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From local to global analytic conjugacies

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Abstract. Let f_1 and f_2 be rational maps with Julia sets J_1 and J_2, and let $\Psi : J_1 \to \mathbb{P}^1$ be any continuous map such that $\Psi \circ f_1 = f_2 \circ \Psi$ on J_1. We show that if Ψ is $\mathbb{C}$-differentiable, with non-vanishing derivative, at some repelling periodic point $z_1 \in J_1$, then Ψ admits an analytic extension to $\mathbb{P}^1 \setminus \mathcal{E}_1$, where $\mathcal{E}_1$ is the exceptional set of f_1. Moreover, this extension is a semiconjugacy. This generalizes a result of Julia (*Ann. Sci. École Norm. Sup.* (3) **40** (1923), 97–150). Furthermore, if $\mathcal{E}_1 = \emptyset$ then the extended map Ψ is rational, and in this situation $\Psi(J_1) = J_2$ and $\Psi^{-1}(J_2) = J_1$, provided that Ψ is not constant. On the other hand, if $\mathcal{E}_1 \neq \emptyset$ then the extended map may be transcendental: for example, when f_1 is a power map (conjugate to $z \mapsto z^{\pm d}$) or a Chebyshev map (conjugate to $\pm\Upsilon_d$ with $\Upsilon_d(z + z^{-1}) = z^d + z^{-d}$), and when f_2 is an integral Lattès example (a quotient of the multiplication by an integer on a torus). Eremenko (*Algebra i Analiz* **1**(4) (1989), 102–116) proved that these are the only such examples. We present a new proof.

0. *Introduction*

Let f be a rational map of degree two or more. It is well known that if the Julia set J contains a smooth free arc then J is itself a real-analytic arc or Jordan curve. Moreover, in this situation f is semiconjugate to a Blaschke product, and J lies on a round circle or a straight line: in fact,

$$f \circ \Psi = \Psi \circ B$$

where B is a Blaschke product and where Ψ is either a Möbius transformation or a quadratic rational map which sends $\mathbb{D}$ and $\mathbb{P}^1 \setminus \overline{\mathbb{D}}$ conformally onto $\mathbb{P}^1 \setminus J$. The proof is carried out in two stages: one begins by promoting the given local property to the entire Julia set, and concludes by deducing the global description of the map itself. For details, see [**S**, Ch. 5].

Our results here are in the same spirit, and our arguments follow a similar outline. The crucial local hypothesis concerns the differentiability, suitably interpreted, of a map

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whose domain may have empty interior. Consider any map $h : Z \to \mathbb{P}^1$ where $Z \subset \mathbb{P}^1$, and let $\zeta \in Z$ be any accumulation point. If $\zeta \neq \infty \neq h(\zeta)$ then we say that h is $\mathbb{C}$-differentiable at ζ if

$$\lim_{z \to \zeta} \frac{h(z) - h(\zeta)}{z - \zeta}$$

exists. Note that the existence of a non-zero finite limit is unaffected by precomposition or postcomposition by local analytic isomorphisms, and in particular by Möbius transformations. Thus, it makes sense to speak of $\mathbb{C}$-differentiability with non-vanishing derivative even when ζ or $h(\zeta)$ is the point at infinity.

In what follows, we use the term *analytic* in the sense of Riemann surface theory. As the relevant Riemann surfaces will all be open subsets of $\mathbb{P}^1$, our usage of this term is identical with the classical usage of *meromorphic*.

THEOREM 1. *Let f_1 and f_2 be rational maps of degrees at least two with Julia sets J_1 and J_2, and let $\Psi : J_1 \to \mathbb{P}^1$ be continuous with $\Psi \circ f_1 = f_2 \circ \Psi$ on J_1. Suppose further that Ψ is $\mathbb{C}$-differentiable, with non-vanishing derivative, at some repelling periodic point $z_1 \in J_1$. Then Ψ admits an analytic extension to $\mathbb{P}^1 \setminus \mathcal{E}_1$, where $\mathcal{E}_1$ is the exceptional set of f_1. Moreover, the extended map Ψ is a semiconjugacy: that is, $\Psi \circ f_1 = f_2 \circ \Psi$ on $\mathbb{P}^1 \setminus \mathcal{E}_1$.*

Here we recall that the exceptional set of any rational map consists of at most two points, and that any non-empty exceptional set is a superattracting cycle.

The proof of Theorem 1 begins with a preliminary step where we show that the assumption on Ψ implies the existence of an analytic extension to a neighborhood of z_1. We then promote this local property to successively more global versions: specifically, we first show that Ψ admits an analytic extension to a neighborhood of J_1, and then argue that Ψ admits a global analytic extension, at least to $\mathbb{P}^1 \setminus \mathcal{E}_1$.

In [**J**], Julia studied the equation $\Psi \circ f_1 = f_2 \circ \Psi$ under the assumption that Ψ is defined and holomorphic in a neighborhood of a repelling point $z_1 \in J_1$. He first proved that Ψ has a maximal analytic extension defined on a Riemann surface Σ_1 which in general is not a subset of $\mathbb{P}^1$. Then, he proved that if Σ_1 is a subset of $\mathbb{P}^1$, then $\mathbb{P}^1 \setminus \mathcal{E}_1 \subset \Sigma_1$ and he gave examples where $\Sigma_1 = \mathbb{P}^1 \setminus \mathcal{E}_1$ (see the discussion of power maps, Chebyshev maps and Lattès examples below). Finally, he proved that if f_1 is a Blaschke product and Ψ has an analytic extension to a neighborhood of J_1, then it has a analytic extension to $\mathbb{P}^1 \setminus \mathcal{E}_1$. Our Theorem 1 asserts that one can get rid of the assumption that f_1 is a Blaschke product and consider an arbitrary rational map f_1.

The possibility of an extension to $\mathbb{P}^1$ is discussed in Theorem 2 below. This complementary result makes use of the following terminology, which is discussed in more detail in §5 and [**M2**].

A *power map* of degree $d \geq 2$ is a rational map analytically conjugate to $z \mapsto z^{\pm d}$.

A *Chebyshev map* of degree $d \geq 2$ is a rational map analytically conjugate to $\pm \Psi_d$ where Ψ_d is the *Chebyshev polynomial* of degree d defined by

$$\Psi_d\left(z + \frac{1}{z}\right) = z^d + \frac{1}{z^d}.$$

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A *Lattès example* is a rational map $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree at least two such that there exists a commutative diagram

$$\begin{array}{ccc} \mathbb{C}/\Lambda = \mathcal{T} & \xrightarrow{L} & \mathcal{T} = \mathbb{C}/\Lambda \\ \downarrow & & \downarrow \pi \\ \mathcal{T}/G_n \simeq \mathbb{P}^1 & \xrightarrow[f]{} & \mathbb{P}^1 \simeq \mathcal{T}/G_n \end{array}$$

where:

- Λ is a discrete additive subgroup of rank two of the complex numbers $\mathbb{C}$;
- $\mathcal{T} = \mathbb{C}/\Lambda$ is the quotient torus;
- G_n is the group of nth roots of unity acting on $\mathcal{T}$ by rotation around a base point, with n equal to either 2, 3, 4 or 6;
- $\mathcal{T}/G_n$ is the quotient space isomorphic to $\mathbb{P}^1$;
- $L : \mathcal{T} \to \mathcal{T}$ is an affine map which commutes with the action of G_n.

If the affine map L is of the form $z \mapsto az + b$ with $a \in \mathbb{Z}$, then we refer to f as an *integral Lattès example*.

THEOREM 2. *Let f_1 and f_2 be rational maps of degrees at least two, and let $\Psi : \mathbb{P}^1 \setminus \mathcal{E}_1 \to \mathbb{P}^1$ be analytic with $\Psi \circ f_1 = f_2 \circ \Psi$ on $\mathbb{P}^1 \setminus \mathcal{E}_1$. Suppose that Ψ does not extend to a rational map $\mathbb{P}^1 \to \mathbb{P}^1$, so that in particular $\mathcal{E}_1 \neq \emptyset$. Then f_1 is either a power map or a Chebyshev map, and f_2 is an integral Lattès example.*

More precisely, there are discrete subgroups $\Gamma \subset \Sigma \subset \mathrm{Isom}(\mathbb{C})$ and an affine map $L : \mathbb{C} \to \mathbb{C}$ with derivative $a \in \mathbb{Z} \setminus \{-1, 0, 1\}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{L} & \mathbb{C} \\ \downarrow & & \downarrow \\ \mathbb{C}/\Gamma \simeq \mathbb{P}^1 \setminus \mathcal{E}_1 & \xrightarrow{f_1} & \mathbb{P}^1 \setminus \mathcal{E}_1 \simeq \mathbb{C}/\Gamma \\ \Psi \downarrow & & \downarrow \Psi \\ \mathbb{C}/\Sigma \simeq \mathbb{P}^1 & \xrightarrow[f_2]{} & \mathbb{P}^1 \simeq \mathbb{C}/\Sigma \end{array}$$

Here $\mathrm{Isom}(\mathbb{C})$ denotes the group of orientation-preserving Euclidean isometries of $\mathbb{C}$, that is, maps of the form $z \mapsto az + b$ with $|a| = 1$. In §5, we will discuss which discrete subgroups $\Gamma \subset \Sigma \subset \mathrm{Isom}(\mathbb{C})$ and which affine maps L can occur.

Taken together, Theorems 1 and 2, along with well-known properties of rational semiconjugacies, have the following consequence, which is perhaps slightly surprising given the apparently local nature of the assumptions.

