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WEIGHTED α-COMPOSITION AND RELATED FUNCTION SPACES

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Abstract

The notion of weighted α-composition was introduced by Ruhan Zhao in the 1990s. In this paper, we study several analytic function spaces that are closely related to weighted α-composition. These include α-Bloch spaces, $F(p, q, s)$ spaces, and Campanato spaces. We obtain derivative-free characterizations for α-Bloch spaces and $F(p, q, s)$ spaces, which improve some previous results in the literature. We also obtain a certain version of Carleson measures for Campanato spaces and $F(p, q, s)$ spaces.

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1. Introduction

Let $\mathbb{D}$ be the open unit disc in the complex plane $\mathbb{C}$ and let $\mathbb{T} = \partial\mathbb{D}$ be the unit circle. We denote by $H(\mathbb{D})$ the space of all analytic functions on $\mathbb{D}$. In this paper, we study several analytic function spaces on $\mathbb{D}$ whose norm is defined in terms of a sup-condition. We begin with the definition of weighted Dirichlet spaces.

For $p > 0$ and $t > -1$, we let $\mathcal{D}_t^p$ denote the space of all functions $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{D}_t^p}^p = \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^t \, dA(z) < \infty,$$

where dA is area measure normalized so that the unit disc has area 1. A norm of f in $\mathcal{D}_t^p$ is $|f(0)| + \|f\|_{\mathcal{D}_t^p}$.

Two special cases of weighted Dirichlet spaces are worth mentioning. First, for $p > 0$ and $t > -1$, it follows from a Hardy–Littlewood type theorem that the space $\mathcal{D}_{p+t}^p$ coincides with the weighted Bergman space A_t^p, consisting of all functions $f \in H(\mathbb{D})$ such that

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$$\int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^t \, dA(z) < \infty.$$

Second, by a classical Littewood–Paley theorem, the space $\mathcal{D}_1^2$ is nothing but the Hardy space H^2. Recall that for any $p > 0$, the Hardy space H^p consists of all functions $f \in H(\mathbb{D})$ such that

$$\|f\|_{H^p}^p = \sup_{0<r<1} \int_0^{2\pi} |f(re^{it})|^p \, dt < \infty.$$

Recall that for any $\alpha > 0$, the α-Bloch space $\mathcal{B}_\alpha$ (see [Zhu2]) consists of all functions $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}_\alpha} = \sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha |f'(z)| < \infty.$$

The special case $\alpha = 1$ results in the classical Bloch space, which we denote by $\mathcal{B}$. As usual, we let BMOA be the space of functions in H^2 whose boundary function is of bounded mean oscillation on $\mathbb{T}$; see [Gi].

Denote by $\mathrm{Aut}(\mathbb{D})$ the set of all Möbius maps (bijective analytic maps) of the unit disc. It is a group under ordinary compositions and it is called the Möbius group or sometimes the automorphism group. Every $\sigma \in \mathrm{Aut}(\mathbb{D})$ can be written in the form

$$\sigma(z) = e^{i\theta}\sigma_a(z) = \sigma_b(e^{it}z),$$

where θ and t are real angles, while a and b are points in $\mathbb{D}$. Here,

$$\sigma_a(z) = \frac{a-z}{1-\overline{a}z}, \quad z \in \mathbb{D}.$$

A subspace X of $H(\mathbb{D})$ endowed with a semi-norm $\|\ \|_X$ is said to be Möbius invariant if $f \circ \sigma \in X$ and $\|f \circ \sigma\|_X \lesssim \|f\|_X$ for all $f \in X$ and $\sigma \in \mathrm{Aut}(\mathbb{D})$. A general construction of Möbius-invariant function spaces is as follows: let Y be any subspace of $H(\mathbb{D})$ endowed with a semi-norm $\|\ \|_Y$ and let $M[Y]$ consist of all functions $f \in Y$ such that

$$\|f\|_{M[Y]} = \sup_{\sigma\in\mathrm{Aut}(\mathbb{D})} \|f \circ \sigma - f(\sigma(0))\|_Y < \infty.$$

Then, $M[Y]$ is a Möbius-invariant function space with the above semi-norm and $M[Y]$ is also called the Möbius-invariant version of Y. For $p > 0$ and $t > -1$, it is known (see [AS, Ax, Gi]) that $M[A_t^p] = \mathcal{B}$ and $M[H^p] = \mathrm{BMOA}$. We refer to [AFP] or [WZ, Ch. 1] for the general theory on Möbius-invariant function spaces.

From [RT], the Bloch space $\mathcal{B}$ is maximal among all Möbius-invariant spaces with an additional mild assumption. For every $\alpha > 0$, is $\mathcal{B}_\alpha$ maximal among some family of analytic function spaces? This question was posed in [Zhu2] and the answer is shown to be positive by Zhao in [Z].

Recall from [Z] that for $f \in H(\mathbb{D})$, $\sigma \in \mathrm{Aut}(\mathbb{D})$, and $\alpha > 0$, the weighted α-composition of f and σ is defined by

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$$f \circ_\alpha \sigma(z) = \int_0^z f'(\sigma(w))[\sigma'(w)]^\alpha \, dw + f(\sigma(0))[\sigma'(0)]^{\alpha-1}.$$

Note that each $\sigma \in \mathrm{Aut}(\mathbb{D})$ is univalent, so $\log \sigma'(z)$, $[\sigma'(z)]^\alpha = \exp[\alpha \log \sigma'(z)]$, and $[\sigma'(z)]^{\alpha-1} = \exp[(\alpha-1)\log \sigma'(z)]$ are all well-defined functions on $\mathbb{D}$. A subspace X of $H(\mathbb{D})$ endowed with a semi-norm $\| \ \|_X$ is said to be α-Möbius invariant if $f \circ_\alpha \sigma \in X$ and $\|f \circ_\alpha \sigma\|_X \lesssim \|f\|_X$ for all $f \in X$ and $\sigma \in \mathrm{Aut}(\mathbb{D})$. In 1996, Zhao [Z] proved that $\mathcal{B}_\alpha$ is maximal among α-Möbius invariant function spaces. In 2021, Bao and Zhu [BZ] identified the minimal nontrivial α-Möbius invariant function space, and proved the existence and uniqueness of a nontrivial α-Möbius invariant semi-Hilbert space of analytic functions on $\mathbb{D}$. We refer to [BLZ, LZ, Zhou] for more investigations on α-Möbius invariant function spaces.

More examples of α-Möbius invariant function spaces are $F(p,q,s)$ spaces. Recall that $F(p,q,s)$, where $p > 0$, $q > -2$, and $q + s > -1$, consists of functions $f \in H(\mathbb{D})$ such that

$$\|f\|^p_{F(p,q,s)} = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q (1-|\sigma_a(z)|^2)^s \, dA(z) < \infty.$$

The spaces $F(p,q,s)$ were introduced and studied in [Z]. In particular, $F(p,q,s)$ is a subset of $\mathcal{B}_{(q+2)/p}$. For $s > 1$, we actually have $F(p,q,s) = \mathcal{B}_{(q+2)/p}$. It is worth mentioning that $F(2,0,s)$ is the well-studied Möbius-invariant space $\mathcal{Q}_s$ (see [AXZ, X1, X2]). It was proved in [Z] that $F(p,q,s)$ is a $(q+2)/p$-Möbius invariant function space. Very recently, Zhou [Zhou] introduced and investigated a new class of α-Möbius invariant function spaces that contains $F(p,q,s)$ spaces as special cases.

We also recall Campanato spaces of analytic functions, which are closely associated with $F(p,q,s)$ spaces. For $p > 0$ and $\lambda > 0$, the Campanato space $\mathcal{H}^{p,\lambda}$ consists of functions $f \in H^p$ such that

$$\|f\|^p_{\mathcal{H}^{p,\lambda}} = \sup_{I\subset\mathbb{T}} \frac{1}{|I|^\lambda} \int_I |f - f_I|^p \, dt < \infty,$$

where the supremum is taken over all arcs $I \subset \mathbb{T}$, $|I|$ denotes the length of I, and f_I is the mean of f over I with respect to dt. We mention some special cases as follows.

- If $0 < \lambda < 1$, then $\mathcal{H}^{p,\lambda}$ is also called a Morrey space.
- If $\lambda = 1$, then $\mathcal{H}^{p,\lambda}$ is BMOA.
- If $\lambda \in (1, 1+p]$, then $\mathcal{H}^{p,\lambda}$ is the analytic Lipschitz space $\Lambda_{(\lambda-1)/p}$.
- If $\lambda \in (1+p, \infty)$, then $\mathcal{H}^{p,\lambda}$ contains constant functions only.

See [XY, page 1003], for example. Thus, the really interesting case for us here is when $0 < \lambda < 1$. Also, $\mathcal{H}^{2,\lambda}$ is equal to $F(2, 1-\lambda, \lambda)$. We refer to [YT] for more investigations on $\mathcal{H}^{p,\lambda}$ and $F(p,q,s)$ spaces.

