$s = \frac{1}{2}(u + v)t$ in the forms (a) $x = \frac{1}{2}(\dot{x}_0 + \dot{x})t$ and (b) $t = \dfrac{2x}{\dot{x}_0 + \dot{x}}$. Differentiating (a) gives $\dot{x} = \frac{1}{2}(\dot{x}_0 + \dot{x}) + \frac{1}{2}\ddot{x}t$ or $\dot{x} = \dot{x}_0 + \ddot{x}t$ and substituting this back into (a) gives $x = \dot{x}_0 t + \frac{1}{2}\ddot{x}t^2$ and $x = \dot{x}t - \frac{1}{2}\ddot{x}t^2$. And differentiating (b) gives $1 = \dfrac{2\dot{x}(\dot{x}_0 + \dot{x}) - 2x\ddot{x}}{(\dot{x}_0 + \dot{x})^2}$ which simplifies to $\dot{x}^2 = \dot{x}_0^2 + 2\ddot{x}x$.

Finally, and I am very grateful to the referee for pointing this out, the five general differential equations under discussion in this note can all be solved by standard methods. Of course, the results above mean that there are no surprises in the solutions obtained for the initial conditions $\dot{x} = u$ and $\ddot{x} = a$ when $t = 0$, but finding them makes a good exercise. With the numbering above:

1. $t\ddot{x} - \dot{x} = -u$ is a linear differential equation in $\dot{x}$;
2. $2x\ddot{x} - \dot{x}^2 = -u^2$ gives a separable differential equation after the substitutions $p = \dot{x}, \ddot{x} = pp'$;
3. $t\dot{x} - 2x = -ut$ is a linear differential equation in x;
4. $t^2\ddot{x} - 2t\dot{x} + 2x = 0$ is an Euler-type differential equation which can be linearised by the usual substitution $t = e^z$;
5. $t^2\ddot{x} - 2x = -2ut$ is also an Euler-type differential equation which can be solved in the same way as (4).

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A rich problem on digital roots

The digital root function is defined by iterating the 'sums of digits' function until a single digit is reached. For example, the successive digit sums $275 \to 14 \to 5$ show that 275 has digital root 5, written $dr(275) = 5$.

In the course of an introductory lesson on proof a Sixth Form student, Max Chester, asked whether it was true that $dr(12n)$ always divides $12n$. I managed to cobble together a proof and what follows is a more considered analysis of the general problem suggested by this specific case, namely:

Which positive integers k have the property that $dr(kn)$ divides kn for all positive integers n?

The mixture of straightforward and subtle patterns described below makes this a rich problem suitable for investigation by a wide range of age groups. For brevity, we shall call an integer k *good* if it has the property sought and *bad* otherwise. Notice that all factors of bad integers are bad and all multiples of good integers are good. We shall use without proof the following standard properties of the digital root function:

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- $dr(n) = \begin{cases} 9 & \text{if 9 divides } n \\ n \pmod 9 & \text{otherwise} \end{cases}$,
- $dr(dr(n)) = dr(n)$,
- $dr(mn) = dr\big(dr(m)\,dr(n)\big)$,
- $dr(m+n) = dr\big(dr(m) + dr(n)\big)$.

(a) If k is odd and 9 does not divide k, then k is bad.

Since 9 does not divide k, the only possible values for $dr(k)$ are 1, 2, 3, 4, 5, 6, 7, 8.

- If $dr(k) = 1, 2, 3, 4$, then $dr(11k) = 2dr(k)$ takes one of the values 2, 4, 6, 8 and so cannot divide $11k$; thus $n = 11$ shows k to be bad.
- If $dr(k) = 5, 6, 7, 8$, then $dr(7k) = dr\big(7dr(k)\big)$ takes the values 8, 6, 4, 2 and so cannot divide $7k$; thus $n = 7$ shows k to be bad.

(b) All powers of 2 are bad.

