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RATIONAL POINTS ON CUBIC HYPERSURFACES THAT SPLIT INTO FOUR FORMS

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Abstract. Let $C \in \mathbb{Z}[x_1, \dots, x_n]$ be a cubic form. Assume that C splits into four forms. Then $C(x_1, \dots, x_n) = 0$ has a non-trivial integer solution provided that $n \geqslant 10$.

§1. *Introduction.* Given a cubic form $C(x_1, \dots, x_n)$ in n variables with integer coefficients, let $X \subseteq \mathbb{P}^{n-1}$ be the corresponding cubic hypersurface defined by $C = 0$. A folklore conjecture predicts that $X(\mathbb{Q}) \neq \emptyset$, or equivalently, C has a non-trivial integer zero, as soon as $n \geqslant 10$.

We say that C, or X, is non-singular if the only solution to the system of equations $\nabla C(\mathbf{x}) = 0$ is $\mathbf{x} = 0$. Heath-Brown [**9**] proved that $n \geqslant 10$ implies $X(\mathbb{Q}) \neq \emptyset$ for non-singular cubic forms. For general forms, Heath-Brown [**10**] verified that $n \geqslant 14$ ensures such a zero. A more general result is $n \geqslant 11 + \sigma_X$ (see [**4**]), where σ_X denotes the projective dimension of the singular locus of X.

The Hasse principle says that $X(\mathbb{Q}) \neq \emptyset$ if and only if $X(\mathbb{R}) \neq \emptyset$ and $X(\mathbb{Q}_p) \neq \emptyset$ for each prime p. Since cubic forms are indefinite, the condition $X(\mathbb{R}) \neq \emptyset$ holds automatically. However, local obstacles may exist when $n \leqslant 9$. See Mordell's example [**13**] for a cubic form in nine variables that fails to have p-adic solutions for some particular p. It is conjectured that the Hasse principle is still true for non-singular hypersurfaces with $n \geqslant 5$. See [**6**] and [**11**] for results in this direction.

We say that C splits off $s - 1$ forms, or C splits into s forms, if C can be written as

$$C = C_1 + \cdots + C_s,$$

where $C_1, \dots, C_s$ have independent variables. Browning [**2**] showed that $X(\mathbb{Q}) \neq \emptyset$ for $n \geqslant 13$ and $s \geqslant 2$. Later, in [**17**], Dai and the first author proved that $X(\mathbb{Q}) \neq \emptyset$ if $n \geqslant 11$ and $s \geqslant 3$. In this paper, we deal with the case $s \geqslant 4$.

THEOREM 1.1. *Let $X \subseteq \mathbb{P}^{n-1}$ be a cubic hypersurface defined by a cubic form that splits into four forms. Then $X(\mathbb{Q}) \neq \emptyset$ as soon as $n \geqslant 10$.*

In view of [**17**, Theorems 1.1 and 1.2], it is sufficient to prove the case $C = C_1 + \cdots + C_4$, where $C_1, \dots, C_4$ have 1, 1, 1, 7 variables, respectively.

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PROPOSITION 1.2. *Let C_0 be an integral cubic form in seven variables. For $c_1, c_2, c_3 \in \mathbb{Z}$, we set*

$$C(x_1, \dots, x_{10}) = c_1x_1^3 + c_2x_2^3 + c_3x_3^3 + C_0(x_4, \dots, x_{10}). \tag{1.1}$$

Then the equation $C(x_1, \dots, x_{10}) = 0$ has a non-trivial integer solution.

Our proof is based on the Hardy–Littlewood circle method. A mean value estimate involving a smooth cubic exponential sum will be used. For more details about applications of the circle method to rational points, we refer the reader to [**3**].

