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SUBNORMALITY AND OPERATOR MULTIDIMENSIONAL MOMENT PROBLEMS

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Abstract

Characterizations of subnormality due to Trent are extended to finite systems of operators, thus generalizing results of Gãvruţã and Suciu (subnormal pairs) and Binzar and Pãunescu (subnormal triplets). Vector and operator multidimensional moment problems with jointly subnormal solutions are characterized entirely in terms of the initial data. Bounded as well as unbounded solutions are taken into consideration.

1. *Introduction*

Among numerous characterizations of a bounded subnormal operator [**14**], there is one due to Trent [**51**] that is formulated in terms of a positive definite operator sequence 'linking' adjoints of powers with powers of the operator in question. Trent's characterization has been generalized by Gãvruţã and Suciu [**16**] to subnormal pairs, and by Binzar and Pãunescu [**4**] to subnormal triplets. In this paper we formulate and prove multi-operator versions of Trent's characterization (cf. Theorem 3.1 and Corollary 3.2) that cover all of the above-mentioned particular cases.

Moment problems considered in this paper were originated in probability theory by Getoor [**17**], who introduced the notion of a normal process. His idea has been used and developed by many authors (cf. [**1**, **11**, **25**, **27**, **32**, **39**]). An independent abstract approach to this kind of moment problem was proposed and solved by Sebestyén in [**34**–**38**]. The present paper deals with, among other things, a vector moment problem which entails determining whether, for a given d-sequence $\{h_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}_+^d}$ of vectors in a Hilbert space $\mathcal{H}$, there exists a subnormal d-tuple $\boldsymbol{S}=(S_1,\dots,S_d)$ composed of bounded operators on $\mathcal{H}$ such that

$$h_{\boldsymbol{n}} = \boldsymbol{S}^{\boldsymbol{n}} h_{\boldsymbol{0}}, \qquad \boldsymbol{n}\in\mathbb{Z}_+^d, \tag{1.1}$$

where $\mathbb{Z}_+=\{0,1,2,\dots\}$, $\mathbb{Z}_+^d=\mathbb{Z}_+\times\ \dots\ \times\mathbb{Z}_+$ (d-times) and $\boldsymbol{S}^{\boldsymbol{n}}=S_1^{n_1}\dots S_d^{n_d}$ for $\boldsymbol{n}=(n_1,\dots,n_d)\in\mathbb{Z}_+^d$. Replacing $\{h_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}_+^d}$ by a d-sequence $\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}_+^d}$ of operators on $\mathcal{H}$, we arrive at the definition of an operator moment problem:

$$A_{\boldsymbol{n}} = \boldsymbol{S}^{\boldsymbol{n}} A_{\boldsymbol{0}}, \qquad \boldsymbol{n}\in\mathbb{Z}_+^d. \tag{1.2}$$

The vector (respectively operator) moment problem (1.1) (respectively (1.2)) has a very natural connection with the theory of orthogonally scattered (respectively quasi-isometric) measures developed by Masani in [**22**] (respectively [**23**]); namely, as shown in Section 4, it generalizes the scalar moment problem in the sense that the

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representing scalar measures are replaced by orthogonally scattered (respectively quasi-isometric) ones.

In view of the fact that the restriction of a subnormal operator to its invariant subspace is still subnormal, there is no loss of generality in assuming that the d-sequence $\{h_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}_+^d}$ (respectively $\{\mathcal{R}(A_{\boldsymbol{n}})\}_{\boldsymbol{n}\in\mathbb{Z}_+^d}$) is linearly dense in the underlying Hilbert space; moreover, this additional assumption guarantees uniqueness of a solution $\boldsymbol{S}$ to (1.1) (respectively (1.2)). Some characterizations of the above-mentioned moment problems were found in [**34**, Theorem C; **36**, Theorem C] for $d=1$, in [**28**, **29**] for $d=2$, and in [**4**] for $d=3$. All of them postulate the existence of an 'extension' of $\{h_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}_+^d}$ (respectively $\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}_+^d}$) to a doubly indexed multisequence satisfying some extra conditions (see Section 9 for more details). In this paper we propose new characterizations written exactly in terms of the sequence $\{h_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}_+^d}$ (respectively $\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}_+^d}$) without any constraints on d. To avoid undue repetition in the statements of our theorems, we subsume both these categories of moment problems into one by allowing the operators $A_{\boldsymbol{n}}$ to act between different Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$. In other words, our general scheme includes the vector moment problem as a particular case (simply through the identification of vectors $h_{\boldsymbol{n}}$ with operators $A_{\boldsymbol{n}}:\mathbb{C}\longrightarrow\mathcal{H}$ via $A_{\boldsymbol{n}}\lambda=\lambda h_{\boldsymbol{n}}$, $\lambda\in\mathbb{C}$). We also allow the d-tuple $\boldsymbol{S}$ appearing in (1.2) to be composed of unbounded subnormal operators. Other classes of operators solving the operator moment problem (1.2), such as normal, selfadjoint, unitary and isometric, are discussed as well.

2. *Preliminaries*

All linear spaces considered in this paper are assumed to be complex. Let $\mathcal{H}$ be a Hilbert space. Given a subset X of $\mathcal{H}$ (respectively a family $\{X_\jmath\}_{\jmath\in J}$ of subsets of $\mathcal{H}$), we denote by $\operatorname{lin} X$ (respectively $\bigvee_{\jmath\in J} X_\jmath$) the linear span of X (respectively the closed linear span of $\bigcup_{\jmath\in J} X_\jmath$). By an operator *in* $\mathcal{H}$, we mean a linear mapping $A:\mathcal{H}\supseteq\mathcal{D}(A)\longrightarrow\mathcal{H}$ defined on a linear subspace $\mathcal{D}(A)$ of $\mathcal{H}$ called the *domain* of A. Denote by $\mathcal{R}(A)$ the range of A. We say that

$$f\in\mathcal{D}^\infty(A)\stackrel{\text{df}}{=}\bigcap_{n=1}^{\infty}\mathcal{D}(A^n)$$

is an *analytic* vector of A if there exists a positive real number t such that

$$\sum_{n=0}^{\infty}\|A^nf\|\frac{t^n}{n!}<\infty;$$

f is said to be a *quasi-analytic* vector of A if

$$\sum_{n=1}^{\infty}\frac{1}{\|A^nf\|^{1/n}}=\infty\quad\left(\text{with }\frac{1}{0}=\infty\right).$$

Denote by $\mathcal{A}(A)$ (respectively $\mathcal{Q}(A)$) the set of all analytic (respectively quasi-analytic) vectors of A. Notice that $\mathcal{A}(A)$ is a linear subspace of $\mathcal{D}(A)$, which is invariant for A, and $\mathcal{A}(A)\subseteq\mathcal{Q}(A)$. The last inclusion may be strict even for essentially selfadjoint operators (cf. [**30**, Section 5]). Moreover, it may happen that A is essentially selfadjoint and $\mathcal{Q}(A)$ is not a linear subspace of $\mathcal{H}$ (cf. [**31**]).

Let $\mathcal{D}$ and $\mathcal{E}$ be dense linear subspaces of Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. Denote by $\boldsymbol{L}(\mathcal{D},\mathcal{E})$ the set of all linear mappings from $\mathcal{D}$ into $\mathcal{E}$ and by $\boldsymbol{B}(\mathcal{D},\mathcal{E})$

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the set of all bounded linear mappings from $\mathcal{D}$ into $\mathcal{E}$. For simplicity, we write $\boldsymbol{L}(\mathcal{D})$ and $\boldsymbol{B}(\mathcal{D})$ instead of $\boldsymbol{L}(\mathcal{D},\mathcal{D})$ and $\boldsymbol{B}(\mathcal{D},\mathcal{D})$, respectively. $\boldsymbol{L}^{\#}(\mathcal{D})$ stands for the $*$-algebra of all $A\in\boldsymbol{L}(\mathcal{D})$ for which there exists (a unique) $A^{\#}\in\boldsymbol{L}(\mathcal{D})$ such that $\langle Af,g\rangle=\langle f,A^{\#}g\rangle$ for all $f,g\in\mathcal{D}$. The mapping $A\longmapsto A^{\#}$ is an involution on $\boldsymbol{L}^{\#}(\mathcal{D})$. The identity operator on $\mathcal{D}$ is denoted by $I=I_{\mathcal{D}}$. It is easily seen that an operator $A:\mathcal{D}\longrightarrow\mathcal{D}$ belongs to $\boldsymbol{L}^{\#}(\mathcal{D})$ if and only if $A^*(\mathcal{D})\subseteq\mathcal{D}$, where A^* stands for the adjoint of A in $\mathcal{H}$; if this is the case, then $A^{\#}=A^*|_{\mathcal{D}}$. If $\mathcal{D}=\mathcal{H}$, then $\boldsymbol{L}^{\#}(\mathcal{D})$ coincides with the C^*-algebra $\boldsymbol{B}(\mathcal{H})$ and $A^{\#}=A^*$ for $A\in\boldsymbol{L}^{\#}(\mathcal{D})$.

We say that $N\in\boldsymbol{L}^{\#}(\mathcal{D})$ is *formally normal* if $\|Nf\|=\|N^{\#}f\|$ for every $f\in\mathcal{D}$, or equivalently $N^{\#}N=NN^{\#}$. Set

$$\{N,N^{\#}\}'=\{M\in\boldsymbol{L}^{\#}(\mathcal{D}):MN=NM \text{ and } MN^{\#}=N^{\#}M\}.$$

LEMMA 2.1 [**41**, Lemma 1.2]. *If $N\in\boldsymbol{L}^{\#}(\mathcal{D})$ is formally normal, then the following hold.*

(i) $\rho_N(f)\overset{\text{df}}{=}\lim_{n\to\infty}\|N^nf\|^{1/n}$ *exists in* $[0,\infty]$ *for every* $f\in\mathcal{D}$.

(ii) $\|Nf\|\leqslant\rho_N(f)\,\|f\|$ *for every* $f\in\mathcal{D}$.

(iii) *For every* $a\geqslant 0$ *and for every* $M\in\{N,N^{\#}\}'$, *the set* $\{f\in\mathcal{D}:\rho_N(f)\leqslant a\}$ *is a linear subspace of* $\mathcal{D}$ *which is invariant for* M *and* $M^{\#}$.

A function $f:X\longrightarrow\mathcal{D}$ defined on a set X is said to be *finitely supported* if its support $\{x\in X:f(x)\neq 0\}$ is finite (in the same manner we can define finitely supported multisequences of vectors). In what follows, X^d stands for the d-fold Cartesian product of X by itself ($d\geqslant 1$). A d-tuple $\boldsymbol{S}=(S_1,\dots,S_d)\in\boldsymbol{L}(\mathcal{D})^d$ is said to be *commuting* if $S_iS_j=S_jS_i$ for all $i,j\in\{1,\dots,d\}$. For such $\boldsymbol{S}$ and $\boldsymbol{n}=(n_1,\dots,n_d)\in\mathbb{Z}_+^d$, we write $\boldsymbol{S}^{\boldsymbol{n}}=S_1^{n_1}\dots S_d^{n_d}$. We say that a d-tuple $\boldsymbol{S}=(S_1,\dots,S_d)\in\boldsymbol{L}(\mathcal{D})^d$ is *formally subnormal* if there exist an inner product space $\mathcal{F}$ and a formally normal d-tuple $\boldsymbol{N}=(N_1,\dots,N_d)\in\boldsymbol{L}^{\#}(\mathcal{F})^d$ (that is, $N_iN_j=N_jN_i$ and $N_iN_j^{\#}=N_j^{\#}N_i$ for all $i,j\in\{1,\dots,d\}$) such that $\boldsymbol{S}\subseteq\boldsymbol{N}$, that is, $\mathcal{D}\subseteq\mathcal{F}$ (isometric embedding) and $S_j\subseteq N_j$ for all $j=1,\dots,d$; such $\boldsymbol{N}$ is called a *formally normal* extension of $\boldsymbol{S}$. It follows from definition that a formally subnormal d-tuple is commuting. As in the single operator case (cf. [**45**, Section 7]), one can show that there always exists a formally normal extension $\boldsymbol{N}$ of $\boldsymbol{S}$ which is *minimal* in the sense that $\mathcal{F}=\operatorname{lin}\bigcup_{\boldsymbol{n}\in\mathbb{Z}_+^d}\boldsymbol{N}^{\#\boldsymbol{n}}(\mathcal{D})$, where $\boldsymbol{N}^{\#}=(N_1^{\#},\dots,N_d^{\#})$.

Let $\boldsymbol{S}=(S_1,\dots,S_d)$ be a d-tuple of operators in a Hilbert space $\mathcal{H}$. Set $\mathcal{D}(\boldsymbol{S})=\mathcal{D}(S_1)\cap\ldots\cap\mathcal{D}(S_d)$ and

$$\mathcal{D}^\infty(\boldsymbol{S})\overset{\text{df}}{=}\bigcap_{n=1}^{\infty}\bigcap_{k_1=1}^{d}\dots\bigcap_{k_n=1}^{d}\mathcal{D}\big(S_{k_1}\dots S_{k_n}\big). \tag{2.1}$$

We say that $\boldsymbol{S}$ is *subnormal* if $\mathcal{D}(\boldsymbol{S})$ is dense in $\mathcal{H}$ and there exist a Hilbert space $\mathcal{L}$ and a normal d-tuple $\boldsymbol{N}=(N_1,\dots,N_d)$ in $\mathcal{L}$ (that is, $N_1,\dots,N_d$ are normal operators in $\mathcal{L}$ whose spectral measures commute (we say also that $N_1,\dots,N_d$ spectrally commute)) such that $\boldsymbol{S}\subseteq\boldsymbol{N}$, that is, $\mathcal{H}\subseteq\mathcal{L}$ (isometric embedding) and $S_i\subseteq N_i$ for all $i=1,\dots,d$; such $\boldsymbol{N}$ is called a *normal extension of* $\boldsymbol{S}$. It is known that if a d-tuple $\boldsymbol{S}$ is subnormal and $\overline{\mathcal{D}^\infty(\boldsymbol{S})}=\mathcal{H}$, then the d-tuple $(S_1|_{\mathcal{D}^\infty(\boldsymbol{S})},\dots,S_d|_{\mathcal{D}^\infty(\boldsymbol{S})})$ is formally subnormal. Notice that a d-tuple $\boldsymbol{S}=(S_1,\dots,S_d)\in\boldsymbol{B}(\mathcal{H})^d$ is subnormal (in our meaning) if and only if there exist a Hilbert space $\mathcal{L}$ and a normal d-tuple $\boldsymbol{N}=(N_1,\dots,N_d)\in\boldsymbol{B}(\mathcal{L})^d$ such that $\boldsymbol{S}\subseteq\boldsymbol{N}$ (this follows from Corollary 5.3 and Ito's theorem [**20**]; see also Theorem 7.3).

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Let $(\Gamma, +, *)$ be a commutative $*$-semigroup with addition $+$, neutral element 0 and involution $*$ (cf. [**3**]). Denote by $\Lambda(\mathcal{D})$ the set of all sesquilinear forms over an inner product space $\mathcal{D}$. A mapping $\Phi : \Gamma \longrightarrow \Lambda(\mathcal{D})$ is said to be *positive definite* if

$$\sum_{\alpha,\beta\in\Gamma} \Phi(\beta^* + \alpha)(f(\alpha), f(\beta)) \geqslant 0 \text{ for every finitely supported } f : \Gamma \longrightarrow \mathcal{D}.$$

Identifying a bounded linear operator A with its sesquilinear form $(f, g) \longmapsto \langle Af, g\rangle$, one can easily adapt the definition of positive definiteness to the context of operator valued functions.

The following unbounded version of the Sz.-Nagy general dilation theorem [**49**] was proved in [**50**] (see also [**24**, Theorem 4.7] for the Banach space case).

THEOREM 2.2. *If $\Phi : \Gamma \longrightarrow \Lambda(\mathcal{D})$ is positive definite, then there exists an inner product space $\mathcal{F}$, a linear mapping $V : \mathcal{D} \longrightarrow \mathcal{F}$ and an involution preserving semigroup homomorphism $\Pi : \Gamma \longrightarrow \boldsymbol{L}^{\#}(\mathcal{F})$ such that the following hold.*

(i) $\Phi(\alpha)(g, h) = \langle \Pi(\alpha)Vg, Vh\rangle$ *for* $g, h \in \mathcal{D}$ *and* $\alpha \in \Gamma$.

(ii) $\mathcal{F}$ *is the linear span of* $\{\Pi(\alpha)Vg : \alpha \in \Gamma, g \in \mathcal{D}\}$.

A mapping $\Phi : \Gamma \longrightarrow \Lambda(\mathcal{D})$ is said to be *weakly positive definite* if $\Phi(\cdot)(f, f) \in \Lambda(\mathbb{C})$ is positive definite for every $f \in \mathcal{D}$. In general, weak positive definiteness does not imply positive definiteness (see [**6**, Theorem 1]). However, weak positive definiteness, when strengthened by an appropriate growth condition, may imply positive definiteness. If Φ is weakly positive definite and $\alpha \in \Gamma$, then we set

$$\mathcal{A}_\Phi(\alpha) = \left\{ g \in \mathcal{D} : \sum_{n=0}^{\infty} \sqrt{\Phi(n(\alpha^* + \alpha))(g, g)}\, \frac{t^n}{n!} < \infty \text{ for some } t = t(g) > 0 \right\}.$$

THEOREM 2.3 [**43**, Corollary 3]. *Let $\Phi : \Gamma \longrightarrow \Lambda(\mathcal{D})$ be a weakly positive definite mapping such that $\mathcal{A}_\Phi(\gamma) = \mathcal{D}$ for every $\gamma \in \mathcal{G}$, where $\mathcal{G}$ is an at most countable subset of Γ which generates the $*$-semigroup Γ. Then Φ is positive definite.*

REMARK 2.4. For each $\alpha \in \Gamma$, the set $\mathcal{A}_\Phi(\alpha)$ is a linear subspace of $\mathcal{D}$. Indeed, since the mapping Φ is weakly positive definite, the sesquilinear form $\Phi(n(\alpha^* + \alpha))$ becomes a semi-inner product, and consequently

$$\sqrt{\Phi(n(\alpha^* + \alpha))(g + h, g + h)} \leqslant \sqrt{\Phi(n(\alpha^* + \alpha))(g, g)} + \sqrt{\Phi(n(\alpha^* + \alpha))(h, h)}$$

for all $g, h \in \mathcal{D}$ and $n \geqslant 1$. This implies that $\mathcal{A}_\Phi(\alpha)$ is an additive subsemigroup of $\mathcal{D}$, which proves our claim.

3. *Subnormality*

Put $\mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$. Let us fix an integer $d \geqslant 1$ and write $\langle \boldsymbol{n}\rangle = (|n_1|, \dots, |n_d|)$ for $\boldsymbol{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d$. Denote by Ω_d the family of all subsets of $\{1, \dots, d\}$. Given $\omega \in \Omega_d$, we set $\widetilde{\omega} = \{1, \dots, d\} \setminus \omega$ and

$$\mathbb{Z}^d_\omega = \{\boldsymbol{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d : n_j \geqslant 0 \text{ for all } j \in \omega \text{ and } n_j < 0 \text{ for all } j \in \widetilde{\omega}\}.$$

It is easily seen that

$$\mathbb{Z}^d = \bigcup_{\omega\in\Omega_d} \mathbb{Z}^d_\omega \quad \text{and} \quad \mathbb{Z}^d_{\omega_1} \cap \mathbb{Z}^d_{\omega_2} = \varnothing \text{ for all } \omega_1, \omega_2 \in \Omega_d \text{ such that } \omega_1 \neq \omega_2. \quad (3.1)$$

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Let $\mathcal{H}$ be a Hilbert space. Given a commuting d-tuple $\boldsymbol{S}=(S_1,\ldots,S_d)\in\boldsymbol{B}(\mathcal{H})^d$, we define

$$\boldsymbol{S}_\omega^{\boldsymbol{n}}=\prod_{j\in\omega}S_j^{n_j},\qquad \omega\in\Omega_d,\ \boldsymbol{n}\in\mathbb{Z}_+^d\qquad \left(\prod_{j\in\varnothing}S_j^{n_j}\stackrel{\mathrm{df}}{=}I\right).$$

Since $\boldsymbol{S}^*\stackrel{\mathrm{df}}{=}(S_1^*,\ldots,S_d^*)$ is a commuting d-tuple, the following definition is justified.

$$\boldsymbol{S}_\omega^{*\boldsymbol{n}}\stackrel{\mathrm{df}}{=}\big(\boldsymbol{S}_\omega^{\boldsymbol{n}}\big)^*=\big(\boldsymbol{S}^*\big)_\omega^{\boldsymbol{n}},\qquad \omega\in\Omega_d,\ \boldsymbol{n}\in\mathbb{Z}_+^d.$$

A d-sequence $\{\Phi_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}^d}\subseteq\boldsymbol{B}(\mathcal{H})$ is said to be *positive definite* if it is positive definite (as a function of $\boldsymbol{n}$) with respect to the $*$-semigroup $(\mathbb{Z}^d,+,\boldsymbol{n}^*=-\boldsymbol{n})$, that is,

$$\sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}^d}\langle\Phi_{\boldsymbol{n}-\boldsymbol{m}}h_{\boldsymbol{n}},h_{\boldsymbol{m}}\rangle\geqslant 0\quad\text{for every finitely supported }\{h_{\boldsymbol{m}}\}_{\boldsymbol{m}\in\mathbb{Z}^d}\subseteq\mathcal{H}.$$

Choosing $k\in\mathbb{Z}_+$ large enough, and replacing the indices $\boldsymbol{m}$ and $\boldsymbol{n}$ by $\boldsymbol{m}+k(1,\ldots,1)$ and $\boldsymbol{n}+k(1,\ldots,1)$, respectively, we see that Φ is positive definite if and only if

$$\sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}_+^d}\langle\Phi_{\boldsymbol{n}-\boldsymbol{m}}h_{\boldsymbol{n}},h_{\boldsymbol{m}}\rangle\geqslant 0\quad\text{for every finitely supported }\{h_{\boldsymbol{m}}\}_{\boldsymbol{m}\in\mathbb{Z}_+^d}\subseteq\mathcal{H}.$$

We are now in a position to formulate and prove two characterizations of bounded subnormal d-tuples.