In 2021, Aleman and Mas [AM] introduced and studied another version of Möbius invariant function spaces. For $\beta > 0$, a subspace X of $H(\mathbb{D})$ endowed with a semi-norm $\| \ \|_X$ is said to be Möbius invariant of index β if $W^\beta_\sigma f \in X$ and $\|W^\beta_\sigma f\|_X \lesssim \|f\|_X$ for all $f \in X$ and $\sigma \in \mathrm{Aut}(\mathbb{D})$, where $W^\beta_\sigma f = f \circ \sigma(\sigma')^\beta$. Roughly speaking, α-Möbius

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invariant spaces place many analytic function spaces defined by the derivative of functions in the same framework, such as α-Bloch spaces, $\mathcal{D}^p_{\alpha p-2}$ spaces, and $F(p, \alpha p-2, s)$ spaces. However, Möbius-invariant function spaces of index β include many analytic function spaces defined without the derivative of functions. For example, as shown in [AM], the Hardy space H^p is Möbius invariant of index $1/p$ and the Bergman space A^p_t is Möbius invariant of index $(2+t)/p$.

When a subspace X of $H(\mathbb{D})$ is defined by a derivative condition, it is natural to seek derivative-free characterizations of X. For instance, it is well known (see [Zhu1]) that an analytic function f on $\mathbb{D}$ belongs to the Bloch space $\mathcal{B}$ if and only if $|f(z)-f(w)| \lesssim \beta(z,w)$ for all $z, w \in \mathbb{D}$, where $\beta(z,w)$ is the Bergman distance between z and w. See [LS, LW] for other derivative-free characterizations of the Bloch space. We refer to [X2, Ch. 2] for a summary of derivative-free characterizations of $\mathcal{Q}_s$ spaces. Recently, Yang and Yuan [YY] gave some derivative-free characterizations of $F(p,q,s)$ spaces.

The notion of Carleson measures was introduced by Carleson [C1, C2] when he studied the interpolation problem and the corona problem for the space of bounded analytic functions on $\mathbb{D}$. He characterized positive Borel measures μ on $\mathbb{D}$ such that $H^p \subseteq L^p(\mathbb{D}, d\mu)$. Carleson measures are useful tools for studying analytic function spaces and related operators on them. In 2008, Xiao [X3] studied a variation of Carleson measures for $\mathcal{Q}_s$ spaces (the embedding from $\mathcal{Q}_s$ to a certain tent-type space), which can be used to characterize the boundedness and compactness of Volterra-type operators on $\mathcal{Q}_s$ spaces. Pau and Zhao [PZ] further studied this variation of Carleson measures for $F(p,q,s)$ spaces.

In this paper, we continue to investigate α-Bloch spaces $\mathcal{B}_\alpha$, Campanato spaces, and $F(p,q,s)$ spaces. In Section 3, we give some derivative-free characterizations of $\mathcal{B}_\alpha$ spaces. Some of them involve the weighted α-composition. In Section 4, we show that the characterizations of $\mathcal{B}_\alpha$ in Section 3 improve a conclusion in [PP] and a result in [PZ]. In Section 5, we consider a variation of Carleson measures for Campanato spaces. In Section 6, we provide a derivative-free characterization and a variation of Carleson measures for $F(p,q,s)$ spaces.

2. Preliminaries

In this section, we collect several technical results that we need multiple times later in the paper. We begin with the following integral estimates.

LEMMA 2.1. *Suppose $t > -1$ and c is real. Then,*

$$\int_{\mathbb{D}} \frac{(1-|z|^2)^t\, dA(z)}{|1-\overline{a}z|^{2+t+c}} \sim \begin{cases} 1, & c<0, \\ \log\dfrac{2}{1-|a|^2}, & c=0, \\ (1-|a|^2)^{-c}, & c>0, \end{cases}$$

for all $a \in \mathbb{D}$.

The result above is well known (see, for example, [Zhu1]).

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LEMMA 2.2. *For $s > -1$, $r > 0$, and $t > 0$ with $r + t > s + 2$, there is a constant $C > 0$ such that*

$$\int_{\mathbb{D}} \frac{(1-|z|^2)^s}{|1-\bar{a}z|^r|1-\bar{b}z|^t}\,dA(z) \leq \begin{cases} \dfrac{C}{|1-\bar{a}b|^{r+t-s-2}}, & r,t < 2+s, \\ \dfrac{C}{(1-|a|^2)^{r-s-2}|1-\bar{a}b|^t}, & t < 2+s < r, \\ \dfrac{C}{(1-|a|^2)^{r-s-2}|1-\bar{a}b|^t} + \dfrac{C}{(1-|b|^2)^{t-s-2}|1-\bar{a}b|^r}, & r,t > 2+s, \end{cases}$$

for all $a, b \in \mathbb{D}$.

This result is slightly less known. It can be found in [OF].

We need the following Hardy–Littlewood type theorem, which can be found in [APR, BWZ]. Here and below, we use $\beta(z,w)$ to denote the hyperbolic (Bergman) metric on $\mathbb{D}$. Let

$$D(a,r) = \{z \in \mathbb{D} : \beta(z,a) < r\}$$

for $a \in \mathbb{D}$ and $r > 0$.

THEOREM 2.3. *Suppose $\omega(z)$ is a positive weight function on $\mathbb{D}$ with the following properties.*

(i) *For any fixed $r > 0$, there exists a positive constant C_1 such that*

$$C_1^{-1} \leq \frac{\omega(z)}{\omega(w)} \leq C_1$$

for all $z, w \in \mathbb{D}$ with $\beta(z,w) < r$.

(ii) *There exist positive numbers γ and C_2 such that*

$$\int_{\mathbb{D}} \frac{\omega(u)\,dA(u)}{|1-z\bar{u}|^{2+\gamma}} \leq \frac{C_2\omega(z)}{(1-|z|^2)^\gamma}$$

for all $z \in \mathbb{D}$.

Then, for any $p > 0$ and $f \in H(\mathbb{D})$, we have

$$\int_{\mathbb{D}} |f(z)-f(0)|^p\omega(z)\,dA(z) \sim \int_{\mathbb{D}} |f'(z)|^p(1-|z|^2)^p\omega(z)\,dA(z).$$

Here, $\sim$ means that each integral is dominated by the other with a positive constant that is independent of f.

The following lemma concerns a particular class of weight functions involving the Bergman kernel.

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LEMMA 2.4. *Suppose $t > -1$, $\sigma < 2 + t$, and $a \in \mathbb{D}$. Then, there exist positive constants C_1 and C_2, independent of a, such that the weight function*

$$\omega(z) = \omega_a(z) = \frac{(1 - |z|^2)^t}{|1 - \overline{a}z|^\sigma}$$

has properties (i) and (ii) in Theorem 2.3.

PROOF. For any fixed radius $r > 0$, it is well known (see [Zhu1] for example) that there exists a constant $C > 0$ such that

$$C^{-1} \le \frac{1 - |z|^2}{1 - |w|^2} \le C$$

for all $z, w \in \mathbb{D}$ with $\beta(z, w) < r$. Furthermore, when $\beta(z, w) < r$, we can find some $u \in D(0, r)$ such that $w = \sigma_z(u)$, which yields

$$1 - \overline{a}w = 1 - \overline{a}\sigma_z(u) = \frac{1 - \overline{a}z}{1 - \overline{z}u} - \frac{u(\overline{z} - \overline{a})}{1 - \overline{z}u}.$$

Thus,

$$\frac{1 - \overline{a}w}{1 - \overline{a}z} = \frac{1}{1 - \overline{z}u}(1 - u\sigma_{\overline{z}}(\overline{a})).$$

Since u belongs to $D(0, r)$ (which is a Euclidean disc centered at 0 with Euclidean radius r' that is strictly less than 1), it follows from the triangle inequality that

$$\frac{1 - r'}{1 + r'} \le \left|\frac{1 - \overline{a}w}{1 - \overline{a}z}\right| \le \frac{1 + r'}{1 - r'}.$$

Consequently, there exists a positive constant C, independent of a, such that

$$C^{-1} \le \frac{\omega(z)}{\omega(w)} = \frac{\omega_a(z)}{\omega_a(w)} \le C$$

for all $z, w \in \mathbb{D}$ with $\beta(z, w) < r$. This verifies condition (i) in Theorem 2.3.

Next, we consider the integral

$$\int_{\mathbb{D}} \frac{\omega(u)\, dA(u)}{|1 - z\overline{u}|^{2+\gamma}} = \int_{\mathbb{D}} \frac{(1 - |u|^2)^t\, dA(u)}{|1 - \overline{a}u|^\sigma |1 - z\overline{u}|^{2+\gamma}}$$

for positive γ.

By Lemma 2.2, if the parameters satisfy

$$t > -1, \quad 0 < \sigma < 2 + t < 2 + \gamma,$$

then there exists another constant $C > 0$, independent of a, such that

$$\int_{\mathbb{D}} \frac{\omega(u)\, dA(u)}{|1 - z\overline{u}|^{2+\gamma}} \le \frac{C}{(1 - |z|^2)^{\gamma - t}|1 - \overline{a}z|^\sigma} = \frac{C\omega(z)}{(1 - |z|^2)^\gamma}$$

for all $z \in \mathbb{D}$. This verifies condition (ii) in Theorem 2.3 under the assumptions $t > -1$, $0 < \sigma < 2 + t$, and $\gamma > t$.

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If $t > -1$, $\sigma = 0$, and $\gamma > t$, it follows from Lemma 2.1 that

$$\int_{\mathbb{D}} \frac{\omega(u)\,dA(u)}{|1-z\overline{u}|^{2+\gamma}} = \int_{\mathbb{D}} \frac{(1-|u|^2)^t\,dA(u)}{|1-z\overline{u}|^{2+\gamma}} \lesssim \frac{\omega(z)}{(1-|z|^2)^\gamma}$$

for all $z \in \mathbb{D}$, which gives condition (ii) in Theorem 2.3.

