Problem Corner

Solutions are invited to the following problems. They should be addressed to **Nick Lord** at **Tonbridge School, Tonbridge, Kent TN9 1JP** (e-mail: **njl@tonbridge-school.org**) and should arrive not later than 10 August 2019.

Proposals for problems are equally welcome. They should also be sent to Nick Lord at the above address and should be accompanied by solutions and any relevant background information.

103.A (Michael Fox)

The point A is $(6,\ 16,\ 12)$, B is $(18,\ -9,\ 12)$ and P is a variable point on the x-axis.

Find the ranges of possible values of (i) $AP + BP$, (ii) $AP - BP$, and (iii) AP / BP.

103.B (Michel Bataille)

Let ABC be a triangle and D, E on the side BC be such that $BD = DE = EC$. Let the circumcircles Γ_b and Γ_c of $\triangle ABD$ and $\triangle ACE$, respectively, intersect at A and A' and let the tangents to Γ_b at D and to Γ_c at E intersect in U.

Prove that the orthogonal projections P, Q, P', Q' of U onto AD, AE, $A'D$, $A'E$, respectively, are collinear and that $PP' = QQ'$.

103.C (K. S. Bhanu and M. N. Deshpande)

Consider the following method of generating a permutation of the integers $(1,\ 2,\ \ldots,\ n)$. We have equal numbers of balls of three colours in a bag. Balls are drawn randomly one by one n times with replacement. For $1 \leqslant k \leqslant n$, the colour of the kth ball drawn is noted and we use this to generate at most three cycles $(a_1, \ldots, a_{k_1})$, $(b_1, \ldots, b_{k_2})$ and $(c_1, \ldots, c_{k_3})$ where the a_i, b_i and c_i denote the draw numbers of balls of the same colour and $k_1 + k_2 + k_3 = n$. From the product of these cycles, we generate a random permutation $(\pi_1, \ldots, \pi_n)$.

Show that, for large n, $\mathrm{E}(\pi_1) \approx 4$ and $\mathrm{E}(\pi_n) \approx 3$.

[For example, if $n = 7$ and the draw number of balls of each colour are $(1, 4)$, $(2,\ 3,\ 6)$ and $(5,\ 7)$, then the generated permutation is $(4,\ 3,\ 6,\ 1,\ 7,\ 2,\ 5)$.]

103.D (Ovidiu Furdui)

Let $b, c, d \in \mathbb{R}$ with $bc > 0$. Calculate

$$\lim_{n\to\infty} \begin{pmatrix} 1 & \sin\dfrac{b}{n} \\ \sin\dfrac{c}{n} & \cos\dfrac{d}{n} \end{pmatrix}^n.$$

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Solutions and comments on **102.E, 102.F, 102.G, 102.H** (July 2018).

102.E (Martin Lukarevski)

Let a, b, c be the lengths of the sides of triangle ABC and let w_a, w_b, w_c be the lengths of the corresponding angle bisectors. Prove that

$$\frac{a^2}{w_a^2} + \frac{b^2}{w_b^2} + \frac{c^2}{w_c^2} \geqslant 4.$$

When does equality hold?

A wide variety of standard inequalities were used by solvers of this intriguing geometrical inequality. All relied on the standard formulae for the length of an angle bisector, $w_a = \dfrac{2bc}{b+c}\cos\tfrac{1}{2}A = \dfrac{2}{b+c}\sqrt{bcs(s-a)}$, where the first formula is quickly proved by expressing the area of triangle ABC as the sum of the areas of the triangles created by w_a and the second follows from the half-angle formula $\cos\tfrac{1}{2}A = \sqrt{\dfrac{s(s-a)}{bc}}$.

A number of solvers argued as follows:

$$\frac{a^2}{w_a^2} = \frac{a^2(b+c)^2}{4bcs(s-a)} \geqslant \frac{a^2}{s(s-a)} \text{ by the AM-GM inequality.}$$

But $\displaystyle\sum \frac{a^2}{s-a} \sum (s-a) \geqslant \left(\sum a\right)^2$ (Cauchy-Schwarz) implies that $\displaystyle\sum \frac{a^2}{s-a} \geqslant 4s$, whence $\displaystyle\sum \frac{a^2}{w_a^2} \geqslant \frac{1}{s}\cdot 4s = 4$, as required.

