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Graphs and flows on surfaces

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Abstract. In 1971, M. M. Peixoto [**15**] introduced an important topological invariant of Morse–Smale flows on surfaces, which he called a distinguished graph X^* associated with a given flow. Here we show how the Peixoto invariant can be essentially simplified and revised by adopting a purely topological point of view connected with the embeddings of arbitrary graphs into compact surfaces. The newly obtained invariant, X^R, is a rotation of a graph X generated by a Morse–Smale flow. (a rotation R is a cyclic order of edges given in every vertex of X.) The invariant X^R 'reads-off' the topological information carried by a flow, being in a one-to-one correspondence with the topological equivalence classes of Morse–Smale flows‡. As a counterpart to the equivalence result we prove a realization theorem for an 'abstractly given' X^R. (Our methods are completely different from those of Peixoto and they clarify the connections between graphs and flows on surfaces.) The idea of 'rotation systems' on graphs can be further exploited in the study of recurrent flows (and foliations) with several disjoint quasiminimal sets on surfaces [**10**].

0. *Introduction*

In his celebrated work [**15**], Peixoto studied an important class of flows, so-called *Morse–Smale flows*, given on compact orientable two-dimensional manifolds. To every flow of such type, he associated a graph endowed with a finite combinatorial structure, and proved that these graphs (later called *Peixoto graphs*) are in one-to-one correspondence with classes of topological equivalence of Morse–Smale flows. Let us recall, in outline, basic definitions and results obtained by Peixoto [**15**].

Denote by M a closed and, until otherwise stated, orientable manifold of dimension 2, which we shall also call *a surface*. Let $\mathcal{X}^r(M)$ be the space of C^r-smooth vector fields $M \to TM$ on M, endowed with the uniform C^r topology. An equivalence relation $\mathcal{E} : x \sim y$, which is an orbital (or, *topological*) conjugacy between trajectories of vector fields $x, y \in \mathcal{X}^r$, is inroduced; it splits $\mathcal{X}^r$ into a certain number of topological classes. As usual, *structural stability* of a vector field $x \in \mathcal{X}^r(M)$ means that there is a neighborhood U of x, such that for all $y \in U$ it holds that $y \sim x$.

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‡ And foliations, see [**3**]. We do not treat the case of foliations, bearing in mind that they are defined by involutive flows on covering manifolds [**9**].

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THEOREM 1. (Andronov and Pontryagin) *In the space $\mathcal{X}^r(M)$ there exists an open and dense subspace $\mathcal{X}_0$ which consists of structural stable vector fields. These are called Morse–Smale vector fields and admit the following simple geometrical description:*

(i) *$\forall x \in \mathcal{X}_0$, the vector field x has a finite number of singular points and closed orbits, all of which are hyperbolic;*

(ii) *there are no separatrix connections between singular points of saddle type;*

(iii) *the α- and ω-limit set of each trajectory is either a singular point or a closed orbit.*

Let $t \in \mathbb{R}$ be a real number. By a *Morse–Smale flow* $\pi^t : M \times \mathbb{R} \to M$ on the surface M, one understands a flow tangent to a Morse–Smale vector field $x \in \mathcal{X}_0$. Every flow π^t defines an oriented graph X (*orgraph*, or *scheme of the flow*) with the vertex set $V_X = (\mathrm{Sing}\,\pi^t) \cup (\mathrm{Cycl}\,\pi^t)$ and the edge set $E_X = \mathrm{Sep}\,\pi^t$, where Sing, Cycl and Sep stand, respectively, for the set of singular points, closed orbits and separatrices of the flow π^t. A fundamental problem, which has been studied by Peixoto, is the following: how exact X defines the topological equivalence classes of Morse–Smale flows on M?