If $t > -1$, $\sigma < 0$, and $\gamma > -\sigma + t$, then we write

$$1 - \overline{a}u = 1 - \overline{a}z + \overline{a}(z-u),$$

use the triangle inequality, and use Lemma 2.1 to estimate

$$\begin{aligned}
\int_{\mathbb{D}} \frac{\omega(u)\,dA(u)}{|1-z\overline{u}|^{2+\gamma}} &= \int_{\mathbb{D}} \frac{(1-|u|^2)^t|1-\overline{a}u|^{-\sigma}\,dA(u)}{|1-z\overline{u}|^{2+\gamma}} \\
&\lesssim \int_{\mathbb{D}} \frac{(1-|u|^2)^t|1-\overline{a}z|^{-\sigma}\,dA(u)}{|1-z\overline{u}|^{2+\gamma}} + \int_{\mathbb{D}} \frac{(1-|u|^2)^t|\overline{a}(z-u)|^{-\sigma}\,dA(u)}{|1-z\overline{u}|^{2+\gamma}} \\
&\lesssim \frac{\omega(z)}{(1-|z|^2)^\gamma} + \int_{\mathbb{D}} \frac{(1-|u|^2)^t\,dA(u)}{|1-z\overline{u}|^{2+\gamma+\sigma}} \\
&\lesssim \frac{\omega(z)}{(1-|z|^2)^\gamma} + \frac{1}{(1-|z|^2)^{\gamma+\sigma-t}} \lesssim \frac{\omega(z)}{(1-|z|^2)^\gamma}
\end{aligned}$$

for all $z \in \mathbb{D}$, where the implicit constants in the above estimates are independent of the choice of a. This completes the verification of condition (ii) in Theorem 2.3 for all $t > -1$ and $\sigma < 2 + t$. □

REMARK 2.5. It is clear that the assumption $t > -1$ is necessary in Lemma 2.4. Also, the condition $\sigma < 2 + t$ in Lemma 2.4 cannot be replaced by $\sigma = 2 + t$. In fact, if $\sigma = 2 + t$, then by [ZLSG, Theorem 3.1],

$$\begin{aligned}
\int_{\mathbb{D}} \frac{\omega(u)\,dA(u)}{|1-z\overline{u}|^{2+\gamma}} &= \int_{\mathbb{D}} \frac{(1-|u|^2)^t\,dA(u)}{|1-\overline{a}u|^{2+t}|1-z\overline{u}|^{2+\gamma}} \\
&\sim \begin{cases}
\dfrac{1}{|1-\overline{a}z|^{2+\gamma}} \log \dfrac{e}{|1-\overline{a}\sigma_a(z)|}, & \gamma < t, \\
\dfrac{1}{|1-\overline{a}z|^{2+t}} \left(\log \dfrac{e}{|1-\overline{a}\sigma_a(z)|} + \log \dfrac{e}{|1-\overline{z}\sigma_z(a)|} \right), & \gamma = t, \\
\dfrac{1}{(1-|z|^2)^{\gamma-t}|1-\overline{a}z|^{2+t}} + \dfrac{1}{|1-\overline{a}z|^{2+\gamma}} \log \dfrac{e}{|1-\overline{a}\sigma_a(z)|}, & \gamma > t,
\end{cases}
\end{aligned}$$

for all a and z in $\mathbb{D}$. This shows that for $\sigma = 2 + t$, we have

$$\sup_{a,z\in\mathbb{D}} \frac{\displaystyle\int_{\mathbb{D}} \frac{\omega_a(u)\,dA(u)}{|1-z\overline{u}|^{2+\gamma}}}{\dfrac{\omega_a(z)}{(1-|z|^2)^\gamma}} = +\infty$$

for any $\gamma > 0$. Thus Lemma 2.4 does not hold when $\sigma = 2 + t$.

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Some special cases of Lemma 2.4 and Theorem 2.3 were known before. By [BP, Lemma 2.1], if $p > 1$, $t > -1$, and $\sigma \geq 0$ with $\sigma < 2 + t$, then

$$\int_{\mathbb{D}} |f(z) - f(0)|^p \frac{(1-|z|^2)^t}{|1-\overline{a}z|^\sigma}\, dA(z) \lesssim \int_{\mathbb{D}} |f'(z)|^p \frac{(1-|z|^2)^{p+t}}{|1-\overline{a}z|^\sigma}\, dA(z)$$

for all $f \in H(\mathbb{D})$ and $a \in \mathbb{D}$. Our result here covers all $p > 0$, $t > -1$, and $\sigma < 2 + t$.

3. Characterizations of α-Bloch spaces

In this section, we consider derivative-free characterizations for $\mathcal{B}_\alpha$ spaces. We need the following description of α-Bloch spaces, which is a special case of [YT, Corollary 9] or [Zhou, Theorem 2].

THEOREM 3.1. *Suppose $s > 0$, $\alpha > 0$, $p > 0$, and $t > p\alpha - 1$. Then, an analytic function f on $\mathbb{D}$ belongs to $\mathcal{B}_\alpha$ if and only if*

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p \frac{(1-|a|^2)^s (1-|z|^2)^t}{|1-\overline{a}z|^{2+s+t-p\alpha}}\, dA(z) < \infty.$$

Using Lemma 2.4, and Theorems 2.3 and 3.1, we now obtain the following derivative-free characterization for $\mathcal{B}_\alpha$ in terms of weighted α-compositions.

THEOREM 3.2. *Suppose $p > 0, \alpha > 0$, and $f \in H(\mathbb{D})$. If the real parameters c and t satisfy*

$$\begin{cases} c < \min[0, p(1-\alpha)], \\ t > \max[-1, p(\alpha-1) - 1], \end{cases} \tag{3-1}$$

then $f \in \mathcal{B}_\alpha$ if and only if

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f \circ_\alpha \sigma_a(w) - f \circ_\alpha \sigma_a(0)|^p \frac{(1-|w|^2)^t}{|1-\bar{a}w|^{2+t+c}}\, dA(w) < \infty.$$

PROOF. By Lemma 2.4 and Theorem 2.3, if c and t satisfy the inequalities

$$t > -1, \quad 2 + t + c < 2 + t,$$

then,

$$\begin{aligned} &\int_{\mathbb{D}} |f \circ_\alpha \sigma_a(z) - f \circ_\alpha \sigma_a(0)|^p \frac{(1-|z|^2)^t}{|1-\overline{a}z|^{2+t+c}}\, dA(z) \\ &\sim \int_{\mathbb{D}} |f'(\sigma_a(z))(\sigma_a'(z))^\alpha|^p \frac{(1-|z|^2)^{p+t}}{|1-\overline{a}z|^{2+t+c}}\, dA(z) \\ &= \int_{\mathbb{D}} |f'(z)|^p \frac{(1-|a|^2)^{p(1-\alpha)-c}(1-|z|^2)^{p+t}}{|1-\overline{a}z|^{2+[p(1-\alpha)-c]+(p+t)-p\alpha}}\, dA(z). \end{aligned}$$

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Combining this with Theorem 3.1, we conclude that $f \in \mathcal{B}_\alpha$ if and only if

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f \circ_\alpha \sigma_a(z) - f \circ_\alpha \sigma_a(0)|^p \frac{(1-|z|^2)^t}{|1-\overline{a}z|^{2+t+c}}\, dA(z) < \infty,$$

provided that the parameters satisfy

$$t > -1, \quad p(1-\alpha) > c, \quad p + t > p\alpha - 1, \quad 2 + t + c < 2 + t.$$

These conditions can be regrouped as (3-1). The proof is complete. □

As was mentioned in Section 1, the Bloch space $\mathcal{B}$ is the Möbius-invariant version of weighted Bergman spaces. More specifically, for $p > 0$ and $t > -1$, a function $f \in H(\mathbb{D})$ belongs to $\mathcal{B}$ if and only if

$$\sup_{\sigma\in \mathrm{Aut}(\mathbb{D})} \|f \circ \sigma - f \circ \sigma(0)\|_{A^p_t} < \infty.$$

It is easy to see that the condition above is equivalent to

$$\sup_{a\in\mathbb{D}} \|f \circ \sigma_a - f \circ \sigma_a(0)\|_{A^p_t} < \infty.$$

Taking $c = -(2 + t)$ in Theorem 3.2, we get the following result immediately, which shows that $\mathcal{B}_\alpha$ is the α-Möbius invariant version of weighted Bergman spaces.

COROLLARY 3.3. *Suppose $t > -1$, $p > 0$, and $\alpha > 0$. If $p(\alpha - 1) < t + 1$, then*

$$\begin{aligned} \|f\|_{\mathcal{B}_\alpha} &\sim \sup_{a\in\mathbb{D}} \|f \circ_\alpha \sigma_a - f \circ_\alpha \sigma_a(0)\|_{A^p_t} \\ &\sim \sup_{\sigma\in \mathrm{Aut}(\mathbb{D})} \|f \circ_\alpha \sigma - f \circ_\alpha \sigma(0)\|_{A^p_t} \end{aligned}$$

for all $f \in H(\mathbb{D})$.