THEOREM 3.1. *If $\boldsymbol{S}\in\boldsymbol{B}(\mathcal{H})^d$ is a commuting d-tuple, then the following conditions are equivalent.*

(i) *$\boldsymbol{S}$ is a subnormal d-tuple.*

(ii) *There exist a Hilbert space $\mathcal{L}\supseteq\mathcal{H}$ and a commuting d-tuple $\boldsymbol{U}\in\boldsymbol{B}(\mathcal{L})^d$ of unitary operators such that*

$$\boldsymbol{S}_\omega^{*\boldsymbol{n}}\boldsymbol{S}_{\widetilde{\omega}}^{\boldsymbol{n}}=\boldsymbol{S}_{\widetilde{\omega}}^{*\boldsymbol{n}}P_{\mathcal{H}}\boldsymbol{U}_{\widetilde{\omega}}^{*\boldsymbol{n}}\boldsymbol{U}_\omega^{\boldsymbol{n}}\big|_{\mathcal{H}}\boldsymbol{S}_\omega^{\boldsymbol{n}},\qquad \boldsymbol{n}\in\mathbb{Z}_+^d,\ \omega\in\Omega_d,\tag{3.2}$$

where $P_{\mathcal{H}}$ is the orthogonal projection of $\mathcal{L}$ onto $\mathcal{H}$.

(iii) *There exists a family $\{A_{\omega,\boldsymbol{n}}:\omega\in\Omega_d,:\boldsymbol{n}\in\mathbb{Z}_+^d\}\subseteq\boldsymbol{B}(\mathcal{H})$ such that*

$$\boldsymbol{S}_\omega^{*\boldsymbol{n}}\boldsymbol{S}_{\widetilde{\omega}}^{\boldsymbol{n}}=\boldsymbol{S}_{\widetilde{\omega}}^{*\boldsymbol{n}}A_{\omega,\boldsymbol{n}}\boldsymbol{S}_\omega^{\boldsymbol{n}},\qquad \boldsymbol{n}\in\mathbb{Z}_+^d,\ \omega\in\Omega_d,\tag{3.3}$$

and the d-sequence $\{\Phi_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}^d}$ defined by

$$\Phi_{\boldsymbol{n}}=A_{\omega,\langle\boldsymbol{n}\rangle},\qquad \boldsymbol{n}\in\mathbb{Z}_\omega^d,\ \omega\in\Omega_d,$$

is positive definite.

Proof. (i)$\Rightarrow$(ii): Let $\boldsymbol{N}=(N_1,\ldots,N_d)\in\boldsymbol{B}(\mathcal{L})^d$ be a normal extension of $\boldsymbol{S}$. It is well known (cf. [**15**]) that $N_1,\ldots,N_d$ have the representations

$$N_j=V_jP_j,\qquad j=1,\ldots,d,\tag{3.4}$$

where $V_1,\ldots,V_d\in\boldsymbol{B}(\mathcal{L})$ are unitary operators and $P_1,\ldots,P_d\in\boldsymbol{B}(\mathcal{L})$ are positive operators such that

$$V_jV_k=V_kV_j,\quad V_jP_k=P_kV_j,\quad\text{for all }j,k\in\{1,\ldots,d\}.\tag{3.5}$$

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Set $\boldsymbol{U}=(V_1^{*2},\dots,V_d^{*2})$. Then, by the Fuglede theorem, (3.4), (3.5) and the unitarity of V_j, we have

$$\begin{aligned}\boldsymbol{N}_\omega^{*\boldsymbol{n}}\boldsymbol{N}_{\widetilde{\omega}}^{\boldsymbol{n}}=\boldsymbol{N}_{\widetilde{\omega}}^{\boldsymbol{n}}\boldsymbol{N}_\omega^{*\boldsymbol{n}}&=\prod_{j\in\widetilde{\omega}}(V_jP_j)^{n_j}\prod_{k\in\omega}(V_k^*P_k)^{n_k}\\&=\prod_{j\in\widetilde{\omega}}\big(V_j^{*n_j}V_j^{2n_j}P_j^{n_j}\big)\prod_{k\in\omega}\big(V_k^{*2n_k}V_k^{n_k}P_k^{n_k}\big)\\&=\prod_{j\in\widetilde{\omega}}\big(V_j^{*n_j}P_j^{n_j}\big)\prod_{j\in\widetilde{\omega}}V_j^{2n_j}\prod_{k\in\omega}V_k^{*2n_k}\prod_{k\in\omega}\big(V_k^{n_k}P_k^{n_k}\big)\\&=\boldsymbol{N}_{\widetilde{\omega}}^{*\boldsymbol{n}}\boldsymbol{U}_{\widetilde{\omega}}^{*\boldsymbol{n}}\boldsymbol{U}_\omega^{\boldsymbol{n}}\boldsymbol{N}_\omega^{\boldsymbol{n}},\qquad\boldsymbol{n}\in\mathbb{Z}_+^d,\,\omega\in\Omega_d.\end{aligned}\tag{3.6}$$

Since $\mathcal{H}$ is invariant for each N_j, $j=1,\dots,d$, we obtain

$$P_{\mathcal{H}}\boldsymbol{N}^{*\boldsymbol{n}}=P_{\mathcal{H}}\boldsymbol{N}^{*\boldsymbol{n}}P_{\mathcal{H}}+P_{\mathcal{H}}\boldsymbol{N}^{*\boldsymbol{n}}(I_{\mathcal{L}}-P_{\mathcal{H}})=P_{\mathcal{H}}\boldsymbol{N}^{*\boldsymbol{n}}P_{\mathcal{H}}=\boldsymbol{S}^{*\boldsymbol{n}}P_{\mathcal{H}},\qquad\boldsymbol{n}\in\mathbb{Z}_+^d.$$

This and (3.6) imply that

$$\begin{aligned}\boldsymbol{S}_\omega^{*\boldsymbol{n}}\boldsymbol{S}_{\widetilde{\omega}}^{\boldsymbol{n}}h&=\boldsymbol{S}_\omega^{*\boldsymbol{n}}P_{\mathcal{H}}\boldsymbol{N}_{\widetilde{\omega}}^{\boldsymbol{n}}h=P_{\mathcal{H}}\boldsymbol{N}_\omega^{*\boldsymbol{n}}\boldsymbol{N}_{\widetilde{\omega}}^{\boldsymbol{n}}h=P_{\mathcal{H}}\boldsymbol{N}_{\widetilde{\omega}}^{*\boldsymbol{n}}\boldsymbol{U}_{\widetilde{\omega}}^{*\boldsymbol{n}}\boldsymbol{U}_\omega^{\boldsymbol{n}}\boldsymbol{N}_\omega^{\boldsymbol{n}}h\\&=\boldsymbol{S}_{\widetilde{\omega}}^{*\boldsymbol{n}}P_{\mathcal{H}}\boldsymbol{U}_{\widetilde{\omega}}^{*\boldsymbol{n}}\boldsymbol{U}_\omega^{\boldsymbol{n}}\boldsymbol{N}_\omega^{\boldsymbol{n}}h=\boldsymbol{S}_{\widetilde{\omega}}^{*\boldsymbol{n}}P_{\mathcal{H}}\boldsymbol{U}_{\widetilde{\omega}}^{*\boldsymbol{n}}\boldsymbol{U}_\omega^{\boldsymbol{n}}\big|_{\mathcal{H}}\boldsymbol{S}_\omega^{\boldsymbol{n}}h,\quad h\in\mathcal{H},\boldsymbol{n}\in\mathbb{Z}_+^d,\omega\in\Omega_d.\end{aligned}$$

(ii) $\Rightarrow$ (iii): Define $A_{\omega,\boldsymbol{n}}=P_{\mathcal{H}}\boldsymbol{U}_{\widetilde{\omega}}^{*\boldsymbol{n}}\boldsymbol{U}_\omega^{\boldsymbol{n}}|_{\mathcal{H}}$ for $\omega\in\Omega_d$ and $\boldsymbol{n}\in\mathbb{Z}_+^d$. Take a finitely supported d-sequence $\{h_{\boldsymbol{m}}\}_{\boldsymbol{m}\in\mathbb{Z}^d}\subseteq\mathcal{H}$. If $\omega\in\Omega_d$ and $\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}^d$ are such that $\boldsymbol{n}-\boldsymbol{m}\in\mathbb{Z}_\omega^d$, then

$$\big\langle\boldsymbol{U}_{\widetilde{\omega}}^{*\langle\boldsymbol{n}-\boldsymbol{m}\rangle}\boldsymbol{U}_\omega^{\langle\boldsymbol{n}-\boldsymbol{m}\rangle}h_{\boldsymbol{n}},h_{\boldsymbol{m}}\big\rangle=\big\langle\boldsymbol{U}_{\widetilde{\omega}}^{*(\boldsymbol{m}-\boldsymbol{n})}\boldsymbol{U}_\omega^{\boldsymbol{n}-\boldsymbol{m}}h_{\boldsymbol{n}},h_{\boldsymbol{m}}\big\rangle=\langle\boldsymbol{U}^{\boldsymbol{n}-\boldsymbol{m}}h_{\boldsymbol{n}},h_{\boldsymbol{m}}\rangle.$$

This and (3.1) give us

$$\begin{aligned}\sum_{\boldsymbol{n},\boldsymbol{m}\in\mathbb{Z}^d}\langle\Phi_{\boldsymbol{n}-\boldsymbol{m}}h_{\boldsymbol{n}},h_{\boldsymbol{m}}\rangle&=\sum_{\omega\in\Omega_d}\sum_{\substack{\boldsymbol{n},\boldsymbol{m}\in\mathbb{Z}^d\\\boldsymbol{n}-\boldsymbol{m}\in\mathbb{Z}_\omega^d}}\langle\Phi_{\boldsymbol{n}-\boldsymbol{m}}h_{\boldsymbol{n}},h_{\boldsymbol{m}}\rangle\\&=\sum_{\omega\in\Omega_d}\sum_{\substack{\boldsymbol{n},\boldsymbol{m}\in\mathbb{Z}^d\\\boldsymbol{n}-\boldsymbol{m}\in\mathbb{Z}_\omega^d}}\big\langle\boldsymbol{U}_{\widetilde{\omega}}^{*\langle\boldsymbol{n}-\boldsymbol{m}\rangle}\boldsymbol{U}_\omega^{\langle\boldsymbol{n}-\boldsymbol{m}\rangle}h_{\boldsymbol{n}},h_{\boldsymbol{m}}\big\rangle\\&=\sum_{\omega\in\Omega_d}\sum_{\substack{\boldsymbol{n},\boldsymbol{m}\in\mathbb{Z}^d\\\boldsymbol{n}-\boldsymbol{m}\in\mathbb{Z}_\omega^d}}\langle\boldsymbol{U}^{\boldsymbol{n}-\boldsymbol{m}}h_{\boldsymbol{n}},h_{\boldsymbol{m}}\rangle\\&=\sum_{\boldsymbol{n},\boldsymbol{m}\in\mathbb{Z}^d}\langle\boldsymbol{U}^{\boldsymbol{n}-\boldsymbol{m}}h_{\boldsymbol{n}},h_{\boldsymbol{m}}\rangle=\Big\|\sum_{\boldsymbol{n}\in\mathbb{Z}^d}\boldsymbol{U}^{\boldsymbol{n}}h_{\boldsymbol{n}}\Big\|^2\geqslant0.\end{aligned}$$

(iii) $\Rightarrow$ (i): Take a finitely supported d-sequence $\{h_{\boldsymbol{m}}\}_{\boldsymbol{m}\in\mathbb{Z}_+^d}\subseteq\mathcal{H}$. If $\omega\in\Omega_d$ and $\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}_+^d$ are such that $\boldsymbol{n}-\boldsymbol{m}\in\mathbb{Z}_\omega^d$, then by (3.3) we have

$$\begin{aligned}\langle\boldsymbol{S}^{\boldsymbol{m}}h_{\boldsymbol{n}},\boldsymbol{S}^{\boldsymbol{n}}h_{\boldsymbol{m}}\rangle&=\big\langle\boldsymbol{S}_{\widetilde{\omega}}^{\boldsymbol{m}-\boldsymbol{n}}\boldsymbol{S}_\omega^{\boldsymbol{m}}\boldsymbol{S}_{\widetilde{\omega}}^{\boldsymbol{n}}h_{\boldsymbol{n}},\boldsymbol{S}_\omega^{\boldsymbol{n}-\boldsymbol{m}}\boldsymbol{S}_\omega^{\boldsymbol{m}}\boldsymbol{S}_{\widetilde{\omega}}^{\boldsymbol{n}}h_{\boldsymbol{m}}\big\rangle\\&=\big\langle\boldsymbol{S}_\omega^{*\langle\boldsymbol{n}-\boldsymbol{m}\rangle}\boldsymbol{S}_{\widetilde{\omega}}^{\langle\boldsymbol{n}-\boldsymbol{m}\rangle}\boldsymbol{S}_\omega^{\boldsymbol{m}}\boldsymbol{S}_{\widetilde{\omega}}^{\boldsymbol{n}}h_{\boldsymbol{n}},\boldsymbol{S}_\omega^{\boldsymbol{m}}\boldsymbol{S}_{\widetilde{\omega}}^{\boldsymbol{n}}h_{\boldsymbol{m}}\big\rangle\\&=\big\langle\boldsymbol{S}_{\widetilde{\omega}}^{*\langle\boldsymbol{n}-\boldsymbol{m}\rangle}A_{\omega,\langle\boldsymbol{n}-\boldsymbol{m}\rangle}\boldsymbol{S}_\omega^{\langle\boldsymbol{n}-\boldsymbol{m}\rangle}\boldsymbol{S}_\omega^{\boldsymbol{m}}\boldsymbol{S}_{\widetilde{\omega}}^{\boldsymbol{n}}h_{\boldsymbol{n}},\boldsymbol{S}_\omega^{\boldsymbol{m}}\boldsymbol{S}_{\widetilde{\omega}}^{\boldsymbol{n}}h_{\boldsymbol{m}}\big\rangle\\&=\big\langle A_{\omega,\langle\boldsymbol{n}-\boldsymbol{m}\rangle}\boldsymbol{S}^{\boldsymbol{n}}h_{\boldsymbol{n}},\boldsymbol{S}^{\boldsymbol{m}}h_{\boldsymbol{m}}\big\rangle.\end{aligned}$$

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This, when combined with (3.1) and positive definiteness of Φ, implies that

$$\begin{aligned}\sum_{\boldsymbol{n},\boldsymbol{m}\in\mathbb{Z}_+^d}\langle \boldsymbol{S}^{\boldsymbol{m}}h_{\boldsymbol{n}},\boldsymbol{S}^{\boldsymbol{n}}h_{\boldsymbol{m}}\rangle &= \sum_{\omega\in\Omega_d}\sum_{\substack{\boldsymbol{n},\boldsymbol{m}\in\mathbb{Z}_+^d\\ \boldsymbol{n}-\boldsymbol{m}\in\mathbb{Z}_\omega^d}}\langle \boldsymbol{S}^{\boldsymbol{m}}h_{\boldsymbol{n}},\boldsymbol{S}^{\boldsymbol{n}}h_{\boldsymbol{m}}\rangle\\ &= \sum_{\omega\in\Omega_d}\sum_{\substack{\boldsymbol{n},\boldsymbol{m}\in\mathbb{Z}_+^d\\ \boldsymbol{n}-\boldsymbol{m}\in\mathbb{Z}_\omega^d}}\big\langle A_{\omega,\langle\boldsymbol{n}-\boldsymbol{m}\rangle}\boldsymbol{S}^{\boldsymbol{n}}h_{\boldsymbol{n}},\boldsymbol{S}^{\boldsymbol{m}}h_{\boldsymbol{m}}\big\rangle\\ &= \sum_{\omega\in\Omega_d}\sum_{\substack{\boldsymbol{n},\boldsymbol{m}\in\mathbb{Z}_+^d\\ \boldsymbol{n}-\boldsymbol{m}\in\mathbb{Z}_\omega^d}}\langle \Phi_{\boldsymbol{n}-\boldsymbol{m}}\boldsymbol{S}^{\boldsymbol{n}}h_{\boldsymbol{n}},\boldsymbol{S}^{\boldsymbol{m}}h_{\boldsymbol{m}}\rangle\\ &= \sum_{\boldsymbol{n},\boldsymbol{m}\in\mathbb{Z}_+^d}\langle \Phi_{\boldsymbol{n}-\boldsymbol{m}}\boldsymbol{S}^{\boldsymbol{n}}h_{\boldsymbol{n}},\boldsymbol{S}^{\boldsymbol{m}}h_{\boldsymbol{m}}\rangle\geqslant 0.\end{aligned}$$

Applying the Ito criterion for subnormality (cf. [**20**]) completes the proof. □

Condition (3.3) (as well as (3.2)) is invariant, in a sense, under the operation of taking adjoints because if $\boldsymbol{S}_\omega^{*\boldsymbol{n}}\boldsymbol{S}_{\widetilde{\omega}}^{\boldsymbol{n}}=\boldsymbol{S}_{\widetilde{\omega}}^{*\boldsymbol{n}}A_{\omega,\boldsymbol{n}}\boldsymbol{S}_\omega^{\boldsymbol{n}}$, where $\omega\in\Omega_d$ and $\boldsymbol{n}\in\mathbb{Z}_+^d$, then $\boldsymbol{S}_{\widetilde{\omega}}^{*\boldsymbol{n}}\boldsymbol{S}_\omega^{\boldsymbol{n}}=\boldsymbol{S}_\omega^{*\boldsymbol{n}}A_{\omega,\boldsymbol{n}}^*\boldsymbol{S}_{\widetilde{\omega}}^{\boldsymbol{n}}$. Thus we can always diminish the number of the equalities appearing in (3.3) at least twice. This can be done by choosing a subset Σ of Ω_d such that $\Omega_d=\Sigma\cup\widetilde{\Sigma}$ and $\Sigma\cap\widetilde{\Sigma}=\varnothing$, where

$$\widetilde{\Sigma}\stackrel{\mathrm{df}}{=}\{\widetilde{\sigma}:\sigma\in\Sigma\}$$

(recall that $\widetilde{\sigma}=\{1,\dots,d\}\backslash\sigma$). Taking $\Sigma_d=\{\sigma:\sigma\subseteq\{1,\dots,d\},\,1\in\sigma\}$ and applying Theorem 3.1, we obtain the following.

COROLLARY 3.2. *If $\boldsymbol{S}\in\boldsymbol{B}(\mathcal{H})^d$ is a commuting d-tuple, then the following conditions are equivalent.*

(i) *$\boldsymbol{S}$ is a subnormal d-tuple.*

(ii) *There exist a Hilbert space $\mathcal{L}\supseteq\mathcal{H}$ and a commuting d-tuple $\boldsymbol{U}\in\boldsymbol{B}(\mathcal{L})^d$ of unitary operators such that*

$$\boldsymbol{S}_\sigma^{*\boldsymbol{n}}\boldsymbol{S}_{\widetilde{\sigma}}^{\boldsymbol{n}}=\boldsymbol{S}_{\widetilde{\sigma}}^{*\boldsymbol{n}}P_{\mathcal{H}}\boldsymbol{U}_{\widetilde{\sigma}}^{*\boldsymbol{n}}\boldsymbol{U}_\sigma^{\boldsymbol{n}}\big|_{\mathcal{H}}\boldsymbol{S}_\sigma^{\boldsymbol{n}},\qquad \boldsymbol{n}\in\mathbb{Z}_+^d,\,\sigma\in\Sigma_d,$$

where $P_{\mathcal{H}}$ is the orthogonal projection of $\mathcal{L}$ onto $\mathcal{H}$.

(iii) *There exists a family $\{A_{\sigma,\boldsymbol{n}}:\sigma\in\Sigma_d,:\boldsymbol{n}\in\mathbb{Z}_+^d\}\subseteq\boldsymbol{B}(\mathcal{H})$ such that*

$$\boldsymbol{S}_\sigma^{*\boldsymbol{n}}\boldsymbol{S}_{\widetilde{\sigma}}^{\boldsymbol{n}}=\boldsymbol{S}_{\widetilde{\sigma}}^{*\boldsymbol{n}}A_{\sigma,\boldsymbol{n}}\boldsymbol{S}_\sigma^{\boldsymbol{n}},\qquad \boldsymbol{n}\in\mathbb{Z}_+^d,\,\sigma\in\Sigma_d,$$

and the d-sequence $\{\Phi_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}^d}\subseteq\boldsymbol{B}(\mathcal{H})$ defined by

$$\Phi_{\boldsymbol{n}}=\begin{cases}A_{\sigma,\langle\boldsymbol{n}\rangle} & \boldsymbol{n}\in\mathbb{Z}_\sigma^d,\ \sigma\in\Sigma_d\\ A_{\sigma,\langle\boldsymbol{n}\rangle}^* & \boldsymbol{n}\in\mathbb{Z}_{\widetilde{\sigma}}^d,\ \sigma\in\Sigma_d\end{cases}$$

is positive definite.