COROLLARY 1. *Let f_1 and f_2 be rational maps of degrees at least two with Julia sets J_1 and J_2, and let $\Psi : J_1 \to J_2$ be a continuous surjection which semiconjugates the restriction f_1 to the restriction of f_2. Assume further that Ψ admits an analytic extension to a neighborhood of some point $z_1 \in J_1$. Then Ψ extends to a rational semiconjugacy from f_1 to f_2. Moreover, if $\Psi : J_1 \to J_2$ is a homeomorphism then Ψ extends to a Möbius conjugacy.*

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Remark. Note that it is easy to give examples of transcendental semiconjugacies from degree one rational maps to higher degree rational maps, namely maximally continued inverse linearizing maps associated to repelling cycles and maximally continued inverse repelling Fatou coordinates associated to parabolic cycles.

Remark. After submitting our manuscript, we were informed by the referee that Theorem 2 was shown earlier by Eremenko [**E**]. We note that our argument differs significantly from Eremenko's. His proof involves the *lamellarity* (invariance under the flow of a suitable vector field) of the equilibrium measure introduced by Brolin, Lyubich and Freire-Lopès-Mañe. Our proof uses techniques available at Julia's time. More specifically, we estimate the number of points in *lateral orbits* for a careful choice of compositions of f_2 and inverse branches of f_2. We show that if f_1 is not a power map or a Chebyshev map or if f_2 is not a Lattès example, then for a generic point $z \in \mathbb{P}^1$, the number of points in $f_2^{-n}(z)$ grows faster than d^n for any integer d. This contradicts the fact that this number is at most D^n where D is the degree of the rational map f_2. The connection with Eremenko's proof presumably lies in the fact that preimages of points equidistribute with respect to the equilibrium measure.

1. *Extension to a neighborhood of the repelling point*

Henceforth, we work under the assumptions of Theorem 1: that is, we assume that $\Psi : J_1 \to \mathbb{P}^1$ is continuous and $\mathbb{C}$-differentiable, with non-vanishing derivative, at a repelling periodic point $z_1 \in J_1$. Note that if z_1 is fixed by the iterate $f_1^{\circ p}$ then $z_2 = \Psi(z_1)$ is fixed by the iterate $f_2^{\circ p}$. Since the passage to iterates does not affect the Julia set or the exceptional set, we may assume without loss of generality that f_i fixes z_i for $i = 1, 2$. Moreover, since the property of $\mathbb{C}$-differentiability with non-vanishing derivative is preserved under locally invertible analytic change of variable, we may further assume that $z_1 = 0 = z_2$.

After these preliminary reductions, we have $\Psi(z) = cz + o(z)$ as $z \to 0$, for some $c \in \mathbb{C}^*$. Moreover, since

$$c \cdot f_1'(0) + o(1) = \frac{\Psi(f_1(z))}{z} = \frac{f_2(\Psi(z))}{z} = f_2'(0) \cdot c + o(1)$$

and since $c \neq 0$, the two maps f_i have the same multiplier at 0. Let λ be this common multiplier, let $\phi_i : U_i \to \mathbb{C}$ be normalized local linearizing maps, so that $\phi_i \circ f_i(z) = \lambda \phi_i(z)$ and $\phi_i'(0) = 1$ for $i = 1, 2$, and set $\widetilde{\Psi} = \phi_2 \circ \Psi \circ \phi_1^{-1}$. Then $\widetilde{\Psi}(z) = cz + o(z)$ as $z \in \phi_1(U_1 \cap J_1)$ tends to 0, and moreover $\widetilde{\Psi}(z/\lambda) = \widetilde{\Psi}(z)/\lambda$ for every $z \in \phi_1(J_1 \cap U_1)$. Furthermore, if $z \in \phi_1(U_1 \cap J_1)$ then $z_n = z/\lambda^n$ also lies in $\phi_1(U_1 \cap J_1)$ for any positive integer n, and clearly

$$\frac{\widetilde{\Psi}(z_n)}{z_n} = \frac{\widetilde{\Psi}(z/\lambda^n)}{z/\lambda^n} = \frac{\widetilde{\Psi}(z)}{z}.$$

As $n \to \infty$, we have $z_n \to 0$ and $\widetilde{\Psi}(z_n)/z_n \to c$, and hence $\widetilde{\Psi}(z) = cz$ for every $z \in \phi_1(U_1 \cap J_1)$. In particular, $\widetilde{\Psi}$ extends analytically to a neighborhood of 0, so the same is true for $\Psi = \phi_2^{-1} \circ \widetilde{\Psi} \circ \phi_1$. We summarize this discussion as follows.

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LEMMA 1. *Let f_1 and f_2 be rational maps of degrees at least two with Julia sets J_1 and J_2, and let $\Psi : J_1 \to \mathbb{P}^1$ be continuous with $\Psi \circ f_1 = f_2 \circ \Psi$ on J_1. Suppose further that Ψ is $\mathbb{C}$-differentiable, with non-vanishing derivative, at some repelling periodic point $z_1 \in J_1$. Then Ψ admits an analytic extension to a neighborhood of z_1.*

2. *Extension to a neighborhood of the Julia set*

We now show that Ψ extends analytically to a neighborhood of J_1.

LEMMA 2. *The set of all points $z \in J_1$ such that Ψ admits an analytic extension to a neighborhood of z is forward invariant under f_1.*

Proof. Assume that Ψ extends analytically to a neighborhood of $z \in J_1$ and set $w = \Psi(z)$. Let V be an open disk containing w, and let U be the connected component of $f_1^{-1}(V)$ containing z. By assumption, if V is sufficiently small then Ψ extends analytically to U. Moreover, in this situation the restriction $f_1 : U \to V$ is a branched cover, unramified except possibly at z. Consequently, there is a cyclic group of analytic isomorphisms $\sigma : U \to U$, all fixing z, such that for $\zeta, \hat{\zeta} \in U$ we have $f_1(\zeta) = f_1(\hat{\zeta})$ if and only if $\hat{\zeta} = \sigma(\zeta)$ for some such σ. Furthermore, since $\Psi(f_1(z)) = f_2(\Psi(z))$ for all $z \in J_1$, it follows that $f_1 = f_1 \circ \sigma$ on U, and hence

$$f_2 \circ \Psi = \Psi \circ f_1 = \Psi \circ f_1 \circ \sigma = f_2 \circ \Psi \circ \sigma$$

on the infinite set $U \cap J_1$. By uniqueness of analytic continuation, it follows that $f_2 \circ \Psi = f_2 \circ \Psi \circ \sigma$ on U, and hence $f_2 \circ \Psi : U \to \mathbb{P}^1$ descends to a map which is continuous on V and analytic on $V \setminus \{w\}$, hence analytic on all of V, and which extends the given map Ψ. □

Now let W be a neighborhood of $z_1 \in J_1$ on which Ψ extends analytically. By Montel's theorem, we have $f_1^{\circ n}(W \cap J_1) = J_1$ for some positive integer n. Consequently, it follows from Lemma 2 that Ψ extends analytically to a neighborhood of any given point in J_1. To obtain the desired extension to a neighborhood of J_1 we apply the following well-known fact.

LEMMA 3. *Let $\Psi : J \to \mathbb{P}^1$ be a map from a closed set $J \subseteq \mathbb{P}^1$, and suppose that for each $z \in J$ the map Ψ extends analytically to some neighborhood of z. Then Ψ extends analytically to a neighborhood of J.*

Proof. If $J = \mathbb{P}^1$ then Ψ is already analytic on all of $\mathbb{P}^1$ and there is nothing further to show. Composing with a Möbius transformation if necessary, we may therefore assume that J is a compact set in $\mathbb{C}$.

For $x \in \mathbb{C}$ and $r > 0$, we denote by $D_x(r)$ the open Euclidean disk with center x and radius r. By assumption, for each $x \in J$ there exists $r_x > 0$ such that Ψ admits an analytic extension Ψ_x defined on the disk $U_x = D_x(r_x)$. Consider the subdisks $V_x = D_x(\frac{1}{3}r_x)$. Note that if $V_x \cap V_y \neq \emptyset$ then $|x - y| < \frac{1}{3}r_x + \frac{1}{3}r_y \leq \frac{2}{3}\max(r_x, r_y)$, and hence if $r_x \leq r_y$ then $V_x \subseteq U_y$. Moreover, in this situation it follows by uniqueness of analytic continuation that if $J \cap V_x$ contains an accumulation point then Ψ_x and Ψ_y have the same restriction to V_x, and hence the restrictions $\Psi_{x|V_x}$ and $\Psi_{y|V_y}$ agree on $V_x \cap V_y$.

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As $J \subset \bigcup_{x\in J} V_x$, it follows by compactness that $J \subset \bigcup_{j=1}^{N} V_{x_j}$ for some $x_1, \ldots, x_N$ in J. We may assume without loss of generality that, for some $n < N$, the intersection $J \cap V_{x_j}$ contains an accumulation point precisely for $1 \le j \le n$. Since the open set $W = \bigcup_{j=1}^{n} V_{x_j}$ contains every accumulation point of J, the complement $I = J \setminus W$ consists of finitely many isolated points of J. It follows from the observations above that if $1 \le j, k \le n$ then $\Psi_{x_j|V_{x_j}}$ and $\Psi_{x_k|V_{x_k}}$ agree on $V_{x_j} \cap V_{x_k}$; hence the maps $\Psi_{x_j|V_{x_j}}$ for $1 \le j \le n$ are all restrictions of some analytic map defined on W, and the map so obtained agrees with Ψ on $J \setminus I$. Finally, for the finitely many points $y \in I$, let $W_y \ni y$ be pairwise disjoint open disks which are also disjoint from W. The map just obtained now extends trivially to an analytic extension of Ψ defined on the open neighborhood $\widehat{W} = W \cup \bigcup_{y\in I} W_y$ of J.