§2. *Definition and notation.* Throughout, we assume that C is given by (1.1). We call a form degenerate if it is equivalent over $\mathbb{Q}$ to a form in fewer variables. It is obvious that a degenerate form has non-trivial integer zeros. So, we always suppose that C is non-degenerate. This means that $c_1c_2c_3 \neq 0$ and C_0 is non-degenerate. Also, we assume that C_0 does not split off forms, otherwise Proposition 1.2 is established by [**17**, Theorem 1.2]. We say that a cubic form is "good" if for any $H \geqslant 1$ and any $\varepsilon > 0$, the upper bound

$$\#\{\mathbf{x} \in \mathbb{Z}^n : |\mathbf{x}| \leqslant H, \operatorname{rank}\mathbf{H}(\mathbf{x}) = r\} \ll H^{r+\varepsilon}$$

holds for each integer $0 \leqslant r \leqslant n$, where $\mathbf{H}(\mathbf{x})$ is the underlying Hessian matrix. If a cubic form has at least three variables and is not "good", then it has non-trivial zeros for "geometric reasons" (see [**8**]). Therefore, we assume that C_0 is "good" as well.

We define the boxes

$$\mathfrak{B}_0 := \{(x_4, \dots, x_{10}) \in \mathbb{R}^7 : |x_i - z_i| \leqslant \varpi\}$$

and

$$\mathfrak{B} := \{(x_1, \dots, x_{10}) \in \mathbb{R}^{10} : 0 \leqslant x_1, x_2 \leqslant 1,\ \varsigma \leqslant x_3 \leqslant 1,\ (x_4, \dots, x_{10}) \in \mathfrak{B}_0\},$$

where $\varpi, \varsigma > 0$ and $z_4, \dots, z_{10}$ are constants depending only on the form C. Let $P\mathfrak{B}$ be the box whose elements x all satisfy $x/P \in \mathfrak{B}$ and $P\mathcal{B}_0$ be similarly defined. Write $e(x) := \mathrm{e}^{2\pi \mathrm{i}x}$ and $e_q(x) := e(x/q)$. We introduce the cubic exponential sums G_i $(i = 1, 2)$, h, S as follows. Let

$$G_i(\alpha) := \sum_{m \in \mathcal{A}} e(\alpha c_i m^3) \quad (i = 1, 2),$$

where

$$\mathcal{A} = \mathcal{A}(P, P^\eta) := \{1 \leqslant m \leqslant P : p|m \Rightarrow p \leqslant P^\eta\}. \tag{2.1}$$

We define

$$h(\alpha) := \sum_{\varsigma P \leqslant x \leqslant P} e(\alpha c_3 x^3)$$

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and

$$F(\alpha) := \sum_{\substack{|x_i/P - z_i| \leqslant \varpi \\ (i=4,\dots,10)}} e(\alpha C_0(\mathbf{x})).$$

The cubic Gauss sums are given by

$$S(q,a) := \sum_{r \pmod q} e_q(ar^3), \qquad S_{G_i}(q,a) := S(q,ac_i) \quad (i=1,2),$$
$$S_h(q,a) := S(q,ac_3),$$
$$S_F(q,a) := \sum_{r_4,\dots,r_{10} \pmod q} e_q(aC_0(r_4,\dots,r_{10}))$$

and

$$S_C(q,a) := S_{G_1}(q,a)S_{G_2}(q,a)S_h(q,a)S_F(q,a).$$

Then we define integrals

$$I_{G_i}(\gamma) := \int_0^1 e(\gamma c_i t_i^3)\,dt_i \quad (i=1,2), \qquad I_h(\gamma) = \int_\varsigma^1 e(\gamma c_3 t_3^3)\,dt_3,$$
$$I_F(\gamma) := \int_{\mathfrak{B}_0} e(\gamma C_0(t_4,\dots,t_{10}))\,dt_4 \dots dt_{10}$$

and

$$I_C(\gamma) := I_{G_1}(\gamma)I_{G_2}(\gamma)I_h(\gamma)I_F(\gamma) = \int_{\mathfrak{B}} e(\gamma C(t_1,\dots,t_{10}))\,dt_1 \dots dt_{10}.$$

We write

$$\mathfrak{M}(X) = \bigcup_{1 \leqslant q \leqslant X} \bigcup_{\substack{a=1 \\ (a,q)=1}}^{q} \left[\frac{a}{q} - \frac{X}{P^3}, \frac{a}{q} + \frac{X}{P^3}\right]$$

and

$$\mathfrak{R}(X) = \bigcup_{1 \leqslant q \leqslant X} \bigcup_{\substack{a=1 \\ (a,q)=1}}^{q} \left[\frac{a}{q} - \frac{X}{qP^3}, \frac{a}{q} + \frac{X}{qP^3}\right].$$