Corollary 3.2 implies characterizations of subnormality due to Trent [**51**], Gãvruţã and Suciu [**16**], and Binzar and Pãunescu [**4**].

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4. *Orthogonally scattered and quasi-isometric measures*

Vector moment problems considered in this paper are intimately related to orthogonally scattered vector measures playing an important role in the theory of stochastic processes. Denote by $\mathfrak{G}_d$ the σ-algebra of all Borel subsets of $\mathbb{C}^d$. Let $\{h_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}_+^d}$ be a d-sequence of vectors in a Hilbert space $\mathcal{H}$ such that $\mathcal{H} = \bigvee_{\boldsymbol{n}\in\mathbb{Z}_+^d} h_{\boldsymbol{n}}$. Suppose first that there exists a subnormal d-tuple $\boldsymbol{S} = (S_1, \ldots, S_d)$ in $\mathcal{H}$ such that $h_{\boldsymbol{n}} = \boldsymbol{S}^{\boldsymbol{n}} h_{\boldsymbol{0}}$ for all $\boldsymbol{n} \in \mathbb{Z}_+^d$. Let $E : \mathfrak{G}_d \longrightarrow \boldsymbol{B}(\mathcal{L})$ be the joint spectral measure of a normal extension $\boldsymbol{N} = (N_1, \ldots, N_d)$ of $\boldsymbol{S}$. Set $\xi(\cdot) = E(\cdot)h_{\boldsymbol{0}}$ and $\mu(\cdot) = \langle E(\cdot)h_{\boldsymbol{0}}, h_{\boldsymbol{0}}\rangle$. Then $\xi : \mathfrak{G}_d \longrightarrow \mathcal{L}$ is an orthogonally scattered measure, $\mu(\cdot) = \|\xi(\cdot)\|^2$ and $\int_{\mathbb{C}^d} |\boldsymbol{z}^{\boldsymbol{n}}|^2 \, d\mu(\boldsymbol{z}) < \infty$ for all $\boldsymbol{n} \in \mathbb{Z}_+^d$. This means that the integral $\int_{\mathbb{C}^d} \boldsymbol{z}^{\boldsymbol{n}} \, d\xi(\boldsymbol{z})$ makes sense (cf. [**22**]) and

$$h_{\boldsymbol{n}} = \int_{\mathbb{C}^d} \boldsymbol{z}^{\boldsymbol{n}} \, d\xi(\boldsymbol{z}), \quad \boldsymbol{n} \in \mathbb{Z}_+^d. \tag{4.1}$$

Conversely, suppose that $\xi : \mathfrak{G}_d \longrightarrow \mathcal{L}$ is an orthogonally scattered measure which satisfies (4.1), where $\mathcal{L}$ is a Hilbert space containing $\mathcal{H}$. Without loss of generality, we can assume that $\mathcal{L} = \bigvee_{\sigma \in \mathfrak{G}_d} \xi(\sigma)$ (because $h_{\boldsymbol{n}} = \int_{\mathbb{C}^d} \boldsymbol{z}^{\boldsymbol{n}} \, d\xi(\boldsymbol{z}) \in \bigvee_{\sigma\in\mathfrak{G}_d} \xi(\sigma)$ for each $\boldsymbol{n} \in \mathbb{Z}_+^d$). Then

$$\mu(\cdot) \stackrel{\text{df}}{=} \|\xi(\cdot)\|^2$$

is a finite Borel measure such that $\langle \xi(\sigma), \xi(\tau)\rangle = \mu(\sigma \cap \tau)$ for all $\sigma, \tau \in \mathfrak{G}_d$ (see [**22**, Theorem 1.8]). This altogether implies that there exists a (unique) unitary isomorphism $U : \mathcal{L} \longrightarrow L^2(\mu)$ such that $U\xi(\sigma) = \chi_\sigma$ for all $\sigma \in \mathfrak{G}_d$, where χ_σ is the indicator function of the set σ. Let $F(\sigma)$ be the operator of multiplication by χ_σ on $L^2(\mu)$ $(\sigma \in \mathfrak{G}_d)$. Define

$$E(\sigma) \stackrel{\text{df}}{=} U^{-1} F(\sigma) U$$

for $\sigma \in \mathfrak{G}_d$. Then $E : \mathfrak{G}_d \longrightarrow \boldsymbol{B}(\mathcal{L})$ is a spectral measure such that $\xi(\cdot) = E(\cdot)\xi(\mathbb{C}^d)$. However, by (4.1), we have $\xi(\mathbb{C}^d) = h_{\boldsymbol{0}}$, and hence $\xi(\cdot) = E(\cdot)h_{\boldsymbol{0}}$. The latter equality and (4.1) imply that $h_{\boldsymbol{0}} \in \mathcal{D}^\infty(\boldsymbol{N})$ and $h_{\boldsymbol{n}} = \int_{\mathbb{C}^d} \boldsymbol{z}^{\boldsymbol{n}} E(d\boldsymbol{z}) h_{\boldsymbol{0}} = \boldsymbol{N}^{\boldsymbol{n}} h_{\boldsymbol{0}}$ for $\boldsymbol{n} \in \mathbb{Z}_+^d$, where $\boldsymbol{N} = (N_1, \ldots, N_d)$ and $N_j = \int_{\mathbb{C}^d} z_j E(d\boldsymbol{z})$. Set $\mathcal{D}(S_j) = \operatorname{lin}\{h_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}_+^d}$ and $S_j = N_j|_{\mathcal{D}(S_j)}$ for $j = 1, \ldots, d$. It is clear that

$$\boldsymbol{S} \stackrel{\text{df}}{=} (S_1, \ldots, S_d)$$

is a subnormal d-tuple in $\mathcal{H}$ (with normal extension $\boldsymbol{N}$) such that $h_{\boldsymbol{n}} = \boldsymbol{S}^{\boldsymbol{n}} h_{\boldsymbol{0}}$ for $\boldsymbol{n} \in \mathbb{Z}_+^d$.

On the basis of results of [**23**], one can adapt the above procedures to the context of operator moment problems, replacing orthogonally scattered measures by quasi-isometric ones. The spectral measure E in this new setting is the Naimark dilation (cf. [**26**, Theorem 6.4]) of an appropriate positive operator valued measure which is an operator counterpart of the scalar measure μ alluded to earlier.

5. *Sufficiency: the unbounded case*

We begin this section with a simple criterion for the existence of so-called shift operators; its proof is left to the reader (cf. [**17**]). In what follows, $\mathcal{D}$ and $\mathcal{E}$ are dense linear subspaces of Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively, and

$$\boldsymbol{e}_j^d \stackrel{\text{df}}{=} (\delta_{j,1}, \ldots, \delta_{j,d}),$$

where $\delta_{k,l}$ stands for the Kronecker symbol $(j = 1, \ldots, d)$.

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LEMMA 5.1. *Let* $\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}_+^d} \subseteq \boldsymbol{L}(\mathcal{D},\mathcal{E})$ *be such that* $\mathcal{E} = \lin\bigcup_{n=0}^{\infty}\mathcal{R}(A_{\boldsymbol{n}})$. *Then the following conditions are equivalent.*

(i) *There exists a (unique)* d*-tuple* $\boldsymbol{S}\in\boldsymbol{L}(\mathcal{E})^d$ *such that* $A_{\boldsymbol{n}} = \boldsymbol{S}^{\boldsymbol{n}}A_{\boldsymbol{0}}$ *for all* $\boldsymbol{n}\in\mathbb{Z}_+^d$.

(ii) *For every integer* $j = 1,\dots,d$ *and for every finitely supported* d*-sequence* $\{f_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}_+^d}\subseteq\mathcal{D}$, *the following implication holds.*

$$\text{If } \sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}_+^d}\langle A_{\boldsymbol{m}}f_{\boldsymbol{m}}, A_{\boldsymbol{n}}f_{\boldsymbol{n}}\rangle = 0, \text{ then } \sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}_+^d}\big\langle A_{\boldsymbol{m}+\boldsymbol{e}_j^d}f_{\boldsymbol{m}}, A_{\boldsymbol{n}+\boldsymbol{e}_j^d}f_{\boldsymbol{n}}\big\rangle = 0.$$

Moreover, if (i) *is satisfied, then* $\boldsymbol{S}$ *is a commuting* d*-tuple.*

A d-sequence $\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}_+^d}\subseteq\boldsymbol{L}(\mathcal{D},\mathcal{E})$ is said to be of *positive definite type* if the mapping $\Phi:\mathbb{Z}_+^d\times\mathbb{Z}_+^d\longrightarrow\Lambda(\mathcal{D})$ given by

$$\Phi(\boldsymbol{m},\boldsymbol{n})(g,h) = \langle A_{\boldsymbol{m}}g, A_{\boldsymbol{n}}h\rangle, \qquad \boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}_+^d,\ g,h\in\mathcal{D}, \tag{5.1}$$

is positive define on the $*$-semigroup $\mathbb{Z}_+^d\times\mathbb{Z}_+^d$ equipped with coordinatewise defined addition and involution $(\boldsymbol{m},\boldsymbol{n})^* = (\boldsymbol{n},\boldsymbol{m})$, that is,

$$\sum_{\substack{\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}_+^d\\ \boldsymbol{p},\boldsymbol{q}\in\mathbb{Z}_+^d}}\langle A_{\boldsymbol{m}+\boldsymbol{q}}g_{\boldsymbol{m},\boldsymbol{n}}, A_{\boldsymbol{n}+\boldsymbol{p}}g_{\boldsymbol{p},\boldsymbol{q}}\rangle \geqslant 0 \ \text{ for every finitely supported } \{g_{\boldsymbol{m},\boldsymbol{n}}\}_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}_+^d}\subseteq\mathcal{D}. \tag{5.2}$$

It is said to be of *weak positive definite type* if Φ is weakly positive definite, that is,

$$\sum_{\substack{\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}_+^d\\ \boldsymbol{p},\boldsymbol{q}\in\mathbb{Z}_+^d}}\langle A_{\boldsymbol{m}+\boldsymbol{q}}g, A_{\boldsymbol{n}+\boldsymbol{p}}g\rangle\lambda_{\boldsymbol{m},\boldsymbol{n}}\overline{\lambda_{\boldsymbol{p},\boldsymbol{q}}} \geqslant 0$$

for every $g\in\mathcal{D}$ and for every finitely supported $\{\lambda_{\boldsymbol{m},\boldsymbol{n}}\}_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}_+^d}\subseteq\mathbb{C}$.

Identifying vectors with rank 1 operators, one can easily adapt the definition of positive definite type to the context of d-sequences of vectors.

LEMMA 5.2. *A* d*-sequence* $\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}_+^d}\subseteq\boldsymbol{L}(\mathcal{D},\mathcal{E})$ *with* $\lin\bigcup_{\boldsymbol{n}\in\mathbb{Z}_+^d}\mathcal{R}(A_{\boldsymbol{n}}) = \mathcal{E}$ *is of positive definite type if and only if there exists a (unique) formally subnormal* d*-tuple* $\boldsymbol{S}\in\boldsymbol{L}(\mathcal{E})^d$ *such that*

$$A_{\boldsymbol{n}} = \boldsymbol{S}^{\boldsymbol{n}}A_{\boldsymbol{0}}, \qquad \boldsymbol{n}\in\mathbb{Z}_+^d. \tag{5.3}$$

Proof. If (5.3) holds and $\boldsymbol{N}\in\boldsymbol{L}^{\#}(\mathcal{F})^d$ is a formally normal extension of $\boldsymbol{S}$, then we have

$$\begin{aligned}\sum_{\substack{\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}_+^d\\ \boldsymbol{p},\boldsymbol{q}\in\mathbb{Z}_+^d}}\langle A_{\boldsymbol{m}+\boldsymbol{q}}f_{\boldsymbol{m},\boldsymbol{n}}, A_{\boldsymbol{n}+\boldsymbol{p}}f_{\boldsymbol{p},\boldsymbol{q}}\rangle &= \sum_{\substack{\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}_+^d\\ \boldsymbol{p},\boldsymbol{q}\in\mathbb{Z}_+^d}}\langle \boldsymbol{S}^{\boldsymbol{m}+\boldsymbol{q}}A_{\boldsymbol{0}}f_{\boldsymbol{m},\boldsymbol{n}}, \boldsymbol{S}^{\boldsymbol{n}+\boldsymbol{p}}A_{\boldsymbol{0}}f_{\boldsymbol{p},\boldsymbol{q}}\rangle\\ &= \sum_{\substack{\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}_+^d\\ \boldsymbol{p},\boldsymbol{q}\in\mathbb{Z}_+^d}}\langle \boldsymbol{N}^{\boldsymbol{m}+\boldsymbol{q}}A_{\boldsymbol{0}}f_{\boldsymbol{m},\boldsymbol{n}}, \boldsymbol{N}^{\boldsymbol{n}+\boldsymbol{p}}A_{\boldsymbol{0}}f_{\boldsymbol{p},\boldsymbol{q}}\rangle\end{aligned}$$

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$$= \sum_{\substack{\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}_+^d\\ \boldsymbol{p},\boldsymbol{q}\in\mathbb{Z}_+^d}} \langle \boldsymbol{N}^{\boldsymbol{m}}\boldsymbol{N}^{\#\boldsymbol{n}}A_{\boldsymbol{0}}f_{\boldsymbol{m},\boldsymbol{n}}, \boldsymbol{N}^{\boldsymbol{p}}\boldsymbol{N}^{\#\boldsymbol{q}}A_{\boldsymbol{0}}f_{\boldsymbol{p},\boldsymbol{q}}\rangle$$

$$= \Big\| \sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}_+^d} \boldsymbol{N}^{\boldsymbol{m}}\boldsymbol{N}^{\#\boldsymbol{n}}A_{\boldsymbol{0}}f_{\boldsymbol{m},\boldsymbol{n}} \Big\|^2 \geqslant 0.$$

Conversely, if $\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}_+^d}$ is of positive definite type, then applying Theorem 2.2 to the positive definite mapping $\Phi : \mathbb{Z}_+^d \times \mathbb{Z}_+^d \longrightarrow \Lambda(\mathcal{D})$ given by (5.1), we find an inner product space $\mathcal{F}$, a linear mapping $V : \mathcal{D} \longrightarrow \mathcal{F}$ and an involution preserving semigroup homomorphism $\Pi : \mathbb{Z}_+^d \times \mathbb{Z}_+^d \longrightarrow \boldsymbol{L}^{\#}(\mathcal{F})$ such that

$$\Phi(\boldsymbol{m},\boldsymbol{n})(g,h) = \langle \Pi(\boldsymbol{m},\boldsymbol{n})Vg, Vh\rangle, \qquad g,h\in\mathcal{D},\ \boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}_+^d, \tag{5.4}$$

$$\mathcal{F} \text{ is the linear span of } \{\Pi(\boldsymbol{m},\boldsymbol{n})Vg : \boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}_+^d,\ g\in\mathcal{D}\}. \tag{5.5}$$

For $i = 1,\dots,d$, we set $N_i = \Pi(\boldsymbol{e}_i^d, \boldsymbol{0}) \in \boldsymbol{L}^{\#}(\mathcal{F})$. Then, by (5.5), we have

$$\mathcal{F} \text{ is the linear span of } \{\boldsymbol{N}^{\#\boldsymbol{n}}\boldsymbol{N}^{\boldsymbol{m}}Vg : \boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}_+^d,\ g\in\mathcal{D}\},$$

where $\boldsymbol{N} \stackrel{\text{df}}{=} (N_1,\dots,N_d)$. It follows from (5.1) and (5.4) that

$$\langle A_{\boldsymbol{m}}g, A_{\boldsymbol{n}}h\rangle = \langle \boldsymbol{N}^{\boldsymbol{m}}Vg, \boldsymbol{N}^{\boldsymbol{n}}Vh\rangle, \qquad \boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}_+^d,\ g,h\in\mathcal{D}. \tag{5.6}$$

Denote by $\mathcal{L}$ the Hilbert space completion of $\mathcal{F}$. According to (5.6), there exists a unitary isomorphism U from $\mathcal{K}$ onto the closed subspace $\mathcal{R}(U)$ of $\mathcal{L}$ such that

$$UA_{\boldsymbol{n}} = \boldsymbol{N}^{\boldsymbol{n}}V, \qquad \boldsymbol{n}\in\mathbb{Z}_+^d. \tag{5.7}$$

Substituting $\boldsymbol{n} = 0$ into (5.7), we get $V = UA_{\boldsymbol{0}}$, which, once more by (5.7), gives us

$$UA_{\boldsymbol{n}} = \boldsymbol{N}^{\boldsymbol{n}}UA_{\boldsymbol{0}}, \qquad \boldsymbol{n}\in\mathbb{Z}_+^d. \tag{5.8}$$

Set $\widetilde{\mathcal{E}} = U(\mathcal{E})$. We claim that $\widetilde{\mathcal{E}}$ is invariant for every operator N_j, $j = 1,\dots,d$. Indeed, if $h = A_{\boldsymbol{n}}f$ with some $f\in\mathcal{D}$ and $\boldsymbol{n}\in\mathbb{Z}_+^d$, then by (5.8) we have

$$N_jUh = \boldsymbol{N}^{\boldsymbol{n}+\boldsymbol{e}_j^d}UA_{\boldsymbol{0}}f = UA_{\boldsymbol{n}+\boldsymbol{e}_j^d}f \in \widetilde{\mathcal{E}}.$$

Since $\mathcal{E}$ is linearly spanned by $\bigcup_{\boldsymbol{n}\in\mathbb{Z}_+^d}\mathcal{R}(A_{\boldsymbol{n}})$, our claim is proved. For $i = 1,\dots,d$, we define $S_i = U^{-1}(N_i|_{\widetilde{\mathcal{E}}})U$. It is clear that the d-tuple

$$\boldsymbol{S} \stackrel{\text{df}}{=} (S_1,\dots,S_d) \in \boldsymbol{L}(\mathcal{E})^d,$$

being unitarily equivalent to the formally subnormal d-tuple $(N_1|_{\widetilde{\mathcal{E}}},\dots,N_d|_{\widetilde{\mathcal{E}}})$, is formally subnormal as well. Multiplying on the left by U^{-1} both sides of (5.8), we get (5.3), which completes the proof. □

Formally subnormal d-tuples can be characterized by the Halmos–Bram–Ito condition (see [**9**, **20**] for the bounded case and [**21**, **44**] for the unbounded one).

COROLLARY 5.3. *A commuting d-tuple $\boldsymbol{S}\in\boldsymbol{L}(\mathcal{D})^d$ is formally subnormal if and only if, for every finitely supported d-sequence $\{f_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}_+^d}\subseteq\mathcal{D}$,*

$$\sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}_+^d} \langle \boldsymbol{S}^{\boldsymbol{m}}f_{\boldsymbol{n}}, \boldsymbol{S}^{\boldsymbol{n}}f_{\boldsymbol{m}}\rangle \geqslant 0. \tag{5.9}$$

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Proof. Set $\mathcal{E}=\mathcal{D}$ and $A_{\boldsymbol{n}}=\boldsymbol{S}^{\boldsymbol{n}}$ for $\boldsymbol{n}\in\mathbb{Z}_+^d$. Then, as is easily verified, (5.9) is equivalent to (5.2), and consequently the conclusion follows from Lemma 5.2. □

In what follows, we will concern ourselves only with the operator moment problem admitting subnormal solutions. It turns out that the positive definiteness is not sufficient for the sequence in question to be a moment sequence of this type.

EXAMPLE 5.4. Let $N\in\boldsymbol{L}^{\#}(\mathcal{F})$ be a non-subnormal formally normal operator such that $\mathcal{F}=\lin\{N^mN^{\#n}\xi\}_{m,n=0}^{\infty}$ for some $\xi\in\mathcal{F}$ (cf. [**40**], see also [**13**, **33**]). Set $\mathcal{D}=\lin\{N^n\xi\}_{n=0}^{\infty}$, $S=N|_{\mathcal{D}}$ and $h_n=S^n\xi$ for $n\geqslant 0$. Clearly, S is formally subnormal and $\lin\{h_n\}_{n=0}^{\infty}=\mathcal{D}$. By Lemma 5.2, $\{h_n\}_{n=0}^{\infty}$ is of positive definite type. We show that S is not subnormal. Suppose, contrary to our claim, that there exist a Hilbert space $\mathcal{T}\supseteq\mathcal{D}$ and a normal operator M in $\mathcal{T}$ such that $S\subseteq M$. By [**41**, Lemma 1.4],

$$\widetilde{\mathcal{F}}\stackrel{\mathrm{df}}{=}\lin\bigcup_{n=0}^{\infty}M^{*n}(\mathcal{D})$$

is invariant for M and M^*, and

$$\widetilde{N}\stackrel{\mathrm{df}}{=}M\big|_{\widetilde{\mathcal{F}}}\in\boldsymbol{L}^{\#}(\widetilde{\mathcal{F}})$$

is formally normal. The operators N and $\widetilde{N}$ are apparently minimal formally normal extensions of S, and, as such, they are unitarily equivalent to each other (cf. [**45**, Section 7]). Since $\widetilde{N}$ is subnormal, so is N, a contradiction.