3. *Extension to the complement of the exceptional set*

Up to now we have shown that Ψ extends analytically to a neighborhood W of J_1. Here we obtain the desired analytic extension to $\mathbb{P}^1 \setminus \mathcal{E}_1$. Note that since J_1 is compact, the neighborhood W contains all but finitely many Fatou components. To treat the remaining components we will make use of the following.

LEMMA 4. *Let $f_1 : \Omega \to \tilde{\Omega}$ be a proper analytic map between open sets $\Omega, \tilde{\Omega} \subseteq \mathbb{P}^1$. Let $\Psi : \Omega \to \mathbb{P}^1$ and $f_2 : \Psi(\Omega) \to \mathbb{P}^1$ be analytic, let $\tilde{\Psi} : U \to \mathbb{P}^1$ be analytic, where $U \neq \emptyset$ is open in $\tilde{\Omega}$, and assume that $\tilde{\Psi} \circ f_1 = f_2 \circ \Psi$ on $f_1^{-1}(U)$. Then $\tilde{\Psi}$ extends analytically to $\tilde{\Omega}$.*

Proof. We claim that, for any point $w \in \tilde{\Omega}$, the value of $f_2 \circ \Psi(z)$ is independent of the choice of preimage $z \in f_1^{-1}(w)$. Note that since f_1 is an open map, the points w for which this is true form a relatively closed set. Consequently, it suffices to verify the claim on a dense subset of $\tilde{\Omega}$. Since f_1 has only countably many critical points, there are only countably many critical values, so the points which are not critical values are dense in $\tilde{\Omega}$: indeed, since $f_1 : \Omega \to \tilde{\Omega}$ is proper, the critical values are discrete in $\tilde{\Omega}$. Thus, we may assume without loss of generality that w is not a critical value.

Fix a point $u \in U$ which is not a critical value, and let γ be a path in $\tilde{\Omega}$ which connects u to w, and which avoids the set of critical values; note that, by the discreteness of critical values, such a path always exists. Now $\tilde{\Psi}$ is defined near u, and moreover we have $\tilde{\Psi} = \tilde{\Psi} \circ f_1 \circ g = f_2 \circ \Psi \circ g$ for any analytic germ g which is locally inverse to f_1 near u. Since $f_1 : \Omega \to \tilde{\Omega}$ is proper, and since there are no critical values on γ, the germ g admits a unique analytic continuation along that path, leading to an analytic germ h defined near w. It follows that the analytic continuation of $\tilde{\Psi}$ along γ is the germ $f_2 \circ \Psi \circ h$. Moreover, the resulting germ is independent of the choice of γ: indeed, the analytic continuation of g along any closed path which is based at u, and which avoids the critical values, is also locally inverse to f_1, and hence the result of the corresponding continuation of the germ of $\tilde{\Psi}$ at u is the same germ $\tilde{\Psi}$.

It follows that the prescription $w \mapsto f_2(\Psi(z))$, where $z \in \Omega$ is any preimage of w, gives a well-defined map $\tilde{\psi} : \tilde{\Omega} \to \mathbb{P}^1$. Moreover, since f_1 is an open map, it follows that $\tilde{\psi}$ is continuous. Furthermore, by construction $\tilde{\psi}$ is analytic away from the set of critical values, and since the latter constitute a discrete set, it follows by continuity that these are

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removable singularities; hence $\tilde{\psi}$ is analytic on $\tilde{\Omega}$. Since $\tilde{\psi}$ and $\tilde{\Psi}$ agree on U, the map $\tilde{\psi}$ is the desired analytic continuation of $\tilde{\Psi}$ to $\tilde{\Omega}$. □

PROPOSITION 1. *Let f_1 and f_2 be rational maps of degrees at least two with Julia sets J_1 and J_2. Let $\Psi : J_1 \to \mathbb{P}^1$ be continuous with $\Psi \circ f_1 = f_2 \circ \Psi$ on J_1, and assume that Ψ admits an analytic extension to a neighborhood of J_1. Then Ψ admits an analytic extension to $\mathbb{P}^1 \setminus \mathcal{E}_1$, where $\mathcal{E}_1$ is the exceptional set of f_1.*

Proof. Assume that $\Psi : W \to \mathbb{P}^1$ is analytic on the open neighborhood $W \supset J_1$. Note that, without loss of generality, we may choose W so that every connected component of W intersects J_1. Consequently, if the restriction $\Psi_{|\partial\Omega}$ extends analytically to a Fatou component Ω, then Ψ itself so extends. In view of Lemma 4 and the fact that W must intersect every Fatou component, if Ψ extends analytically to a Fatou component Ω then Ψ further extends to the image component $f_1(\Omega)$. If Ω is not totally invariant under f_1 then the backward orbit of Ω consists of infinitely many Fatou components, and all but finitely many lie in the given neighborhood where Ψ is already analytic. It follows that Ψ extends analytically to every such component Ω.

Suppose now that Ω is totally invariant under f_1. Such a component is necessarily a (super)attracting or parabolic basin, with associated fixed point ζ. In this situation we apply a similar argument to an appropriate collection of nested open subsets of Ω. Let $U_k = \Omega \setminus f_1^{-k}(\overline{D})$, where $D \subset \Omega$ is an open disk chosen as follows. In the (super)attracting case, we take any $D \ni \zeta$ which is small enough that $f_1(D)$ lies compactly in D; in the parabolic case, we take D to be an attracting petal, so here $\zeta \in \partial D$. By construction, the restriction of f_1 maps U_{k+1} properly to U_k, and $U_{k+1} \subset U_k$, for every integer $k \geq 0$. Moreover, if k is sufficiently large then $U_k \subset W$, and hence Ψ is analytic on such U_k, and furthermore Ψ semiconjugates f_1 to f_2 on U_{k+1}. Starting from the restriction of Ψ to such a set U_k, and applying Lemma 4 inductively, we deduce that the restriction $\Psi_{|\partial\Omega}$ admits successive analytic extensions to the sets U_j with $j < k$, each such extension giving a semiconjugacy on U_{j+1}. In particular, $\Psi_{|\partial\Omega}$ extends to an analytic map on $\Omega \setminus \overline{D}$. Furthermore, since $\Omega \setminus \overline{D}$ is connected with the boundary containing $\partial\Omega$, replacing D by a smaller disk yields a further analytic extension. Thus, shrinking $\overline{D}$ to ζ, we obtain an analytic extension of $\Psi_{|\partial\Omega}$ to $\Omega \setminus \{\zeta\}$, and hence, as above, it follows that Ψ extends analytically to $\Omega \setminus \{\zeta\}$, giving a semiconjugacy on $\Omega \setminus \{\zeta\}$.

In the parabolic case we have $\zeta \in \partial\Omega$, so there is nothing left to show. If $\zeta \notin \mathcal{E}_1$ then there exists a preimage $\xi \neq \zeta$, and $\xi \in \Omega$ by total invariance. If D is sufficiently small then the component $B \ni \xi$ of $f_1^{-1}(D)$ is contained in $\Omega \setminus \{\zeta\}$. Since $\Psi \circ f_1 = f_2 \circ \Psi$ on $B \setminus \{\xi\}$, it follows from Lemma 4 that Ψ extends analytically to Ω. □

4. *Rational semiconjugacies between rational maps*

Here we review certain well-known properties of rational semiconjugacies, and then deduce Corollary 1 from Theorems 1 and 2.

LEMMA 5. *Let f_1 and f_2 be rational maps with Julia sets J_1 and J_2, and let Ψ be a rational map with $\Psi \circ f_1 = f_2 \circ \Psi$. If Ψ is not constant then f_1 and f_2 have the same degree, and if this degree is two or more then $\Psi(J_1) = J_2$ and $\Psi^{-1}(J_2) = J_1$.*

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Proof. To see that the degrees of f_1 and f_2 are equal, observe that

$$\deg \Psi \cdot \deg f_1 = \deg(\Psi \circ f_1) = \deg(f_2 \circ \Psi) = \deg f_2 \cdot \deg \Psi$$

and note that $\deg \Psi \neq 0$ since Ψ is not constant. Assuming now that the common degree is at least two, we first show that $\Psi(J_1) \subseteq J_2$, and hence $J_1 \subseteq \Psi^{-1}(J_2)$, and then show conversely that $\Psi^{-1}(J_2) \subseteq J_1$, and hence $J_2 \subseteq \Psi(J_1)$.

Recall that if f_1 fixes z_1 then f_2 fixes $z_2 = \Psi(z_1)$. Moreover, it follows from the chain rule that $\lambda_2 = \lambda_1^{\delta}$, where λ_i are the corresponding multipliers and where δ is the local degree of Ψ at z_1; thus, Ψ sends repelling fixed points of f_1 to repelling fixed points of f_2. Applying this observation to iterates, we deduce more generally that Ψ sends repelling periodic points of f_1 to repelling periodic points of f_2. Since the Julia sets J_1 and J_2 are the closures of the corresponding sets of repelling periodic points, it follows that $\Psi(J_1) \subseteq J_2$.

Conversely, let w be periodic for f_2 and let z be any point in the preimage $\Psi^{-1}(z)$. Note that such a point z need not be periodic for f_1; however, since the forward orbit of z under f_1 lies in the Ψ-preimage of the forward orbit of w by f_2, such a point z is at least preperiodic. Considering multipliers as above, we deduce that z eventually maps to a repelling periodic cycle. Applying the density of repelling periodic points once again, we conclude that $\Psi^{-1}(J_2) \subseteq J_1$. □

Note that the same considerations show that a rational semiconjugacy sends attracting periodic points to attracting periodic points of f_2. In fact, with a bit more care it may be shown that parabolic, indifferently linearizable, and indifferently non-linearizable periodic points are sent to periodic points of the same character.