The unit interval $\mathbb{T}$ are split into major arcs $\mathfrak{M} = \mathfrak{M}(P^\Delta)$ and minor arcs $\mathfrak{m} = \mathbb{T}\backslash\mathfrak{M}$, where Δ is a fixed small positive number. We further write $\mathfrak{m} = \mathfrak{m}_1 \cup \mathfrak{m}_2$, where

$$\mathfrak{m}_1 = \mathfrak{m} \cap \mathfrak{R}(P), \qquad \mathfrak{m}_2 = \mathfrak{m}\backslash\mathfrak{R}(P).$$

As usual, the letter ε will denote a sufficiently small positive number, while the value of ε may vary from sentence to sentence.

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§3. *The minor arc estimate.* In the next three lemmas, let us abbreviate G_i as G, z_i as z and c_i as c. The following result is due to Wooley [16].

LEMMA 3.1. *For each $\varepsilon > 0$, there exists an $\eta = \eta(\varepsilon) > 0$ such that*

$$\int_0^1 |h(\alpha)^2 G(\alpha)^4| \, d\alpha \ll P^{3+(\sqrt{2833}-43)/41+\varepsilon}.$$

Remark 1. $(\sqrt{2833} - 43)/41 = 0.24941301\ldots < 1/4$.

In particular, one has the following lemma.

LEMMA 3.2. *There exists an absolute constant $\eta > 0$ such that*

$$\int_0^1 |h(\alpha)^2 G(\alpha)^4| \, d\alpha \ll P^{3.2495}.$$

Throughout the paper, we assume that η is a fixed small positive number such that Lemma 3.2 is valid. The following result is proved using Lemma 3.1 of Brüdern and Wooley [5].

LEMMA 3.3. *Suppose that $2 \leqslant X \leqslant P$. There exists a constant $B > 0$ such that*

$$\int_{\mathfrak{R}(X)} |h(\alpha)^2 G(\alpha)^4| \, d\alpha \ll P^3 (\log X)^B + P^{2+\varepsilon} X.$$

Proof. We define the multiplicative function $\omega(q)$ determined by

$$\omega(p^{3u+v}) = \begin{cases} 2p^{-u-1/2} & \text{when } u \geqslant 0 \text{ and } v = 1, \\ p^{-u-1} & \text{when } u \geqslant 0 \text{ and } 2 \leqslant v \leqslant 3. \end{cases}$$

When $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a, q) = 1$, one has $q^{-1} S(q, a) \ll \omega(q)$. In particular, one has

$$q^{-1} S(q, ac) \ll_c \omega(q),$$

where the implied constant depends on c. By [15, Lemmas 6.1 and 6.2], we have

$$h(\alpha) \ll \frac{\omega(q) P}{1 + |\beta| P^3} + X^{1/2+\varepsilon}. \tag{3.1}$$

Hence,

$$\begin{aligned} &\int_{\mathfrak{R}(X)} |h(\alpha)^2 G(\alpha)^4| \, d\alpha \\ &\quad \ll P^2 \sum_{q \leqslant X} \omega(q)^2 \int_{-X/(qP^3)}^{X/(qP^3)} (1 + |\beta| P^3)^{-2} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} |G(a/q + \beta)|^4 \, d\beta \\ &\qquad + X^{1+\varepsilon} \int_0^1 |G(\alpha)|^4 \, d\alpha. \end{aligned}$$

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On applying [5, Lemma 3.1], there exists $B > 0$ uniformly for $2 \leqslant X \leqslant P$ such that

$$\sum_{q \leqslant X} \omega(q)^2 \int_{-1/(qP^2)}^{1/(qP^2)} (1+|\beta|P^3)^{-2} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} |G(a/q+\beta)|^2 \, d\beta \ll P^{-1}(\log X)^B.$$

The proof is completed by noting that

$$\int_0^1 |G(\alpha)|^4 \, d\alpha \ll P^{2+\varepsilon}. \qquad \square$$

LEMMA 3.4. *For any sufficiently small $\varepsilon > 0$, we have*

$$P^{3/2+\varepsilon} \left(\int_{\mathfrak{m}} |F(\alpha)|^2 \, d\alpha \right)^{1/2} = o(P^7). \tag{3.2}$$