Unfortunately, there is no hope that orgraphs X, taken as such, are enough to establish the required one-to-one correspondence with equivalence classes of Morse–Smale flows; see [**15**, p. 392], or Figures 4 and 5 of this article. One finds there an example of two inequivalent Morse–Smale flows on S^2, which have, up to an orgraph isomorphism, the same X†. It was Peixoto's main motivation to introduce an extra structure on X, called a 'distinguished graph' X^*. (A 'distinguished graph' is a graph with a full list of 'distinguished subgraphs' and their adjacency structure, or 'distinguished structure'.) Now an isomorphism $\cong$ on graphs should respect the 'distinguished structure' on graphs. Peixoto proved the following two beautiful statements.

THEOREM 2. (Peixoto) *Let $x, y \in \mathcal{X}_0$. Then $x \sim y$ if and only if $X^* \cong Y^*$, where X^* and Y^* are distinguished graphs of the vector fields x and y.*

THEOREM 3. (Peixoto) *Let X^* be an abstract distinguished graph. Then X^* is realizable as a Peixoto (distinguished) graph of a Morse–Smale flow on M. The Euler characteristic $\chi(M)$ of M coincides with the Euler characteristic of X^* (by which one understands a difference between the number of nodes and the number of saddles in V(X)).*

The above theorems imply that the X^* are *complete* topological invariants, since they stay in one-to-one correspondence with topological conjugacy classes of Morse–Smale flows. To this end, Peixoto orgraphs give a definitive and final solution to a rather classical problem of the qualitative theory of flows on manifolds.

Peixoto formulated and proved these results by a remarkable amount of intuition and inventiveness. However, the notion of a distinguished set, see [**15**, p. 395], looks rather artificial, and as a result of this the proof of Theorems 2 and 3 are too long. It should be emphasized that Theorems 2 and 3 are intimately connected with the question of embedding of a given graph in a 2-manifold. This sort of problem is a subject of topological graph theory and has a rich background; see [**5**] for an excellent introduction. Apparently, Heffter [**6**] was the first to understand the role of rotation

† A relevant example for Morse–Smale foliations was constructed in [**3**].

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systems R on a graph X for embedding of X into a surface M. Next, Edmonds [**4**] formalized the technique and proved that each rotation system R on X defines a unique embedding $X \to M$ (the so-called *Heffter–Edmonds theorem*). Seemingly, this theorem was unfamiliar to the author of [**15**].

The aim of the present paper is to fill in a gap existing between the two theories by giving a new interpretation to Peixoto's results from the viewpoint of topological graph theory. Such insights yield a surprisingly simple proof of the equivalence and realization theorems of Peixoto. (Peixoto's original proof contains, in a hidden form, the proof of a very special case of the Heffter–Edmonds theorem, and also Maier's statement on the structure of Morse–Smale flows [**7**], thus making the presentation somewhat lengthy and vague.) It provides us with a precise understanding of the topological information which is carried by Morse–Smale flows, this information being expressed in terms of a 'rotation system' associated with the graph X.

The structure of this paper is as follows. In §1 rotation systems on graphs are defined and the Heffter–Edmonds theorem is formulated. The equivalence (Theorem 6) and realization (Theorem 7) statements are formulated in §2 and proved in §§3.1 and 3.2. Finally, in §4 a classical Peixoto example for a pair of inequivalent Morse–Smale flows on a sphere is treated in detail.

1. *Preliminaries*

In the two subsections which follow we introduce the notion of a rotation system associated with a finite graph and we discuss a decomposition theorem for Morse–Smale flows. The definitions and statements below are extremely important for understanding the rest of the paper.

1.1. *Rotation systems.*

It has already been mentioned that every Morse–Smale flow $\pi^t : M \times \mathbb{R} \to M$ on a surface M defines a special kind of graph embedded into M. Problems related to the embeddings of graphs have been well investigated as a subject of topological graph theory; see [**5**] for an elementary introduction.

Definition 1. An (oriented) graph X is defined to be a set of vertices V_X and a set of edges E_X such that an (ordered) pair of endpoints of every edge $e \in E_X$ lies in V_X. In other words, graph X is the set $E_X \cup V_X$ endowed with an 'incidence structure' $I_X = \{V_X(e) \mid e \in E_X\}$. For brevity, an oriented graph is called an orgraph.