Successive powers of 2 give a periodic sequence of digital roots 2, 4, 8, 7, 5, 1 implying that $dr\big(2^{6r+4}\big) = 7$ for all $r \geqslant 0$. To verify this, note that $2^{6r+4} = 16 \times 64^r$ so that $dr\big(2^{6r+4}\big) = dr\big(dr(16) \times dr(64)^r\big) = 7$. Thus $dr\big(2^{6r+4}\big) = 7$ does not divide 2^{6r+4} which suffices to establish (b).

(c) If 9 divides k then k is good.

For if 9 divides k then $dr(kn) = 9$ divides kn for all n.

(d) If 6 divides k then k is good. (This covers our motivating example where $k = 12$.)

In this case, either kn is a multiple of 9 in which case $dr(kn) = 9$ divides kn, or it is a multiple of 3 with $dr(kn) = 3$ or 6 so that certainly $dr(kn)$ divides 6 which in turn divides kn.

(e) If 280 divides k then k is good.

For, in general, $dr(kn) = \begin{cases} 1, 2, 4, 5, 7, 8 & \text{if 3 does not divide } kn \\ 3, 6 & \text{if 3 divides } kn \text{ and 9 does not} \\ 9 & \text{if 9 divides } kn \end{cases}$.

Since 280 is the lowest common multiple of 2, 4, 5, 7, 8, we easily check that $dr(kn)$ divides kn in all cases.

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(f) There are no other good even values of k.

Suppose that $k = 2^r M$ with $r \geqslant 1$, M odd and assume that 3 does not divide M (because the case where 3 divides M is covered by (c), (d) above). Then $dr(M) = 1, 2, 4, 5, 7, 8$ for all of which $dr(M^6) \equiv 1 \pmod 9$. Let $n = 2^s M^5$ and consider $kn = 2^{r+s} M^6$. By (b), we can choose s so that $dr(2^{r+s})$ is 5 or 7 whence $dr(kn)$ is 5 or 7; thus k is bad unless 35 divides M, say $M = 35N$ with N odd and, as above, $dr(N^6) \equiv 1 \pmod 9$.

- If $r = 1$, we would then have $k = 70N$. Such a k is bad because $n = 14N^5$ gives $dr(kn) = dr(980N^6) = 8$ which does not divide $980N^6$.
- If $r = 2$, we would then have $k = 140N$. Such a k is bad because $n = 7N^5$ gives $dr(kn) = dr(980N^6) = 8$ which does not divide $980N^6$.

We conclude that $r \geqslant 3$, so that 280 divides k which has already been dealt with in (e).

Finally, as Max commented in our initial discussion, we can consider the same problem in other number bases.

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Feedback

On 75.12 Alan Beardon writes: Nick Lord challenged the reader to supply possible generalisations of the identity

$$(1 + 2 + \ldots + n)^2 = 1^3 + 2^3 + \ldots + n^3,$$

and asked whether any reader could find positive integers a, b, c with $b > 1$ such that

$$\left(1^a + 2^a + \ldots + n^a\right)^b = 1^c + 2^c + \ldots + n^c$$

holds for all n? In fact, this is true only in the given case, namely $(a, b, c) = (1, 2, 3)$. However, if we write

$$\sigma_k(n) = 1^k + 2^k + \ldots + n^k,$$

so that the given relation is $\sigma_1^2 = \sigma_3$, there are many other interesting relations between the quantities $\sigma_k(n)$ for various k (and all n). For example,

$$\sigma_1^3 = \tfrac{1}{4}\sigma_3 + \tfrac{3}{4}\sigma_5;$$

$$8\sigma_1^3 = -\sigma_1^2 + 9\sigma_2^2;$$

$$81\sigma_2^4 = 18\sigma_2^2\sigma_3 - \sigma_3^2 + 64\sigma_3^3;$$

$$16\sigma_3^3 = \sigma_3^2 + 6\sigma_3\sigma_5 + 9\sigma_5^2.$$

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