Proof. Recall that C_0 is "good". On applying [17, Lemma 4.6(ii)] with $n = 7$, $t = 3/2$, $\Lambda = 7$, we find that $\Xi = 0$, $\varpi_2 = -1/3$, $\pi_2 = 3$ and (3.2) holds (see [17] for the definition of these parameters). $\square$

LEMMA 3.5. *We have*

$$\left(\int_{\mathfrak{m}_2} |F(\alpha)|^2 \, d\alpha \right)^{1/2} = o(P^{43/8}).$$

Proof. Taking $n = 7, t = 0, \Lambda = 43/8$ in [17, Lemma 4.6(i)], we obtain $\Xi = 19/44$, $\varpi_2 = -1/3$, $\pi_2 = 8/3$ and

$$P^{\varepsilon} \left(\int_{\mathfrak{m} \setminus \mathfrak{N}} |F(\alpha)|^2 \, d\alpha \right)^{1/2} = o(P^{43/8}),$$

where $\mathfrak{N}$ is the set of $\alpha \in \mathfrak{m}$ with the representation

$$\alpha = \frac{a}{q} + \beta, \qquad 0 \leqslant a \leqslant q \leqslant P^{19/44+K\delta},$$
$$(a,q) = 1, \quad |\beta| \leqslant q^{1/3} P^{-8/3+\delta}.$$

Here $K > 0$ is an absolute constant and $\delta > 0$ is a real number depending only on ε. In the above range, we have

$$|\beta| \leqslant q^{1/3} P^{-8/3+\delta} \leqslant (qP^2)^{-1}$$

and so $\mathfrak{N} \subseteq \mathfrak{R}(P)$. Recalling that $\mathfrak{m}_2 = \mathfrak{m} \setminus \mathfrak{R}(P)$, Lemma 3.5 now follows. $\square$

LEMMA 3.6. *One has*

$$\int_{\mathfrak{m}} h(\alpha) G_1(\alpha) G_2(\alpha) F(\alpha) \, d\alpha = o(P^7).$$

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Proof. By the Cauchy–Schwarz inequality, for $i = 1, 2$, we have

$$\begin{aligned}&\int_{\mathfrak{m}_i} h(\alpha)G_1(\alpha)G_2(\alpha)F(\alpha)\,d\alpha\\ &\ll \left(\int_{\mathfrak{m}_i} |h(\alpha)^2G_1(\alpha)^4|\,d\alpha\right)^{1/4}\\ &\quad\times\left(\int_{\mathfrak{m}_i} |h(\alpha)^2G_2(\alpha)^4|\,d\alpha\right)^{1/4}\left(\int_{\mathfrak{m}_i} |F(\alpha)|^2\,d\alpha\right)^{1/2}.\end{aligned}$$

Since $\mathfrak{m}_1 \subseteq \mathfrak{R}(P)$, we apply Lemmas 3.3 and 3.4 to conclude that

$$\begin{aligned}&\int_{\mathfrak{m}_1} h(\alpha)G_1(\alpha)G_2(\alpha)F(\alpha)\,d\alpha\\ &\ll \left(\int_{\mathfrak{R}(P)} |h(\alpha)^2G_1(\alpha)^4|\,d\alpha\right)^{1/4}\\ &\quad\times\left(\int_{\mathfrak{R}(P)} |h(\alpha)^2G_2(\alpha)^4|\,d\alpha\right)^{1/4}\left(\int_{\mathfrak{m}} |F(\alpha)|^2\,d\alpha\right)^{1/2}\\ &\ll P^{(3+\varepsilon)/4}\cdot P^{(3+\varepsilon)/4}\cdot\left(\int_{\mathfrak{m}} |F(\alpha)|^2\,d\alpha\right)^{1/2}\\ &= o(P^7).\end{aligned}$$