Proof of Corollary 1. It follows from Theorem 1 that Ψ admits an analytic extension to a semiconjugacy defined on $\mathbb{P}^1 \setminus \mathcal{E}_1$, and we claim that this extension is in fact rational. Indeed, if Ψ has an essential singularity then it follows from Theorem 2 that $\mathcal{E}_1 \neq \emptyset$, and hence J_1 is nowhere dense, and moreover that $J_2 = \mathbb{P}^1$. On the other hand, since Ψ is a non-constant analytic map, the image $\Psi(J_1)$ must be a meagre subset of $\mathbb{P}^1$, and hence $J_2 = \Psi(J_1) \neq \mathbb{P}^1$ for a contradiction. Thus, Ψ must be a rational semiconjugacy, and hence $\Psi^{-1}(J_2) = J_1$ by Lemma 5. Consequently, if Ψ restricts to a homeomorphism between J_1 and J_2 then every $z \in J_2$ has a unique preimage under Ψ. Since J_2 is infinite, it follows that Ψ has degree one, and hence Ψ is a Möbius transformation. □

5. *Some transcendental semiconjugacies between rational maps*

Following Milnor [**M2**], we will say that a rational map of degree $d \geq 2$ is a *power map* if it is analytically conjugate to either $p_d : z \mapsto z^d$ or $p_{-d} : z \mapsto z^{-d}$. The power maps are the rational maps f whose exceptional set $\mathcal{E}_f$ contains two points which are either fixed (when it is conjugate to p_d) or exchanged (when it is conjugate to p_{-d}). The restriction of a power map f to $\mathbb{P}^1 \setminus \mathcal{E}_f$ is a covering map of degree d. The Julia set of $p_{\pm d}$ is the unit circle.

A rational map of degree $d \geq 2$ will be called a *Chebyshev map* if it is conjugate to $\pm \Psi_d$ where Ψ_d is the degree d *Chebyshev polynomial*, defined by the equation

$$\Psi_d\left(z + \frac{1}{z}\right) = z^d + \frac{1}{z^d}.$$

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For example,

$$\Psi_2(z) = z^2 - 2, \quad \Psi_3(z) = z^3 - 3z \quad \text{and} \quad \Psi_4(z) = z^4 - 4z^2 + 2.$$

The Julia set of $\pm\Psi_d$ is the segment $[-2, 2]$. The postcritical set of $\pm\Psi_d$ consists of the three points $\{-2, 2, \infty\}$. If d is even, $-\Psi_d$ is conjugate to Ψ_d by $-\mathrm{Id}$. However, when d is odd, Ψ_d fixes the postcritical set whereas $-\Psi_d$ permutes the two points of the postcritical set, and hence cannot be conjugate to Ψ_d.

The following construction due to Lattès [**L**] is well-known (see also [**M2**] for a detailed discussion on Lattès examples). If Λ is a discrete group of translations of rank two, then the quotient surface $\mathbb{C}/\Lambda$ is a torus $\mathcal{T}$. Moreover, if G_n is the group of nth roots of unity acting on $\mathcal{T}$ by rotation around a base point with n equal to either 2, 3, 4 or 6, then the quotient surface $\mathcal{T}/G_n$ is a Riemann surface isomorphic to $\mathbb{P}^1$. If $L : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda$ is an affine map $z \mapsto az + b$ which commutes with the action of G_n, then there is an induced rational map $L/G_n : \mathcal{T}/G_n \to \mathcal{T}/G_n$ from the quotient surface to itself. This rational map has degree $|a|^2$, its Julia set is $\mathcal{T}/G_n$ and its postcritical set is the set of critical values for the projection $\mathcal{T} \to \mathcal{T}/G_n$.

A rational map of degree $D \geq 2$ is said to be a *Lattès example* if it is conformally conjugate to a map of the form $L/G_n : \mathcal{T}/G_n \to \mathcal{T}/G_n$. We say that f is an *integral Lattès example* if $D = d^2$ with $d \geq 2$ an integer and $L(z) = az + b$ with $a = \pm d$. By definition, for such a map f there is a discrete subgroup $\Sigma \subset \mathrm{Isom}(\mathbb{C})$ and an affine map $L : z \mapsto az + b$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{L} & \mathbb{C} \\ \downarrow & & \downarrow \\ \mathbb{C}/\Sigma \simeq \mathbb{P}^1 & \xrightarrow[f]{} & \mathbb{P}^1 \simeq \mathbb{C}/\Sigma \end{array}$$

The subgroup $\Lambda \subset \Sigma$ of translations is a normal subgroup of rank two, Σ is an extension of Λ of degree n and $\Sigma/\Lambda \simeq G_n$ acts on the torus $\mathcal{T} = \mathbb{C}/\Lambda$ by rotation around a base point.

Note that if $\Gamma_1 \subset \Lambda$ is a subgroup of rank one then the projection $\mathbb{C} \to \mathbb{C}/\Sigma$ factors as $\mathbb{C} \to \mathbb{C}/\Gamma_1 \to \mathbb{C}/\Sigma$. If f is an integral Lattès example, so that $L(z) = az + b$ with $a \in \mathbb{Z} \setminus \{-1, 0, 1\}$, then we have the following commutative diagram:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{L} & \mathbb{C} \\ \downarrow & & \downarrow \\ \mathbb{C}^* \simeq \mathbb{C}/\Gamma_1 & \xrightarrow{f_1} & \mathbb{C}/\Gamma_1 \simeq \mathbb{C}^* \\ \Psi_1 \downarrow & & \downarrow \Psi_1 \\ \mathbb{P}^1 \simeq \mathbb{C}/\Sigma & \xrightarrow[f]{} & \mathbb{C}/\Sigma \simeq \mathbb{P}^1 \end{array}$$

Here f_1 is a power map analytically conjugate to $z \mapsto z^a$. This construction shows the existence of transcendental semiconjugacies between power maps and integral Lattès examples.

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Note that the subgroup Λ of Σ has index n. If n is even, necessarily 2, 4 or 6, then Γ_1 is an index 2 subgroup of some $\Gamma_2 \subset \Sigma$. The projection $\mathbb{C} \to \mathbb{C}/\Sigma$ factors as $\mathbb{C} \to \mathbb{C}/\Gamma_2 \to \mathbb{C}/\Sigma$. In this situation, if f is an integral Lattès example, so that $L(z) = az + b$ with $a \in \mathbb{Z} \setminus \{-1, 0, 1\}$, then we have the following commutative diagram:

$$\begin{array}{ccc}
\mathbb{C} & \xrightarrow{L} & \mathbb{C} \\
\downarrow & & \downarrow \\
\mathbb{C} \simeq \mathbb{C}/\Gamma_2 & \xrightarrow{f_2} & \mathbb{C}/\Gamma_2 \simeq \mathbb{C} \\
{\scriptstyle \Psi_2}\downarrow & & \downarrow{\scriptstyle \Psi_2} \\
\mathbb{P}^1 \simeq \mathbb{C}/\Sigma & \xrightarrow[f]{} & \mathbb{C}/\Sigma \simeq \mathbb{P}^1
\end{array}$$

Here f_2 is a Chebyshev map. If G_n acts on $\mathbb{C}/\Lambda$ by rotation around a fixed point of L, then f_2 is conjugate to $\mathcal{T}_d$ with $d = |a|$. This is always the case when a is odd. Otherwise, f_2 is conjugate to $-\mathcal{T}_d$. This construction shows the existence of transcendental semiconjugacies between Chebyshev maps and integral Lattès examples for which Λ has even index in the extension Σ.

Note that if Ψ is a semiconjugacy between two rational maps f and g, then $\Psi \circ f^{\circ k}$ is another such semiconjugacy, for any positive integer k. This observation relates to the above discussion as follows.

If $\Gamma_1 \subset \Gamma_1'$ are rank-one subgroups of Λ, then $\Gamma_1'/\Gamma_1 \simeq \mathbb{Z}/k\mathbb{Z}$ for some positive integer k. The following commutative diagram shows that the semiconjugacy Ψ_1 associated to Γ_1 may be viewed as $\Psi_1' \circ f_1^{\circ k}$, where Ψ_1' is the semiconjugacy associated to Γ_1':

$$\begin{array}{ccc}
\mathbb{C} & \xrightarrow{L} & \mathbb{C} \\
\downarrow & & \downarrow \\
\mathbb{C}/\Gamma_1 & \xrightarrow{f_1} & \mathbb{C}/\Gamma_1 \\
{\scriptstyle f_1^{\circ k}}\downarrow & & \downarrow{\scriptstyle f_1^{\circ k}} \\
\mathbb{C}/\Gamma_1' \simeq \mathbb{C}/\Gamma_1 & \xrightarrow[f_1]{} & \mathbb{C}/\Gamma_1 \simeq \mathbb{C}/\Gamma_1' \\
{\scriptstyle \Psi_1'}\downarrow & & \downarrow{\scriptstyle \Psi_1'} \\
\mathbb{C}/\Sigma & \xrightarrow[f]{} & \mathbb{C}/\Sigma
\end{array}$$

Similarly, if Σ contains an involution and if $\Gamma_2 \subset \Gamma_2' \subset \Sigma$ are subgroups such that Γ_1 has index 2 in Γ_2 and Γ_1' has index 2 in Γ_2', then the semiconjugacy Ψ_2 associated to Γ_2 may be viewed as $\Psi_2' \circ f_2^{\circ k}$, where Ψ_2' is the semiconjugacy associated to Γ_2'.