Imposing a growth condition on a sequence of positive definite type may guarantee that it is a moment sequence in the above-mentioned sense.

THEOREM 5.5. *Let $\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}_+^d}\subseteq\boldsymbol{L}(\mathcal{D},\mathcal{E})$ be such that $\mathcal{E}=\lin\bigcup_{\boldsymbol{n}\in\mathbb{Z}_+^d}A_{\boldsymbol{n}}(\mathfrak{Q}_j)$ for every $j=1,\dots,d$, where*

$$\mathfrak{Q}_j=\left\{f\in\mathcal{D}:\sum_{k=1}^{\infty}\frac{1}{\big\|A_{k\boldsymbol{e}_j^d}f\big\|^{1/k}}=\infty\right\}\quad\left(\text{with }\frac{1}{0}=\infty\right).$$

Then the following conditions are equivalent.

(i) *There exists a (unique) subnormal d-tuple $\boldsymbol{S}\in\boldsymbol{L}(\mathcal{E})^d$ such that*

$$A_{\boldsymbol{n}}=\boldsymbol{S}^{\boldsymbol{n}}A_{\boldsymbol{0}},\qquad\boldsymbol{n}\in\mathbb{Z}_+^d.\tag{5.10}$$

(ii) *$\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}_+^d}$ is of positive definite type.*

Proof. (i)$\Rightarrow$(ii): The implication (i)$\Rightarrow$(ii) follows from Lemma 5.2 due to the fact that each subnormal d-tuple $\boldsymbol{S}\in\boldsymbol{L}(\mathcal{E})^d$ is formally subnormal.

(ii)$\Rightarrow$(i): According to Lemma 5.2, there exists a formally subnormal d-tuple $\boldsymbol{S}=(S_1,\dots,S_d)\in\boldsymbol{L}(\mathcal{E})^d$ which satisfies (5.10). It follows from (5.10) that $A_{\boldsymbol{0}}(\mathfrak{Q}_j)\subseteq\mathcal{Q}(S_j)$. Applying (5.10) once more, we see that $\bigcup_{\boldsymbol{n}\in\mathbb{Z}_+^d}A_{\boldsymbol{n}}(\mathfrak{Q}_j)\subseteq\bigcup_{\boldsymbol{n}\in\mathbb{Z}_+^d}\boldsymbol{S}^{\boldsymbol{n}}(\mathcal{Q}(S_j))$, which, by our main assumption, implies that the linear span of $\bigcup_{\boldsymbol{n}\in\mathbb{Z}_+^d}\boldsymbol{S}^{\boldsymbol{n}}(\mathcal{Q}(S_j))$ is equal to $\mathcal{E}$ for every $j=1,\dots,d$. By virtue of Corollary 5.3 and [**43**, Theorem 10′], the d-tuple $\boldsymbol{S}$ is subnormal. □

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Our next goal is to weaken Theorem 5.5(ii) so as to preserve equivalence (i)⇔(ii) (but under a stronger growth condition on $\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}_+^d}$). Recall that (by Lemmas 5.1 and 5.2) positive definiteness implies weak positive definiteness and Lemma 5.1(ii).

THEOREM 5.6. *Let $\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}_+^d}\subseteq \boldsymbol{L}(\mathcal{D},\mathcal{E})$ be such that $\mathcal{E}=\lin\bigcup_{\boldsymbol{n}\in\mathbb{Z}_+^d}A_{\boldsymbol{n}}(\mathfrak{A}_j)$ for every $j=1,\dots,d$, where*

$$\mathfrak{A}_j=\bigcup_{t>0}\left\{f\in\mathcal{D}:\sum_{k=0}^{\infty}\left\|A_{k\boldsymbol{e}_j^d}f\right\|\frac{t^k}{k!}<\infty\right\}. \tag{5.11}$$

Then the following conditions are equivalent.

(i) *There exists a (unique) subnormal d-tuple $\boldsymbol{S}\in\boldsymbol{L}(\mathcal{E})^d$ such that $A_{\boldsymbol{n}}=\boldsymbol{S}^{\boldsymbol{n}}A_{\boldsymbol{0}}$ for all $\boldsymbol{n}\in\mathbb{Z}_+^d$.*

(ii) *$\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}_+^d}$ is of weak positive definite type and it satisfies condition Lemma* 5.1(ii).

Moreover, if (i) *holds, then $\mathfrak{A}_j=\mathcal{D}$ for every $j=1,\dots,d$. On the other hand, if $\mathfrak{A}_j=\mathcal{D}$ for every $j=1,\dots,d$, then condition* (i) *is equivalent to the following.*

(iii) *$\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}_+^d}$ is of weak positive definite type.*

Proof. (i)⇒(ii), (i)⇒(iii): The implications (i)⇒(ii) and (i)⇒(iii) follow from Lemmas 5.1 and 5.2.

(ii)⇒(i): By Lemma 5.1, $A_{\boldsymbol{n}}=\boldsymbol{S}^{\boldsymbol{n}}A_{\boldsymbol{0}}$ for $\boldsymbol{n}\in\mathbb{Z}_+^d$ with some $\boldsymbol{S}=(S_1,\dots,S_d)\in\boldsymbol{L}(\mathcal{E})^d$. Let $\Gamma=\mathbb{Z}_+^d\times\mathbb{Z}_+^d$ be the $*$-semigroup equipped with coordinatewise defined addition and involution $(\boldsymbol{m},\boldsymbol{n})^*=(\boldsymbol{n},\boldsymbol{m})$, and let $\Phi:\Gamma\longrightarrow\Lambda(\mathcal{D})$ be defined by (5.1). Set $\gamma_j=(\boldsymbol{e}_j^d,\boldsymbol{0})$ for $j=1,\dots,d$, and $\mathcal{G}=\{\gamma_1,\dots,\gamma_d\}$. Applying the Schwarz inequality (cf. [**49**, inequality (1)])

$$|\Phi(\beta^*+\alpha)(f,f)|^2\leqslant\Phi(\beta^*+\beta)(f,f)\,\Phi(\alpha^*+\alpha)(f,f),\qquad \alpha,\beta\in\Gamma,\ f\in\mathcal{D},$$

we obtain, for all $k\geqslant 1$, $\boldsymbol{n}\in\mathbb{Z}_+^d$ and $f\in\mathcal{D}$,

$$\begin{aligned}\left\|S_j^kA_{\boldsymbol{n}}f\right\|^2&=\left\|S_j^k\boldsymbol{S}^{\boldsymbol{n}}A_{\boldsymbol{0}}f\right\|^2=\Phi\big(\big(k\boldsymbol{e}_j^d,k\boldsymbol{e}_j^d\big)^*+(\boldsymbol{n},\boldsymbol{n})\big)(f,f)\\&\leqslant\sqrt{\Phi\big(2k\boldsymbol{e}_j^d,2k\boldsymbol{e}_j^d\big)(f,f)}\,\sqrt{\Phi(2\boldsymbol{n},2\boldsymbol{n})(f,f)}=\left\|A_{2k\boldsymbol{e}_j^d}f\right\|\,\|A_{2\boldsymbol{n}}f\|.\end{aligned} \tag{5.12}$$

This, the inequality $\limsup_{k\to\infty}\sqrt[2k]{(2k)!/(k!)^2}<\infty$ (use the Stirling formula), and the Cauchy–Hadamard formula for the radius of convergence of power series lead to

$$\begin{aligned}\limsup_{k\to\infty}\sqrt[k]{\frac{\left\|S_j^kA_{\boldsymbol{n}}f\right\|}{k!}}&\leqslant\limsup_{k\to\infty}\sqrt[k]{\frac{\left\|A_{2k\boldsymbol{e}_j^d}f\right\|^{1/2}}{k!}}\\&=\limsup_{k\to\infty}\sqrt[2k]{\frac{\left\|A_{2k\boldsymbol{e}_j^d}f\right\|}{(2k)!}}\cdot\sqrt[2k]{\frac{(2k)!}{(k!)^2}}<\infty,\qquad \boldsymbol{n}\in\mathbb{Z}_+^d,\ f\in\mathfrak{A}_j.\end{aligned}$$

Applying once more the Cauchy–Hadamard formula, we get $A_{\boldsymbol{n}}(\mathfrak{A}_j)\subseteq\mathcal{A}(S_j)$. Since $\mathcal{A}(S_j)$ is a linear subspace of $\mathcal{E}$, we conclude that $\mathcal{E}=\lin\bigcup_{\boldsymbol{n}\in\mathbb{Z}_+^d}A_{\boldsymbol{n}}(\mathfrak{A}_j)\subseteq\mathcal{A}(S_j)\subseteq\mathcal{E}$. In consequence, we have $\mathcal{A}_\Phi(\gamma_j)=\mathfrak{A}_j=A_{\boldsymbol{0}}^{-1}(\mathcal{A}(S_j))=A_{\boldsymbol{0}}^{-1}(\mathcal{E})=\mathcal{D}$ for all $j=1,\dots,d$. By Theorem 2.3, the mapping Φ is positive definite. Since $\mathfrak{A}_j\subseteq\mathfrak{Q}_j$, Theorem 5.5 completes the proof of implication (ii)⇒(i).

(iii)⇒(i): An argument similar to that used at the very end of the proof of implication (ii)⇒(i) gives us (iii)⇒(i). ∎

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REMARK 5.7. Theorem 5.6 is closely connected with [**43**, Theorem 11]. In the case of finite systems of operators, the latter theorem follows from the former one with $A_{\boldsymbol{n}} = \boldsymbol{S}^{\boldsymbol{n}}$, $\boldsymbol{n} \in \mathbb{Z}_+^d$. In fact, our proof of Theorem 5.6 applies to arbitrary families of operators. In consequence, the statement of [**43**, Theorem 11] remains valid if we replace its condition (ii) by the following one.

(ii′) For every $\sigma \in \Sigma$, $\mathcal{E}$ is a linear subspace spanned by the set

$$\{\boldsymbol{S}^{\alpha} f : \alpha \in A, f \in \mathcal{A}(S_\sigma)\}.$$

6. *Necessity and sufficiency: the unbounded case*

In this section we continue investigations of the operator moment problem (1.2) admitting unbounded subnormal solutions $\boldsymbol{S}$. However, we now concentrate on necessary and sufficient conditions for the sequence in question to be a moment one. Recall that positive definiteness is not sufficient for subnormality of $\boldsymbol{S}$, and hence it has to be replaced by a stronger condition that is in general more complicated. As before, $\mathcal{D}$ and $\mathcal{E}$ are dense linear subspaces of Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively.

THEOREM 6.1. *If $\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}_+^d} \subseteq \boldsymbol{L}(\mathcal{D},\mathcal{E})$ is such that $\mathcal{E} = \lin \bigcup_{\boldsymbol{n}\in\mathbb{Z}_+^d} \mathcal{R}(A_{\boldsymbol{n}})$, then the following conditions are equivalent.*

(i) *There exists a (unique) subnormal d-tuple $\boldsymbol{S} \in \boldsymbol{L}(\mathcal{E})^d$ such that $A_{\boldsymbol{n}} = \boldsymbol{S}^{\boldsymbol{n}} A_{\boldsymbol{0}}$ for all $\boldsymbol{n} \in \mathbb{Z}_+^d$.*

(ii) *If $\{a_{\boldsymbol{p},\boldsymbol{q}}^{i,j}\}_{\boldsymbol{p},\boldsymbol{q}\in\mathbb{Z}_+^d}^{i,j\in\mathbb{Z}_+} \subseteq \mathbb{C}$ is a finitely supported $2(d+1)$-sequence such that $\sum_{i,j\geqslant 0} \sum_{\boldsymbol{p},\boldsymbol{q}\in\mathbb{Z}_+^d} a_{\boldsymbol{p},\boldsymbol{q}}^{i,j} \boldsymbol{\lambda}^{\boldsymbol{p}} \bar{\boldsymbol{\lambda}}^{\boldsymbol{q}} z_i \bar{z}_j \geqslant 0$ for all $\boldsymbol{\lambda} \in \mathbb{C}^d$ and $z_0, z_1, \ldots \in \mathbb{C}$, then*

$$\sum_{i,j\geqslant 0} \sum_{\boldsymbol{k},\boldsymbol{l}\in\mathbb{Z}_+^d} \sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}_+^d} \sum_{\boldsymbol{p},\boldsymbol{q}\in\mathbb{Z}_+^d} a_{\boldsymbol{p},\boldsymbol{q}}^{i,j} \big\langle A_{\boldsymbol{k}+\boldsymbol{m}+\boldsymbol{p}} h_{\boldsymbol{l},\boldsymbol{m}}^{j}, A_{\boldsymbol{l}+\boldsymbol{n}+\boldsymbol{q}} h_{\boldsymbol{k},\boldsymbol{n}}^{i} \big\rangle \geqslant 0 \tag{6.1}$$

for each finitely supported $(2d+1)$-sequence $\{h_{\boldsymbol{k},\boldsymbol{n}}^{i}\}_{\boldsymbol{k},\boldsymbol{n}\in\mathbb{Z}_+^d}^{i\in\mathbb{Z}_+} \subseteq \mathcal{D}$.

Proof. (ii) $\Rightarrow$ (i): We claim that $\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}_+^d}$ is of positive definite type. Substituting

$$a_{\boldsymbol{p},\boldsymbol{q}}^{i,j} = \begin{cases} 1 & i = j = 0,\ \boldsymbol{p} = \boldsymbol{q} = \boldsymbol{0} \\ 0 & \text{otherwise} \end{cases}$$

into (6.1) and setting $g_{\boldsymbol{m},\boldsymbol{n}} = h_{\boldsymbol{n},\boldsymbol{m}}^{0}$, we get

$$\sum_{\boldsymbol{m},\boldsymbol{l}\in\mathbb{Z}_+^d} \sum_{\boldsymbol{n},\boldsymbol{k}\in\mathbb{Z}_+^d} \langle A_{\boldsymbol{m}+\boldsymbol{k}} g_{\boldsymbol{m},\boldsymbol{l}}, A_{\boldsymbol{l}+\boldsymbol{n}} g_{\boldsymbol{n},\boldsymbol{k}} \rangle \geqslant 0,$$

which proves our claim. By Lemma 5.2, there exists $\boldsymbol{S} \in \boldsymbol{L}(\mathcal{E})^d$ such that $A_{\boldsymbol{n}} = \boldsymbol{S}^{\boldsymbol{n}} A_{\boldsymbol{0}}$ for all $\boldsymbol{n} \in \mathbb{Z}_+^d$. Let $\{f_{\boldsymbol{k}}^{i}\}_{\boldsymbol{k}\in\mathbb{Z}_+^d}^{i\in\mathbb{Z}_+} \subseteq \mathcal{E}$ be a finitely supported $(d+1)$-sequence. Then there exists a finitely supported $(2d+1)$-sequence $\{h_{\boldsymbol{k},\boldsymbol{n}}^{i}\}_{\boldsymbol{k},\boldsymbol{n}\in\mathbb{Z}_+^d}^{i\in\mathbb{Z}_+} \subseteq \mathcal{D}$ such that

$$f_{\boldsymbol{k}}^{i} = \sum_{\boldsymbol{n}\in\mathbb{Z}_+^d} A_{\boldsymbol{n}} h_{\boldsymbol{k},\boldsymbol{n}}^{i} = \sum_{\boldsymbol{n}\in\mathbb{Z}_+^d} \boldsymbol{S}^{\boldsymbol{n}} A_{\boldsymbol{0}} h_{\boldsymbol{k},\boldsymbol{n}}^{i}, \qquad i \geqslant 0,\ \boldsymbol{k} \in \mathbb{Z}_+^d. \tag{6.2}$$

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This, when combined with (6.1), gives us

$$\sum_{i,j\geqslant 0}\sum_{\boldsymbol{k},\boldsymbol{l}\in\mathbb{Z}_+^d}\sum_{\boldsymbol{p},\boldsymbol{q}\in\mathbb{Z}_+^d} a_{\boldsymbol{p},\boldsymbol{q}}^{i,j}\langle \boldsymbol{S}^{\boldsymbol{k}+\boldsymbol{p}} f_{\boldsymbol{l}}^j, \boldsymbol{S}^{\boldsymbol{l}+\boldsymbol{q}} f_{\boldsymbol{k}}^i\rangle$$
$$= \sum_{i,j\geqslant 0}\sum_{\boldsymbol{k},\boldsymbol{l}\in\mathbb{Z}_+^d}\sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}_+^d}\sum_{\boldsymbol{p},\boldsymbol{q}\in\mathbb{Z}_+^d} a_{\boldsymbol{p},\boldsymbol{q}}^{i,j}\langle A_{\boldsymbol{k}+\boldsymbol{m}+\boldsymbol{p}} h_{\boldsymbol{l},\boldsymbol{m}}^j, A_{\boldsymbol{l}+\boldsymbol{n}+\boldsymbol{q}} h_{\boldsymbol{k},\boldsymbol{n}}^i\rangle \geqslant 0,$$

which means that the 'if' part of the conclusion of [**44**, Theorem 3] is satisfied. ([**44**, Theorem 3] is true for finite systems of operators because its proof is based on results of Borchers and Yngvason stated in [**8**, Section 5] for finite systems of operators.) Applying this theorem completes the proof of implication (ii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii): Since $\boldsymbol{S}$ is subnormal, it satisfies the 'only if' part of the conclusion of [**44**, Theorem 3]. After substitutions like those in (6.2) have been made, [**44**, Theorem 3(ii)] becomes (6.1), which completes the proof. □

COROLLARY 6.2. *Suppose that* $\{A_{\sigma,\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}_+^d}\subseteq \boldsymbol{L}(\mathcal{D},\mathcal{E})$, $\sigma\in\Sigma$, *is a net of* d*-sequences of operators of the form* $A_{\sigma,\boldsymbol{n}}=\boldsymbol{S}_\sigma^{\boldsymbol{n}}A_{\sigma,\boldsymbol{0}}$, *where* $\{\boldsymbol{S}_\sigma\}_{\sigma\in\Sigma}\subseteq \boldsymbol{L}(\mathcal{E})^d$ *is a net of subnormal* d*-tuples. Let* $\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}_+^d}\subseteq \boldsymbol{L}(\mathcal{D},\mathcal{E})$ *be such that* $\mathcal{E}=\operatorname{lin}\bigcup_{\boldsymbol{n}\in\mathbb{Z}_+^d}\mathcal{R}(A_{\boldsymbol{n}})$ *and* $\lim_{\sigma\in\Sigma}A_{\sigma,\boldsymbol{n}}g=A_{\boldsymbol{n}}g$ *for all* $g\in\mathcal{D}$ *and* $\boldsymbol{n}\in\mathbb{Z}_+^d$. *Then there exists a subnormal* d*-tuple* $\boldsymbol{S}\in\boldsymbol{L}(\mathcal{E})^d$ *such that* $A_{\boldsymbol{n}}=\boldsymbol{S}^{\boldsymbol{n}}A_{\boldsymbol{0}}$ *for all* $\boldsymbol{n}\in\mathbb{Z}_+^d$.

Proof. Applying Theorem 6.1 to the d-sequence $\{A_{\sigma,\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}_+^d}\subseteq \boldsymbol{L}(\mathcal{D},\mathcal{E}_\sigma)$ with

$$\mathcal{E}_\sigma \stackrel{\text{df}}{=} \operatorname{lin}\bigcup_{\boldsymbol{n}\in\mathbb{Z}_+^d}\mathcal{R}(A_{\sigma,\boldsymbol{n}}),$$

we see that $\{A_{\sigma,\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}_+^d}$ satisfies (6.1). After passage to the limit with σ, we conclude that the limit sequence $\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}_+^d}$ satisfies (6.1). Applying Theorem 6.1 completes the proof. (In the case $d=1$, Corollary 6.2 also follows from Theorem 6.4.) □

A vector counterpart of Theorem 6.1 stated below is not a direct consequence of this theorem.

PROPOSITION 6.3. *If* $\{h_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}_+^d}\subseteq\mathcal{D}$ *is such that* $\mathcal{D}=\operatorname{lin}\{h_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}_+^d}$, *then the following conditions are equivalent.*

(i) *There exists a (unique) subnormal* d*-tuple* $\boldsymbol{S}\in\boldsymbol{L}(\mathcal{D})^d$ *such that* $h_{\boldsymbol{n}}=\boldsymbol{S}^{\boldsymbol{n}}h_{\boldsymbol{0}}$ *for all* $\boldsymbol{n}\in\mathbb{Z}_+^d$.

(ii) *If* $\{a_{\boldsymbol{k},\boldsymbol{l}}\}_{\boldsymbol{k},\boldsymbol{l}\in\mathbb{Z}_+^d}$ *is a finitely supported* $2d$*-sequence of complex numbers such that* $\sum_{\boldsymbol{k},\boldsymbol{l}\in\mathbb{Z}_+^d}a_{\boldsymbol{k},\boldsymbol{l}}\boldsymbol{\lambda}^{\boldsymbol{k}}\overline{\boldsymbol{\lambda}}^{\boldsymbol{l}}\geqslant 0$ *for each* $\boldsymbol{\lambda}\in\mathbb{C}^d$, *then* $\sum_{\boldsymbol{k},\boldsymbol{l}\in\mathbb{Z}_+^d}a_{\boldsymbol{k},\boldsymbol{l}}\langle h_{\boldsymbol{k}},h_{\boldsymbol{l}}\rangle\geqslant 0$.