We deduce from Lemmas 3.2 and 3.5 that

$$\begin{aligned}&\int_{\mathfrak{m}_2} h(\alpha)G_1(\alpha)G_2(\alpha)F(\alpha)\,d\alpha\\ &\ll \left(\int_0^1 |h(\alpha)^2G_1(\alpha)^4|\,d\alpha\right)^{1/4}\\ &\quad\times\left(\int_0^1 |h(\alpha)^2G_2(\alpha)^4|\,d\alpha\right)^{1/4}\left(\int_{\mathfrak{m}_2} |F(\alpha)|^2\,d\alpha\right)^{1/2}\\ &\ll P^{3.2495/4}\cdot P^{3.2495/4}\cdot P^{43/8}\\ &= o(P^7).\end{aligned}$$

Now the desired estimate follows from above by observing that $\mathfrak{m} = \mathfrak{m}_1 \cup \mathfrak{m}_2$. □

§4. *The major arcs estimate.*

LEMMA 4.1. *Suppose that $\alpha = a/q + \beta$ with $1 \leqslant a \leqslant q \leqslant P^{\Delta}$, $(a, q) = 1$ and $|\beta| \leqslant P^{-3+\Delta}$. Then*

$$F(\alpha) = q^{-7}P^7S_F(q, a)I_F(\beta P^3) + O(P^{6+2\Delta}). \tag{4.1}$$

Moreover, for any $\varepsilon > 0$, one has

$$S_F(q, a) \ll q^{35/6+\varepsilon} \tag{4.2}$$

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and

$$I_F(\gamma) \ll (1+|\gamma|)^{-7/8+\varepsilon}. \tag{4.3}$$

Proof. One can obtain (4.1) from [**1**, Lemma 8.2]. Recalling that C_0 is good, the estimate (4.2) follows from [**10**, (7.1)]. As for (4.3), see [**1**, Lemma 8.8] for an analog. □

LEMMA 4.2. *Write* $\mathcal{L} = (\log P)^{1/100}$. *Then there exists a constant* $\varrho > 0$ *such that*

$$\int_{\mathfrak{M}} h(\alpha)G_1(\alpha)G_2(\alpha)F(\alpha)\,d\alpha = \varrho^2 P^7 \mathfrak{S}(\mathcal{L})\mathfrak{J}(\mathcal{L}) + O(P^7 \log^{-1/1000} P), \tag{4.4}$$

where

$$\mathfrak{S}(X) := \sum_{q\leqslant X} q^{-10} \sum_{\substack{a=1\\(a,q)=1}}^{q} S_C(q,a)$$

and

$$\mathfrak{J}(X) := \int_{-X}^{X} I_C(\beta)\,d\beta = \int_{-X}^{X}\int_{\mathcal{B}} e(\beta C(t_1,\dots,t_{10}))\,dt_1\dots dt_{10}\,d\beta.$$

Proof. Suppose that $2 \leqslant Q \leqslant P^{\Delta}$. For $\alpha \in \mathfrak{M}(2Q)\backslash\mathfrak{M}(Q)$, there exist a, q and β such that

$$\alpha = \frac{a}{q} + \beta, \qquad 1 \leqslant a \leqslant q \leqslant 2Q, \quad (a,q) = 1, \quad |\beta| \leqslant 2QP^{-3}.$$

By Lemma 4.1, we have

$$F(\alpha) \ll P^7 q^{-7/6+\varepsilon}(1+|\beta|P^3)^{-7/8+\varepsilon}. \tag{4.5}$$

Since $\alpha \notin \mathfrak{M}(Q)$, one has either $q > Q$ or $|\beta| > QP^{-3}$. It follows from above that

$$F(\alpha) \ll P^7 Q^{-1/9} q^{-19/18+\varepsilon}(1+|\beta|P^3)^{-55/72+\varepsilon} \tag{4.6}$$

for $\alpha \in \mathfrak{M}(2Q)\backslash\mathfrak{M}(Q)$ and, therefore,

$$\int_{\mathfrak{M}(2Q)\backslash\mathfrak{M}(Q)} |F(\alpha)|^2\,d\alpha \ll P^{14}Q^{-2/9}$$
$$\times \sum_{q\leqslant 2Q} \sum_{\substack{a=1\\(a,q)=1}}^{q} \int_{|\beta|\leqslant 2Q/P^3} q^{-19/9}(1+P^3|\beta|)^{-3/2}\,d\beta \ll P^{11}Q^{-2/9}.$$