We now present a list of the semiconjugacies which may be obtained through admissible choices of $\Gamma \subset \Sigma \subset \mathrm{Isom}(\mathbb{C})$. In the following section, we will show that up to conjugacies with Möbius transformations, these are the only transcendental semiconjugacies between rational maps of degrees at least two.

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For each Lattès example $f : \mathbb{P}^1 \to \mathbb{P}^1$ there is an associated 'ramification index' function $r : \mathbb{P}^1 \to \mathbb{N}$ which may be characterized as the unique function satisfying the identity

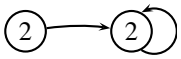
$$r(f(z)) = \deg_f(z) \cdot r(z)$$

for which $r(z) = 1$ outside the postcritical set (see, for example, [**DH**] or [**M2**]). If $f : \mathbb{P}^1 \to \mathbb{P}^1$ is conformally conjugate to the quotient map $L/G_n : \mathcal{T}/G_n \to \mathcal{T}/G_n$ and if z corresponds to $\Theta(\tau_0)$ with $\tau_0 \in \mathcal{T}$ and $\Theta : \mathcal{T} \to \mathcal{T}/G_n \simeq \mathbb{P}^1$ the projection, then $r(z) = \deg_\Theta(\tau_0)$. The possible collections of ramification indices are:

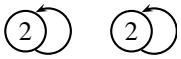
- $\{2, 2, 2, 2\}$ if $n = 2$;
- $\{3, 3, 3\}$ if $n = 3$;
- $\{2, 4, 4\}$ if $n = 4$; and
- $\{2, 3, 6\}$ if $n = 6$.

We organize the integral Lattès examples into four classes according to which group of nth roots of unity acts on the torus $\mathcal{T} = \mathbb{C}/\Lambda$, equivalently on the set of ramification indices. We subdivide each class according to whether the derivative a of the affine map L is even or odd. If a is odd and $n = 2$ or $n = 4$, we may further distinguish the cases in which G_n does, or does not, act on $\mathcal{T} = \mathbb{C}/\Lambda$ by rotation around a fixed point of $L : \mathcal{T} \to \mathcal{T}$. Finally, in each case, we may describe the postcritical dynamics of f in terms of the ramification indices: for example, $2 \mapsto 6$ means that a point with ramification index 2 is mapped to a point with ramification index 6. In all cases besides $\{3, 3, 3\}$, replacing L by $-L$ yields the same Lattès example. We obtain the results in Table 1.

We use the notation $d = |a|$. Recall that when a is even, Ψ_d is conjugate to $-\Psi_d$ and the dynamics in the postcritical set is



When a is odd we need to distinguish two cases: for Ψ_d, the dynamics in the postcritical set of Ψ_d is



and for $-\Psi_d$, the dynamics in the postcritical set of $-\Psi_d$ is



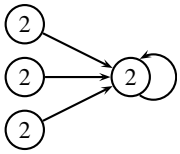
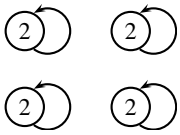
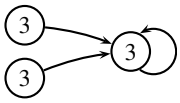
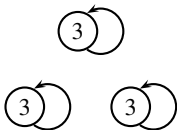
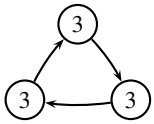
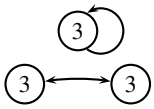
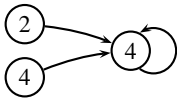
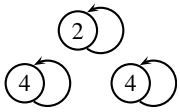
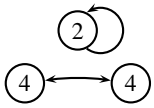
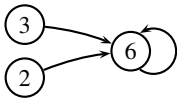
In Table 2, we list the power and Chebyshev maps which are semiconjugate to the various Lattès examples.

6. *Transcendental semiconjugacies*

For the remainder of this paper, we work under the assumptions of Theorem 2: that is, we assume that $\Psi : \mathbb{P}^1 \setminus \mathcal{E}_1 \to \mathbb{P}^1$ is a transcendental semiconjugacy from f_1 to f_2. In particular, the exceptional set $\mathcal{E}_1$ is non-empty, and at least one point in $\mathcal{E}_1$ is an essential

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TABLE 1. A classification of Lattès examples.

Ramification indices	G_n acts on $\mathcal{T}$ by rotation around a fixed point of L	G_n does not act on $\mathcal{T}$ by rotation around a fixed point of L
$\{2, 2, 2, 2\}$ a even		No Lattès example
$\{2, 2, 2, 2\}$ a odd		 
$\{3, 3, 3\}$ $a = 0 \mod 3$		No Lattès example
$\{3, 3, 3\}$ $a = 1 \mod 3$		
$\{3, 3, 3\}$ $a = 2 \mod 3$		No Lattès Example
$\{2, 4, 4\}$ a even		No Lattès example
$\{2, 4, 4\}$ a odd		
$\{2, 3, 6\}$ $a = 0 \mod 6$		No Lattès example

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TABLE 1. Continued.

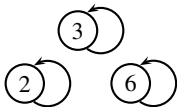
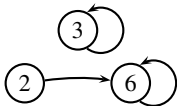
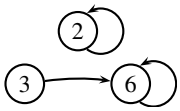
Ramification indices	G_n acts on $\mathcal{T}$ by rotation around a fixed point of L	G_n does not act on $\mathcal{T}$ by rotation around a fixed point of L
$\{2, 3, 6\}$ $a = 1 \mod 6$ $a = 5 \mod 6$		No Lattès example
$\{2, 3, 6\}$ $a = 2 \mod 6$ $a = 4 \mod 6$		No Lattès example
$\{2, 3, 6\}$ $a = 3 \mod 6$		No Lattès example

TABLE 2. Power maps and Chebyshev maps conjugate to Lattès examples.

Ramification indices	G_n acts on $\mathcal{T}$ by rotation around a fixed point of L	G_n does not act on $\mathcal{T}$ by rotation around a fixed point of L
$\{2, 2, 2, 2\}$	$p_d, p_{-d}, \mathcal{T}_d$	$p_d, p_{-d}, -\mathcal{T}_d$
$\{3, 3, 3\}$ $a > 0$	p_d	p_d
$\{3, 3, 3\}$ $a < 0$	p_{-d}	p_{-d}
$\{2, 4, 4\}$	$p_d, p_{-d}, \mathcal{T}_d$	$p_d, p_{-d}, -\mathcal{T}_d$
$\{2, 3, 6\}$	$p_d, p_{-d}, \mathcal{T}_d$	No Lattès example

singularity of Ψ. We will first consider the case where f_1 is a polynomial and Ψ has an essential singularity at infinity. We will then deduce the general case.

Henceforth we write p for f_1 and F for f_2. We let d be the degree of p and D the degree of F. We denote by K_p the filled-in Julia set of p, and by J_F the Julia set of F. Consider the Green's function G_p associated to p, namely

$$G_p(z) = \lim_{n\to\infty} \frac{1}{d^n} \log^+ |p^{\circ n}(z)|.$$

Recall that if η is greater than

$$\eta_p = \sup_{p'(c)=0} G(c)$$

then the corresponding equipotential

$$\{z \in \mathbb{C} : G_p(z) = \eta\}$$

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is a Jordan curve: indeed, the normalized Böttcher coordinate bot_p extends analytically and univalently to the set Ω_p where G_p exceeds η_p, and there we have $G_p = \log|\mathrm{bot}_p|$.

Let $\mathbb{H}_p$ be the right half-plane

$$\mathbb{H}_p := \{z \in \mathbb{C} \,:\, \mathrm{Re}(z) > \eta_p\}$$

and define $\Phi : \mathbb{H}_p \to \mathbb{P}^1$ by

$$\Phi := \Psi \circ \mathrm{bot}_p^{-1} \circ \exp.$$

The map $\mathrm{bot}_p^{-1} \circ \exp : \mathbb{H}_p \to \Omega_p$ is a universal covering. For all $z \in \mathbb{H}_p$,

$$\Phi(z + 2i\pi) = \Phi(z) \quad \text{and} \quad \Phi(d \cdot z) = F(\Phi(z)).$$

In particular, for any integer k, we have the following equality:

$$F^{\circ k}(\Phi(z)) = F^{\circ k}(\Phi(z + 2i\pi d^{-k})).$$

PROPOSITION 2. *Assume that $z_1 \in \mathbb{H}_p$ and $z_2 \in \mathbb{H}_p$ are not critical points of Φ. Then there exists an affine map $A : z \mapsto az + b$ which sends z_1 to z_2 and such that $\Phi \circ A = \Phi$ in a neighborhood of z_1.*

Proof. Since Φ is locally univalent at z_1 and z_2, there exist a neighborhood U of $\Phi(z_1) = \Phi(z_2)$, and neighborhoods $U_1 \ni z_1$ and $U_2 \ni z_2$, such that the restrictions $\Phi : U_1 \to U$ and $\Phi : U_2 \to U$ are homeomorphisms. We wish to show that the homeomorphism

$$A := (\Phi|_{U_2})^{-1} \circ (\Phi|_{U_1}) : U_1 \to U_2$$

is an affine map.

Consider the following vector fields defined on U:

$$X_1 := (\Phi|_{U_1})_* \frac{1}{2i\pi} \frac{d}{dz} \quad \text{and} \quad X_2 := (\Phi|_{U_2})_* \frac{1}{2i\pi} \frac{d}{dz}.$$

If

$$X_1 = \xi_1(z) \frac{d}{dz} \quad \text{and} \quad X_2 = \xi_2(z) \frac{d}{dz}$$

then

$$[X_1, X_2] = (\xi_1(z)\xi_2'(z) - \xi_1'(z)\xi_2(z)) \frac{d}{dz}.$$

The above commutator vanishes if and only if X_1 and X_2 are equal up to multiplication by a constant, in which case the derivative of A is constant. Thus, A is an affine map if and only if X_1 and X_2 commute.