Proof. We can repeat all but one of the arguments involved in the proof of Theorem 6.1, using [**44**, Proposition 3] instead of [**44**, Theorem 3]. ([**44**, Proposition 3] is true for finite cyclic systems of operators because its proof is based on Haviland's characterization of the multidimensional Hamburger moment problem, cf. [**18**, **19**].) Only the fact that condition (ii) implies that $\{h_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}_+^d}$ is of positive definite type requires a proof. For this, take a finitely supported

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$2d$-sequence $\{\mu_{\boldsymbol{m},\boldsymbol{n}}\}_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}_+^d}$ of complex numbers. After setting

$$a_{\boldsymbol{k},\boldsymbol{l}} = \sum_{\substack{\boldsymbol{m},\boldsymbol{q}\in\mathbb{Z}_+^d\\ \boldsymbol{m}+\boldsymbol{q}=\boldsymbol{k}}} \sum_{\substack{\boldsymbol{n},\boldsymbol{p}\in\mathbb{Z}_+^d\\ \boldsymbol{n}+\boldsymbol{p}=\boldsymbol{l}}} \mu_{\boldsymbol{m},\boldsymbol{n}}\bar{\mu}_{\boldsymbol{p},\boldsymbol{q}}, \qquad \boldsymbol{k},\boldsymbol{l}\in\mathbb{Z}_+^d,$$

we see that

$$\sum_{\boldsymbol{k},\boldsymbol{l}\in\mathbb{Z}_+^d} a_{\boldsymbol{k},\boldsymbol{l}}\boldsymbol{\lambda}^{\boldsymbol{k}}\bar{\boldsymbol{\lambda}}^{\boldsymbol{l}} = \sum_{\boldsymbol{m},\boldsymbol{n},\boldsymbol{p},\boldsymbol{q}\in\mathbb{Z}_+^d} \mu_{\boldsymbol{m},\boldsymbol{n}}\bar{\mu}_{\boldsymbol{p},\boldsymbol{q}}\boldsymbol{\lambda}^{\boldsymbol{m}+\boldsymbol{q}}\bar{\boldsymbol{\lambda}}^{\boldsymbol{n}+\boldsymbol{p}} = \left|\sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}_+^d} \mu_{\boldsymbol{m},\boldsymbol{n}}\boldsymbol{\lambda}^{\boldsymbol{m}}\bar{\boldsymbol{\lambda}}^{\boldsymbol{n}}\right|^2 \geqslant 0,$$

which, by virtue of condition (ii), yields

$$\sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}_+^d} \sum_{\boldsymbol{p},\boldsymbol{q}\in\mathbb{Z}_+^d} \langle h_{\boldsymbol{m}+\boldsymbol{q}}, h_{\boldsymbol{n}+\boldsymbol{p}}\rangle \mu_{\boldsymbol{m},\boldsymbol{n}}\bar{\mu}_{\boldsymbol{p},\boldsymbol{q}} = \sum_{\boldsymbol{k},\boldsymbol{l}\in\mathbb{Z}_+^d} a_{\boldsymbol{k},\boldsymbol{l}}\langle h_{\boldsymbol{k}}, h_{\boldsymbol{l}}\rangle \geqslant 0.$$

This precisely means that $\{h_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}_+^d}$ is of positive definite type. □

The next criterion is of a different nature, that is, it is based on an 'equality implies inequality' type condition. To simplify notation, we write $\{\lambda_{m,n}\}_{m+n\geqslant 0}$ below in place of

$$\{\lambda_{m,n}\}_{\substack{m,n\in\mathbb{Z}.\\ m+n\geqslant 0}}$$

A similar convention applies to summation symbols appearing in (6.3) and (6.4). Given $z=(z_1,\dots,z_r)\in\mathbb{C}^r$ and $w=(w_1,\dots,w_r)\in\mathbb{C}^r$, we write $\langle z,w\rangle = \sum_{j=1}^r z_j\bar{w}_j$.

THEOREM 6.4. *If $\{A_n\}_{n=0}^\infty \subseteq \boldsymbol{L}(\mathcal{D},\mathcal{E})$ is such that $\mathcal{E} = \operatorname{lin}\bigcup_{n=0}^\infty \mathcal{R}(A_n)$, then the following conditions are equivalent.*

(i) *There exists a (unique) subnormal operator $S\in\boldsymbol{L}(\mathcal{E})$ such that $A_n = S^nA_0$ for all $n\in\mathbb{Z}_+$.*

(ii) *For all positive integers $r\geqslant s$, for all finitely supported 2-sequences*

$$\{\lambda_{m,n}^{(1)}\}_{m+n\geqslant 0},\dots,\{\lambda_{m,n}^{(s)}\}_{m+n\geqslant 0}$$

in $\mathbb{C}^r$ such that

$$\sum_{\substack{m+n,p+q\geqslant 0\\ m+q=i,n+p=j}} \langle\lambda_{m,n}^{(k)},\lambda_{p,q}^{(l)}\rangle = 0, \qquad i+j\geqslant 0,\ i\cdot j<0,\ k,l=1,\dots,s, \tag{6.3}$$

and for each sequence $\{g_j\}_{j=1}^s$ in $\mathcal{D}$, the following inequality holds.

$$\sum_{\substack{m+n,p+q\geqslant 0\\ m+q,n+p\geqslant 0}} \sum_{k,l=1}^s \langle A_{m+q}g_k, A_{n+p}g_l\rangle\langle\lambda_{m,n}^{(k)},\lambda_{p,q}^{(l)}\rangle \geqslant 0. \tag{6.4}$$

Proof. (ii) ⇒ (i): We first show that the sequence $\{A_n\}_{n=0}^\infty$ is of positive definite type. Take a finitely supported non-zero 2-sequence $\{f_{m,n}\}_{m,n=0}^\infty \subseteq \mathcal{D}$. Let $\{g_k\}_{k=1}^s$ be a Hamel basis of $\operatorname{lin}\{f_{m,n} : m,n\geqslant 0\}$. Then there exist finitely supported 2-sequences

$$\{\vartheta_{m,n}^{(1)}\}_{m,n=0}^\infty,\dots,\{\vartheta_{m,n}^{(s)}\}_{m,n=0}^\infty$$

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of complex numbers such that

$$f_{m,n} = \sum_{k=1}^{s} \vartheta_{m,n}^{(k)} g_k \quad \text{for all } m,n \in \mathbb{Z}_+.$$

Set $r = s$. It is obvious that the 2-sequences

$$\{\lambda_{m,n}^{(1)}\}_{m+n\geqslant 0}, \dots, \{\lambda_{m,n}^{(s)}\}_{m+n\geqslant 0}$$

in $\mathbb{C}^r$ defined by

$$\lambda_{m,n}^{(k)} = \begin{cases} \vartheta_{m,n}^{(k)} \boldsymbol{e}_1^r & m,n \geqslant 0 \\ 0 & m+n \geqslant 0, m\cdot n < 0 \end{cases} \qquad k = 1,\dots,s$$

satisfy (6.3) and in consequence (6.4), that is,

$$\sum_{\substack{m,n\geqslant 0\\ p,q\geqslant 0}} \langle A_{m+q} f_{m,n}, A_{n+p} f_{p,q}\rangle = \sum_{\substack{m+n,p+q\geqslant 0\\ m+q,n+p\geqslant 0}} \sum_{k,l=1}^{s} \langle A_{m+q} g_k, A_{n+p} g_l\rangle \langle \lambda_{m,n}^{(k)}, \lambda_{p,q}^{(l)}\rangle \geqslant 0,$$

which completes the proof of our claim. By Lemma 5.2, there exists an operator $S \in \boldsymbol{L}(\mathcal{E})$ such that $A_n = S^n A_0$ for all $n \in \mathbb{Z}_+$. It is a matter of direct verification that the operator S satisfies condition (iv) of [**47**, Theorem 37] with the 'cyclic' space $\mathcal{F} = \mathcal{R}(A_0)$. By this theorem, S is subnormal, which completes the proof of (ii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii): The implication (i) $\Rightarrow$ (ii) can be inferred from implication (i) $\Rightarrow$ (iv) of [**47**, Theorem 37]; the reader may ensure, by verifying the proof of [**47**, Theorem 29], that implication (i) $\Rightarrow$ (iv) of [**47**, Theorem 37] is valid with no restriction on sequences $\{e_j\}_{j=1}^{\beta} \subseteq \mathcal{F}$ (in other words, $\{e_j\}_{j=1}^{\beta}$ need not be orthonormal). This completes the proof. □

In the vector counterpart of Theorem 6.4, condition (ii) is equivalent to the following.

(ii′) *For every positive integer r and for every finitely supported 2-sequence $\{\lambda_{m,n}\}_{m+n\geqslant 0}$ in $\mathbb{C}^r$ such that*

$$\sum_{\substack{m+n,p+q\geqslant 0\\ m+q=i,n+p=j}} \langle \lambda_{m,n}, \lambda_{p,q}\rangle = 0, \qquad i+j \geqslant 0,\ i\cdot j < 0,$$

the following inequality holds.

$$\sum_{\substack{m+n,p+q\geqslant 0\\ m+q,n+p\geqslant 0}} \langle h_{m+q}, h_{n+p}\rangle \langle \lambda_{m,n}, \lambda_{p,q}\rangle \geqslant 0.$$

According to Example 5.4, there is no hope of solving the operator moment problem in terms of positive definiteness only. However, in some specific cases, this is possible. Following [**12**, **40**], we say that a complex polynomial p in $2d$ (complex) variables is of *type* A$^{\text{o}}$ if, for every inner product space $\mathcal{D}$ and for every formally normal d-tuple $\boldsymbol{N} \in \boldsymbol{L}^{\#}(\mathcal{D})^d$ satisfying $p(\boldsymbol{N}, \boldsymbol{N}^{\#}) = 0$, there exists a normal d-tuple $\boldsymbol{T}$ in a Hilbert space $\mathcal{K} \supseteq \mathcal{D}$ (isometric embedding) such that $\boldsymbol{N} \subseteq \boldsymbol{T}$. The reader may consult [**7**, **12**, **40**, **47**, **48**] about the type A$^{\text{o}}$ and the related type A. In particular, p is of type A$^{\text{o}}$ if the set $\{\boldsymbol{z} \in \mathbb{C}^d : p(\boldsymbol{z}, \bar{\boldsymbol{z}}) = 0\}$ is compact (cf. [**12**, Theorem 40]) or if $p(z,w) = 1 + zq(z,w)$, $z,w \in \mathbb{C}$ (cf. [**47**, Theorem 52]); in the latter case, the set $\{z \in \mathbb{C} : p(z,\bar{z}) = 0\}$ may be unbounded.

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PROPOSITION 6.5. *Let* $\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}_+^d} \subseteq \boldsymbol{L}(\mathcal{D},\mathcal{E})$ *be such that*

$$\mathcal{E} = \lin \bigcup_{\boldsymbol{n}\in\mathbb{Z}_+^d} \mathcal{R}(A_{\boldsymbol{n}})$$

and let $\{a_{\boldsymbol{k},\boldsymbol{l}}\}_{\boldsymbol{k},\boldsymbol{l}\in\mathbb{Z}_+^d} \subseteq \mathbb{C}$ *be a finitely supported* $2d$*-sequence such that*

$$\sum_{\boldsymbol{i},\boldsymbol{j}\in\mathbb{Z}_+^d} \sum_{\boldsymbol{k},\boldsymbol{l}\in\mathbb{Z}_+^d} a_{\boldsymbol{i},\boldsymbol{j}}\bar{a}_{\boldsymbol{k},\boldsymbol{l}}\langle A_{\boldsymbol{i}+\boldsymbol{l}}g, A_{\boldsymbol{j}+\boldsymbol{k}}g\rangle = 0, \qquad g\in\mathcal{D}. \tag{6.5}$$

If

$$p(\boldsymbol{z},\boldsymbol{w}) \stackrel{\text{df}}{=} \sum_{\boldsymbol{k},\boldsymbol{l}\in\mathbb{Z}_+^d} a_{\boldsymbol{k},\boldsymbol{l}}\boldsymbol{z}^{\boldsymbol{k}}\boldsymbol{w}^{\boldsymbol{l}}$$

is of type A$^{\text{o}}$*, then the following are equivalent.*

(i) *There exists a (unique) subnormal* d*-tuple* $\boldsymbol{S}\in\boldsymbol{L}(\mathcal{E})^d$ *such that* $A_{\boldsymbol{n}} = \boldsymbol{S}^{\boldsymbol{n}}A_{\boldsymbol{0}}$ *for all* $\boldsymbol{n}\in\mathbb{Z}_+^d$.

(ii) $\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}_+^d}$ *is of positive definite type.*

Note that condition (6.5) *can be replaced by the following one* (*cf.* [**47**, *Lemma* 50]).

$$\sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}_+^d} a_{\boldsymbol{m},\boldsymbol{n}}\langle A_{\boldsymbol{m}+\boldsymbol{k}}g, A_{\boldsymbol{n}+\boldsymbol{l}}g\rangle = 0, \qquad \boldsymbol{k},\boldsymbol{l}\in\mathbb{Z}_+^d,\ g\in\mathcal{D}.$$

Proof. We only need to prove the implication (ii)$\Rightarrow$(i). By Lemma 5.2, there exists a formally normal d-tuple $\boldsymbol{N}\in\boldsymbol{L}^{\#}(\mathcal{F})^d$ such that $\mathcal{E}\subseteq\mathcal{F} = \lin\bigcup_{\boldsymbol{n}\in\mathbb{Z}_+^d}\boldsymbol{N}^{\#\boldsymbol{n}}(\mathcal{E})$ and $A_{\boldsymbol{n}} = \boldsymbol{N}^{\boldsymbol{n}}A_{\boldsymbol{0}}$ for all $\boldsymbol{n}\in\mathbb{Z}_+^d$. This and (6.5) imply that

$$\|p(\boldsymbol{N},\boldsymbol{N}^{\#})A_{\boldsymbol{0}}g\|^2 = \sum_{\boldsymbol{i},\boldsymbol{j}\in\mathbb{Z}_+^d} \sum_{\boldsymbol{k},\boldsymbol{l}\in\mathbb{Z}_+^d} a_{\boldsymbol{i},\boldsymbol{j}}\bar{a}_{\boldsymbol{k},\boldsymbol{l}}\langle A_{\boldsymbol{i}+\boldsymbol{l}}g, A_{\boldsymbol{j}+\boldsymbol{k}}g\rangle = 0, \qquad g\in\mathcal{D},$$

which, by formal normality of $\boldsymbol{N}$, yields

$$p(\boldsymbol{N},\boldsymbol{N}^{\#})A_{\boldsymbol{n}} = p(\boldsymbol{N},\boldsymbol{N}^{\#})\boldsymbol{N}^{\boldsymbol{n}}A_{\boldsymbol{0}} = \boldsymbol{N}^{\boldsymbol{n}}p(\boldsymbol{N},\boldsymbol{N}^{\#})A_{\boldsymbol{0}} = 0, \qquad \boldsymbol{n}\in\mathbb{Z}_+^d.$$

Since $\mathcal{E} = \lin\bigcup_{\boldsymbol{n}\in\mathbb{Z}_+^d}\mathcal{R}(A_{\boldsymbol{n}})$ and $\mathcal{F} = \lin\bigcup_{\boldsymbol{n}\in\mathbb{Z}_+^d}\boldsymbol{N}^{\#\boldsymbol{n}}(\mathcal{E})$, we obtain $p(\boldsymbol{N},\boldsymbol{N}^{\#}) = 0$. As p is of type A$^{\text{o}}$, the d-tuple $\boldsymbol{N}$ has a normal extension $\boldsymbol{M} = (M_1,\dots,M_d)$. This means that the d-tuple

$$\boldsymbol{S} \stackrel{\text{df}}{=} (M_1|_{\mathcal{E}},\dots,M_d|_{\mathcal{E}})$$

satisfies (i). □

A reasoning similar to that in Example 5.4, supported by [**40**, Theorem 6.3], enables us to show that implication (ii)$\Rightarrow$(i) of Proposition 6.5 is no longer true for $p(z,w) = (z-w)(z-w-iq(z+w))$, $z,w\in\mathbb{C}$, where q is a nonnegative nonzero polynomial in a single variable such that $\{x\in\mathbb{R}: q(x) = 0\}\neq\varnothing$.

It is straightforward to formulate the vector counterpart of Proposition 6.5, replacing 'type A$^{\text{o}}$' by 'type A'. The latter notion seems to be weaker than the former one (see [**12**, **40**] for definitions and properties of types A and A$^{\text{o}}$).

Picking up the thread of Proposition 6.5, we show that the two-sided operator moment problem can be completely solved for a subnormal unknown by means of positive definiteness.

THEOREM 6.6. *If* $\{A_n\}_{n=-\infty}^{\infty}\subseteq\boldsymbol{L}(\mathcal{D},\mathcal{E})$ *is such that* $\mathcal{E} = \lin\bigcup_{n=-\infty}^{\infty}\mathcal{R}(A_n)$, *then the following conditions are equivalent.*

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(i) *There exists an injective subnormal operator* $S \in \boldsymbol{L}(\mathcal{E})$ *such that* $S^{-1} \in \boldsymbol{L}(\mathcal{E})$ *and* $A_n = S^n A_0$ *for all* $n \in \mathbb{Z}$.

(ii) *For every finitely supported* $\{g_{m,n}\}_{m,n \in \mathbb{Z}} \subseteq \mathcal{D}$,

$$\sum_{m,n \in \mathbb{Z}} \sum_{p,q \in \mathbb{Z}} \langle A_{m+q} g_{m,n}, A_{n+p} g_{p,q} \rangle \geqslant 0.$$

Sketch of the proof. The proof of Theorem 6.6 proceeds along the same lines as proofs of other theorems in this section except for some necessary changes. First, mimicking the proof of Lemma 5.2 (replace in it the $*$-semigroup $\mathbb{Z}_+ \times \mathbb{Z}_+$ by the $*$-group $\mathbb{Z} \times \mathbb{Z}$ with involution $(m,n)^* = (n,m)$), we show that a sequence $\{A_n\}_{n=-\infty}^{\infty} \subseteq \boldsymbol{L}(\mathcal{D}, \mathcal{E})$ with $\mathcal{E} = \operatorname{lin} \bigcup_{n=-\infty}^{\infty} \mathcal{R}(A_n)$ satisfies (ii) if and only if there exists a formally subnormal injective operator $S \in \boldsymbol{L}(\mathcal{E})$ such that $S^{-1} \in \boldsymbol{L}(\mathcal{E})$ and $A_n = S^n A_0$ for all $n \in \mathbb{Z}$. Hence if (ii) holds, then the obtained operator S satisfies condition (iii) of [**47**, Theorem 39] with the 'bilaterally cyclic' space

$$\mathcal{F} \stackrel{\text{df}}{=} \mathcal{R}(A_0),$$

by this theorem, S is subnormal, as required in (i). Implication (i) $\Rightarrow$ (ii) follows from implication (i) $\Rightarrow$ (iii) of [**47**, Theorem 39]. □

In fact, Theorem 6.6 can also be proved with the help of the operator version of Bisgaard's theorem (cf. [**47**, Theorem 24]), while its vector counterpart can be proved with the help of Bisgaard's theorem itself (cf. [**5**, Theorem 1]).

7. *Necessity and sufficiency*: *the bounded case*

In this section we formulate versions of Theorems 5.5 and 5.6 for bounded operators. Just as in Section 5, $\mathcal{D}$ and $\mathcal{E}$ are assumed to be dense linear subspaces of Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. We begin by formulating the boundedness propagation property for d-sequences of operators acting between $\mathcal{H}$ and $\mathcal{K}$ which are of positive definite type. This property can be deduced from [**47**, Lemma 18] via (5.1).

PROPOSITION 7.1. *If* $\{A_{\boldsymbol{n}}\}_{\boldsymbol{n} \in \mathbb{Z}_+^d} \subseteq \boldsymbol{L}(\mathcal{H}, \mathcal{K})$ *is of positive definite type and* $A_{\boldsymbol{0}}$ *is bounded, then each* $A_{\boldsymbol{n}}$, $\boldsymbol{n} \in \mathbb{Z}_+^d$, *is bounded.*

We now discuss the question of the existence of the limit $\lim_{k \to \infty} \|A_{k\boldsymbol{e}_j^d} g\|^{1/k}$ for sequences $\{A_{\boldsymbol{n}}\}_{\boldsymbol{n} \in \mathbb{Z}_+^d}$ of operators which are of weak positive definite type. Recall that an operator $S \in \boldsymbol{L}(\mathcal{D})$ is said to be *paranormal* if $\|Sg\|^2 \leqslant \|g\| \cdot \|S^2 g\|$ for all $g \in \mathcal{D}$. It is known that formally subnormal operators are paranormal (cf. [**46**]).