REMARK 3.4. When $p \geq 1$, Corollary 3.3 is known as [Zhou, Lemma 2.9]. The result here covers all $p > 0$. For $t > -1$ and $p > 0$, if $0 < \alpha \leq 1$, the condition $p(\alpha - 1) < t + 1$ in Corollary 3.3 always holds. If $\alpha > 1$, the condition $p(\alpha - 1) < t + 1$ is best possible. In fact, the function

$$f(z) = \sum_{k=1}^{\infty} 2^{k(\alpha-1)} z^{2^k}$$

belongs to $\mathcal{B}_\alpha$ (see, for example, [Zhu2]). If

$$\sup_{\sigma\in \mathrm{Aut}(\mathbb{D})} \|f \circ_\alpha \sigma - f \circ_\alpha \sigma(0)\|_{A^p_t} < \infty,$$

then

$$\|f \circ_\alpha \sigma - f \circ_\alpha \sigma(0)\|^p_{A^p_t} \approx \int_{\mathbb{D}} |f'(\sigma(z))|^p |\sigma'(z)|^{p\alpha} (1-|z|^2)^{p+t}\, dA(z) < \infty \qquad (3\text{-}2)$$

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for $f \in H(\mathbb{D})$ and $\sigma \in \mathrm{Aut}(\mathbb{D})$. Taking $\sigma(z) = z$ in (3-2) yields $f' \in A^p_{p+t}$, which is known (see [Zhu3, Theorem 18]) to be equivalent to

$$\sum_{k=1}^{\infty} 2^{k[p(\alpha-1)-1-t]} < \infty.$$

This means $p(\alpha - 1) < t + 1$ and hence this assumption in Corollary 3.3 is necessary.

REMARK 3.5. Note that for $\alpha \le 1$, the conditions in (3-1) simplify to $t > -1$ and $c < 0$. The second condition in (3-1) is necessary for Theorem 3.2. In fact, $t > -1$ is clearly necessary, and by an analysis similar to that in Remark 3.4, the condition $t > p(\alpha - 1) - 1$ is also necessary. We are not sure if the first condition in (3-1) is necessary for Theorem 3.2.

Finally, in this section, we prove a characterization for α-Bloch spaces based on integrals involving $|f(z) - f(a)|^p$, which occurs naturally in many situations.

THEOREM 3.6. *Suppose that $p > 0$, $c > 0$, $t > -1$, and $0 < \alpha \le 1$. Then, a function $f \in H(\mathbb{D})$ belongs to $\mathcal{B}_\alpha$ if and only if*

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f(z) - f(a)|^p \frac{(1-|a|^2)^c (1-|z|^2)^t \, dA(z)}{|1-\overline{a}z|^{2+t+p(1-\alpha)+c}} < \infty.$$

PROOF. Let us use $I(f, a)$ to denote the integral above. By a change of variables and a subharmonic integral estimate, there exist two positive constants c and c' (independent of f and a) such that

$$\begin{aligned}
I(f,a) &= \int_{\mathbb{D}} |f\circ\sigma_a(z) - f(a)|^p \frac{(1-|a|^2)^{2+c}(1-|\sigma_a(z)|^2)^t \, dA(z)}{|1-\overline{a}\sigma_a(z)|^{2+t+p(1-\alpha)+c}|1-\overline{a}z|^4} \\
&= (1-|a|^2)^{p(\alpha-1)} \int_{\mathbb{D}} |f\circ\sigma_a(z) - f(a)|^p \frac{(1-|z|^2)^t \, dA(z)}{|1-\overline{a}z|^{2+t+p(\alpha-1)-c}} \\
&\ge (1-|a|^2)^{p(\alpha-1)} \int_{|z|<0.5} |f\circ\sigma_a(z) - f(a)|^p \frac{(1-|z|^2)^t \, dA(z)}{|1-\overline{a}z|^{2+t+p(\alpha-1)-c}} \\
&\ge c(1-|a|^2)^{p(\alpha-1)} \int_{|z|<0.5} |(f\circ\sigma_a)'(z)|^p \, dA(z) \\
&\ge c'(1-|a|^2)^{p(\alpha-1)} |(f\circ\sigma_a)'(0)|^p \\
&= c'(1-|a|^2)^{p\alpha} |f'(a)|^p.
\end{aligned}$$

This shows that the boundedness of $I(f, a)$ as a function of a implies that $f \in \mathcal{B}_\alpha$. Note that the proof of this direction works for all $\alpha > 0$ (instead of just $0 < \alpha \le 1$).

To prove the other direction, we first consider the case $\alpha = 1$. If $f \in \mathcal{B}_1 = \mathcal{B}$, then, in terms of the Bergman/hyperbolic metric,

$$|f\circ\sigma_a(z) - f(a)| \le \|f\|_{\mathcal{B}} \, \beta(z, 0) = \frac{\|f\|_{\mathcal{B}}}{2} \log \frac{1+|z|}{1-|z|} \lesssim (1-|z|^2)^{-\varepsilon/p}$$

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for all $a, z \in \mathbb{D}$, where $0 < \varepsilon < \min(c, t+1)$. It follows from Lemma 2.1 that

$$\begin{aligned} I(f,a) &= \int_{\mathbb{D}} |f \circ \sigma_a(z) - f(a)|^p \frac{(1-|z|^2)^t \, dA(z)}{|1-\overline{a}z|^{2+t-c}} \\ &\lesssim \int_{\mathbb{D}} \frac{(1-|z|^2)^{t-\varepsilon} \, dA(z)}{|1-\overline{a}z|^{2+t-c}} \lesssim 1. \end{aligned}$$

Next, we assume $0 < \alpha < 1$ and $f \in \mathcal{B}_\alpha$. It is well known (see [Zhu2] for example) that $\mathcal{B}_\alpha = \Lambda_{1-\alpha}$, the analytic Lipschitz space of order $1-\alpha$. Thus,

$$|f(z) - f(a)| \le C|z-a|^{1-\alpha}$$

for some positive constant C and all $z, a \in \mathbb{D}$. It follows from Lemma 2.1 that

$$\begin{aligned} I(f,a) &= \int_{\mathbb{D}} \left| \frac{f(z)-f(a)}{(1-\overline{a}z)^{1-\alpha}} \right|^p \frac{(1-|a|^2)^c (1-|z|^2)^t \, dA(z)}{|1-\overline{a}z|^{2+t+c}} \\ &\lesssim \int_{\mathbb{D}} \left| \frac{a-z}{1-\overline{a}z} \right|^{p(1-\alpha)} \frac{(1-|a|^2)^c (1-|z|^2)^t \, dA(z)}{|1-\overline{a}z|^{2+t+c}} \\ &\le \int_{\mathbb{D}} \frac{(1-|a|^2)^c (1-|z|^2)^t \, dA(z)}{|1-\overline{a}z|^{2+t+c}} \\ &\lesssim 1. \end{aligned}$$

This completes the proof of the theorem. □

REMARK 3.7. When $p > 0$, $c > 0$, $t \ge 0$, and $\alpha = 1$, the conclusion of Theorem 3.6 is already known. See [LW, Theorem 1].

THEOREM 3.8. *Suppose that $p > 0$, $c > 0$, $t > -1$, and $\alpha > 1$. If, in addition,*

$$p(\alpha - 1) < \min(c, t+1),$$

then a function $f \in H(\mathbb{D})$ belongs to $\mathcal{B}_\alpha$ if and only if

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f(z) - f(a)|^p \frac{(1-|a|^2)^c (1-|z|^2)^t \, dA(z)}{|1-\overline{a}z|^{2+t+p(1-\alpha)+c}} < \infty.$$

PROOF. Once again, let $I(f,a)$ denote the integral above. By the first part of the proof of Theorem 3.6, we only need to prove $f \in \mathcal{B}_\alpha$ implies that $I(f,a)$ is bounded as a function of a. To do this, we recall (see, for example, [Zhu2]) that for $\alpha > 1$, a function $f \in H(\mathbb{D})$ belongs to $\mathcal{B}_\alpha$ if and only if the function $(1-|z|^2)^{\alpha-1} f(z)$ is bounded on $\mathbb{D}$.

Fix some $f \in \mathcal{B}_\alpha$ and, once again, let $I(f,a)$ denote the integral in question. It is clear that we have $I(f,a) \lesssim I_1(f,a) + I_2(f,a)$, where

$$I_1(f,a) = (1-|a|^2)^c \int_{\mathbb{D}} |f(z)|^p \frac{(1-|z|^2)^t \, dA(z)}{|1-\overline{a}z|^{2+t+p(1-\alpha)+c}}$$

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and

$$I_2(f,a) = (1-|a|^2)^c |f(a)|^p \int_{\mathbb{D}} \frac{(1-|z|^2)^t\, dA(z)}{|1-\overline{a}z|^{2+t+p(1-\alpha)+c}}.$$

Since $|f(z)| \le C(1-|z|^2)^{1-\alpha}$ for some constant $C > 0$ and all $z \in \mathbb{D}$, we have

$$I_1(f,a) \lesssim (1-|a|^2)^c \int_{\mathbb{D}} \frac{(1-|z|^2)^{t+p(1-\alpha)}\, dA(z)}{|1-\overline{a}z|^{2+t+p(1-\alpha)+c}}.$$

Since $p(\alpha-1) < t+1$ or $t+p(1-\alpha) > -1$, it follows from Lemma 2.1 that $I_1(f,a)$ is a bounded function of a. Similarly,

$$I_2(f,a) \lesssim (1-|a|^2)^{p(1-\alpha)+c} \int_{\mathbb{D}} \frac{(1-|z|^2)^t\, dA(z)}{|1-\overline{a}z|^{2+t+p(1-\alpha)+c}},$$

which implies (by Lemma 2.1 and the assumption $p(\alpha-1) < c$) that $I_2(f,a)$ is a bounded function of $a \in \mathbb{D}$. This completes the proof of the theorem. □

We note that the assumption $p(\alpha-1) < t+1$ is necessary in Theorem 3.8. In fact, for any $\alpha > 0$, as was mentioned in Remark 3.4, the function

$$f(z) = \sum_{k=1}^{\infty} 2^{k(\alpha-1)} z^{2^k}$$

belongs to $\mathcal{B}^\alpha$. For $p > 0$ and $t > -1$, it follows from [Zhu3, Theorem 18] that $f \in A^p_t$ if and only if

$$\sum_{k=1}^{\infty} 2^{k[p(\alpha-1)-1-t]} < \infty,$$

that is, $p(\alpha-1) < t+1$. Note that if $I(f,a)$ is a bounded function of a, then in particular $f \in A^p_t$. Consequently, the condition $p(\alpha-1) < t+1$ in Theorem 3.8 is necessary.