Given any open set V compactly contained in U, if ε is sufficiently small, the flows $\phi_1(t, z)$ of X_1 and $\phi_2(t, z)$ of X_2 are defined on $[-\varepsilon, \varepsilon] \times V$ and the maps

$$\exp(tX_1) : z \mapsto \phi_1(t, z) \quad \text{and} \quad \exp(tX_2) : z \mapsto \phi_2(t, z)$$

are univalent on V.

Given univalent maps $f : U \to \mathbb{C}$ and $g : U \to \mathbb{C}$, we will denote by $\langle f, g \rangle$ the map $g^{-1} \circ f^{-1} \circ g \circ f$ where defined.

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LEMMA 6. *Assume that f and g are analytic in $[-\varepsilon, \varepsilon] \times V$ for some $\varepsilon > 0$ and some open set $V \subset \mathbb{C}$, with*

$$f_s(z) := f(s, z) = z + s \cdot \xi_1(z) + o(s) \quad \text{and} \quad g_t(z) := g(t, z) = z + t \cdot \xi_2(z) + o(t)$$

for some holomorphic vector fields $\xi_1(z)\, d/dz$ and $\xi_2(z)\, d/dz$. Then

$$\langle f_s, g_t \rangle(z) = z + st \cdot \xi_3(z) + o(st)$$

where $\xi_3(z)\, d/dz = [\xi_1(z)\, d/dz, \xi_2(z)\, d/dz]$.

Proof. Note that $\langle f_s, g_t \rangle - \mathrm{Id}$ depends analytically on s and t and identically vanishes when $s = 0$ or $t = 0$. Thus,

$$\langle f_s, g_t \rangle(z) = z + st \cdot \xi_3(z) + o(st)$$

for some holomorphic vector field ξ_3. We have

$$\begin{aligned}
g_t \circ f_s(z) &= z + s \cdot \xi_1(z) + t \cdot \xi_2(z) + st \cdot \xi_2'(z)\xi_1(z) + o(s) + o(t), \\
f_s^{-1} \circ g_t \circ f_s(z) &= z + s \cdot \xi_1(z) + t \cdot \xi_2(z) + st \cdot \xi_2'(z)\xi_1(z) \\
&\quad - s \cdot \xi_1(s) - st \cdot \xi_1'(z)\xi_2(z) + o(s) + o(t) \\
&= z + t \cdot \xi_2(z) + st \cdot \xi_2'(z)\xi_1(z) - st \cdot \xi_1'(z)\xi_2(z) + o(s) + o(t) \\
\langle f_s, g_t \rangle(z) &= z + t \cdot \xi_2(z) + st \cdot \xi_2'(z)\xi_1(z) - st \cdot \xi_1'(z)\xi_2(z) \\
&\quad - t\xi_2(z) + o(s) + o(t) \\
&= z + st \cdot (\xi_1(z)\xi_2'(z) - \xi_1'(z)\xi_2(z)) + o(s) + o(t).
\end{aligned}$$

It follows that

$$st \cdot (\xi_1(z)\xi_2'(z) - \xi_1'(z)\xi_2(z)) - st \cdot \xi_3 = o(s) + o(t)$$

and hence $\xi_3(z)\, d/dz = [\xi_1(z)\, d/dz, \xi_2(z)\, d/dz]$. □

Given $i_1, \ldots, i_n \in \{1, 2\}$, if ε is sufficiently small, the map ψ defined by

$$\psi(t_1, \ldots, t_n, z) := \langle\langle \exp(t_1 X_{i_1}), \exp(t_2 X_{i_2}) \rangle, \ldots \rangle, \exp(t_n X_{i_n}) \rangle(z)$$

is defined and analytic on $[-\varepsilon, \varepsilon]^n \times V$. It follows from the previous lemma and by induction on n that

$$\psi(t_1, \ldots, t_n, z) = z + t_1 t_2 \cdots t_n \cdot \upsilon_n(z) + o(t_1 t_2 \cdots t_n)$$

where

$$\upsilon_n(z) \frac{d}{dz} = [[[X_{i_1}, X_{i_2}], \ldots], X_{i_n}] = Y_n.$$

We will denote by ψ_t the map $z \mapsto \psi(t, \ldots, t, z)$. If t is sufficiently small, then ψ_t is analytic and univalent on V. Moreover, we have

$$\psi_t(z) = z + t^n \cdot \upsilon_n(z) + o(t^n).$$

LEMMA 7. *If Y_n does not identically vanish on U, then there exist $c > 0$ and $z_0 \in U$ such that, for all sufficiently small t, the orbit of z_0 under forward iteration of ψ_t contains at least c/t^n points.*

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Proof. Choose a point z_0 which is not a zero of Y_n and work in a local coordinate around z_0 in which $Y_n = d/dz$. If $r > 0$ and $\varepsilon > 0$ are sufficiently small, then for all $t \in [-\varepsilon, \varepsilon]$ the map ψ_t is defined and univalent on $D(z_0, r)$, with

$$|\psi_t(z) - z - t^n| \leq \tfrac{1}{2}t^n.$$

Since the image of any point $z \in D(z_0, r)$ lies strictly to the right of z, the orbit of z_0 consists of distinct points. Moreover, it follows by induction that for $j \leq 2r/3t^n$ we have

$$\psi_t^{\circ j}(z_0) \in D(z_0 + jt^n, jt^n/2) \subseteq D(z_0, r)$$

and hence we may take $c = 2r/3$. □

LEMMA 8. *If k is sufficiently large, then $F^{\circ k} \circ \psi_{d^{-k}} = F^{\circ k}$ on V.*

Proof. We claim that if W is compactly contained in U, then for all sufficiently large k we have $F^{\circ k} \circ \phi_j(\pm d^{-k}, z) = F^{\circ k}(z)$ for $j = 1, 2$ and any $z \in W$. Indeed, if $\Phi(z_1) = z = \Phi(z_2)$ where $z_1 \in U_1$ and $z_2 \in U_2$, then we have

$$F^{\circ k} \circ \Phi(z_j \pm 2i\pi d^{-k}) = F^{\circ k} \circ \Phi(z_j) = F^{\circ k}(z)$$

for $j = 1, 2$ and every positive integer k, so if k is large enough so that $\phi_j(\pm d^{-k}, z)$ is defined, then we have

$$F^{\circ k} \circ \phi_j(\pm d^{-k}, z) = F^{\circ k} \circ \Phi(z_j \pm 2i\pi d^{-k}) = F^{\circ k}(z).$$

The assertion now follows by a straightforward induction on the size of the relevant commutator. □

Consequently, if Y_n does not identically vanish on U, then for large enough k we may find $c \cdot d^{kn}$ distinct points having the same image by $F^{\circ k}$, and hence $c \cdot d^{kn} \leq D^k$, and thus

$$d^n \leq D \quad \text{and} \quad n \leq \frac{\log D}{\log d}.$$

LEMMA 9. *If the vector fields X_1 and X_2 do not commute, then for every integer $n \geq 2$ there exist $i_1, \ldots, i_n \in \{1, 2\}$ such that*

$$Y_n := [[[X_{i_1}, X_{i_2}], \ldots], X_{i_n}]$$

does not identically vanish on U.

Proof. We will prove that we can find such a commutator by induction on n. The property holds for $n = 2$ by the assumption that X_1 and X_2 do not commute. Let us assume that Y_n has already been produced. Now Y_n cannot commute with both X_1 and X_2, since otherwise we would have $Y_n = c_i X_i$ for some $c_i \neq 0$, in which case X_1 and X_2 would commute. If Y_n does not commute with X_1 we choose $Y_{n+1} = [Y_n, X_1]$, and otherwise we choose $Y_{n+1} = [Y_n, X_2]$. □

We thereby deduce that X_1 and X_2 commute, and hence A is an affine map. This completes the proof of the proposition. □

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PROPOSITION 3. *The map* $\Phi : \mathbb{H}_p \to \mathbb{P}^1$ *admits an analytic extension* $\widehat{\Phi} : \mathbb{C} \to \mathbb{P}^1$. *Moreover,* $\widehat{\Phi}(z_1) = \widehat{\Phi}(z_2)$ *if and only if there is an isometry* $A : \mathbb{C} \to \mathbb{C}$ *sending* z_1 *to* z_2 *such that* $\widehat{\Phi} \circ A = \widehat{\Phi}$.

Proof. Since Ψ has an essential singularity at infinity, there are at most two values which are omitted near infinity. Since Φ has isolated critical points in $\mathbb{H}_p$, the set of critical values is countable. Thus, we may find a point $w \in \mathbb{P}^1$ which is not a critical value of Φ, and points $z_1, z_2 \in \mathbb{H}_p$ with $\mathrm{Re}(z_2) > \mathrm{Re}(z_1)$ and $\Phi(z_1) = w = \Phi(z_2)$. In view of Proposition 2, there exists an affine map $A : \mathbb{C} \to \mathbb{C}$ sending z_1 to z_2 such that $\Phi \circ A = \Phi$ in a neighborhood of z_1.

Note that $V = A^{-1}(\mathbb{H}_p) \cap \mathbb{H}_p$ is non-empty, since it contains z_1, and is also connected: indeed, V is either a sector (if $a \notin \mathbb{R}^-$) or a strip (if $a \in \mathbb{R}^-$). Since $\Phi \circ A = \Phi$ near z_1, it follows by uniqueness of analytic continuation that $\Phi \circ A = \Phi$ on V. There are various cases to consider.