PROPOSITION 7.2. *Let* $\{A_{\boldsymbol{n}}\}_{\boldsymbol{n} \in \mathbb{Z}_+^d} \subseteq \boldsymbol{L}(\mathcal{D}, \mathcal{E})$. *Then the set* $\mathfrak{B}_{j,c}$ *defined by*

$$\mathfrak{B}_{j,c} = \left\{ g \in \mathcal{D} : \limsup_{k \to \infty} \left\| A_{k\boldsymbol{e}_j^d} g \right\|^{1/k} \leqslant c \right\}$$

is a linear subspace of $\mathcal{D}$ *for every real* $c \geqslant 0$. *If* $\{A_{\boldsymbol{n}}\}_{\boldsymbol{n} \in \mathbb{Z}_+^d}$ *is of weak positive definite type, then the limit* $\lim_{k \to \infty} \|A_{k\boldsymbol{e}_j^d} g\|^{1/k}$ *exists (finite or not) for all* $j = 1, \ldots, d$ *and* $g \in \mathcal{D}$. *If* $\{A_{\boldsymbol{n}}\}_{\boldsymbol{n} \in \mathbb{Z}_+^d}$ *is of weak positive definite type,* $\mathcal{D} = \mathcal{E}$, *and* $A_{\boldsymbol{n}} = \boldsymbol{S}^{\boldsymbol{n}}$ *for all* $\boldsymbol{n} \in \mathbb{Z}_+^d$, *where* $\boldsymbol{S} = (S_1, \ldots, S_d) \in \boldsymbol{L}(\mathcal{D})^d$, *then the operators* $S_1, \ldots, S_d$ *are paranormal.*

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Proof. We show that $\mathfrak{B}_{j,c}$ is an additive subsemigroup of $\mathcal{D}$. Take $\epsilon>0$ and $f,g\in\mathfrak{B}_{j,c}$. Then there exists $k_0\geqslant 1$ such that $\|A_{k\boldsymbol{e}_j^d}h\|\leqslant(c+\epsilon)^k$ for $h=f,g$ and $k\geqslant k_0$. Hence $\|A_{k\boldsymbol{e}_j^d}(f+g)\|\leqslant 2(c+\epsilon)^k$ for all $k\geqslant k_0$, and consequently $\limsup_{k\to\infty}\|A_{k\boldsymbol{e}_j^d}(f+g)\|^{1/k}\leqslant c+\epsilon$, which implies that $f+g\in\mathfrak{B}_{j,c}$.

Assume now that $\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}_+^d}$ is of weak positive definite type. Let $\Gamma=\mathbb{Z}_+^d\times\mathbb{Z}_+^d$ be the $*$-semigroup equipped with coordinatewise defined addition and involution $(\boldsymbol{m},\boldsymbol{n})^*=(\boldsymbol{n},\boldsymbol{m})$, and let Φ be defined by (5.1). Set $\gamma_j=(\boldsymbol{e}_j^d,\boldsymbol{0})$. Since $\Phi(\cdot)(g,g)\in\Lambda(\mathbb{C})$ is positive definite and $\|A_{k\boldsymbol{e}_j^d}g\|^2=\Phi(k(\gamma_j^*+\gamma_j))(g,g)$ for all integers $k\geqslant 1$, the limit $\lim_{k\to\infty}\|A_{k\boldsymbol{e}_j^d}g\|^{1/k}$ exists (cf. [**42**]).

If moreover $A_{\boldsymbol{n}}=\boldsymbol{S}^{\boldsymbol{n}}$, then applying (5.12) with $\boldsymbol{n}=\boldsymbol{0}$ and $k=1$ shows that S_j is a paranormal operator on $\mathcal{D}$. □

The main criterion for solvability of the operator moment problem (5.3) with bounded subnormal d-tuple $\boldsymbol{S}$ is formulated below.

THEOREM 7.3. *Let $\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}_+^d}\subseteq\boldsymbol{L}(\mathcal{D},\mathcal{E})$ be such that $\mathcal{E}=\lin\bigcup_{\boldsymbol{n}\in\mathbb{Z}_+^d}\mathcal{R}(A_{\boldsymbol{n}})$. Then the following conditions are equivalent.*

(i) *There exists a (unique) subnormal d-tuple $\boldsymbol{S}=(S_1,\dots,S_d)\in\boldsymbol{B}(\mathcal{K})^d$ such that $A_{\boldsymbol{n}}=\boldsymbol{S}^{\boldsymbol{n}}A_{\boldsymbol{0}}$ for all $\boldsymbol{n}\in\mathbb{Z}_+^d$.*

(ii) *$\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}_+^d}$ is of positive definite type and*

$$\max_{j=1}^{d}\sup_{g\in\mathcal{D}}\lim_{k\to\infty}\|A_{k\boldsymbol{e}_j^d}g\|^{1/k}<\infty.$$

Moreover, if (i) *holds, then*

$$\|S_j\|=\sup_{g\in\mathcal{D}}\lim_{k\to\infty}\big\|A_{k\boldsymbol{e}_j^d}g\big\|^{1/k},\qquad j=1,\dots,d. \tag{7.1}$$

Proof. (i)$\Rightarrow$(ii): The implication (i)$\Rightarrow$(ii) follows from Lemma 5.2 because the space $\mathcal{E}=\lin\bigcup_{\boldsymbol{n}\in\mathbb{Z}_+^d}\mathcal{R}(A_{\boldsymbol{n}})$ is invariant for $S_1,\dots,S_d$, and

$$\lim_{k\to\infty}\big\|A_{k\boldsymbol{e}_j^d}g\big\|^{1/k}\leqslant\|S_j\|.$$

(ii)$\Rightarrow$(i): Applying Lemma 5.2, we find a formally subnormal d-tuple $\boldsymbol{T}=(T_1,\dots,T_d)\in\boldsymbol{L}(\mathcal{E})^d$ such that $A_{\boldsymbol{n}}=\boldsymbol{T}^{\boldsymbol{n}}A_{\boldsymbol{0}}$ for $\boldsymbol{n}\in\mathbb{Z}_+^d$. Let $\boldsymbol{M}=(M_1,\dots,M_d)\in\boldsymbol{L}^{\#}(\mathcal{F})^d$ be a minimal formally normal extension of $\boldsymbol{T}$. Then

$$A_{\boldsymbol{n}}=\boldsymbol{M}^{\boldsymbol{n}}A_{\boldsymbol{0}},\qquad\boldsymbol{n}\in\mathbb{Z}_+^d. \tag{7.2}$$

Fix $j=1,\dots,d$. Set $\widehat{a}_j=\sup_{g\in\mathcal{D}}\lim_{k\to\infty}\|A_{k\boldsymbol{e}_j^d}g\|^{1/k}$ and (cf. Lemma 2.1)

$$\mathcal{B}_c(M_j)=\big\{h\in\mathcal{F}:\rho_{M_j}(h)\leqslant c\big\},\qquad 0\leqslant c<\infty. \tag{7.3}$$

It follows from (7.2) and $\widehat{a}_j<\infty$ that

$$\mathcal{R}(A_{\boldsymbol{0}})\subseteq\mathcal{B}_{\widehat{a}_j}(M_j). \tag{7.4}$$

Since $\boldsymbol{M}^{\boldsymbol{n}}\in\{M_j,M_j^{\#}\}'$ for $\boldsymbol{n}\in\mathbb{Z}_+^d$, Lemma 2.1 implies that $\mathcal{B}_{\widehat{a}_j}(M_j)$ is a linear subspace of $\mathcal{F}$ which is invariant for $\boldsymbol{M}^{\boldsymbol{n}}$ and $\boldsymbol{M}^{\#\boldsymbol{n}}$. This, (7.2) and (7.4) lead to

$$\lin\bigcup_{\boldsymbol{n}\in\mathbb{Z}_+^d}\boldsymbol{M}^{\#\boldsymbol{n}}(\mathcal{E})=\lin\bigcup_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}_+^d}\boldsymbol{M}^{\#\boldsymbol{n}}\boldsymbol{M}^{\boldsymbol{m}}(\mathcal{R}(A_{\boldsymbol{0}}))\subseteq B_{\widehat{a}_j}(M_j).$$

Since $\boldsymbol{M}$ is a minimal formally normal extension of $\boldsymbol{T}$, we get $\mathcal{F}=B_{\widehat{a}_j}(M_j)$. Applying Lemma 2.1(ii), we conclude that the closure $\bar{M}_j$ of M_j (considered as

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an operator in the Hilbert space completion $\mathcal{L}$ of $\mathcal{F}$) is bounded and normal, and $\|\bar{M}_j\| \leqslant \widehat{a}_j$. Summarizing, $\bar{M}_1, \ldots, \bar{M}_d$ are commuting bounded normal operators on $\mathcal{L}$. Since $\mathcal{K} = \bar{\mathcal{E}}$ and $\bar{T}_j \subseteq \bar{M}_j$ for $j = 1, \ldots, d$, (7.2) implies (i) with

$$\boldsymbol{S} \stackrel{\text{df}}{=} (\bar{T}_1, \ldots, \bar{T}_d).$$

In particular, we have $\|S_j\| \leqslant \|\bar{M}_j\| \leqslant \widehat{a}_j$ for $j = 1, \ldots, d$, which completes the proof of (7.1) (because $\boldsymbol{S}$ in (i) is unique). □

Notice that Theorem 7.3(ii) implies that $\mathfrak{Q}_j = \mathcal{D}$ for every $j = 1, \ldots, d$, which by Theorem 5.5 guarantees the existence of a subnormal d-tuple $\boldsymbol{S} \in \boldsymbol{L}(\mathcal{E})^d$ such that $A_{\boldsymbol{n}} = \boldsymbol{S}^{\boldsymbol{n}} A_{\boldsymbol{0}}$ for $\boldsymbol{n} \in \mathbb{Z}_+^d$. This means that the essence of the proof of Theorem 7.3 lies in showing that $\boldsymbol{S}$ is composed of bounded operators.

The result which follows can be deduced from Theorem 7.3 with $\mathcal{D} = \mathcal{H}$ and $\mathcal{E} = \operatorname{lin} \bigcup_{\boldsymbol{n} \in \mathbb{Z}_+^d} \mathcal{R}(A_{\boldsymbol{n}})$. The details are left to the reader.

COROLLARY 7.4. *Let $\{A_{\boldsymbol{n}}\}_{\boldsymbol{n} \in \mathbb{Z}_+^d} \subseteq \boldsymbol{B}(\mathcal{H}, \mathcal{K})$ be such that $\mathcal{K} = \bigvee_{\boldsymbol{n} \in \mathbb{Z}_+^d} \mathcal{R}(A_{\boldsymbol{n}})$. Then the following conditions are equivalent.*

(i) *There exists a (unique) subnormal d-tuple $\boldsymbol{S} = (S_1, \ldots, S_d) \in \boldsymbol{B}(\mathcal{K})^d$ such that $A_{\boldsymbol{n}} = \boldsymbol{S}^{\boldsymbol{n}} A_{\boldsymbol{0}}$ for all $\boldsymbol{n} \in \mathbb{Z}_+^d$.*

(ii) *$\{A_{\boldsymbol{n}}\}_{\boldsymbol{n} \in \mathbb{Z}_+^d}$ is of positive definite type, and for every integer $j = 1, \ldots, d$, there exist $a_j, c_j \geqslant 0$ such that*

$$\left\|A_{k\boldsymbol{e}_j^d}\right\| \leqslant c_j a_j^k, \qquad k \in \mathbb{Z}_+. \tag{7.5}$$

Moreover, if (i) *holds, then*

$$\|S_j\| = \min\{a_j \geqslant 0 : (7.5) \text{ is valid for some } c_j \geqslant 0\},$$
$$\|A_{\boldsymbol{0}}\| = \min\{c_j \geqslant 0 : (7.5) \text{ is valid for some } a_j \geqslant 0\}.$$

The following corollary is a version of Corollary 6.2 for bounded operators.

COROLLARY 7.5. *Suppose that $\{A_{\sigma,\boldsymbol{n}}\}_{\boldsymbol{n} \in \mathbb{Z}_+^d} \subseteq \boldsymbol{B}(\mathcal{H}, \mathcal{K})$, $\sigma \in \Sigma$, is a net of d-sequences of operators of the form $A_{\sigma,\boldsymbol{n}} = \boldsymbol{S}_\sigma^{\boldsymbol{n}} A_{\sigma,\boldsymbol{0}}$, where $\{\boldsymbol{S}_\sigma\}_{\sigma \in \Sigma} \subseteq \boldsymbol{B}(\mathcal{K})^d$ is a net of subnormal d-tuples. Let $\{A_{\boldsymbol{n}}\}_{\boldsymbol{n} \in \mathbb{Z}_+^d} \subseteq \boldsymbol{B}(\mathcal{H}, \mathcal{K})$ be a d-sequence such that $\mathcal{K} = \bigvee_{\boldsymbol{n} \in \mathbb{Z}_+^d} \mathcal{R}(A_{\boldsymbol{n}})$, $\max_{j=1}^d \limsup_{k \to \infty} \|A_{k\boldsymbol{e}_j^d}\|^{1/k} < \infty$ and $\lim_\sigma A_{\sigma,\boldsymbol{n}} g = A_{\boldsymbol{n}} g$ for all $g \in \mathcal{H}$ and $\boldsymbol{n} \in \mathbb{Z}_+^d$. Then there exists a subnormal d-tuple $\boldsymbol{S} \in \boldsymbol{B}(\mathcal{K})^d$ such that $A_{\boldsymbol{n}} = \boldsymbol{S}^{\boldsymbol{n}} A_{\boldsymbol{0}}$ for all $\boldsymbol{n} \in \mathbb{Z}_+^d$.*

Proof. Applying Lemma 5.2 to the sequence $\{A_{\sigma,\boldsymbol{n}}\}_{\boldsymbol{n} \in \mathbb{Z}_+^d} \subseteq \boldsymbol{L}(\mathcal{H}, \mathcal{E}_\sigma)$ with $\mathcal{E}_\sigma = \operatorname{lin} \bigcup_{\boldsymbol{n} \in \mathbb{Z}_+^d} \mathcal{R}(A_{\sigma,\boldsymbol{n}})$, we see that $\{A_{\sigma,\boldsymbol{n}}\}_{\boldsymbol{n} \in \mathbb{Z}_+^d}$ satisfies (5.2). After passage to the limit with σ, we conclude that the d-sequence $\{A_{\boldsymbol{n}}\}_{\boldsymbol{n} \in \mathbb{Z}_+^d} \subseteq \boldsymbol{B}(\mathcal{H}, \mathcal{E})$ satisfies (5.2) with $\mathcal{E} = \operatorname{lin} \bigcup_{\boldsymbol{n} \in \mathbb{Z}_+^d} \mathcal{R}(A_{\boldsymbol{n}})$. The application of Theorem 7.3 completes the proof. □

An argument similar to that used in the proof of Corollary 6.2 shows that after removing the condition $\max_{j=1}^d \limsup_{k \to \infty} \|A_{k\boldsymbol{e}_j^d}\|^{1/k} < \infty$ from assumptions of Corollary 7.5, we come to the same conclusion, except that now $\boldsymbol{S} \in \boldsymbol{L}(\mathcal{E})^d$ with $\mathcal{E} = \operatorname{lin} \bigcup_{\boldsymbol{n} \in \mathbb{Z}_+^d} \mathcal{R}(A_{\boldsymbol{n}})$ ($\boldsymbol{S}$ is still subnormal).

Our next aim is to prove a 'bounded' counterpart of Theorem 5.6.

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THEOREM 7.6. *Let* $\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}_+^d} \subseteq \boldsymbol{L}(\mathcal{D},\mathcal{E})$ *be such that* $\mathcal{E} = \lin\bigcup_{\boldsymbol{n}\in\mathbb{Z}_+^d} \mathcal{R}(A_{\boldsymbol{n}})$. *Then the following conditions are equivalent.*

(i) *There exists a (unique) subnormal* d*-tuple* $\boldsymbol{S} = (S_1,\dots,S_d) \in \boldsymbol{B}(\mathcal{K})^d$ *such that* $A_{\boldsymbol{n}} = \boldsymbol{S}^{\boldsymbol{n}} A_{\boldsymbol{0}}$ *for all* $\boldsymbol{n}\in\mathbb{Z}_+^d$.

(ii) $\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}_+^d}$ *is of weak positive definite type, it satisfies Lemma* 5.1(ii), *and for every* $j = 1,\dots,d$, *there exists* $a_j \geqslant 0$ *such that* $\mathcal{E} = \lin\bigcup_{n\in\mathbb{Z}_+^d} A_{\boldsymbol{n}}(\mathfrak{B}_{j,a_j})$.

(iii) $\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}_+^d}$ *is of weak positive definite type and*

$$\max_{j=1}^{d} \sup_{g\in\mathcal{D}} \varlimsup_{k\to\infty} \big\|A_{k\boldsymbol{e}_j^d} g\big\|^{1/k} < \infty.$$

Moreover, if (i) *holds, then*

$$\|S_j\| = \min\{a_j \geqslant 0 : \mathcal{E} = \lin \bigcup_{n\in\mathbb{Z}_+^d} A_{\boldsymbol{n}}(\mathfrak{B}_{j,a_j})\}, \qquad j = 1,\dots,d. \tag{7.6}$$

Proof. (i) $\Rightarrow$ (ii), (i) $\Rightarrow$ (iii): The implications (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii) follow from $\mathfrak{B}_{j,\|S_j\|} = \mathcal{D}$.

(ii) $\Rightarrow$ (i): Since $\mathfrak{B}_{j,a_j} \subseteq \mathfrak{A}_j$, we infer from Theorem 5.6 that there exists a formally subnormal d-tuple $\boldsymbol{T} = (T_1,\dots,T_d) \in \boldsymbol{L}(\mathcal{E})^d$ such that $A_{\boldsymbol{n}} = \boldsymbol{T}^{\boldsymbol{n}} A_{\boldsymbol{0}}$ for $\boldsymbol{n}\in\mathbb{Z}_+^d$. Let $\boldsymbol{M} = (M_1,\dots,M_d) \in \boldsymbol{L}^{\#}(\mathcal{F})^d$ be a minimal formally normal extension of $\boldsymbol{T}$. Arguing as in the proof of Theorem 7.3, we first see that $A_{\boldsymbol{0}}(\mathfrak{B}_{j,a_j}) \subseteq \mathcal{B}_{a_j}(M_j)$, then that $\mathcal{E} = \lin\bigcup_{n\in\mathbb{Z}_+^d} A_{\boldsymbol{n}}(\mathfrak{B}_{j,a_j}) \subseteq \mathcal{B}_{a_j}(M_j)$ (because $A_{\boldsymbol{n}} = \boldsymbol{M}^{\boldsymbol{n}} A_{\boldsymbol{0}}$ and $\boldsymbol{M}^{\boldsymbol{n}} \in \{M_j, M_j^{\#}\}'$), and finally that $\mathcal{F} = \lin\bigcup_{\boldsymbol{n}\in\mathbb{Z}_+^d} \boldsymbol{M}^{\#\boldsymbol{n}}(\mathcal{E}) \subseteq \mathcal{B}_{a_j}(M_j)$ (due to minimality of $\boldsymbol{M}$). Repetition of the arguments used in the final part of the proof of Theorem 7.3 completes the proof of implication (ii) $\Rightarrow$ (i) as well as that of equality (7.6).

(iii) $\Rightarrow$ (i): Apply Theorem 5.6, Lemma 5.2 and finally Theorem 7.3. □

When d-sequences of bounded Hilbert space operators are considered, Theorem 7.6 takes the following form.

COROLLARY 7.7. *Let* $\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}_+^d} \subseteq \boldsymbol{B}(\mathcal{H},\mathcal{K})$ *be such that* $\mathcal{K} = \bar{\mathcal{E}}$, *where* $\mathcal{E} = \lin\bigcup_{\boldsymbol{n}\in\mathbb{Z}_+^d} \mathcal{R}(A_{\boldsymbol{n}})$. *Then the following conditions are equivalent.*

(i) *There exists a (unique) subnormal* d*-tuple* $\boldsymbol{S} = (S_1,\dots,S_d) \in \boldsymbol{B}(\mathcal{K})^d$ *such that* $A_{\boldsymbol{n}} = \boldsymbol{S}^{\boldsymbol{n}} A_{\boldsymbol{0}}$ *for all* $\boldsymbol{n}\in\mathbb{Z}_+^d$.

(ii) $\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}_+^d}$ *is of weak positive definite type,* $A_{\boldsymbol{n}} = \boldsymbol{T}^{\boldsymbol{n}} A_{\boldsymbol{0}}$ *for all* $\boldsymbol{n}\in\mathbb{Z}_+^d$, *where* $\boldsymbol{T} = (T_1,\dots,T_d) \in \boldsymbol{L}(\mathcal{E})^d$ *consists of closable operators in* $\mathcal{K}$ (*all this is implied by positive definiteness because formally subnormal operators* (*appearing in Lemma* 5.2) *are always closable in the Hilbert space in which they act*), *and for each* $j = 1,\dots,d$, *there exists* $a_j \geqslant 0$ *such that* $\mathcal{K} = \bigvee_{n\in\mathbb{Z}_+^d} A_{\boldsymbol{n}}(\bigcap_{j=1}^d \mathfrak{B}_{j,a_j})$.

(iii) $\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}_+^d}$ *is of weak positive definite type, it satisfies Lemma* 5.1(ii) *with* $\mathcal{D} = \mathcal{H}$, *and for each* $j = 1,\dots,d$, *there exists* $a_j \geqslant 0$ *such that* $\mathcal{E} = \lin\bigcup_{n\in\mathbb{Z}_+^d} A_{\boldsymbol{n}}(\mathfrak{B}_{j,a_j})$.