Under the assumption $p(\alpha-1) < t+1$, we have

$$\begin{aligned}
&\int_{\mathbb{D}} |f(z)|^p \frac{(1-|a|^2)^c (1-|z|^2)^t\, dA(z)}{|1-\overline{a}z|^{2+t+p(1-\alpha)+c}} \\
&\quad \lesssim (1-|a|^2)^c \int_{\mathbb{D}} \frac{(1-|z|^2)^{t+p(1-\alpha)}\, dA(z)}{|1-\overline{a}z|^{2+t+p(1-\alpha)+c}} \\
&\quad \sim 1
\end{aligned}$$

for all functions $f \in \mathcal{B}_\alpha$. Therefore, the boundedness of $I(f,a)$ in a is equivalent to the boundedness of the function

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$$a \mapsto (1-|a|^2)^c |f(a)|^p \int_{\mathbb{D}} \frac{(1-|z|^2)^t \, dA(z)}{|1-\overline{a}z|^{2+t+p(1-\alpha)+c}}$$
$$= (1-|a|^2)^{p(\alpha-1)} |f(a)|^p \int_{\mathbb{D}} \frac{(1-|a|^2)^{p(1-\alpha)+c}(1-|z|^2)^t \, dA(z)}{|1-\overline{a}z|^{2+t+p(1-\alpha)+c}}.$$

By Lemma 2.1, the boundedness of the above as a function of a is equivalent to $p(1-\alpha)+c>0$ for any function $f \in \mathcal{B}_\alpha \setminus \mathcal{B}_{\alpha,0}$, where $\mathcal{B}_{\alpha,0}$ is the space of functions g in $H(\mathbb{D})$ such that

$$\lim_{|z|\to 1^-} (1-|z|^2)^\alpha |g'(z)| = 0.$$

This shows that the assumption $p(\alpha-1)<c$ in Theorem 3.8 is also necessary.

It is clear that Theorems 3.6 and 3.8 can be combined into a single theorem, because the additional assumption $p(\alpha-1)<\min(c,t+1)$ in Theorem 3.8 is automatically satisfied in the case $0<\alpha\le 1$.

4. A question of Xiao's and some related problems

In [PP, page 551], Pau and Peláez obtained the following estimate: for any $0<s<1$, there exists a constant $C>0$ such that

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left|\frac{f(z)-f(a)}{1-\overline{a}z}\right|^2 (1-|\sigma_a(z)|^2)^s \, dA(z) \le C\|f\|^2_{\mathcal{Q}_s} \tag{4-1}$$

for all $f \in \mathcal{Q}_s$. In [X3, page 2080], Xiao extended this estimate from $0<s<1$ to $0<s<\infty$. Xiao also asked in [X3, page 2081] whether we actually have

$$\|f\|^2_{\mathcal{Q}_s} \sim \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left|\frac{f(z)-f(a)}{1-\overline{a}z}\right|^2 (1-|\sigma_a(z)|^2)^s \, dA(z)\,? \tag{4-2}$$

By Theorem 3.6 (with $p=2$, $c=t=s>0$, and $\alpha=1$),

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left|\frac{f(z)-f(a)}{1-\overline{a}z}\right|^2 (1-|\sigma_a(z)|^2)^s \, dA(z)$$
$$= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f(z)-f(a)|^2 \frac{(1-|a|^2)^s(1-|z|^2)^s \, dA(z)}{|1-\overline{a}z|^{2+2s}}$$
$$\sim \|f\|_{\mathcal{B}}.$$

It is well known that $\mathcal{Q}_s=\mathcal{B}$ for $s>1$. So, the answer to Xiao's question is positive when $s>1$. However, it is also well known that (see [AXZ] for example) for $0<s\le 1$, the space $\mathcal{Q}_s$ is strictly contained in the Bloch space $\mathcal{B}$ and $\|f\|_{\mathcal{Q}_s} \nsim \|f\|_{\mathcal{B}}$. Thus, for $0<s\le 1$, the answer to Xiao's question above is negative, that is, the relation in (4-2) does not hold. We mention that Theorem 3.6 also improves (4-1).

In [PZ], Pau and Zhao extended the estimate (4-1) to a more general family of $F(p,q,s)$ spaces. Recall that $\mathcal{Q}_s=F(2,0,s)$ for $s>0$. Also, the case $q=\alpha p-2$ with $\alpha>0$ is special. In fact, it is known that (see [Z] for example) $F(p,\alpha p-2,s)\subset\mathcal{B}_\alpha$ for

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all $s > 0$ and $F(p, \alpha p - 2, s) = \mathcal{B}^\alpha$ when $s > 1$. Furthermore, $F(p, \alpha p - 2, s)$ contains only constant functions if $s + \alpha p \le 1$ (see [Z, Proposition 2.12]). Therefore, to ensure $F(p, \alpha p - 2, s)$ is nontrivial, we always assume $s + \alpha p > 1$.

It was proved in [PZ, Proposition 2.8] that, for

$$p > 0, \quad s > 0, \quad \alpha \ge 1, \quad s + p\alpha > 1,$$

there exists a positive constant C such that

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left|\frac{f(z) - f(a)}{1 - \bar{a}z}\right|^p (1 - |\sigma_a(z)|^2)^{s+p(\alpha-1)} (1 - |a|^2)^{p(\alpha-1)} (1 - |z|^2)^{p-2}\, dA(z)$$
$$\le C\|f\|^p_{F(p,\alpha p-2,s)} \tag{4-3}$$

for all $f \in H(\mathbb{D})$. It is also natural to ask whether we actually have

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left|\frac{f(z) - f(a)}{1 - \bar{a}z}\right|^p (1 - |\sigma_a(z)|^2)^{s+p(\alpha-1)} (1 - |a|^2)^{p(\alpha-1)} (1 - |z|^2)^{p-2}\, dA(z)$$
$$\sim \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^{p\alpha-2} (1 - |\sigma_a(z)|^2)^s\, dA(z). \tag{4-4}$$

By Theorems 3.6 and 3.8, the left-hand side of (4-4) is finite if and only if $f \in \mathcal{B}_\alpha$, provided that $s + 2p(\alpha - 1) > 0$, $s + p\alpha > 1$, and

$$p(\alpha - 1) < \min[s + 2p(\alpha - 1), s + p\alpha - 1]. \tag{4-5}$$

In particular, these conditions will all be satisfied if $s + p > 1$ and $\alpha \ge 1$. Also, Condition (4-5) clearly implies $s + p > 1$.

Consequently, when $p > 0$, $s > 0$, $\alpha \ge 1$, and $s + p > 1$, we can find a positive constant C such that

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left|\frac{f(z) - f(a)}{1 - \bar{a}z}\right|^p (1 - |\sigma_a(z)|^2)^{s+p(\alpha-1)} (1 - |a|^2)^{p(\alpha-1)} (1 - |z|^2)^{p-2}\, dA(z)$$
$$\le C\|f\|^p_{\mathcal{B}_\alpha}$$

for all $f \in H(\mathbb{D})$, which is usually stronger than the estimate in (4-3), because for all $s > 0$, we have $F(p, p\alpha - 2, s) \subset \mathcal{B}_\alpha$ with

$$\|f\|_{\mathcal{B}_\alpha} \lesssim \|f\|_{F(p,p\alpha-2,s)}, \quad f \in H(\mathbb{D}).$$

Note here that we need the condition $s + p > 1$, which is stronger than $s + p\alpha > 1$ when $\alpha > 1$. It is not clear what happens when

$$\alpha > 1, \quad s + p \le 1, \quad s + p\alpha > 1. \tag{4-6}$$

Since the parameter ranges in Theorems 3.6 and 3.8 are optimal, the Pau–Zhao estimate in the case of (4-6) is probably stating something that is not well understood by us so far!

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By Theorem 3.6, when $0 < \alpha < 1$, $p > 0$, $s + 2p(\alpha - 1) > 0$, and $s + p\alpha > 1$, the left-hand side of (4-3) is finite if and only if $f \in \mathcal{B}_\alpha$. Since $F(p, p\alpha - 2, s) \subset \mathcal{B}_\alpha$ in this case, we see that (4-3) also holds for $0 < \alpha < 1$, $p > 0$, $s + 2p(\alpha - 1) > 0$, and $s + p\alpha > 1$.