As Lienhard Wimmer observed, a method of proof also used by several other solvers yields the generalisation

$$\sum \left(\frac{a}{w_a}\right)^n \geqslant 3^{1-\frac{n}{2}}.2^n \text{ for } n > 0.$$

For

$$\sum \left(\frac{a}{w_a}\right)^n \geqslant 3\left(\frac{abc}{w_a w_b w_c}\right)^{n/3} \text{ by the AM-GM inequality,}$$

$$= 3\left[\frac{(b+c)(a+c)(a+b)}{8abc\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C}\right]^{\frac{n}{3}} \text{ from the first formula for } w_a,$$

$$\geqslant \frac{3}{\left[\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C\right]^{n/3}} \text{ by the AM-GM inequality.}$$

But Jensen's inequality applied to the concave function $\log\cos x$ gives

$$\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C \leqslant \cos^3\frac{A+B+C}{6} = \frac{3\sqrt{3}}{8}$$

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so that $$\sum \left(\frac{a}{w_a}\right)^n > 3\left(\frac{8}{3\sqrt{3}}\right)^{\frac{n}{3}} = 3^{1-\frac{n}{2}}.2^n.$$

(This also includes the result for $n = 1$ set by the proposer as Problem 11945 in the American Mathematical Monthly with solution by Dmitry Fleischman using this argument in the October 2018 issue, pp. 758-759.)

Finally, M. G. Elliott proved the stronger result $\sum \frac{a^2}{m_a^2} \geqslant 4$ involving the medians m_a, m_b, m_c which implies the result in **102.E** because $w_a \leqslant m_a$, $w_b \leqslant m_b, w_c \leqslant m_c$.

Correct solutions were received from: C. R. Aditya, M. Bataille, M. V. Channakeshava, N. Curwen, S. Dolan, M. G. Elliott, GCHQ Problem Solving Group, G. Howlett, J. A. Mundie, M. Nadapana, N. Neha, A. Ng, P. Nüetsch, V. Schindler, I. D. Sfikas, P. Shiu, G. B. Trustrum, J. T. Varsha, L. Wimmer (2 solutions), S. Yip and the proposer Martin Lukarevski.

102.F (Michał Kremzer)

Find all pairs of positive real numbers (a, b) such that $\ln(\lfloor a + b \rfloor) = \lfloor a \rfloor + \lfloor b \rfloor$, where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.

Answer: The solutions (a, b) comprise the triangular region of the plane given by $0 < a, b < 1$ and $a + b \geqslant 1$.

The solvers of this attractive problem generally argued along similar lines to Stan Dolan and Ioannis Sfikas whose succinct solution follows.

Writing $a = \lfloor a \rfloor + \delta$, $b = \lfloor b \rfloor + \varepsilon$ with $0 \leqslant \delta, \varepsilon < 1$, we see that $\lfloor a + b \rfloor = \lfloor a \rfloor + \lfloor b \rfloor + \lfloor \delta + \varepsilon \rfloor \leqslant \lfloor a \rfloor + \lfloor b \rfloor + 1$. Now $e^x \geqslant 1 + x$ with equality if, and only if, $x = 0$. Thus

$$e^{\lfloor a \rfloor + \lfloor b \rfloor} \geqslant (1 + \lfloor a \rfloor)(1 + \lfloor b \rfloor) \geqslant 1 + \lfloor a \rfloor + \lfloor b \rfloor \geqslant \lfloor a + b \rfloor.$$

The condition postulated in the problem then implies equality throughout which occurs if, and only if, $\lfloor a \rfloor = \lfloor b \rfloor = 0$ and $\lfloor a + b \rfloor = 1$ so that $0 < a, b < 1$ and $a + b \geqslant 1$.