Further, we usually omit subscripts for the vertex and the edge sets. If both E and V are finite, graph X is said to be finite. Every finite graph X can be geometrically represented as a drawing (dot-and-line structure), where dots correspond to the vertices $v \in V$ and lines (segments) correspond to the edges $e \in E$ such that the incidence structure I is respected. (Although drawings are easily visualized, they should be treated carefully since a drawing is not a graph, but only a representation of its incidence structure.)

A drawing of a finite graph X can be endowed with a natural Hausdorff topology which turns X into a one-dimensional CW-complex $K^1(X)$. It is usually performed as

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follows [**8**]. The 'topological structure' on a graph consists of a Hausdorff space X and a subspace X^0 such that the following conditions are satisfied:

(i) X^0 is a discrete, closed subspace of X. Points of X^0 are called 'vertices of X'.

(ii) $X\backslash X^0$ is a disjoint union of open subsets e_i, where each e_i is homeomorphic to an open interval $I \subset \mathbb{R}$ of the real line $\mathbb{R}$. The sets e_i are called 'edges of X'.

(iii) For each edge e_i its boundary ∂e_i is a subset of X^0 consisting either of one or two points. In the case when ∂e_i consists of two points, the set $\overline{e}_i$ is homeomorphic to a closed interval $\overline{I} = [0, 1] \subset \mathbb{R}$; in the case when ∂e_i consists of one point, the set $\overline{e}_i$ is homeomorphic to the unit circle S^1.

(iv) X can be endowed with a weak topology: a subset $A \subset X$ is closed (open) if and only if $A \cap \overline{e}_i$ is closed (open) for all $e_i \in E_X$.

(Clearly, properties (i)–(iii) are verified, property (iv) holds when X is a finite graph.) Note that, in general, $K^1(X)$ cannot be supplied with the structure of a manifold, having only the structure of a CW-complex.

Let X and X' be two finite graphs. A graph map $f : X \to X'$ consists of a vertex function $V_X \to V_{X'}$ and an edge function $E_X \to E_{X'}$ such that the incidence structure is preserved (it means that for every $e \in E$, the vertex function f_V and the edge function f_E are such that the endpoints a, b of e are sent to the endpoints p, q of $f_E(e) : p = f_V(a), q = f_V(b)$). For orgraphs we also demand that f preserves the orientation.

Definition 2. A graph map $f : X \to X'$ between two graphs X and X' is called an isomorphism if both its vertex function f_V and edge function f_E are one-to-one and onto (surjective). Two graphs X and X' are called isomorphic if there exists an isomorphism $f : X \to X'$.

Let $u, v \in V$ be vertices of a graph X. A *walk* w on X from u to v of length n means an alternating sequence of vertices and directed edges, such that the initial vertex $v_0 = u$ and the final vertex $v_n = v$ for $i = 1, \ldots, n$. If $u \neq v$ a walk is called open, otherwise it is closed. An open walk is called a *path* if all its vertices are distinct. A path on X such that its initial vertex coincides with its final vertex is called a *cycle* on X.

A central concern of topological graph theory is the problem of placement (or embedding) of a graph on a surface. A classical problem is, for example, the embedding of a graph into the plane (or 'planarity' of a graph). It is known, for example, that the complete graph K_5 is not planar; that is, the drawing of the graph in the plane (or sphere) always has self-intersections. It is easy to see, however, how to avoid intersections on the higher genus surfaces: it is sufficient to glue up an extra handle and to draw one edge over 'the bridge', the other under it. Therefore each graph can be embedded into a surface of a sufficiently high genus.

Definition 3. An embedding $i : X \to M$ of a graph X into a surface M is a one-to-one continuous map of the topological space X, taken as a 1-complex $K^1(X)$, into the topological space M†. Two embeddings i_1 and i_2 of X in a surface M are equivalent if there exists a homeomorphism $h : M \to M$ such that $h \circ i_1 = i_2$ (in other words, h brings the image $i_1(X)$ to the image $i_2(X)$).