Although Xiao's original question on Q_s has a negative answer for $0 < s \le 1$, it still leads us to some other related and interesting problems. Thus, we consider some integrals that are more general than those related to Q_s and $F(p, q, s)$ spaces.

THEOREM 4.1. *Suppose $f \in H(\mathbb{D})$, $p > 0$, and*

$$1 < 2 + t < b. \tag{4-7}$$

Then,

$$\int_{\mathbb{D}} |f(z) - f(a)|^p \frac{(1 - |z|^2)^t}{|1 - \overline{a}z|^b}\, dA(z) \sim \int_{\mathbb{D}} |f'(z)|^p \frac{(1 - |z|^2)^{p+t}}{|1 - \overline{a}z|^b}\, dA(z). \tag{4-8}$$

Here, $\sim$ means that either side is dominated by the other with a positive constant that is independent of a and f.

PROOF. Let $I(f, a)$ denote the first integral above. By a change of variables,

$$\begin{aligned} I(f, a) &= \int_{\mathbb{D}} |f \circ \sigma_a(z) - f(a)|^p \frac{(1 - |a|^2)^2 (1 - |\sigma_a(z)|^2)^t}{|1 - \overline{a}\sigma_a(z)|^b |1 - \overline{a}z|^4}\, dA(z) \\ &= \int_{\mathbb{D}} |f \circ \sigma_a(z) - f \circ \sigma_a(0)|^p \frac{(1 - |a|^2)^{2+t-b} (1 - |z|^2)^t\, dA(z)}{|1 - \overline{a}z|^{4+2t-b}}. \end{aligned}$$

The inequalities in (4-7) guarantee that we can apply Lemma 2.4 to the weight functions

$$\omega(z) = \frac{(1 - |z|^2)^t}{|1 - \overline{a}z|^{4+2t-b}}.$$

Thus, by Theorem 2.3,

$$\begin{aligned} I(f, a) &\sim \int_{\mathbb{D}} |(f \circ \sigma_a)'(z)|^p \frac{(1 - |a|^2)^{2+t-b} (1 - |z|^2)^{p+t}\, dA(z)}{|1 - \overline{a}z|^{4+2t-b}} \\ &= \int_{\mathbb{D}} |f'(\sigma_a(z))|^p \frac{(1 - |a|^2)^{2+t+p-b} (1 - |z|^2)^{p+t}\, dA(z)}{|1 - \overline{a}z|^{4+2t+2p-b}} \\ &= \int_{\mathbb{D}} |f'(z)|^p \frac{(1 - |a|^2)^{t+p-b} (1 - |\sigma_a(z)|^2)^{p+t}\, dA(z)}{|1 - \overline{a}\sigma_a(z)|^{2t+2p-b}} \\ &= \int_{\mathbb{D}} |f'(z)|^p \frac{(1 - |z|^2)^{p+t}}{|1 - \overline{a}z|^b}\, dA(z). \end{aligned}$$

This completes the proof of the theorem. □

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REMARK 4.2. By Lemma 2.4 and Theorem 2.3, if $p > 0$, $t > -1$, and $b < 2 + t$, then

$$\int_{\mathbb{D}} |f(z) - f(0)|^p \frac{(1-|z|^2)^t}{|1-\overline{a}z|^b}\, dA(z) \sim \int_{\mathbb{D}} |f'(z)|^p \frac{(1-|z|^2)^{p+t}}{|1-\overline{a}z|^b}\, dA(z) \tag{4-9}$$

for all $f \in H(\mathbb{D})$ and $a \in \mathbb{D}$. Note that $|f(z) - f(a)|$ in (4-8) is different from $|f(z) - f(0)|$ in (4-9).

5. A version of Carleson measures for Campanato spaces

In this section, we prove a certain version of Carleson measures for Campanato spaces $\mathcal{H}^{p,\lambda}$. More precisely, for a positive Borel measure μ on $\mathbb{D}$, we investigate the embedding of $\mathcal{H}^{p,\lambda}$ into the space of functions f in $L^p(\mathbb{D}, d\mu)$ such that

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f(z) - f(a)|^p \frac{(1-|a|^2)^{2-\lambda}}{|1-\overline{a}z|^2}\, d\mu(z) < \infty.$$

By [XY, Theorem 1.1],

$$\|f\|_{\mathcal{H}^{p,\lambda}} \approx \sup_{a\in\mathbb{D}} (1-|a|^2)^{(1-\lambda)/p} \|f \circ \sigma_a - f(a)\|_{H^p} \tag{5-1}$$

for $0 < \lambda < 2$ and $p > 1$. The space defined by (5-1) is also called a Hardy–Morrey space in the literature. Thus, it is also natural to study the so-called Bergman–Morrey space of functions $f \in H(\mathbb{D})$ such that

$$\sup_{a\in\mathbb{D}} (1-|a|^2)^c \|f \circ \sigma_a - f \circ \sigma_a(0)\|_{A^p_t} < \infty.$$

Some Bergman–Morrey spaces are special cases of $F(p, q, s)$ spaces. See [XLL, YL] for instance.

More generally, it is natural to study the space of functions $f \in H(\mathbb{D})$ such that

$$\sup_{a\in\mathbb{D}} (1-|a|^2)^c \|f \circ_\alpha \sigma_a - f \circ_\alpha \sigma_a(0)\|_{A^p_t} < \infty.$$

By the Hardy–Littlewood theorem for Bergman spaces and a change of variables, such spaces are then special cases of the family of spaces defined by the condition

$$\sup_{a\in\mathbb{D}} (1-|a|^2)^c \int_{\mathbb{D}} |f'(z)|^p \frac{(1-|z|^2)^t}{|1-\overline{a}z|^b}\, dA(z) < \infty.$$

By Theorem 3.1, these are in some sense generalizations of α-Bloch spaces. They obviously cover $F(p, q, s)$ spaces as special cases as well.

Recall that a positive Borel measure μ on $\mathbb{D}$ is called an s-Carleson measure, $s > 0$, if there exists a positive constant C such that $\mu(S_I) \le C|I|^s$ for all arcs $I \subset \mathbb{T}$, where S_I

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is the standard Carleson box associated to I. When $s = 1$, these measures are simply called Carleson measures (see [C1, C2, Du]).

The main result of this section is the following theorem.

THEOREM 5.1. *Let μ be a nonnegative Borel measure on $\mathbb{D}$ and suppose that the parameters p and λ satisfy one of the following assumptions:*

(a) $p > 1$ *and* $0 \le \lambda \le 2$;
(b) $p = 1$ *and* $0 < \lambda < 2$;
(c) $0 < p < 1$ *and* $1 - p \le \lambda \le p + 1$.

Then, the following two conditions are equivalent.

(i) *There exists a constant $C > 0$ such that*

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f(z) - f(a)|^p \frac{(1-|a|^2)^{2-\lambda}}{|1-\overline{a}z|^2}\, d\mu(z) \le C \sup_{a\in\mathbb{D}} (1-|a|^2)^{1-\lambda} \|f\circ\sigma_a - f(a)\|_{H^p}^p \tag{5-2}$$

for all $f \in H(\mathbb{D})$.

(ii) *μ is a Carleson measure, that is, there exists a positive constant C such that $\mu(S_I) \le C|I|$ for all arcs $I \subset \mathbb{T}$.*

PROOF. First, assume that Condition (5-2) holds for all $f \in H(\mathbb{D})$. For any $I \subset \mathbb{T}$ with $|I| < 1/9$, let $e^{i\theta}$, $\theta \in [0, 2\pi)$, be the midpoint of I and let $w = (1 - 2|I|)e^{i\theta}$. For any $z \in S(I)$,

$$|1 - \overline{w}z| \ge 1 - |w||z| \ge 1 - (1 - 2|I|) = 2|I|.$$

It is also clear that

$$\begin{aligned} |1 - \overline{w}z| &= |1 - ze^{-i\theta} + 2|I|ze^{-i\theta}| \le |z - e^{i\theta}| + 2|I||z| \\ &\le |z - (z/|z|)| + |(z/|z|) - e^{i\theta}| + 2|I| \\ &\le |I| + |I|/2 + 2|I| = (7/2)|I|. \end{aligned}$$

Therefore,

$$2|I| \le |1 - \overline{w}z| \le (7/2)|I|. \tag{5-3}$$

Consider the function

$$f_w(z) = \frac{1}{1 - \overline{w}z}, \quad z \in \mathbb{D},$$

which also appeared in [XY, Lemma 4.5] as a test function in $\mathcal{H}^{q,\eta}$ for $q > 1$ and $0 < \eta < 2$. For $a \in \mathbb{D}$, we deduce that