Peter Johnson investigated the effect of removing the restriction that a, b are positive and showed that the solution set is then given by all the translates of the triangular region above by translations of the form $\begin{pmatrix} n \\ -n \end{pmatrix}$, $n \in \mathbb{Z}$. The problem is also worth investigating in other bases.

Correct solutions were received from: M. Bataille, M. V. Channakeshava, S. Dolan, M. G. Elliott, GCHQ Problem Solving Group, P. F. Johnson, A. Ng, I. D. Sfikas, P. Shiu, G. B. Trustrum, L. Wimmer and the proposer Michał Kremzer.

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102.G (John Melville)

Show that $\displaystyle\int_0^1 \frac{\ln\left(x + \sqrt{1 - x^2}\right)}{x}\,dx = \frac{\pi^2}{16}$.

A wide variety of substitutions and other integration techniques were employed to evaluate this integral which attracted many appreciative comments from respondents. The solution below utilises ideas from several solvers but most closely follows that given by M. G. Elliott.

Let $\displaystyle I = \int_0^1 \frac{\ln\left(x + \sqrt{1 - x^2}\right)}{x}\,dx = \int_0^{\pi/2} \frac{\ln(\sin\theta + \cos\theta)}{\sin\theta} \cdot \cos\theta\, d\theta,$

on substituting $x = \sin\theta$.

The switch $\theta \to \frac{\pi}{2} - \theta$ shows also that

$\displaystyle I = \int_0^{\pi/2} \frac{\ln(\sin\theta + \cos\theta)}{\cos\theta} \cdot \sin\theta\, d\theta$, so, on adding,

$$2I = \int_0^{\pi/2} \frac{\ln(\sin\theta + \cos\theta)}{\sin\theta\cos\theta}\,d\theta = \int_0^{\pi/2} \frac{\ln(\sin\theta + \cos\theta)^2}{2\sin\theta\cos\theta}\,d\theta$$

$$= \int_0^{\pi/2} \frac{\ln(1 + \sin 2\theta)}{\sin 2\theta}\,d\theta$$

and

$$I = \frac{1}{2}\int_0^{\pi/2} \frac{\ln(1 + \sin 2\theta)}{\sin 2\theta}\,d\theta = \int_0^{\pi/4} \frac{\ln(1 + \sin 2\theta)}{\sin 2\theta}\,d\theta,$$

by symmetry.

The substitution $t = \tan\theta$ then gives

$$I = \int_0^1 \frac{\ln\left(1 + \dfrac{2t}{1 + t^2}\right)}{\dfrac{2t}{1 + t^2}} \cdot \frac{1}{1 + t^2}\,dt = \int_0^1 \frac{\ln(1 + t)}{t} - \frac{\ln\left(1 + t^2\right)}{2t}dt.$$

But, setting $u = t^2$, shows that $\displaystyle\int_0^1 \frac{\ln\left(1 + t^2\right)}{2t}dt = \int_0^1 \frac{\ln(1 + u)}{4u}\,du$ so that

$$I = \frac{3}{4}\int_0^1 \frac{\ln(1 + t)}{t}\,dt$$

$$= \frac{3}{4}\int_0^1 \left(\sum_{n=1}^{\infty} (-1)^{n-1}\frac{t^{n-1}}{n}\right)dt$$

$$= \frac{3}{4}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}, \text{ on switching the sum and integral}$$

$$= \frac{3}{4}\left[\sum_{n=1}^{\infty}\frac{1}{n^2} - 2\sum_{n=1}^{\infty}\frac{1}{(2n)^2}\right]$$

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$$= \frac{3}{8}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$= \frac{\pi^2}{16}, \text{ since } \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

Another technique used quite frequently was differentiation under the integral sign and we illustrate this with the integral used by the GCHQ Problem Solving Group; similar arguments were given by Andrew Ng and Ioannis Sfikas. We have

$$I = \int_0^{\pi/4}\frac{\ln(1+\sin 2\theta)}{\sin 2\theta}\,d\theta = \frac{1}{2}\int_0^{\pi/2}\frac{\ln(1+\sin\theta)}{\sin\theta}\,d\theta$$

$$= \frac{1}{4}\int_0^{\pi}\frac{\ln(1+\sin\theta)}{\sin\theta}\,d\theta.$$

Consider $f(\lambda) = \int_0^{\pi}\ln\left(\frac{1+\sin\theta}{1+\cos\lambda\sin\theta}\right)\frac{d\theta}{\sin\theta}$ for $|\lambda| < \pi$.