† That is, the CW-topology of $K^1(X)$ coincides with the induced topology on $i(X)$.

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If one takes an embedding $i : X \to M$ of a connected graph X in M, then the set $M \backslash i(X)$ is the union of open regions V_m. Clearly, by gluing up handles to each V_m, it is possible to obtain embeddings of X into the surfaces of an arbitrary high genus. An embedding $i : X \to M$ is called *2-cell* (or *cellular*) if all open regions V_m are homeomorphic to an open disc. Throughout this paper, only cellular embeddings are considered. There exists a graph which is intimately connected with a 2-cell embedding $X \to M$. This is a *dual graph* $X^{\#}$ of the embedding $X \to M$.

Definition 4. Denote by $i : X \to M$ a 2-cell embedding of an orgraph X into a surface M. Let $N \backslash i(X) = F_1 \cup \cdots \cup F_m$ be a union of open disc regions in M. A dual graph $X^{\#}$, associated to i, is a graph with the vertex set $V_{X^{\#}} = \{F_1, \ldots, F_m\}$. An edge $e^{\#} \in E_{X^{\#}}$ between F_i and F_j should be drawn (the case $i = j$ is not excluded) if and only if there is an edge $e \in E_X$ between F_i and F_j, i.e. $e \subseteq \overline{F}_i \cap \overline{F}_j$.

Definition 5. Let X be a finite graph and let $v \in V(X)$ be its vertex. A local rotation of a vertex v is an oriented cyclic order (defined up to the cyclic permutations) of all edges incident to v. (Local rotation of 1-valent vertices is uniquely defined and is called trivial.) A rotation system R (or, simply, a rotation) of a graph X is a union of all local rotations over all vertices of X.

Rotations give rise to a certain system of faces ($\equiv$ cycles) on X. The following so-called face tracing algorithm allows us to determine all faces of a graph X, endowed with rotation R.

Face tracing algorithm (see [**5**]). Let X be a finite graph and let R be its rotation system. Take an arbitrary vertex $v_1 \in V(X)$ and an edge a_{v_1}, incident to v_1. Let v_2 be a vertex of X, connected to v_1 by the edge a_{v_2}, and let b_{v_2} be an edge of the vertex v_2 which lies *to the right*† in the cyclic order from a_{v_1}. Moving along the edge b_{v_2} to a vertex v_3, we shall define an edge c_{v_3} which lies *to the right* from b_{v_2}. Proceeding inductively, we stop the process at an edge z_{v_n} if two forthcoming edges will be again a_{v_1} and b_{v_2}. Hence, a cycle $a_{v_1}, b_{v_2}, \ldots, z_{v_n}$ of length n, which defines a face F_1 on X, will be traced. For tracing the next face F_2, one should start with an edge which lies *to the right* of any edge of the face F_1 and such that a corner between them did not occur in F_1, and apply the above construction. All faces $F_1, F_2, \ldots, F_m$ on X will be traced when no unused corners remain.

The fundamental theorem, proved independently by Heffter [**6**] and Edmonds [**4**], shows how rotations and embeddings of X are related. Because of the exceptional importance of the Heffter–Edmonds statement, we will give its proof in §3.3.

THEOREM 4. (Heffter–Edmonds) *Let X be a finite graph endowed with a rotation system R. Then there exists a 2-cell embedding of X into an orientable surface M such that one of two rotations, induced by this embedding, coincides with R. Moreover, two embeddings are equivalent if and only if they have equivalent rotation systems.*

† For the 1-valent vertices v_i with the edge e_i, the edge lying to the right of e_i will again be e_i. In other words, such rotation is trivial, see Definition 5.

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Remark 1. Each embedding $i : X \to M$ induces a pair of rotation systems R and R^*, where R^* is a mirror image of R (i.e. can be obtained from R by reversing the cyclic order of all local rotations in R). The corresponding embeddings $i(X)$ and $i^*(X)$ are conjugate by a homeomorphism $h : M \to M$, which is not homotopic to id_M.