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$$\begin{aligned}\|f_w\circ\sigma_a - f_w(a)\|_{H^p}^p &= \int_{\partial\mathbb{D}} |f_w(\xi)-f_w(a)|^p \frac{1-|a|^2}{|1-\overline{a}\xi|^2}\,\frac{|d\xi|}{2\pi}\\ &= \frac{|w|^p}{|1-\overline{w}a|^p}\int_{\partial\mathbb{D}} \frac{|\xi-a|^p}{|1-\overline{w}\xi|^p}\frac{1-|a|^2}{|1-\overline{a}\xi|^2}\,\frac{|d\xi|}{2\pi}\\ &= \frac{|w|^p}{|1-\overline{w}a|^p}\int_{\partial\mathbb{D}} \frac{|\sigma_a(\xi)-a|^p}{|1-\overline{w}\sigma_a(\xi)|^p}\,\frac{|d\xi|}{2\pi}\\ &= \frac{|w|^p(1-|a|^2)^p}{|1-\overline{w}a|^{2p}}\int_{\partial\mathbb{D}} \frac{1}{|1-\sigma_{\overline{a}}(\overline{w})\xi|^p}\,\frac{|d\xi|}{2\pi}.\end{aligned}$$

It is well known (see [WZ]) that

$$\int_{\partial\mathbb{D}} \frac{1}{|1-\sigma_{\overline{a}}(\overline{w})\xi|^p}\,\frac{|d\xi|}{2\pi} \sim \begin{cases}\dfrac{1}{(1-|\sigma_{\overline{a}}(\overline{w})|^2)^{p-1}}, & p>1,\\ \log\dfrac{2}{1-|\sigma_{\overline{a}}(\overline{w})|^2}, & p=1,\\ 1, & p<1.\end{cases}$$

Consequently, for $p>1$ and $0\le\lambda\le 2$,

$$\begin{aligned}\sup_{a\in\mathbb{D}}(1-|a|^2)^{1-\lambda}\|f_w\circ\sigma_a - f_w(a)\|_{H^p}^p &\sim \sup_{a\in\mathbb{D}}\frac{|w|^p(1-|a|^2)^{2-\lambda}}{|1-\overline{w}a|^2(1-|w|^2)^{p-1}}\\ &\lesssim \frac{1}{(1-|w|)^{p-1+\lambda}} \sim \frac{1}{|I|^{p-1+\lambda}}.\end{aligned}$$

If $p=1$ and $0<\lambda<2$, then there exists $\epsilon>0$ such that

$$1-p+\epsilon\le\lambda\le 1+p-\epsilon.$$

It follows that

$$\begin{aligned}&\sup_{a\in\mathbb{D}}(1-|a|^2)^{1-\lambda}\|f_w\circ\sigma_a - f_w(a)\|_{H^p}^p\\ &\quad\sim \sup_{a\in\mathbb{D}}\frac{|w|^p(1-|a|^2)^{p+1-\lambda}}{|1-\overline{w}a|^{2p}}\log\frac{2}{1-|\sigma_{\overline{a}}(\overline{w})|^2}\\ &\quad\lesssim \sup_{a\in\mathbb{D}}\frac{|w|^p(1-|a|^2)^{p+1-\lambda}}{|1-\overline{w}a|^{2p}}(1-|\sigma_{\overline{a}}(\overline{w})|^2)^{-\epsilon}\\ &\quad\lesssim \sup_{a\in\mathbb{D}}\frac{(1-|a|^2)^{p+1-\lambda-\epsilon}(1-|w|^2)^{-\epsilon}}{|1-\overline{w}a|^{2p-2\epsilon}}\\ &\quad\lesssim \frac{1}{(1-|w|)^{p-1+\lambda}} \sim \frac{1}{|I|^{p-1+\lambda}}.\end{aligned}$$

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If $0 < p < 1$ and $1 - p \le \lambda \le p + 1$, then

$$\sup_{a\in\mathbb{D}}(1-|a|^2)^{1-\lambda}\|f_w\circ\sigma_a - f_w(a)\|_{H^p}^p \sim \sup_{a\in\mathbb{D}}\frac{|w|^p(1-|a|^2)^{p+1-\lambda}}{|1-\overline{w}a|^{2p}} \lesssim \frac{1}{(1-|w|)^{p-1+\lambda}} \sim \frac{1}{|I|^{p-1+\lambda}}.$$

Therefore, when p and λ satisfy assumptions (a), (b), or (c), we always have

$$\sup_{a\in\mathbb{D}}(1-|a|^2)^{1-\lambda}\|f_w\circ\sigma_a - f_w(a)\|_{H^p}^p \lesssim \frac{1}{|I|^{p-1+\lambda}}. \tag{5-4}$$

Note that the assumption $|I| < 1/9$ implies

$$1-|w|^2 = (1-|w|)(1+|w|) \ge 2|I|(2-2|I|) = 4|I|(1-|I|) > (32/9)|I|.$$

This, together with the triangle inequality and (5-3), gives

$$\begin{aligned}
&\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f_w(z)-f_w(a)|^p\frac{(1-|a|^2)^{2-\lambda}}{|1-\overline{a}z|^2}\,d\mu(z)\\
&\quad\ge \int_{S(I)}\left|\frac{1}{1-\overline{w}z}-\frac{1}{1-|w|^2}\right|^p\frac{(1-|w|^2)^{2-\lambda}}{|1-\overline{w}z|^2}\,d\mu(z)\\
&\quad\gtrsim \frac{1}{|I|^\lambda}\int_{S(I)}\left(\frac{1}{(7/2)|I|}-\frac{1}{(32/9)|I|}\right)^p d\mu(z)\\
&\quad= \left(\frac{1}{224}\right)^p\frac{\mu(S(I))}{|I|^{p+\lambda}}.
\end{aligned}$$

Combining this with (5-2) and (5-4), we obtain $\mu(S(I)) \lesssim |I|$, which shows that μ is a Carleson measure.

Conversely, for $f \in H(\mathbb{D})$ and $a \in \mathbb{D}$, we consider the function

$$F_a(z) = (f(z)-f(a))\frac{(1-|a|^2)^{(2-\lambda)/p}}{(1-\overline{a}z)^{2/p}}, \quad a\in\mathbb{D}.$$

It is easy to see that

$$\begin{aligned}
\sup_{a\in\mathbb{D}}\|F_a\|_{H^p}^p &= \sup_{a\in\mathbb{D}}(1-|a|^2)^{1-\lambda}\int_0^{2\pi}|f(e^{i\theta})-f(a)|^p\frac{1-|a|^2}{|1-\overline{a}e^{i\theta}|^2}\,\frac{d\theta}{2\pi}\\
&= \sup_{a\in\mathbb{D}}(1-|a|^2)^{1-\lambda}\|f\circ\sigma_a - f(a)\|_{H^p}^p.
\end{aligned}$$

If μ is a Carleson measure, then H^p is continuously contained in $L^p(\mathbb{D}, d\mu)$, so there is a constant $C > 0$ such that

$$\int_{\mathbb{D}}|F_a(z)|^p\,d\mu(z) \le C\|F_a\|_{H^p}^p = C(1-|a|^2)^{1-\lambda}\|f\circ\sigma_a - f(a)\|_{H^p}^p$$

for all $a \in \mathbb{D}$, which gives (5-2) and completes the proof of the theorem. □

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In view of (5-1), Theorem 5.1 establishes a certain version of Carleson measures for Campanato spaces $\mathcal{H}^{p,\lambda}$ when $0 < \lambda < 2$ and $p > 1$. Taking $\lambda = 1$ in Theorem 5.1, we obtain the following special case, which is [Ya, Theorem 2].

COROLLARY 5.2. *Let μ be a nonnegative Borel measure on $\mathbb{D}$ and $p > 0$. Then, there exists a constant $C > 0$ such that*

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f(z) - f(a)|^p \frac{1-|a|^2}{|1-\overline{a}z|^2}\, d\mu(z) \le C\|f\|^p_{\mathrm{BMOA}}$$

holds for all $f \in$ BMOA if and only if μ is a Carleson measure.

6. New characterizations for $F(p, q, s)$

In this section, we obtain a derivative-free characterization and a new version of Carleson measures for $F(p, q, s)$ spaces.

Suppose $p > 0$ and $q + s > p - 1$. Setting $b = 2s$ and $t = s + q - p$ in Theorem 4.1,

$$\begin{aligned}
&\int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q (1-|\sigma_a(z)|^2)^s\, dA(z)\\
&\quad = (1-|a|^2)^s \int_{\mathbb{D}} |f'(z)|^p \frac{(1-|z|^2)^{s+q}}{|1-\overline{a}z|^{2s}}\, dA(z)\\
&\quad = (1-|a|^2)^s \int_{\mathbb{D}} |f'(z)|^p \frac{(1-|z|^2)^{p+t}}{|1-\overline{a}z|^{b}}\, dA(z)\\
&\quad \sim (1-|a|^2)^s \int_{\mathbb{D}} |f(z)-f(a)|^p \frac{(1-|z|^2)^{t}}{|1-\overline{a}z|^{b}}\, dA(z)\\
&\quad \sim (1-|a|^2)^s \int_{\mathbb{D}} |f(z)-f(a)|^p \frac{(1-|z|^2)^{s+q-p}}{|1-\overline{a}z|^{2s}}\, dA(z),
\end{aligned}$$

provided the parameters also satisfy the assumptions in Theorem 4.1. Thus, we have the following result.