Then $f'(\lambda) = \int_0^{\pi}\frac{\sin\lambda}{1+\cos\lambda\sin\theta}\,d\theta = 2\lambda$, on evaluating the integral using substitution $t = \tan\frac{1}{2}\theta$. It follows that, since $f(0) = 0, f(\lambda) = \lambda^2$ so that

$$f\left(\frac{\pi}{2}\right) = \int_0^{\pi}\frac{\ln(1+\sin\theta)}{\sin\theta}\,d\theta = \frac{\pi^2}{4} \text{ and } I = \tfrac{1}{4}f\left(\frac{\pi}{2}\right) = \frac{\pi^2}{16}.$$

Correct solutions were received from: M. Bataille, N. Curwen, S. Dolan, M. G. Elliott, GCHQ Problem Solving Group, R. Gordon, G. Howlett, J. D. Mahony, A. Ng, V. Schindler, I. D. Sfikas, P. Shiu, I. Sofair and the proposer John Melville.

102.H (Chris Starr)

Consider the ellipse given by the parametric equations $x = a\cos\theta$, $y = b\sin\theta$ and let $P(p, q)$ be a point not on the ellipse with $p, q \neq 0$.

(a) Show the value of θ giving the minimum distance from P to the ellipse is given by a quartic in $t = \tan\frac{1}{2}\theta$.

(b) Show that rational, non-zero values of a, b, p, q can be found such that the quartic in part (a) factorises as the product of two quadratics with rational coefficients.

(c) Find the exact value of the minimum distance from $P\left(-\frac{15}{4}, 1\right)$ to the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

Answer: (c) The minimum distance is $\frac{1}{20}\sqrt{7965 - 1520\sqrt{19}}$.

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Parts (a) and (c) of this problem were tackled in much the same way by all solvers.

(a) The distance D from $P(p, q)$ to $(a\cos\theta, b\sin\theta)$ is given by

$$D^2 = (p - a\cos\theta)^2 + (q - b\sin\theta)^2.$$

This is minimised when $\dfrac{dD^2}{d\theta} = 0$ giving

$$(a^2 - b^2)\sin\theta\cos\theta - ap\sin\theta + bq\cos\theta = 0,$$

which is the condition for P to lie on the normal to the ellipse at $(a\sin\theta, b\cos\theta)$. Substituting $t = \tan\frac{1}{2}\theta$ with $\sin\theta = \dfrac{2t}{1+t^2}$, $\cos\theta = \dfrac{1-t^2}{1+t^2}$ and simplifying gives the quartic

$$bqt^4 + 2(ap + a^2 - b^2)t^3 + 2(ap - a^2 + b^2)t - bq = 0. \qquad (*)$$

(c) Here, $a = 2, b = 3, p = \dfrac{-15}{4}, q = 1$ and the quartic from part (a) is

$$3t^4 - 25t^3 - 5t - 3 = 0 \text{ or } (3t^2 - t + 1)(t^2 - 8t - 3) = 0.$$

Geometrically, it is clear that the minimising point on the ellipse is in the second quadrant, corresponding to a positive, real value of t, so the relevant root is $t = 4 + \sqrt{19}$ (from the second factor) giving $\cos\theta = \dfrac{-1 - 2\sqrt{19}}{10}$, $\sin\theta = \dfrac{-2 + \sqrt{19}}{10}$ and $D = \frac{1}{20}\sqrt{7965 - 1520\sqrt{19}}$.

(b) Obviously, part (c) gives one example where the quartic in part (b) factorises in the required manner, but respondents rose magnificently to the challenge of finding other solutions.