(1) If A is a translation then, since $\mathrm{Re}(z_2) > \mathrm{Re}(z_1)$, the orbit of any point in $\mathbb{C}$ under forward iteration of A eventually lies in $\mathbb{H}_p$. Thus, Φ extends to $\mathbb{C}$ by the formula $\Phi(z) = \Phi \circ A^{\circ n}(z)$, where n is chosen sufficiently large so that $A^{\circ n}(z) \in \mathbb{H}_p$.

(2) If $A = az + b$ with $a \neq 1$ then A has a fixed point $\alpha = b/(1-a)$.

(i) If $\alpha \in \mathbb{H}_p$ then $\alpha \in V$, and hence $\Phi \circ A = \Phi$ in a neighborhood of α. Exchanging z_1 and z_2 and replacing A by A^{-1} if necessary, we may assume without loss of generality that $|a| \leq 1$, so that V contains the forward orbit under A of any point sufficiently close to α. Since Φ is not constant, such an orbit is necessarily finite, and hence a is a root of unity and A is a rational rotation. Consequently, the orbit of any point in $\mathbb{C}$ must intersect $\mathbb{H}_p$, and hence Φ extends to $\mathbb{C}$ by the formula $\Phi(z) = \Phi \circ A^{\circ n}(z)$, where n is chosen so that $A^{\circ n}(z) \in \mathbb{H}_p$.

(ii) If $a < 0$ then the fixed point α lies in the segment $[z_1, z_2] \subset \mathbb{H}_p$. It follows from (i) that $a = -1$, and hence Φ extends to $\mathbb{C}$.

(iii) If $\alpha \notin \mathbb{H}_p$ and $a > 0$ then since $\mathrm{Re}(z_2) > \mathrm{Re}(z_1)$ we must have $a > 1$, so $A(\mathbb{H}_p) \subseteq \mathbb{H}_p$; hence $V = \mathbb{H}_p$, and thus $\Phi \circ A = \Phi$ on $\mathbb{H}_p$. It follows that Φ takes the same value at z and the points

$$A^{-k}(A^k(z) + 2i\pi) = z + \frac{2i\pi}{a^k}$$

for positive integers k. Since these points accumulate in $\mathbb{H}_p$, and since Φ is not constant, this case cannot occur.

(iv) If $\alpha \notin \mathbb{H}_p$ and $a \notin \mathbb{R}$ then we may replace z_2 by $z_2 + 2i\pi k$ and A by $A_k = A + 2i\pi k$, so that the fixed point becomes

$$\alpha_k = \frac{b + 2i\pi k}{1 - a}.$$

If $a \notin \mathbb{R}$ is and $-k \cdot \mathrm{Im}(a)$ is sufficiently large, then $\alpha_k \in \mathbb{H}_p$. It follows from (i) that Φ extends to $\mathbb{C}$, and that A is a rational rotation.

We denote by $\widehat{\Phi}$ the analytic extension of Φ to $\mathbb{C}$. If $\widehat{\Phi}(z_1) = w = \widehat{\Phi}(z_2)$ then, by construction of $\widehat{\Phi}$, for $j = 1, 2$ there exist isometries A_j such that $z_j' = A_j(z_j) \in \mathbb{H}_p$ and

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such that $\widehat{\Phi} \circ A_j = \widehat{\Phi}$. Now if there is an isometry $A : \mathbb{C} \to \mathbb{C}$ sending z_1' to z_2' such that $\Phi \circ A = \Phi$, then the isometry $A_2^{-1} \circ A \circ A_1$ sends z_1 to z_2, and

$$\widehat{\Phi} \circ A_2^{-1} \circ A \circ A_1 = \Phi \circ A \circ A_1 = \Phi \circ A_1 = \widehat{\Phi}$$

near z_1, and hence everywhere. Thus, we may assume without loss of generality that z_1 and z_2 belong to $\mathbb{H}_p$.

In any neighborhood of w, we may find a point w^* which is not a critical value of Φ and which has a preimage $z_1^* \in \mathbb{H}_p$ near z_1 and a preimage $z_2^* \in \mathbb{H}_p$ near z_2. Moreover, if w^* is sufficiently close to w then we may arrange that z_1 is the preimage of w closest to z_1^*, and that z_2 is the preimage of w closest to z_2^*. Since neither z_1^* nor z_2^* are critical points of Φ, there exists an isometry A sending z_1^* to z_2^* such that $\Phi \circ A = \Phi$. This isometry permutes the preimages of w, and sends the one closest to z_1^* to the one closest to z_2^*. Thus, we have $A(z_1) = z_2$, and hence A is the required isometry. □

Let Σ be the group of isometries $A : \mathbb{C} \to \mathbb{C}$ such that $\Phi \circ A = \Phi$. Each such isometry acts as a permutation of $\Phi^{-1}(w)$, for any $w \in \mathbb{P}^1$; note that if such a set is non-empty it is necessarily infinite, as Φ commutes with translation by $2i\pi$. Since the preimage of any point $w \in \mathbb{P}^1$ is a discrete set, and since any point $z \in \mathbb{C}$ is fixed by only finitely many elements of Σ, it follows that Σ is a discrete group. We denote by $\pi : \mathbb{C} \to \mathbb{C}/\Sigma$ the canonical projection from $\mathbb{C}$ to the orbifold $\mathbb{C}/\Sigma$.

Since for any points z_1 and z_2 such that $\Phi(z_1) = \Phi(z_2)$ there exists $A \in \Sigma$ sending z_1 to z_2, it follows that $\widehat{\Phi} : \mathbb{C} \to \mathbb{P}^1$ induces an injective map $\iota : \mathbb{C}/\Sigma \to \mathbb{P}^1$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{C} & & \\ {\scriptstyle\pi}\downarrow & \searrow^{\widehat{\Phi}} & \\ \mathbb{C}/\Sigma & \xrightarrow[\iota]{} & \mathbb{P}^1 \end{array}$$

Moreover, since $\Phi(d \cdot z) = F(\Phi(z))$ on $\mathbb{H}_p$, we also have the following commutative diagram:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{z \mapsto d \cdot z} & \mathbb{C} \\ {\scriptstyle \iota\circ\pi}\downarrow & & \downarrow{\scriptstyle \iota\circ\pi} \\ \mathbb{C}/\Sigma \simeq \mathbb{P}^1 & \xrightarrow[F]{} & \mathbb{P}^1 \simeq \mathbb{C}/\Sigma \end{array}$$

PROPOSITION 4. *The rational map F is a Lattès example of degree $D = d^2$ induced by multiplication by d on a torus.*

Proof. We have seen that Σ is a discrete group of isometries of $\mathbb{C}$. Let $\Lambda \subseteq \Sigma$ be the subgroup consisting of translations. Note that Λ contains the translation by $2i\pi$. We claim that Λ has rank two. Indeed, if Σ were of rank one then:

- either σ would be generated by $z \mapsto z + 2i\pi$, and each point $w \in \mathbb{P}^1$ would have at most one preimage by Φ modulo $2i\pi$;
- or σ would be generated by $z \mapsto z + 2i\pi$ and some order two rotation, and each point $w \in \mathbb{P}^1$ would have at most two preimages by Φ modulo $2i\pi$.

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However, using Picard's theorem we have seen that there are points $w \in \mathbb{P}^1$ whose preimages under Φ have arbitrarily large real parts.

Since Σ has rank two, it follows that $\mathbb{C}/\Lambda$ is a torus $\mathcal{T}$. One may show (see, for example, [**M2**]) that $G = \Sigma/\Lambda$ is a cyclic group of order $n \in \{2, 3, 4, 6\}$, which acts on $\mathcal{T}$ by rotation around a base point. Multiplication by d in $\mathbb{C}$ induces an affine map $L : \mathcal{T} \to \mathcal{T}$ which commutes with the action of G. Moreover, there is an analytic map $\Theta : \mathcal{T} \to \mathbb{P}^1$, necessarily of finite degree, such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{L} & \mathcal{T} \\ {\scriptstyle\Theta}\downarrow & & \downarrow{\scriptstyle\Theta} \\ \mathcal{T}/G \simeq \mathbb{P}^1 & \xrightarrow[F]{} & \mathbb{P}^1 \simeq \mathcal{T}/G \end{array}$$

It follows that F is a Lattès example. Since F is induced by multiplication by d on the torus $\mathcal{T}$, the degree of F is $D = d^2$. □

PROPOSITION 5. *The polynomial p is conjugate to either $z \mapsto z^d$ or to $\pm\Psi_d$. Moreover, if p is conjugate to $\pm\Psi_d$ then $n \neq 3$.*

Proof. We may assume without loss of generality that p is not conjugate to a power map, and hence Ψ is defined on $\mathbb{C}$, and every point $z \in \mathbb{C}$ has infinite backward orbit under p.

