. Sémin. Lothar. Comb. **82**(2019), Article #59, 12 pp.
[12] S. Dahlberg, A. Foley, and S. van Willigenburg, *Resolving Stanley's e-positivity of claw-contractible-free graphs*. J. Eur. Math. Soc. **22**(2020), no. 8, 2673–2696.
[13] S. Dahlberg and S. van Willigenburg, *Chromatic symmetric functions in noncommuting variables revisited*. Adv. Appl. Math. **112**(2020), 101942.
[14] R. Diestel, *Graph theory*. 5th ed., volume 173 of Graduate Texts in Mathematics, Springer, Berlin, 2017.
[15] V. Gasharov, *Incomparability graphs of $(3+1)$-free posets are s-positive*. Discret. Math. **157**(1996), nos. 1–3, 193–197.
[16] D. D. Gebhard and B. E. Sagan, *A chromatic symmetric function in noncommuting variables*. J. Algebr. Comb. **13**(2001), no. 3, 227–255.
[17] M. Guay-Paquet, *A modular relation for the chromatic symmetric functions of $(3+1)$-free posets*. Preprint, 2013. arXiv:1306.2400.
[18] J. Haglund and A. T. Wilson, *Macdonald polynomials and chromatic quasisymmetric functions*. Electron. J. Comb. **27**(2020), no. 3, Article no. 3.37, 21 pp.
[19] T. Hikita, *A proof of the Stanley-Stembridge conjecture*. Preprint, 2024. arXiv:2410.12758.
[20] B.-H. Hwang, *Chromatic quasisymmetric functions and noncommutative P-symmetric functions*. Trans. Amer. Math. Soc. **377**(2024), no. 4, 2855–2896.
[21] S. Iwao, *Grothendieck polynomials and the Boson-Fermion correspondence*. Algebr. Comb. **3**(2020), no. 5, 1023–1040.
[22] T. Lam and P. Pylyavskyy, *Combinatorial Hopf algebras and K-homology of Grassmannians*. Int. Math. Res. Notices **24**(2007), Article no. ID rnm125, 48 pp.
[23] A. Lascoux and H. Naruse, *Finite sum Cauchy identity for dual Grothendieck polynomials*. Proc. Japan Acad. Ser. A Math. Sci. **90**(2014), no. 7, 87–91.
[24] C. Lenart, *Combinatorial aspects of the K-theory of Grassmannians*. Ann. Comb. **4**(2000), 67–82.
[25] I. G. Macdonald, *Symmetric functions and Hall polynomials*, Oxford University Press, Oxford, 1998.
[26] L. Manivel, *Symmetric functions, Schubert polynomials and degeneracy loci*, volume 6 of SMF/AMS Texts and Monographs, American Mathematical Society, Société Mathématique de France, Providence, RI, 2001. Translated from the 1998 French original by John R. Swallow, Cours Spécialisés, 3.
[27] E. Marberg, *Kromatic quasisymmetric functions*. Electron. J. Comb. **32**(2025), no. 1, P1–11.
[28] C. Monical, O. Pechenik, and D. Searles, *Polynomials from combinatorial K-theory*. Canad. J. Math. **73**(2021), no. 1, 29–62.
[29] G. Nenashev and B. Shapiro, *"K-theoretic" analog of Postnikov-Shapiro algebra distinguishes graphs*. J. Comb. Theory. Ser. A **148**(2017), 316–332.
[30] S. D. Noble and D. J. A. Welsh, *A weighted graph polynomial from chromatic invariants of knots*. Ann. Inst. Fourier **49**(1999), 1057–1087.
[31] R. Orellana and G. Scott, *Graphs with equal chromatic symmetric functions*. Discret. Math. **320**(2014), 1–14.
[32] O. Pechenik and A. Yong, *Genomic tableaux*. J. Algebr. Comb. **45**(2017), no. 3, 649–685.
[33] L. Pierson, *Counting induced subgraphs with the Kromatic symmetric function*. Preprint, 2024. arXiv:2403.15929.
[34] L. Pierson, *A power sum expansion for the Kromatic symmetric function*. Discret. Math. **349**(2026), no. 5, Article no. 114957, 10 pp.

https://doi.org/10.4153/S0008439526101830 Published online by Cambridge University Press

[35] L. Pierson and S. Samanta, *On graphs with equal and different Kromatic symmetric functions*. Preprint, 2025. arXiv:2508.17682.
[36] A. Postnikov and B. Shapiro, *Trees, parking functions, syzygies, and deformations of monomial ideals*. Trans. Amer. Math. Soc. **356**(2004), no. 8, 3109–3142.
[37] B. Shapiro, I. Smirnov, and A. Vaintrob, *Deformed graphical zonotopal algebras*. Preprint, 2022. arXiv:2204.11331.
[38] J. Shareshian and M. L. Wachs, *Chromatic quasisymmetric functions*. Adv. Math. **295**(2016), 497–551.
[39] R. P. Stanley, *A symmetric function generalization of the chromatic polynomial of a graph*. Adv. Math. **111**(1995), no. 1, 166–194.
[40] R. P. Stanley, *Graph colorings and related symmetric functions: Ideas and applications a description of results, interesting applications, & notable open problems*. Discret. Math. **193**(1998), nos. 1–3, 267–286.
[41] R. P. Stanley and S. Fomin, *Enumerative combinatorics*. Vol. **2**, volume 62 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 1999.
[42] R. P. Stanley and J. R. Stembridge, *On immanants of Jacobi-Trudi matrices and permutations with restricted position*. J. Comb. Theory. Ser. A **62**(1993), no. 2, 261–279.
[43] V. Tewari, A. T. Wilson, and P. B. Zhang, *Chromatic nonsymmetric polynomials of Dyck graphs are slide-positive*. Proc. Amer. Math. Soc. **150**(2022), no. 5, 1873–1888.
[44] H. Thomas and A. Yong, *A jeu de taquin theory for increasing tableaux, with applications to K-theoretic Schubert calculus*. Algebra Number Theory **3**(2009), no. 2, 121–148.
[45] F. Tom, *Private communication to L. Crew and S. Spirkl*. 2021.
[46] M. L. Wachs, *Flagged Schur functions, Schubert polynomials, and symmetrizing operators*. J. Comb. Theory, Ser. A **40**(1985), no. 2, 276–289.
[47] D. B. West, *Combinatorial mathematics*, Cambridge University Press, Cambridge, 2021.

University of Waterloo, Canada
e-mail: logan.crew@uwaterloo.ca oliver.pechenik@uwaterloo.ca sspirkl@uwaterloo.ca

https://doi.org/10.4153/S0008439526101830 Published online by Cambridge University Press