Stan Dolan, Peter Shiu and the GCHQ Problem Solving Group observed that when $a = b$ (so the ellipse is a circle), $(*)$ becomes

$$aqt^4 + 2apt^3 + 2apt - aq = a(t^2 + 1)(qt^2 + 2pt - q)$$

which, for rational a, p, q, provides a solution.

Russell Gordon and Volkhard Schindler gave the following general approach. The monic quartic corresponding to $(*)$ is

$$t^4 + \frac{2}{bq}(ap + a^2 - b^2)t^3 + \frac{2}{bq}(ap - a^2 + b^2)t - 1 = 0$$

for which we seek a factorisation

$$(t^2 + ct + d)\left(t^2 + \frac{1 - d^2}{cd}t - \frac{1}{d}\right) = 0$$

involving rational numbers c, d.

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Equating coefficients shows that

$$A \equiv c + \frac{1 - d^2}{cd} = \frac{2}{bq}\left(ap + a^2 - b^2\right)$$

$$B \equiv -\frac{c}{d} + \frac{1 - d^2}{c} = \frac{2}{bq}\left(ap - a^2 + b^2\right).$$

Then $Abq = 2ap + 2a^2 - 2b^2, Bbq = 2ap - 2a^2 + 2b^2$ solves to give $q = \dfrac{4\left(b^2 - a^2\right)}{b\left(B - A\right)}, p = \dfrac{\left(B + A\right)\left(b^2 - a^2\right)}{a\left(B - A\right)}$.

Hence, given rational values for a, b, we can choose rational values for c, d and then use the above equations to find $P(p, q)$ with rational coordinates.

Other, specific solutions with a, b, p, q (where, by symmetry, we may suppose that a, b, p, $q > 0$) included:

- $a = m$, $b = n$, $p = 2n$, $q = 2m$: a good way of generating examples with integer parameters. (J. A. Mundie)
- The sub-case $c = d = \frac{1}{\lambda}$ $(\lambda \neq 0, \pm 1, -2)$ with $a = \frac{1}{8}\left(\lambda^2 + 2\lambda + 6\right)$, $b = \frac{1}{8}\left(2 + 6\lambda - \lambda^2\right)$, $ap = \frac{1}{4}\lambda(\lambda - 1)(\lambda + 2)$, $bq = \lambda$. (M. G. Elliott)
- Two families with $a - b = \pm 1$ and single rational parameter $Q > 1$:
  $a = \frac{1}{4}\left(Q^2 + 3\right), b = \frac{1}{4}\left(Q^2 - 1\right), p = \dfrac{8Q\left(Q^2 + 1\right)}{\left(Q^2 - 1\right)\left(Q^2 + 3\right)}, q = \dfrac{4Q}{Q^2 - 1}$
  and
  $a = \frac{1}{4}\left(Q^2 - 1\right), b = \frac{1}{4}\left(Q^2 + 3\right), p = \dfrac{8Q\left(Q^2 + 1\right)}{\left(Q^2 - 1\right)^2}, q = \dfrac{4Q}{Q^2 + 3}$
  (where $Q = 3$ corresponds to part (c) of **102.H**). (G. Howlett)
- Integer solutions of the Pell-type equation $b^2 - 2a^2 = q^2$ with $p = \dfrac{4bq\left(a^2 - b^2\right)}{a\left(b^2 - q^2\right)}$, where $a = 2, b = 3, q = 1$ gives part (c) of **102.H**. (Chris Starr, the proposer)

Russell Gordon commented that the evolute of an ellipse is an astroid. If P is inside the astroid, the quartic $(*)$ has four real roots: outside (as in part (c)) it has just two real roots. And H. Azad and A. Laradji showed in the November 2004 *Gazette* (pp. 548-551) that, in general, the construction of the minimising point on the ellipse is not possible with ruler and compasses.

Correct solutions were received from: M. V. Channakeshava, N. Curwen, S. Dolan, M. G. Elliott, GCHQ Problem Solving Group, R. Gordon, G. T. Q. Hoare, G. Howlett, J. D. Mahony, J. A. Mundie, V. Schindler, I. D. Sfikas, P. Shiu, L. Wimmer and the proposer Chris Starr.

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