Noting that $\mathfrak{M}(2Q) \subseteq \mathfrak{R}(4Q^2)$, we conclude from Lemma 3.3 and the Cauchy–Schwarz inequality that

$$\int_{\mathfrak{M}(2Q)} |h(\alpha)G_1(\alpha)G_2(\alpha)|^2\,d\alpha \ll P^3(\log Q)^B.$$

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We apply the Cauchy–Schwarz inequality again to obtain

$$\begin{aligned}\int_{\mathfrak{M}(2Q)\setminus\mathfrak{M}(Q)} & h(\alpha)G_1(\alpha)G_2(\alpha)F(\alpha)\,d\alpha\\ &\leqslant \left(\int_{\mathfrak{M}(2Q)} |h(\alpha)G_1(\alpha)G_2(\alpha)|^2\,d\alpha\right)^{1/2}\left(\int_{\mathfrak{M}(2Q)\setminus\mathfrak{M}(Q)} |F(\alpha)|^2\,d\alpha\right)^{1/2}\\ &\ll P^7Q^{-1/9}(\log Q)^B \ll P^7Q^{-1/10}.\end{aligned}$$

Then, by a dyadic argument, we have

$$\int_{\mathfrak{M}(P^\Delta)\setminus\mathfrak{M}(\mathcal{L})} h(\alpha)G_1(\alpha)G_2(\alpha)F(\alpha)\,d\alpha \ll P^7\mathcal{L}^{-1/10}. \tag{4.7}$$

In view of [**14**, Lemma 5.4], there exists a positive number ϱ, depending at most on η, such that whenever $1 \leqslant a \leqslant q \leqslant \mathcal{L}$ and $|\beta| \leqslant \mathcal{L}P^{-3}$, one has for $j = 1, 2$ that

$$G_j(a/q+\beta) = \varrho q^{-1}PS_{G_j}(q,a)I_{G_i}(\beta P^3) + O(P^7(\log P)^{-1/2}). \tag{4.8}$$

It follows from [**15**, Theorem 4.1] that

$$h(a/q+\beta) = q^{-1}PS_h(q,a)I_h(\beta P^3) + O(\log P). \tag{4.9}$$

Now we can deduce from Lemma 4.1, (4.8) and (4.9) that

$$\int_{\mathfrak{M}(\mathcal{L})} h(\alpha)G_1(\alpha)G_2(\alpha)F(\alpha)\,d\alpha = \varrho^2P^{10}\mathfrak{S}(\mathcal{L})\mathcal{I}(\mathcal{L}) + O(P^7(\log P)^{-1/4}), \tag{4.10}$$

where

$$\mathcal{I}(\mathcal{L}) = \int_{-\mathcal{L}P^{-3}}^{\mathcal{L}P^{-3}} I_h(\beta P^3)I_{G_1}(\beta P^3)I_{G_2}(\beta P^3)I_F(\beta P^3)\,d\beta.$$

A change of variables gives

$$\mathcal{I}(\mathcal{L}) = P^{-3}\mathfrak{J}(\mathcal{L}). \tag{4.11}$$

Then (4.10) and (4.11) yield

$$\int_{\mathfrak{M}(\mathcal{L})} h(\alpha)G_1(\alpha)G_2(\alpha)F(\alpha)\,d\alpha = \varrho^2P^7\mathfrak{S}(\mathcal{L})\mathfrak{J}(\mathcal{L}) + O(P^7\log^{-1/4}P). \tag{4.12}$$

We complete the proof by combining (4.7) and (4.10). □

The singular series is defined to be

$$\mathfrak{S} := \lim_{X\to+\infty}\mathfrak{S}(X) = \sum_{q=1}^{\infty} q^{-10}\sum_{\substack{a=1\\(a,q)=1}}^{q} S_C(q,a).$$

The limit exists, thanks to the following lemma.

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LEMMA 4.3. *The singular series* $\mathfrak{S}$ *is absolutely convergent and* $\mathfrak{S}(X) = \mathfrak{S} + O(X^{-1/6+\varepsilon})$ *for any* $\varepsilon > 0$.