THEOREM 6.1. *Suppose $p > 0$, $f \in H(\mathbb{D})$, and $1 < 2 + s + q - p < 2s$. Then, $f \in F(p, q, s)$ if and only if*

$$\sup_{a\in\mathbb{D}} (1-|a|^2)^s \int_{\mathbb{D}} |f(z)-f(a)|^p \frac{(1-|z|^2)^{s+q-p}}{|1-\overline{a}z|^{2s}}\, dA(z) < \infty$$

or

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f(z)-f(a)|^p (1-|z|^2)^{q-p} (1-|\sigma_a(z)|^2)^s\, dA(z) < \infty$$

or

$$\sup_{a\in\mathbb{D}} (1-|a|^2)^{p/2} \int_{\mathbb{D}} \left|\frac{f(z)-f(a)}{1-\overline{a}z}\right|^p (1-|z|^2)^{q-(p/2)} (1-|\sigma_a(z)|^2)^{s-(p/2)}\, dA(z) < \infty.$$

Note that the assumptions in Theorem 6.1 imply $s > 1/2$. If $p = 2$ and $q = 0$, then the parameter conditions in Theorem 6.1 yield $s > 1$. Therefore, the characterization

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for $F(p,q,s)$ spaces in Theorem 6.1 does NOT yield a new characterization for Q_s spaces when $0 < s < 1$.

The following result shows that $F(p,q,s)$ spaces are the α-Möbius invariant version of weighted Dirichlet spaces.

PROPOSITION 6.2. *Suppose $0 < p, s, \alpha < \infty$, and $s + p\alpha > 1$. Then,*

$$\|f\|_{F(p,p\alpha-2,s)} = \sup_{a\in\mathbb{D}} \|f \circ_\alpha \sigma_a - f \circ_\alpha \sigma_a(0)\|_{\mathcal{D}^p_{p\alpha-2+s}}$$

for $f \in F(p, p\alpha-2, s)$.

PROOF. For $a \in \mathbb{D}$, by a change of variables,

$$\begin{aligned}
&\|f \circ_\alpha \sigma_a - f \circ_\alpha \sigma_a(0)\|^p_{\mathcal{D}^p_{p\alpha-2+s}} \\
&\quad = \int_{\mathbb{D}} |f'(\sigma_a(z))(\sigma_a'(z))^\alpha|^p (1-|z|^2)^{p\alpha-2+s}\, dA(z) \\
&\quad = \int_{\mathbb{D}} |f'(w)|^p (1-|w|^2)^{p\alpha-2} (1-|\sigma_a(w)|^2)^s\, dA(w).
\end{aligned}$$

This proves the desired result. □

We also recall the following characterization for certain special $F(p,q,s)$ spaces; see [YT, Corollary 9].

THEOREM 6.3. *Suppose $0 < p, s, t, \alpha < \infty$, and $s + p\alpha > 1$. Then, $f \in H(\mathbb{D})$ belongs to $F(p, p\alpha-2, s)$ if and only if*

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p\alpha-2} \frac{(1-|a|^2)^t (1-|z|^2)^s}{|1-\overline{a}z|^{t+s}}\, dA(z) < \infty.$$

Our next result below can be considered as a characterization for some version of Carleson measures for certain $F(p, p\alpha-2, s)$ spaces.

THEOREM 6.4. *Let μ be a nonnegative Borel measure on $\mathbb{D}$ and let $0 < p, s < \infty$ with*

$$\alpha > 1, \quad s + p > 1, \quad t > p(\alpha-1).$$

If $0 < p \le 1$ or $1 < p \le 2$ with $s + p(\alpha-1) = 1$ or $1 < p < \infty$ with $s + p(\alpha-1) > 1$, then there exists a constant $C > 0$ such that

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f(z) - f(a)|^p \frac{(1-|a|^2)^t}{|1-\overline{a}z|^{t+s}}\, d\mu(z) \le C\|f\|^p_{F(p,p\alpha-2,s)} \tag{6-1}$$

holds for all $f \in F(p, p\alpha-2, s)$ if and only if μ is a $(p\alpha-p+s)$-Carleson measure on $\mathbb{D}$.

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PROOF. Suppose μ is a $(p\alpha - p + s)$-Carleson measure on $\mathbb{D}$. By [Wu, Theorem 1] and [PZ, page 504], μ is a Carleson measure for $\mathcal{D}^p_{s+p\alpha-2}$, that is, there exists a constant $C > 0$ such that

$$\int_{\mathbb{D}} |f(z)|^p \, d\mu(z) \le C[|f(0)| + \|f\|_{\mathcal{D}^p_{s+\alpha p-2}}]^p \tag{6-2}$$

for all $f \in \mathcal{D}^p_{s+p\alpha-2}$.

For $f \in F(p, p\alpha - 2, s)$ and $a \in \mathbb{D}$, we consider the function

$$J_a(z) = (f(z) - f(a)) \frac{(1 - |a|^2)^{t/p}}{(1 - \overline{a}z)^{(t+s)/p}}, \quad z \in \mathbb{D}.$$

By Theorem 6.3,

$$\begin{aligned}
[|J_a(0)| + \|J_a\|_{\mathcal{D}^p_{s+\alpha p-2}}]^p &\lesssim |J_a(0)|^p + \int_{\mathbb{D}} |J_a'(z)|^p (1 - |z|^2)^{s+\alpha p-2} \, dA(z) \\
&\lesssim (1 - |a|^2)^t |f(a) - f(0)|^p \\
&\quad + \int_{\mathbb{D}} |f'(z)|^p \frac{(1 - |a|^2)^t (1 - |z|^2)^{s+\alpha p-2}}{|1 - \overline{a}z|^{t+s}} \, dA(z) \\
&\quad + \int_{\mathbb{D}} \left| \frac{f(z) - f(a)}{1 - \overline{a}z} \right|^p \frac{(1 - |a|^2)^t (1 - |z|^2)^{s+p\alpha-2}}{|1 - \overline{a}z|^{t+s}} \, dA(z) \\
&\lesssim (1 - |a|^2)^t |f(a) - f(0)|^p + \|f\|^p_{F(p,\alpha p-2,s)} \\
&\quad + \int_{\mathbb{D}} \left| \frac{f(z) - f(a)}{1 - \overline{a}z} \right|^p \frac{(1 - |a|^2)^t (1 - |z|^2)^{s+p\alpha-2}}{|1 - \overline{a}z|^{t+s}} \, dA(z).
\end{aligned}$$

Since $p > 0$, $t > p(\alpha - 1)$, $s + p > 1$, Theorem 3.8 yields

$$\int_{\mathbb{D}} \left| \frac{f(z) - f(a)}{1 - \overline{a}z} \right|^p \frac{(1 - |a|^2)^t (1 - |z|^2)^{s+p\alpha-2}}{|1 - \overline{a}z|^{t+s}} \, dA(z) \sim \|f\|^p_{\mathcal{B}_\alpha}.$$

Because $F(p, \alpha p - 2, s) \subset \mathcal{B}^\alpha$,

$$|f(a) - f(0)| \lesssim \|f\|_{F(p,\alpha p-2,s)} (1 - |a|^2)^{1-\alpha}.$$

This implies

$$(1 - |a|^2)^t |f(a) - f(0)|^p \lesssim \|f\|^p_{F(p,\alpha p-2,s)} (1 - |a|^2)^{t+p-p\alpha} \le \|f\|^p_{F(p,\alpha p-2,s)}$$

as $t \ge p\alpha - p$. Consequently, for $t > p(\alpha - 1)$ and $s + p > 1$,

$$\sup_{a \in \mathbb{D}} \|J_a\|^p_{\mathcal{D}^p_{s+\alpha p-2}} \lesssim \|f\|^p_{F(p,\alpha p-2,s)}.$$

Combining this with (6-2), we see that

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f(z) - f(a)|^p \frac{(1 - |a|^2)^t}{|1 - \overline{a}z|^{t+s}} \, d\mu(z) \lesssim \|f\|^p_{F(p,\alpha p-2,s)}$$

for all $f \in F(p, p\alpha - 2, s)$.

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Conversely, suppose (6-1) holds. For any $I \subset \mathbb{T}$ with $|I| < 1/9$, let $e^{i\theta}$, $\theta \in [0, 2\pi)$, be the mid-point of I and let $w = (1 - 2|I|)e^{i\theta}$. For $\alpha \neq 1$ and $b \in \mathbb{D}$, we consider the function

$$f_{b,\alpha}(z) = \frac{1}{(1-\alpha)(1-\overline{b}z)^{\alpha-1}}.$$

By [PZ, Lemma 2.6],

$$\sup_{b\in\mathbb{D}} \|f_{b,\alpha}\|_{F(p,p\alpha-2,s)} < \infty. \tag{6-3}$$

Since $\alpha > 1$, it follows from (6-1), (6-3), and (5-3) that

$$\begin{aligned}
1 &\gtrsim \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f_{w,\alpha}(z) - f_{w,\alpha}(a)|^p \frac{(1-|a|^2)^t}{|1-\overline{a}z|^{t+s}}\, d\mu(z) \\
&\gtrsim \int_{S(I)} \left| \frac{1}{(1-\overline{w}z)^{\alpha-1}} - \frac{1}{(1-|w|^2)^{\alpha-1}} \right|^p \frac{(1-|w|^2)^t}{|1-\overline{w}z|^{t+s}}\, d\mu(z) \\
&\gtrsim \frac{1}{|I|^s} \int_{S(I)} \left(\frac{1}{[(7/2)|I|]^{\alpha-1}} - \frac{1}{[(32/9)|I|]^{\alpha-1}} \right)^p d\mu(z) \\
&\gtrsim \frac{\mu(S(I))}{|I|^{p\alpha-p+s}}.
\end{aligned}$$

Thus, μ is a $(p\alpha - p + s)$-Carleson measure on $\mathbb{D}$. □

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