1.2. *Elementary cells of Morse–Smale flows.* Here a structure theorem for Morse–Smale flows on surfaces is formulated. (It will give us a description of 'elementary cells' of flows, which are 'constructing blocks' in the proof of the realization theorem.) The early results of this kind are due to Andronov and Pontryagin [**1**] and Maier [**7**]. They were developed further by Peixoto [**14**] and Neumann and O'Brien [**12**]. For other references, see [**2**] and [**11**].

Let $\pi^t : M \times \mathbb{R} \to M$ be a Morse–Smale flow on a surface M. Let us consider the set $M \setminus \mathrm{Sep}\,\pi^t$ which is the union of open connected regions U_i of M. Disjoint regions U_i are called *elementary cells of a Morse–Smale flow*.

Definition 6. By a polar cell, standard cell and the Andronov cell, one understands cells formed, respectively, by a polar flow, a flow with 1 source and 1 sink bounded by 4 separatrices and a flow with 1 source and 1 sink bounded by 3 separatrices; see Figure 1. The corresponding orgraphs are pictured in Figure 2, and are called, respectively, a polar molecule, a standard molecule, and the Andronov molecule.

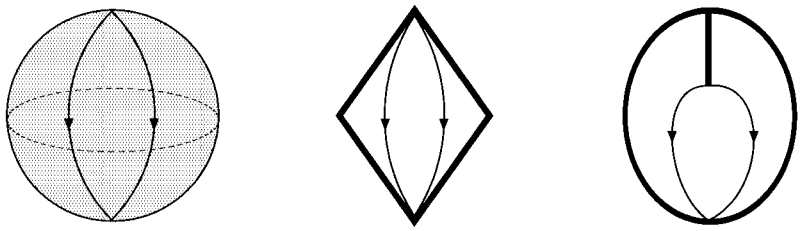


FIGURE 1. (a) Polar, (b) standard and (c) Andronov cells.

THEOREM 5. (Maier) *Every Morse–Smale flow on an orientable surface M can be decomposed into a finite number of polar, standard or Andronov cells. Each standard and Andronov cell is homeomoprhic to a disc region. Each polar cell is homeomorphic either to a sphere or to a disc or, else, to an annulus. In the latter case, the polar cell is a sphere if both of its critical elements (a source and a sink) are nodes. The polar cell is a disc if one of its critical elements is a node and the other is a cycle; see Theorem 1. Finally, the polar cell is an annulus if both of its critical elements are cycles.*

Proof. See [**7**], [**14**] and [**12**]. □

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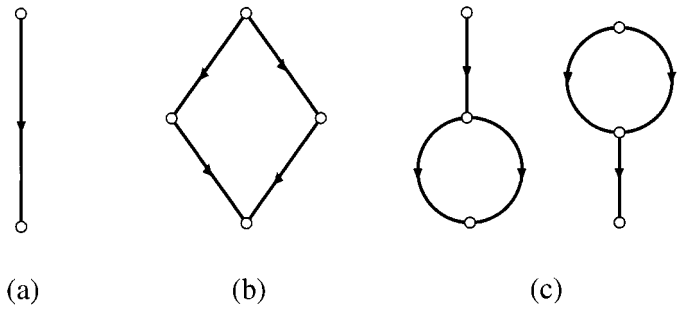


FIGURE 2. (a) Polar, (b) standard and (c) Andronov molecules.

2. *The main result*

Let $\pi^t : M \times \mathbb{R} \to M$ be a Morse–Smale flow on a compact orientable surface M. Let $(\operatorname{Sing} \pi^t) \cup (\operatorname{Cycl} \pi^t)$ be a set of critical elements of π^t, consisting of m singular points and n limit cycles.