Proof. It is shown in [**15**, Theorem 4.2] that whenever $(a, q) = 1$, one has

$$S_h(q, a), \qquad S_{G_1}(q, a), \qquad S_{G_2}(q, a) \ll q^{2/3}. \tag{4.13}$$

It follows from (4.2) and (4.13) that

$$\sum_{q>X} q^{-10} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} S_h(q, a) S_{G_1}(q, a) S_{G_2}(q, a) S_F(q, a) \ll \sum_{q>X} q^{-7/6+\varepsilon} \ll X^{-1/6+\varepsilon}.$$

The lemma easily follows. □

The singular integral is defined to be

$$\mathfrak{J} := \lim_{X\to+\infty} \mathfrak{J}(X) = \int_{-\infty}^{\infty} \int_{\mathfrak{B}} e(\beta C(t_1, \dots, t_{10}))\, dt_1 \dots dt_{10}\, d\beta.$$

The following lemma shows that the above limit exists.

LEMMA 4.4. *The singular integral* $\mathfrak{J}$ *exists and, for* $X \geqslant 1$, *one has*

$$\mathfrak{J}(X) = \mathfrak{J} + O(X^{-7/8+\varepsilon})$$

for any $\varepsilon > 0$.

Proof. We conclude from [**15**, Lemma 2.8] that

$$I_h(\gamma), \qquad I_{G_1}(\gamma), \qquad I_{G_2}(\gamma) \ll (1 + |\gamma|)^{-1/3}. \tag{4.14}$$

Then we deduce by (4.3) and (4.14) that

$$\int_{|\beta|>X} I_h(\beta) I_{G_1}(\beta) I_{G_2}(\beta) I_F(\beta)\, d\beta \ll \int_{X}^{+\infty} |\beta|^{-15/8+\varepsilon}\, d\beta \ll X^{-7/8+\varepsilon}.$$

This completes the proof. □

Lemmas 4.2–4.4 yield the following conclusion.

LEMMA 4.5. *There exists a constant* $\varrho > 0$ *such that*

$$\int_{\mathfrak{M}} h(\alpha) G_1(\alpha) G_2(\alpha) F(\alpha)\, d\alpha = \varrho^2 P^7 \mathfrak{S}\, \mathfrak{J} + O(P^7 \log^{-1/1000} P).$$

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§5. *The proof of Theorem 1.1.* As pointed out in §1, Theorem 1.1 follows from Proposition 1.2 thanks to [**17**, Theorems 1.1 and 1.2]. In view of the discussion in §2, we assume that $c_1c_2c_3 \neq 0$ and C_0 is non-degenerate. We also assume that C_0 is "good". Let

$$\begin{aligned}\mathcal{N}(P) = \#\{(x_1, \dots, x_{10}) \in \mathbb{Z}^{10} : \ & C(x_1, \dots, x_{10}) = 0, \\ & x_1, x_2 \in \mathcal{A},\ \varsigma P < x_3 \leqslant P,\ |x_i/P - z_i| \leqslant \varpi\ (i = 4, \dots, 10)\},\end{aligned} \tag{5.1}$$

where z_i $(4 \leqslant i \leqslant 10)$ and ϖ will be chosen later. By orthogonality, one has

$$\mathcal{N}(P) = \int_0^1 h(\alpha)G_1(\alpha)G_2(\alpha)F(\alpha)\,d\alpha.$$

Recalling the definition of $\mathfrak{M}$ and $\mathfrak{m}$, we have

$$\mathcal{N}(P) = \int_{\mathfrak{M}} h(\alpha)G_1(\alpha)G_2(\alpha)F(\alpha)\,d\alpha + \int_{\mathfrak{m}} h(\alpha)G_1(\alpha)G_2(\alpha)F(\alpha)\,d\alpha.$$

On applying Lemmas 3.6 and 4.5, we obtain

$$\mathcal{N}(P) = \varrho^2 P^7 \mathfrak{S}\mathfrak{J} + o(P^7)$$

for some positive number ϱ. Standard arguments show that both $\mathfrak{S}$ and $\mathfrak{J}$ are positive, provided that ϖ is chosen sufficiently small (see [**7**, Lemma 7.3], [**8**, §16] and [**12**, §20.2] for details). We conclude that

$$\mathcal{N}(P) \gg P^7.$$

This completes the proof of Proposition 1.2, and therefore Theorem 1.1 is established.

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