Consider the ramification function $r_F : \mathbb{P}^1 \to \mathbb{N}$ associated to the Lattès map F. Recall that r_F takes the value 1 outside of the postcritical set, and that

$$r_F(F(w)) = \deg_F(w) \cdot r_F(w)$$

for every $w \in \mathbb{P}^1$. These properties determine r_F uniquely (for details, see [**DH**], [**M**] or [**M2**]). Note further that

$$r_F(\widehat{\Phi}(z)) = \deg_{\widehat{\Phi}}(w)$$

for every $z \in \mathbb{C}$. Now consider the function $r_p : \mathbb{C} \to \mathbb{Q}$ given by

$$r_p(z) = \frac{r_F(\Psi(z))}{\deg_\Psi(z)}.$$

Since $F \circ \Psi = \Psi \circ p$, we have

$$\deg_F(\Psi(z)) \cdot \deg_\Psi(z) \cdot r_p(z) = r_F(F \circ \Psi(z)) = r_F(\Psi \circ p(z)) = \deg_\Psi(p(z)) \cdot r_p(p(z))$$

and

$$\deg_F(\Psi(z)) \cdot \deg_\Psi(z) = \deg_\Psi(p(z)) \cdot \deg_p(z),$$

and hence

$$r_p(p(z)) = \deg_p(z) \cdot r_p(z). \tag{1}$$

Note that if $r_p(z) \neq 1$ then $r_F(\Psi(z)) \neq 1$ or $\deg_\Psi(z) \neq 1$. Thus, the points where r_p differs from 1 are either critical points of Ψ or preimages of the finitely many points where r_F differs from 1; note that any bounded region contains only finitely many such points. Since any point $z \in \mathbb{C}$ has infinite backward orbit under p, there exist backward orbit points where r_p takes the value 1. It follows that r_p is an integer-valued function.

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The map $\mathrm{bot}_p^{-1} \circ \exp$ is locally univalent near any $z \in \mathbb{H}_p$, so if $w = \mathrm{bot}_p^{-1} \circ \exp(z)$ then

$$r_F(\Psi(z)) = r_F(\Phi(z)) = \deg_\Phi(z) = \deg_\Psi(w).$$

It follows that r_p is identically 1 in the region where G_p exceeds η_p. Consequently, the set $S = \{z \in \mathbb{C} : r_p(z) > 1\}$ is finite. Note that S contains the postcritical set $\mathcal{P}_p \subset \mathbb{C}$ of p, by equation (1) and the fact that r_p is integer valued. Thus, p is postcritically finite. Moreover, as the only way to have $r_p(z) = 1$ and $r_p(p(z)) > 1$ is for z to be a critical point of p, we in fact have $S = \mathcal{P}_p$, and furthermore

$$p^{-1}(\mathcal{P}_p) = \mathcal{P}_p \cup \mathcal{C}_p$$

where $\mathcal{C}_p \subset \mathbb{C}$ denotes the set of (finite) critical points of p. Applying the Riemann–Hurwitz formula to $p : \mathbb{C} \setminus (\mathcal{P}_p \cup \mathcal{C}_p) \to \mathbb{C} \setminus \mathcal{P}_p$, we deduce that

$$1 - \mathrm{card}(\mathcal{P}_p \cup \mathcal{C}_p) = d \cdot (1 - \mathrm{card}(\mathcal{P}_p)). \tag{2}$$

Since p has $d - 1$ critical points counting multiplicity, we have $k \leq d - 1$ where k is the number not counting multiplicity. Setting $\ell = \mathrm{card}(\mathcal{P}_p) \geq 1$, we deduce from equation (2) that $1 - (\ell + m) \leq d \cdot (1 - \ell)$, and hence $(d - 1)(\ell - 1) \leq m \leq d - 1$. Thus, we must have $\ell \leq 2$.

If $\ell = 1$, equation (2) implies that $\mathrm{card}(\mathcal{P}_p \cup \mathcal{C}_p) = 1$, and hence p has only one critical point, and this point is fixed by p. But then p is conjugate to $z \mapsto z^d$, contrary to assumption. Thus, we must have $\ell = 2$, and hence equation (2) implies that $\mathrm{card}(\mathcal{P}_p \cup \mathcal{C}_p) = d + 1$. It follows that p has $d - 1$ simple critical points, none of which is postcritical. Conjugating with an affine map if necessary, we may assume that $\mathcal{P}_p = \{-2, 2\}$.

Let $\pi : \mathbb{P}^1 \to \mathbb{P}^1$ be the degree-two ramified covering given by

$$\pi(z) = z + \frac{1}{z}.$$

Note that π has critical points at -1 and 1, with images -2 and 2, and that $\pi : \mathbb{P}^1 \setminus \{-1, 1\} \to \mathbb{P}^1 \setminus \{-2, 2\}$ is a covering map of degree two. The map $p \circ \pi : \mathbb{P}^1 \to \mathbb{P}^1$ has simple critical points at all the preimages of $+1$ and -1, along with $d-1$ critical points at 0 and $d - 1$ critical points at ∞. It follows that there exists a (unique) lift $\tilde{p}$ fixing ∞ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{P}^1 & \xrightarrow{\tilde{p}} & \mathbb{P}^1 \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ \mathbb{P}^1 & \xrightarrow[p]{} & \mathbb{P}^1 \end{array}$$

The degree d rational map $\tilde{p} : \mathbb{P}^1 \to \mathbb{P}^1$ has $d - 1$ critical points at the fixed point ∞, and hence $\tilde{p}$ polynomial. Since the preimages of 0 by $\tilde{p}$ are preimages of ∞ by $p \circ \pi$, we deduce that $\tilde{p}^{-1}(0) = \{0\}$, and hence we must have $\tilde{p}(z) = az^d$ for some $a \in \mathbb{C}^*$. Finally, since $\pi(\tilde{p}(1)) = p \circ \pi(1) \in \mathcal{P}_p = \{-2, 2\}$, it follows that $\tilde{p}(1) = \pm 1$, and hence $\tilde{p}(z) = \pm z^d$. If $\tilde{p}(z) = z^d$, then $p = \curlyvee_d$, and otherwise $p = -\curlyvee_d$.

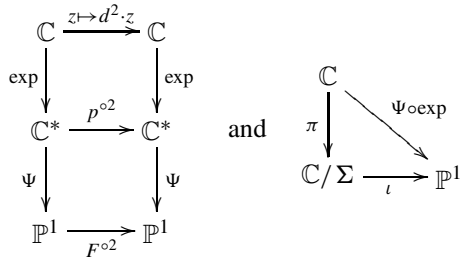
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To complete the proof of the proposition, note than if p is conjugate to $\pm\Psi_d$ then $r_p(\pm 2) = 2$, and hence $r_F(\Psi(\pm 2))$ must be even. On the other hand, if $n = 3$ then r_F takes the value 1 at all but three points where the value is 3 (see [**M2**]). □

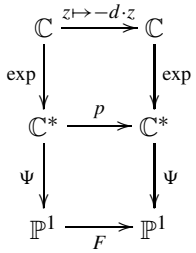
As a corollary of Propositions 4 and 5, we deduce that if f_1 and f_2 are rational maps related by a transcendental semiconjugacy $\Psi : \mathbb{P}^1 \setminus \mathcal{E}_1 \to \mathbb{P}^1$, and if f_1 fixes $\mathcal{E}_1$ pointwise, then f_1 is a power map or a Chebyshev map and f_2 is a Lattès example. To complete the proof of Theorem 2, we must consider the case where f_1 exchanges the exceptional points: note that up to Möbius conjugacy we then have $f_1(z) = z^{-d}$ for some integer $d \geq 2$.

PROPOSITION 6. *Let F be a rational map. If there exists a transcendental semiconjugacy $\Psi : \mathbb{C}^* \to \mathbb{P}^1$ between the power map $z \mapsto z^{-d}$ and F, then F is a Lattès example induced by an affine map with derivative $-d$.*

Proof. Let p be the power map $z \mapsto z^{-d}$. Since Ψ is a transcendental semiconjugacy between $p^{\circ 2}$ and $F^{\circ 2}$, we have the following commutative diagrams:



Here Σ is a discrete group of isometries of $\mathbb{C}$, the map $\pi : \mathbb{C} \to \mathbb{C}/\Sigma$ is the canonical projection, and the map $\iota : \mathbb{C}/\Sigma \to \mathbb{P}^1$ is an isomorphism. Since $\exp : \mathbb{C} \to \mathbb{C}$ conjugates multiplication by $-d$ to p and since Ψ semiconjugates p to F, we also have the following commutative diagram:



It follows that F is a Lattès example induced by an affine map with derivative $-d$. □

PROPOSITION 7. *Let F be a Lattès example of type* $\{3, 3, 3\}$.

- *If there exists a transcendental semiconjugacy between $z \mapsto z^d$ and F, then F is not induced by an affine map with negative derivative.*
- *If there exists a transcendental semiconjugacy between $z \mapsto z^{-d}$ and F, then F is not induced by an affine map with positive derivative.*

Proof. Let a be the derivative of an affine map inducing F. We claim that the fixed point multipliers of F belong to the set $\{a, e^{2i\pi/3}a, e^{-2i\pi/3}a, a^3\}$. Indeed, if $z \in \mathbb{P}^1$ is a fixed

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point of F and $\tau \in \mathbb{C}$ lies above z, then $L(\tau)$ also lies above z. Thus, we must have $\sigma \circ L(\tau) = \tau$ for some isometry $\sigma \in \Sigma$. The derivative of $\sigma \circ L$ at τ necessarily belongs to the set $\{a, e^{2i\pi/3}a, e^{-2i\pi/3}a\}$. Now the projection $\mathbb{C} \to \mathbb{C}/\Sigma$ semiconjugates $\sigma \circ L$ to F. If τ is not a critical point of the projection then the multiplier of F at z is equal to the multiplier of $\sigma \circ L$ at τ. On the other hand, if τ is a critical point of the projection, then the projection has local degree three at τ, and hence the multiplier of F at z is a^3.

Now if $z \mapsto z^\delta$ is semiconjugate to F, where $\delta = \pm d$, then $|a| = d = |\delta|$. Moreover, any fixed point $z \in \mathbb{C}^*$ of the power map has multiplier δ. The image under the semiconjugacy is a fixed point of F with multiplier δ^m, where m is the local degree of the semiconjugacy at z. Consequently, we must have $a = \delta$. □

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