Remark 2. If a flow $\pi^t : M \times \mathbb{R} \to M$ has n hyperbolic cycles, then it can be replaced by a flow $\varepsilon^t : N \times \mathbb{R} \to N$ on a surface N which is produced from M by cutting along each of the n cycles and by the consequent squeezing of the banks of each cut into a point. The flow ε^t has $2n$ more nodal singular points than π^t, and has no cycles. The manifold N is, in general, disconnected.

Further, we consider flows $\varepsilon^t : N \times \mathbb{R} \to M$ which we obtain from π^t by the above modification. Therefore we have the following chain of Morse–Smale flows $(M, \pi^t) \to (N, \varepsilon^t)$, where $\operatorname{Cycl} \varepsilon^t = \emptyset$.

Definition 7. By an orgraph of a Morse–Smale flow π^t one understands a tripartite orgraph X with the vertex set $V_X = \operatorname{Sing} \varepsilon^t$ and the edge set $E_X = \operatorname{Sep} \varepsilon^t$. An upper level, which we denote by α, and a lower level, which we denote by ω, are filled, correspondingly, with sources and sinks of the flow ε^t; a middle level, which we denote by σ, is filled with 4-separatrix saddles. An oriented incidence structure I_X is established as follows: for every separatrix of a saddle it corresponds to an edge which joins the saddle on a σ-level with a source (a sink) on an $\alpha(\omega)$-level and so that the direction on the edge coincides with the direction of the flow ε^t on the corresponding separatrix. Moreover,

(i) one marks on X disjoint pairs of sources (sinks), which correspond to unstable (stable) cycles of the flow π^t (see Remark 2);

(ii) if a connected component $N_0 \subseteq N$ of a flow (N, ε^t) has no saddles, then the corresponding subgraph $X_0 \subseteq X$ is a source–sink pair, joined by an edge; in this particular case $N_0 = S^2$ and the restriction of ε^t to N_0 is a polar flow.

Unfortunately, orgraphs X built upon Morse–Smale flows are insufficient for classification of such flows; see [**15**]. They lose a part of 'topological' information on the equivalence classes of π^t. Again, let $(M, \pi^t) \to (N, \varepsilon^t)$ be a Morse–Smale flow on M. Fix a clockwise (counter-clockwise) direction in each singular point $p \in N$ of the flow ε^t. Also fix a clockwise (counter-clockwise) direction in the corresponding vertices $v \in V_X$ of the orgraph X. The flow ε^t induces in v a *local rotation* (see Definition 5).

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The cyclic order on edges, incident to v, is the same as for the separatrices, provided a clockwise (counter-clockwise) direction is fixed on N.

Let $R(R^*)$ be a clockwise (counter-clockwise) rotation system, induced by ε^t in the vertices of X. An orgraph X of a flow π^t taken together with a rotation system $R(R^*)$ is denoted by $X^R(X^{R^*})$ and is called a *Peixoto orgraph of the flow* π^t. An isomorphism $\cong$ on X^R is an isomorphism of the underlying orgraphs X, taken together with the equivalence of their rotation systems. (Recall (§1.1) that the equivalence of rotation systems in two isomorphic graphs means that local rotations in isomorphic vertices either coincide, or, if they are mirror symmetric in one vertex, then they are mirror symmetric in all of them.)

THEOREM 6. *Morse–Smale flows π_1^t and π_2^t are topologically conjugate if and only if their Peixoto orgraphs are isomorphic, $X^{R_1} \cong X^{R_2}$.*

The above theorem reduces a problem of the classification of Morse–Smale flows to a problem of the classification of special graphs, endowed with rotation systems. Consider an inverse problem: which conditions should a rotation system R on an abstractly given graph X satisfy so that there exists a Morse–Smale flow π^t (on an orientable surface M) with X^R as the topological invariant?