(iv) $\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}_+^d}$ *is of weak positive definite type, and for each* $j = 1,\dots,d$, *there exists* $a_j \geqslant 0$ *such that, for every finitely supported* $\{f_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}_+^d} \subseteq \mathcal{H}$,

$$\sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}_+^d} \big\langle A_{\boldsymbol{m}+\boldsymbol{e}_j^d} f_{\boldsymbol{m}}, A_{\boldsymbol{n}+\boldsymbol{e}_j^d} f_{\boldsymbol{n}}\big\rangle \leqslant a_j^2 \sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}_+^d} \langle A_{\boldsymbol{m}} f_{\boldsymbol{m}}, A_{\boldsymbol{n}} f_{\boldsymbol{n}}\rangle. \tag{7.7}$$

(v) $\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}_+^d}$ *is of weak positive definite type and* $\limsup_{k\to\infty} \|A_{k\boldsymbol{e}_j^d}\|^{1/k} < \infty$.

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Proof. Implications (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii) follow from $\mathfrak{B}_{j,\|S_j\|} = \mathcal{H}$. The proofs of (i) $\Rightarrow$ (iv) and (i) $\Rightarrow$ (v) are straightforward (with $a_j = \|S_j\|$). Implications (iii) $\Rightarrow$ (i) and (v) $\Rightarrow$ (i) are fairly direct consequences of Theorem 7.6 (with $\mathcal{D} = \mathcal{H}$).

(ii) $\Rightarrow$ (i): By Proposition 7.2,

$$\mathcal{D}_{\mathrm{b}} \stackrel{\mathrm{df}}{=} \bigcap_{j=1}^{d} \mathfrak{B}_{j,a_j}$$

is a linear subspace of $\mathcal{H}$. Set $\mathcal{E}_{\mathrm{b}} = \lin \bigcup_{n\in\mathbb{Z}_+^d} A_{\boldsymbol{n}}(\mathcal{D}_{\mathrm{b}})$. Applying Theorem 7.6 to the d-sequence $\{A_{\boldsymbol{n}}|_{\mathcal{D}_{\mathrm{b}}}\}_{\boldsymbol{n}\in\mathbb{Z}_+^d} \subseteq \boldsymbol{B}(\mathcal{D}_{\mathrm{b}}, \mathcal{E}_{\mathrm{b}})$, we deduce that $(T_1|_{\mathcal{E}_{\mathrm{b}}}, \ldots, T_d|_{\mathcal{E}_{\mathrm{b}}}) \in \boldsymbol{B}(\mathcal{E}_{\mathrm{b}})^d$ is a subnormal d-tuple and $\|T_j|_{\mathcal{E}_{\mathrm{b}}}\| \leqslant a_j$ for all $j = 1, \ldots, d$. Since $\mathcal{E}_{\mathrm{b}}$ is dense in $\mathcal{K}$ and T_j is closable in $\mathcal{K}$, we conclude that

$$\boldsymbol{S} \stackrel{\mathrm{df}}{=} (\bar{T}_1, \ldots, \bar{T}_d) \in \boldsymbol{B}(\mathcal{K})^d$$

is a subnormal d-tuple and $\|S_j\| \leqslant a_j$ for all $j = 1, \ldots, d$, which completes the proof of implication (ii) $\Rightarrow$ (i).

(iv) $\Rightarrow$ (i): Since $\mathcal{K} = \bigvee_{\boldsymbol{n}\in\mathbb{Z}_+^d} \mathcal{R}(A_{\boldsymbol{n}})$, (7.7) implies that there exists a (unique) d-tuple $\boldsymbol{S} = (S_1, \ldots, S_d) \in \boldsymbol{B}(\mathcal{K})^d$ such that $A_{\boldsymbol{n}} = \boldsymbol{S}^{\boldsymbol{n}} A_{\boldsymbol{0}}$ for all $\boldsymbol{n} \in \mathbb{Z}_+^d$ and $\|S_j\| \leqslant a_j$ for all $j = 1, \ldots, d$. The application of (ii) $\Rightarrow$ (i) completes the proof of (iv) $\Rightarrow$ (i). $\square$

REMARK 7.8. Notice that if $A_{\boldsymbol{n}} \in \boldsymbol{B}(\mathcal{H}, \mathcal{K})$ and $A_{\boldsymbol{n}} = \boldsymbol{S}^{\boldsymbol{n}} A_{\boldsymbol{0}}$ for $\boldsymbol{n} \in \mathbb{Z}_+^d$, where $\boldsymbol{S} = (S_1, \ldots, S_d) \in \boldsymbol{B}(\mathcal{K})^d$ is a subnormal d-tuple, and $\mathcal{K} = \bigvee_{\boldsymbol{n}\in\mathbb{Z}_+^d} \mathcal{R}(A_{\boldsymbol{n}})$, then

$$\|S_j\| = \min\left\{a_j \geqslant 0 : \mathcal{K} = \bigvee_{n\in\mathbb{Z}_+^d} A_{\boldsymbol{n}}(\mathfrak{B}_{j,a_j})\right\} = \lim_{k\to\infty} \sqrt[k]{\left\|A_{k\boldsymbol{e}_j^d}\right\|}, \qquad j = 1, \ldots, d. \tag{7.8}$$

Indeed, if $\mathcal{K} = \bigvee_{n\in\mathbb{Z}_+^d} A_{\boldsymbol{n}}(\mathfrak{B}_{j,a_j})$ and $\boldsymbol{N} \in \boldsymbol{B}(\mathcal{L})^d$ is a normal extension of $\boldsymbol{S}$, then

$$\widetilde{\mathcal{E}} \stackrel{\mathrm{df}}{=} \lin \bigcup_{n\in\mathbb{Z}_+^d} \boldsymbol{N}^{\boldsymbol{n}} A_{\boldsymbol{0}}\big(\mathfrak{B}_{j,a_j}\big) \subseteq \lin \bigcup_{n\in\mathbb{Z}_+^d} \boldsymbol{N}^{\boldsymbol{n}}\big(\mathcal{B}_{a_j}(N_j)\big) \subseteq \mathcal{B}_{a_j}(N_j),$$

which by Lemma 2.1 yields $\|S_j g\| = \|N_j g\| \leqslant a_j \|g\|$ for all $g \in \widetilde{\mathcal{E}}$. Since $\widetilde{\mathcal{E}}$ is dense in $\mathcal{K}$, we conclude that $\|S_j\| \leqslant a_j$, which together with $\mathfrak{B}_{j,\|S_j\|} = \mathcal{H}$ implies the first equality in (7.8). Finally, (7.1) gives us

$$\|S_j\| = \sup_{g\in\mathcal{H}} \lim_{k\to\infty} \sqrt[k]{\left\|A_{k\boldsymbol{e}_j^d} g\right\|} \leqslant \liminf_{k\to\infty} \sqrt[k]{\left\|A_{k\boldsymbol{e}_j^d}\right\|} \leqslant \limsup_{k\to\infty} \sqrt[k]{\left\|A_{k\boldsymbol{e}_j^d}\right\|} \leqslant \|S_j\|.$$

8. *Particular solutions*

In this section we shall investigate the operator moment problem (1.2) for specific classes of unknowns $\boldsymbol{S}$. For simplicity, we mostly confine ourselves to discussing the case of bounded operators.

8.1. *Normal solutions*

In this particular case, the minimality requirement $\mathcal{K} = \bigvee_{\boldsymbol{n}\in\mathbb{Z}_+^d} \mathcal{R}(A_{\boldsymbol{n}})$ turns out to be much too restrictive. To illustrate this, let us consider the multiplication

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operators $S_1, \ldots, S_d$ by independent complex variables $z_1, \ldots, z_d$ on $L^2(\Delta)$, where Δ is a nonempty compact subset of $\mathbb{C}^d$. Then $\boldsymbol{S} = (S_1, \ldots, S_d)$ is a d-tuple of commuting normal bounded operators on $L^2(\Delta)$ such that $\boldsymbol{z}^{\boldsymbol{n}} = \boldsymbol{S}^{\boldsymbol{n}} \boldsymbol{z}^{\boldsymbol{0}}$ for all $\boldsymbol{n} \in \mathbb{Z}_+^d$ (with $\boldsymbol{z}^{\boldsymbol{n}} = z_1^{n_1} \ldots z_d^{n_d}$). The minimality requirement is now that the set of all analytic polynomials in d complex variables is dense in $L^2(\Delta)$. Unfortunately, this is rarely the case. If we drop the minimality requirement, then it may still happen that Corollary 7.4(ii) does not imply the existence of a normal solution $\boldsymbol{S} \in \boldsymbol{B}(\mathcal{K})^d$ of (1.2).

EXAMPLE 8.1. Let μ be a normalized Lebesgue measure on

$$\mathbb{T} \stackrel{\text{df}}{=} \{z \in \mathbb{C} : |z| = 1\},$$

and let $\mathcal{H}$ be the closed linear subspace of $L^2(\mathbb{T}, \mu)$ spanned by the set $\{z^n\}_{n=-1}^{\infty}$. Then the sequence $h_n \stackrel{\text{df}}{=} z^n$, $n \geqslant 0$, satisfies the vector counterpart of Corollary 7.4(ii), although there is no normal solution $S \in \boldsymbol{B}(\mathcal{H})$ of (1.1). Suppose, contrary to our claim, that there exists such S. It is clear that $\bigvee_{n=0}^{\infty} h_n$ coincides with the Hardy space H^2 and $S|_{H^2}$ is the shift operator. Set

$$\mathcal{K} \stackrel{\text{df}}{=} \bigvee_{n=0}^{\infty} S^{*n} H^2.$$

Since $S|_{\mathcal{K}}$ and the operator of multiplication by the independent variable z on $L^2(\mathbb{T}, \mu)$ are minimal normal extensions of $S|_{H^2}$, we conclude that $\infty = \dim L^2(\mathbb{T}, \mu) \ominus H^2 = \dim \mathcal{K} \ominus H^2 \leqslant \dim \mathcal{H} \ominus H^2 = 1$, a contradiction.

However, if the space $\mathcal{H} \ominus \bigvee_{n=0}^{\infty} h_n$ is infinite-dimensional, then the situation described in Example 8.1 can never happen.

PROPOSITION 8.2. *Let* $\{A_{\boldsymbol{n}}\}_{\boldsymbol{n} \in \mathbb{Z}_+^d} \subseteq \boldsymbol{B}(\mathcal{H}, \mathcal{K})$ *be such that* $\dim \mathcal{K} \ominus \mathcal{K}_0 \geqslant \dim \mathcal{K}_0$, *where*

$$\mathcal{K}_0 \stackrel{\text{df}}{=} \bigvee_{\boldsymbol{n} \in \mathbb{Z}_+^d} \mathcal{R}(A_{\boldsymbol{n}}).$$

There exists a normal d*-tuple* $\boldsymbol{S} \in \boldsymbol{B}(\mathcal{K})^d$ *such that* $A_{\boldsymbol{n}} = \boldsymbol{S}^{\boldsymbol{n}} A_{\boldsymbol{0}}$ *for all* $\boldsymbol{n} \in \mathbb{Z}_+^d$ *if and only if* $\{A_{\boldsymbol{n}}\}_{\boldsymbol{n} \in \mathbb{Z}_+^d}$ *satisfies Corollary* 7.4(ii).

Proof. It is enough to prove the 'if' part of the conclusion. Suppose that $\{A_{\boldsymbol{n}}\}_{\boldsymbol{n} \in \mathbb{Z}_+^d}$ satisfies Corollary 7.4(ii). By this corollary, there exists a subnormal d-tuple $\boldsymbol{T} \in \boldsymbol{B}(\mathcal{K}_0)^d$ such that $A_{\boldsymbol{n}} = \boldsymbol{T}^{\boldsymbol{n}} A_{\boldsymbol{0}}$ for all $\boldsymbol{n} \in \mathbb{Z}_+^d$. Let $\boldsymbol{M} \in \boldsymbol{B}(\mathcal{L})^d$ be a minimal normal extension of $\boldsymbol{T}$ (minimality means that $\mathcal{L} = \bigvee_{\boldsymbol{n} \in \mathbb{Z}_+^d} \boldsymbol{M}^{\boldsymbol{n}*}(\mathcal{K}_0)$). A reasoning similar to that in the proof of [**45**, Proposition 1(δ)] shows that $\dim \mathcal{L} = \dim \mathcal{K}_0$. This implies that $\dim \mathcal{L} \ominus \mathcal{K}_0 \leqslant \dim \mathcal{K}_0$. Since $\dim \mathcal{K} \ominus \mathcal{K}_0 \geqslant \dim \mathcal{K}_0$, we conclude that there exists a closed linear subspace $\widetilde{\mathcal{L}}$ of $\mathcal{K}$ such that $\mathcal{K}_0 \subseteq \widetilde{\mathcal{L}}$ and $\dim \widetilde{\mathcal{L}} \ominus \mathcal{K}_0 = \dim \mathcal{L} \ominus \mathcal{K}_0$. Let $U : \widetilde{\mathcal{L}} \longrightarrow \mathcal{L}$ be a unitary isomorphism such that $Uf = f$ for all $f \in \mathcal{K}_0$. Set $S_j = I_{\mathcal{K} \ominus \widetilde{\mathcal{L}}} \oplus U^{-1} M_j U$ for $j = 1, \ldots, d$. It is a matter of direct verification that

$$\boldsymbol{S} \stackrel{\text{df}}{=} (S_1, \ldots, S_d) \in \boldsymbol{B}(\mathcal{K})^d$$

is a normal d-tuple that solves the moment problem (1.2). ∎

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8.2. *Selfadjoint solutions*

A 'symmetric' counterpart of Lemma 5.2 takes the following form.

LEMMA 8.3. *If* $\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}_+^d} \subseteq \boldsymbol{L}(\mathcal{D},\mathcal{E})$ *is such that* $\mathcal{E} = \lin\bigcup_{\boldsymbol{n}\in\mathbb{Z}_+^d} \mathcal{R}(A_{\boldsymbol{n}})$, *then the following two conditions are equivalent.*

(i) *There exists a* (*unique*) *commuting* d*-tuple* $\boldsymbol{S}\in\boldsymbol{L}^{\#}(\mathcal{E})^d$ *of symmetric operators such that* $A_{\boldsymbol{n}} = \boldsymbol{S}^{\boldsymbol{n}}A_{\boldsymbol{0}}$ *for all* $\boldsymbol{n}\in\mathbb{Z}_+^d$.

(ii) $\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}_+^d}$ *is of positive definite type and*

$$\big\langle A_{\boldsymbol{n}}g, A_{\boldsymbol{e}_j^d}h\big\rangle = \big\langle A_{\boldsymbol{n}+\boldsymbol{e}_j^d}g, A_{\boldsymbol{0}}h\big\rangle, \qquad \boldsymbol{n}\in\mathbb{Z}_+^d,\ g,h\in\mathcal{D},\ j=1,\dots,d. \tag{8.1}$$

(*Notice that if* $d=1$, *then the symmetric operator* S *appearing in* (i) *has a selfadjoint extension possibly in a larger Hilbert space* (*cf.* [**2**, §111, Theorem 1]). *In consequence, in this particular case, condition* (ii) *itself guarantees the existence of a subnormal solution of the operator moment problem under consideration.*)

Proof. (ii) $\Rightarrow$ (i): By Lemma 5.2, there exists a formally subnormal d-tuple $\boldsymbol{S}\in\boldsymbol{L}(\mathcal{E})^d$ such that $A_{\boldsymbol{n}} = \boldsymbol{S}^{\boldsymbol{n}}A_{\boldsymbol{0}}$ for all $\boldsymbol{n}\in\mathbb{Z}_+^d$. Let $\boldsymbol{N}\in\boldsymbol{L}^{\#}(\mathcal{F})^d$ be a formally normal extension of $\boldsymbol{S}$ and let $j=1,\dots,d$ be fixed. By (8.1), we have

$$\langle \boldsymbol{S}^{\boldsymbol{n}}A_{\boldsymbol{0}}g, N_jA_{\boldsymbol{0}}h\rangle = \big\langle A_{\boldsymbol{n}+\boldsymbol{e}_j^d}g, A_{\boldsymbol{0}}h\big\rangle = \langle S_j\boldsymbol{S}^{\boldsymbol{n}}A_{\boldsymbol{0}}g, A_{\boldsymbol{0}}h\rangle = \langle \boldsymbol{S}^{\boldsymbol{n}}A_{\boldsymbol{0}}g, N_j^{\#}A_{\boldsymbol{0}}h\rangle$$

for all $\boldsymbol{n}\in\mathbb{Z}_+^d$ and $g,h\in\mathcal{D}$. Since $\mathcal{E} = \lin\bigcup_{\boldsymbol{n}\in\mathbb{Z}_+^d} \boldsymbol{S}^{\boldsymbol{n}}A_{\boldsymbol{0}}(\mathcal{D})$, we conclude that

$$\langle f, N_jA_{\boldsymbol{0}}h\rangle = \langle f, N_j^{\#}A_{\boldsymbol{0}}h\rangle, \qquad f\in\mathcal{E}, h\in\mathcal{D}.$$

Hence $N_jA_{\boldsymbol{0}} = PN_j^{\#}A_{\boldsymbol{0}}$, where P is the orthogonal projection of $\mathcal{L}$ (which equals the Hilbert space completion of $\mathcal{F}$) onto $\mathcal{K}$. This and the formal normality of N_j lead to

$$\|N_jA_{\boldsymbol{0}}h\| = \|PN_j^{\#}A_{\boldsymbol{0}}h\| \leqslant \|N_j^{\#}A_{\boldsymbol{0}}h\| = \|N_jA_{\boldsymbol{0}}h\|, \qquad h\in\mathcal{D}.$$

Thus $N_j^{\#}A_{\boldsymbol{0}}h\in\mathcal{K}$ for all $h\in\mathcal{D}$, which together with $N_jA_{\boldsymbol{0}} = PN_j^{\#}A_{\boldsymbol{0}}$ gives us $N_jA_{\boldsymbol{0}} = N_j^{\#}A_{\boldsymbol{0}}$. This and the formal normality of $\boldsymbol{N}$ yield

$$N_j\boldsymbol{S}^{\boldsymbol{n}}A_{\boldsymbol{0}}h = \boldsymbol{N}^{\boldsymbol{n}}N_jA_{\boldsymbol{0}}h = \boldsymbol{N}^{\boldsymbol{n}}N_j^{\#}A_{\boldsymbol{0}}h = N_j^{\#}\boldsymbol{S}^{\boldsymbol{n}}A_{\boldsymbol{0}}h, \qquad \boldsymbol{n}\in\mathbb{Z}_+^d,\ h\in\mathcal{D}.$$

Using once more the equality $\mathcal{E} = \lin\bigcup_{\boldsymbol{n}\in\mathbb{Z}_+^d} \boldsymbol{S}^{\boldsymbol{n}}A_{\boldsymbol{0}}(\mathcal{D})$, we get $N_jf = N_j^{\#}f$ for all $f\in\mathcal{E}$. Finally, we obtain $\langle S_jf, g\rangle = \langle N_jf, g\rangle = \langle N_j^{\#}f, g\rangle = \langle f, S_jg\rangle$ for all $f,g\in\mathcal{E}$, which is equivalent to $S_j = S_j^{\#}$.

(i) $\Rightarrow$ (ii): The proof of (i) $\Rightarrow$ (ii) is straightforward. □

In view of Lemma 8.3, most of the results of Sections 5 and 7 can be restated for spectrally commuting d-tuples of selfadjoint operators $\boldsymbol{S}$ simply by strengthening the positive definiteness by condition (8.1). There is an independent way of dealing with the symmetric case. That is, one can verify that (still under the assumption that $\mathcal{E} = \lin\bigcup_{\boldsymbol{n}\in\mathbb{Z}_+^d} \mathcal{R}(A_{\boldsymbol{n}})$) Lemma 8.3(i) is equivalent to the conjunction of Lemma 5.1(ii) and

$$\big\langle A_{\boldsymbol{n}}g, A_{\boldsymbol{m}+\boldsymbol{e}_j^d}h\big\rangle = \big\langle A_{\boldsymbol{n}+\boldsymbol{e}_j^d}g, A_{\boldsymbol{m}}h\big\rangle, \qquad \boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}_+^d,\ g,h\in\mathcal{D},\ j=1,\dots,d. \tag{8.2}$$

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It is obvious that (8.2) implies (8.1). Continuing this kind of argumentation, we see that if $\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}^d_+} \subseteq \boldsymbol{B}(\mathcal{H},\mathcal{K})$ is such that $\mathcal{K} = \bigvee_{\boldsymbol{n}\in\mathbb{Z}^d_+} \mathcal{R}(A_{\boldsymbol{n}})$, then there exists a commuting d-tuple $\boldsymbol{S} \in \boldsymbol{B}(\mathcal{K})^d$ of selfadjoint operators such that $A_{\boldsymbol{n}} = \boldsymbol{S}^{\boldsymbol{n}} A_{\boldsymbol{0}}$ for all $\boldsymbol{n} \in \mathbb{Z}^d_+$ if and only if condition (8.2) holds and there exists $a \geqslant 0$ such that

$$\sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}^d_+} \big\langle A_{\boldsymbol{m}+\boldsymbol{e}^d_j} f_{\boldsymbol{m}}, A_{\boldsymbol{n}+\boldsymbol{e}^d_j} f_{\boldsymbol{n}} \big\rangle \leqslant a^2 \sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}^d_+} \langle A_{\boldsymbol{m}} f_{\boldsymbol{m}}, A_{\boldsymbol{n}} f_{\boldsymbol{n}} \rangle$$

for each finitely supported d-sequence $\{f_{\boldsymbol{m}}\}_{\boldsymbol{m}\in\mathbb{Z}^d_+} \subseteq \mathcal{H}$ and for all $j = 1, \ldots, d$.

8.3. *Unitary solutions: the 'bilateral' case*

Mimicking proofs of Lemmas 5.2 and 8.3, one can show that if $\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}^d} \subseteq \boldsymbol{B}(\mathcal{H},\mathcal{K})$ is such that $\mathcal{K} = \bigvee_{\boldsymbol{n}\in\mathbb{Z}^d} \mathcal{R}(A_{\boldsymbol{n}})$, then there exists a commuting d-tuple $\boldsymbol{U} \in \boldsymbol{B}(\mathcal{K})^d$ of unitary operators such that $A_{\boldsymbol{n}} = \boldsymbol{U}^{\boldsymbol{n}} A_{\boldsymbol{0}}$ for all $\boldsymbol{n} \in \mathbb{Z}^d$ if and only if $\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}^d}$ is of positive definite type (that is, (5.2) holds, with $\mathbb{Z}^d_+$ replaced by $\mathbb{Z}^d$) and

$$\begin{aligned} \big\langle A_{\boldsymbol{n}} g, A_{-\boldsymbol{e}^d_j} h \big\rangle &= \big\langle A_{\boldsymbol{n}+\boldsymbol{e}^d_j} g, A_{\boldsymbol{0}} h \big\rangle, \\ \big\| A_{\boldsymbol{e}^d_j} g \big\| &= \|A_{\boldsymbol{0}} g\|, \end{aligned} \qquad \boldsymbol{n} \in \mathbb{Z}^d,\ g, h \in \mathcal{H},\ j = 1, \ldots, d. \tag{8.3}$$

Another way of solving this question is as follows. If $\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}^d} \subseteq \boldsymbol{B}(\mathcal{H},\mathcal{K})$ is such that $\mathcal{K} = \bigvee_{\boldsymbol{n}\in\mathbb{Z}^d} \mathcal{R}(A_{\boldsymbol{n}})$, then there exists a commuting d-tuple $\boldsymbol{U} \in \boldsymbol{B}(\mathcal{K})^d$ of unitary operators such that $A_{\boldsymbol{n}} = \boldsymbol{U}^{\boldsymbol{n}} A_{\boldsymbol{0}}$ for all $\boldsymbol{n} \in \mathbb{Z}^d$ if and only if (cf. [**35**, Corollary B])

$$\langle A_{\boldsymbol{n}} g, A_{\boldsymbol{m}} h \rangle = \big\langle A_{\boldsymbol{n}+\boldsymbol{e}^d_j} g, A_{\boldsymbol{m}+\boldsymbol{e}^d_j} h \big\rangle, \qquad \boldsymbol{m}, \boldsymbol{n} \in \mathbb{Z}^d,\ g, h \in \mathcal{H},\ j = 1, \ldots, d. \tag{8.4}$$

Clearly (8.4) implies (8.3).

8.4. *Isometric solutions*

It is easily seen that if $\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}^d_+} \subseteq \boldsymbol{B}(\mathcal{H},\mathcal{K})$ is such that $\mathcal{K} = \bigvee_{\boldsymbol{n}\in\mathbb{Z}^d_+} \mathcal{R}(A_{\boldsymbol{n}})$, then there exists a d-tuple $\boldsymbol{V} \in \boldsymbol{B}(\mathcal{K})^d$ of commuting isometries such that $A_{\boldsymbol{n}} = \boldsymbol{V}^{\boldsymbol{n}} A_{\boldsymbol{0}}$ for all $\boldsymbol{n} \in \mathbb{Z}^d_+$ if and only if $\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}^d_+}$ satisfies condition (8.4), with $\mathbb{Z}^d$ replaced by $\mathbb{Z}^d_+$.

8.5. *Unitary solutions: the 'unilateral' case*

Subsection 8.4 suggests the question of how the operator moment 'unilateral' sequences with unitary solutions (cf. Subsection 8.3) can be characterized. As in Subsection 8.1, we see that in the present setting, the minimality requirement $\mathcal{K} = \bigvee_{\boldsymbol{n}\in\mathbb{Z}^d_+} \mathcal{R}(A_{\boldsymbol{n}})$ is still too restrictive (see also Example 8.1). However, a result analogous to Proposition 8.2 holds true.

PROPOSITION 8.4. *Let* $\{A_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}^d_+} \subseteq \boldsymbol{B}(\mathcal{H},\mathcal{K})$ *be such that* $\dim \mathcal{K} \ominus \mathcal{K}_0 \geqslant \dim \mathcal{K}_0$, *where*

$$\mathcal{K}_0 \stackrel{\text{df}}{=} \bigvee_{\boldsymbol{n}\in\mathbb{Z}^d_+} \mathcal{R}(A_{\boldsymbol{n}}).$$

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There exists a commuting d-tuple $\boldsymbol{U} \in \boldsymbol{B}(\mathcal{K})^d$ of unitary operators such that $A_{\boldsymbol{n}} = \boldsymbol{U}^{\boldsymbol{n}} A_{\boldsymbol{0}}$ for all $\boldsymbol{n} \in \mathbb{Z}_+^d$ if and only if $\{A_{\boldsymbol{n}}\}_{\boldsymbol{n} \in \mathbb{Z}_+^d}$ satisfies (8.4) *with $\mathbb{Z}^d$ replaced by $\mathbb{Z}_+^d$.*

Proof. Suppose that $\{A_{\boldsymbol{n}}\}_{\boldsymbol{n} \in \mathbb{Z}_+^d}$ satisfies (8.4) with $\mathbb{Z}^d$ replaced by $\mathbb{Z}_+^d$. According to Subsection 8.4, there exists a d-tuple $\boldsymbol{V} \in \boldsymbol{B}(\mathcal{K}_0)^d$ of commuting isometries such that $A_{\boldsymbol{n}} = \boldsymbol{V}^{\boldsymbol{n}} A_{\boldsymbol{0}}$ for all $\boldsymbol{n} \in \mathbb{Z}_+^d$. It follows from the Ito–Brehmer theorem (cf. [**10**, **20**]) that there exists a d-tuple $\boldsymbol{W} \in \boldsymbol{B}(\mathcal{L})^d$ of commuting unitary operators such that $V_j \subseteq W_j$ for all $j = 1, \ldots, d$. Without loss of generality, we can assume that $\boldsymbol{W}$ is a minimal extension of $\boldsymbol{V}$ (minimality now reads $\mathcal{L} = \bigvee_{\boldsymbol{n} \in \mathbb{Z}_+^d} \boldsymbol{W}^{-\boldsymbol{n}}(\mathcal{K}_0)$). Arguing as in the proof of Proposition 8.2, we get the 'if' part of the conclusion. □

9. *The bounded case revised*

In this section we show how to infer from Theorem 3.1 and Corollary 7.4 a multioperator version of some results relating to vector and operator moment problems obtained by Sebestyén in [**34**, **36**], Pãunescu in [**28**, **29**] and Binzar and Pãunescu in [**4**]. As usual, $\mathcal{H}$ and $\mathcal{K}$ stand for Hilbert spaces.

PROPOSITION 9.1. *If $\{A_{\boldsymbol{n}}\}_{\boldsymbol{n} \in \mathbb{Z}_+^d} \subseteq \boldsymbol{B}(\mathcal{H}, \mathcal{K})$ is such that $\mathcal{K} = \bigvee_{\boldsymbol{n} \in \mathbb{Z}_+^d} \mathcal{R}(A_{\boldsymbol{n}})$, then the following conditions are equivalent.*

(i) *There exists a subnormal d-tuple $\boldsymbol{S} = (S_1, \ldots, S_d) \in \boldsymbol{B}(\mathcal{K})^d$ such that $A_{\boldsymbol{n}} = \boldsymbol{S}^{\boldsymbol{n}} A_{\boldsymbol{0}}$ for all $\boldsymbol{n} \in \mathbb{Z}_+^d$.*

(ii) *There exist $a, c \geqslant 0$ and $\{A_{\boldsymbol{m},\boldsymbol{n}}\}_{\boldsymbol{m},\boldsymbol{n} \in \mathbb{Z}_+^d} \subseteq \boldsymbol{B}(\mathcal{H}, \mathcal{K})$ such that the following hold.*

(a) $\|A_{k\boldsymbol{e}_j^d}\| \leqslant c a^k$ *for all $k \in \mathbb{Z}_+$ and $j = 1, \ldots, d$.*

(b) $A_{\boldsymbol{n},\boldsymbol{0}} = A_{\boldsymbol{n}}$ *for all $\boldsymbol{n} \in \mathbb{Z}_+^d$.*

(c) $\langle A_{\boldsymbol{m},\boldsymbol{n}} g, A_{\boldsymbol{p}} h \rangle = \langle A_{\boldsymbol{m}} g, A_{\boldsymbol{n}+\boldsymbol{p}} h \rangle$ *for all $g, h \in \mathcal{H}$ and $\boldsymbol{m}, \boldsymbol{n}, \boldsymbol{p} \in \mathbb{Z}_+^d$.*

(d) *For every finitely supported $2d$-sequence $\{f_{\boldsymbol{m},\boldsymbol{n}}\}_{\boldsymbol{m},\boldsymbol{n} \in \mathbb{Z}_+^d} \subseteq \mathcal{H}$,*

$$\left\| \sum_{\boldsymbol{m},\boldsymbol{n} \in \mathbb{Z}_+^d} A_{\boldsymbol{m},\boldsymbol{n}} f_{\boldsymbol{m},\boldsymbol{n}} \right\|^2 \leqslant \sum_{\boldsymbol{m},\boldsymbol{n} \in \mathbb{Z}_+^d} \sum_{\boldsymbol{p},\boldsymbol{q} \in \mathbb{Z}_+^d} \langle A_{\boldsymbol{m}+\boldsymbol{q}} f_{\boldsymbol{m},\boldsymbol{n}}, A_{\boldsymbol{n}+\boldsymbol{p}} f_{\boldsymbol{p},\boldsymbol{q}} \rangle.$$

Proof. (i) $\Rightarrow$ (ii): By Theorem 3.1, there exist a Hilbert space $\mathcal{L} \supseteq \mathcal{K}$ and a commuting d-tuple $\boldsymbol{U} = (U_1, \ldots, U_d) \in \boldsymbol{B}(\mathcal{L})^d$ of unitary operators such that

$$\boldsymbol{S}_{\omega}^{*\boldsymbol{n}} \boldsymbol{S}_{\widetilde{\omega}}^{\boldsymbol{n}} = \boldsymbol{S}_{\widetilde{\omega}}^{*\boldsymbol{n}} P_{\mathcal{K}} \boldsymbol{U}_{\widetilde{\omega}}^{*\boldsymbol{n}} \boldsymbol{U}_{\omega}^{\boldsymbol{n}} |_{\mathcal{K}} \boldsymbol{S}_{\omega}^{\boldsymbol{n}}, \qquad \boldsymbol{n} \in \mathbb{Z}_+^d, \, \omega \in \Omega_d, \tag{9.1}$$

where $P_{\mathcal{K}}$ is the orthogonal projection of $\mathcal{L}$ onto $\mathcal{K}$ (Ω_d and $\widetilde{\omega}$ are defined in Section 3). Define the multisequence $\{A_{\boldsymbol{m},\boldsymbol{n}}\}_{\boldsymbol{m},\boldsymbol{n} \in \mathbb{Z}_+^d} \subseteq \boldsymbol{B}(\mathcal{H}, \mathcal{K})$ via

$$A_{\boldsymbol{m},\boldsymbol{n}} = \boldsymbol{S}^{*\boldsymbol{n}} \boldsymbol{S}^{\boldsymbol{m}} A_{\boldsymbol{0}}, \qquad \boldsymbol{m}, \boldsymbol{n} \in \mathbb{Z}_+^d. \tag{9.2}$$

Conditions (a), (b) and (c) are easily seen to be true. To prove (d), notice first that if $g, h \in \mathcal{H}$, $\omega \in \Omega_d$, $\boldsymbol{m}, \boldsymbol{n}, \boldsymbol{p}, \boldsymbol{q} \in \mathbb{Z}_+^d$ and $\boldsymbol{n} - \boldsymbol{q} \in \mathbb{Z}_{\omega}^d$, then by (9.1)

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we have

$$\begin{aligned}
&\langle \boldsymbol{U}^{\boldsymbol{n}}\boldsymbol{S}^{\boldsymbol{m}+\boldsymbol{n}}A_{\boldsymbol{0}}g, \boldsymbol{U}^{\boldsymbol{q}}\boldsymbol{S}^{\boldsymbol{p}+\boldsymbol{q}}A_{\boldsymbol{0}}h\rangle\\
&\quad= \langle \boldsymbol{U}^{*\boldsymbol{q}}\boldsymbol{U}^{\boldsymbol{n}}\boldsymbol{S}^{\boldsymbol{m}+\boldsymbol{n}}A_{\boldsymbol{0}}g, \boldsymbol{S}^{\boldsymbol{p}+\boldsymbol{q}}A_{\boldsymbol{0}}h\rangle\\
&\quad= \big\langle P_{\mathcal{K}}\boldsymbol{U}_{\widetilde{\omega}}^{*\langle \boldsymbol{n}-\boldsymbol{q}\rangle}\boldsymbol{U}_{\omega}^{\langle \boldsymbol{n}-\boldsymbol{q}\rangle}\boldsymbol{S}_{\omega}^{\langle \boldsymbol{n}-\boldsymbol{q}\rangle}\boldsymbol{S}_{\omega}^{\boldsymbol{q}}\boldsymbol{S}_{\widetilde{\omega}}^{\boldsymbol{n}}\boldsymbol{S}^{\boldsymbol{m}}A_{\boldsymbol{0}}g, \boldsymbol{S}_{\widetilde{\omega}}^{\langle \boldsymbol{n}-\boldsymbol{q}\rangle}\boldsymbol{S}_{\widetilde{\omega}}^{\boldsymbol{n}}\boldsymbol{S}_{\omega}^{\boldsymbol{q}}\boldsymbol{S}^{\boldsymbol{p}}A_{\boldsymbol{0}}h\big\rangle\\
&\quad= \big\langle \boldsymbol{S}_{\widetilde{\omega}}^{*\langle \boldsymbol{n}-\boldsymbol{q}\rangle}P_{\mathcal{K}}\boldsymbol{U}_{\widetilde{\omega}}^{*\langle \boldsymbol{n}-\boldsymbol{q}\rangle}\boldsymbol{U}_{\omega}^{\langle \boldsymbol{n}-\boldsymbol{q}\rangle}\boldsymbol{S}_{\omega}^{\langle \boldsymbol{n}-\boldsymbol{q}\rangle}\boldsymbol{S}_{\omega}^{\boldsymbol{q}}\boldsymbol{S}_{\widetilde{\omega}}^{\boldsymbol{n}}\boldsymbol{S}^{\boldsymbol{m}}A_{\boldsymbol{0}}g, \boldsymbol{S}_{\widetilde{\omega}}^{\boldsymbol{n}}\boldsymbol{S}_{\omega}^{\boldsymbol{q}}\boldsymbol{S}^{\boldsymbol{p}}A_{\boldsymbol{0}}h\big\rangle\\
&\quad= \big\langle \boldsymbol{S}_{\omega}^{*\langle \boldsymbol{n}-\boldsymbol{q}\rangle}\boldsymbol{S}_{\widetilde{\omega}}^{\langle \boldsymbol{n}-\boldsymbol{q}\rangle}\boldsymbol{S}_{\omega}^{\boldsymbol{q}}\boldsymbol{S}_{\widetilde{\omega}}^{\boldsymbol{n}}\boldsymbol{S}^{\boldsymbol{m}}A_{\boldsymbol{0}}g, \boldsymbol{S}_{\widetilde{\omega}}^{\boldsymbol{n}}\boldsymbol{S}_{\omega}^{\boldsymbol{q}}\boldsymbol{S}^{\boldsymbol{p}}A_{\boldsymbol{0}}h\big\rangle\\
&\quad= \langle \boldsymbol{S}^{\boldsymbol{m}+\boldsymbol{q}}A_{\boldsymbol{0}}g, \boldsymbol{S}^{\boldsymbol{n}+\boldsymbol{p}}A_{\boldsymbol{0}}h\rangle. \qquad (9.3)
\end{aligned}$$

Substituting $\omega = \{1, \dots, d\}$ into (9.1), we get

$$\boldsymbol{S}^{*\boldsymbol{n}} = P_{\mathcal{K}}\boldsymbol{U}^{\boldsymbol{n}}|_{\mathcal{K}}\boldsymbol{S}^{\boldsymbol{n}}, \qquad \boldsymbol{n} \in \mathbb{Z}_+^d. \qquad (9.4)$$

Using (9.2), (9.4), (3.1) and (9.3), we see that

$$\begin{aligned}
\Big\| \sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}_+^d} A_{\boldsymbol{m},\boldsymbol{n}}f_{\boldsymbol{m},\boldsymbol{n}} \Big\|^2 &= \Big\| \sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}_+^d} P_{\mathcal{K}}\boldsymbol{U}^{\boldsymbol{n}}\boldsymbol{S}^{\boldsymbol{n}}\boldsymbol{S}^{\boldsymbol{m}}A_{\boldsymbol{0}}f_{\boldsymbol{m},\boldsymbol{n}} \Big\|^2\\
&\leqslant \Big\| \sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}_+^d} \boldsymbol{U}^{\boldsymbol{n}}\boldsymbol{S}^{\boldsymbol{m}+\boldsymbol{n}}A_{\boldsymbol{0}}f_{\boldsymbol{m},\boldsymbol{n}} \Big\|^2\\
&= \sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}_+^d} \sum_{\boldsymbol{p},\boldsymbol{q}\in\mathbb{Z}_+^d} \langle \boldsymbol{U}^{\boldsymbol{n}}\boldsymbol{S}^{\boldsymbol{m}+\boldsymbol{n}}A_{\boldsymbol{0}}f_{\boldsymbol{m},\boldsymbol{n}}, \boldsymbol{U}^{\boldsymbol{q}}\boldsymbol{S}^{\boldsymbol{p}+\boldsymbol{q}}A_{\boldsymbol{0}}f_{\boldsymbol{p},\boldsymbol{q}}\rangle\\
&= \sum_{\omega\in\Omega_d} \sum_{\substack{\boldsymbol{n},\boldsymbol{q}\in\mathbb{Z}_+^d\\ \boldsymbol{n}-\boldsymbol{q}\in\mathbb{Z}_\omega^d}} \sum_{\boldsymbol{m},\boldsymbol{p}\in\mathbb{Z}_+^d} \langle \boldsymbol{U}^{\boldsymbol{n}}\boldsymbol{S}^{\boldsymbol{m}+\boldsymbol{n}}A_{\boldsymbol{0}}f_{\boldsymbol{m},\boldsymbol{n}}, \boldsymbol{U}^{\boldsymbol{q}}\boldsymbol{S}^{\boldsymbol{p}+\boldsymbol{q}}A_{\boldsymbol{0}}f_{\boldsymbol{p},\boldsymbol{q}}\rangle\\
&= \sum_{\omega\in\Omega_d} \sum_{\substack{\boldsymbol{n},\boldsymbol{q}\in\mathbb{Z}_+^d\\ \boldsymbol{n}-\boldsymbol{q}\in\mathbb{Z}_\omega^d}} \sum_{\boldsymbol{m},\boldsymbol{p}\in\mathbb{Z}_+^d} \langle \boldsymbol{S}^{\boldsymbol{m}+\boldsymbol{q}}A_{\boldsymbol{0}}f_{\boldsymbol{m},\boldsymbol{n}}, \boldsymbol{S}^{\boldsymbol{n}+\boldsymbol{p}}A_{\boldsymbol{0}}f_{\boldsymbol{p},\boldsymbol{q}}\rangle\\
&= \sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}_+^d} \sum_{\boldsymbol{p},\boldsymbol{q}\in\mathbb{Z}_+^d} \langle \boldsymbol{S}^{\boldsymbol{m}+\boldsymbol{q}}A_{\boldsymbol{0}}f_{\boldsymbol{m},\boldsymbol{n}}, \boldsymbol{S}^{\boldsymbol{n}+\boldsymbol{p}}A_{\boldsymbol{0}}f_{\boldsymbol{p},\boldsymbol{q}}\rangle\\
&= \sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}_+^d} \sum_{\boldsymbol{p},\boldsymbol{q}\in\mathbb{Z}_+^d} \langle A_{\boldsymbol{m}+\boldsymbol{q}}f_{\boldsymbol{m},\boldsymbol{n}}, A_{\boldsymbol{n}+\boldsymbol{p}}f_{\boldsymbol{p},\boldsymbol{q}}\rangle,
\end{aligned}$$

which completes the proof of (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (i): The implication (ii) $\Rightarrow$ (i) is an immediate consequence of Corollary 7.4. □

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