Definition 8. An abstract orgraph is a tripartite finite orgraph $\mathcal{X}$ (disconnected, in general) with 4-valent vertices located at the middle level σ. Every σ-vertex has a pair of incoming and a pair of outgoing edges. Each vertex of σ-level is connected arbitrarily with vertices of upper (α) and lower (ω) levels with the help of incoming and outgoing edges, respectively. Furthermore:

(i) in the set of vertices of α- and ω-levels there is a set (empty, maybe) of a disjoint pairs of vertices;

(ii) if a connected component $\mathcal{X}_0 \subseteq \mathcal{X}$ of the orgraph $\mathcal{X}$ has no vertices of σ-level, then it consists of a unique α- and a unique ω-vertex, joined by a unique edge.

Definition 9. An abstract orgraph $\mathcal{X}$ is called an abstract Peixoto orgraph $\mathcal{X}^{\mathcal{R}}$, if there exists a rotation system $\mathcal{R}$ such that every cycle traced on $\mathcal{X}$ by $\mathcal{R}$ is isomorphic either to a standard or to an Andronov molecule; see §1.2.

THEOREM 7. *Let X^R be a finite orgraph, endowed with a rotation system R. Then X^R is realized as a Peixoto orgraph of a Morse–Smale flow on an orientable surface M if (and only if) it is isomorphic to an abstract Peixoto orgraph $\mathcal{X}^{\mathcal{R}}$.*

3. *Proofs*

3.1. *Proof of Theorem 6.*

Necessary conditions. Let $(M, \pi^t) \to (N, \varepsilon^t)$ be a chain, introduced in §2. It is evident that if two Morse–Smale flows π_1^t and π_2^t are conjugate by a homeomorphism h, then $X^{R_1} \cong X^{R_2}$. Indeed, consider a restriction of $h : N \to N$ to the set $\operatorname{Sep}\varepsilon_1^t \mapsto \operatorname{Sep}\varepsilon_2^t$, which establishes the isomorphism of respective orgraphs, $X_1 \cong X_2$. The equivalence of rotation systems $R_2 \sim R_1$ or $R_2 \sim R_1^*$ follows from the fact that h preserves 'the corner structure' of embeddings $i : X \to N$.

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Sufficient conditions. Suppose that $X^{R_1} \cong X^{R_2}$. One must build a homeomorphism $h : N \to N$ which conjugates flows ε_1^t and ε_2^t. As a first step, one should construct h, which establishes the equivalence of two embeddings $i_1, i_2 : X \to N$, defined by equivalent rotation systems $R_1 \sim R_2$.

LEMMA 1. *Let $i_1, i_2 : X \to N$ be two embeddings of X with the corresponding rotation systems R_1 and R_2. If $R_2 \sim R_1$ or $R_2 \sim R_1^*$, then dual graphs $X_1^\#$ and $X_2^\#$, defined by i_1 and i_2, are isomorphic, $X_1^\# \cong X_2^\#$.*

Proof of Lemma 1. According to the face tracing algorithm, rotations R_1 and R_2 generate the systems of cycles† $F_1^{(i)}, \ldots, F_m^{(i)}, i = 1, 2$, on X, which are given by the table

$$\begin{array}{ll} F_1^{(1)} = a_1^{(1)} b_1^{(1)} \ldots z_1^{(1)} & F_1^{(2)} = a_1^{(2)} b_1^{(2)} \ldots z_1^{(2)} \\ F_2^{(1)} = a_2^{(1)} b_2^{(1)} \ldots z_2^{(1)} & F_2^{(2)} = a_2^{(2)} b_2^{(2)} \ldots z_2^{(2)} \\ \vdots & \vdots \\ F_m^{(1)} = a_m^{(1)} b_m^{(1)} \ldots z_m^{(1)} & F_m^{(2)} = a_m^{(2)} b_m^{(2)} \ldots z_m^{(2)}. \end{array}$$

Since $R_2 \sim R_1(R_1^*)$, two systems of cycles $F_j^{(1)}$ and $F_j^{(2)}$ (which are the vertices of $X_1^\#$ and $X_2^\#$) will be the same, so that one easily establishes a vertex function $f_V : F_i^{(1)} \to F_j^{(2)}$ on the dual graphs.

Now take a cycle $F_k^{(1)} = a_k^{(1)} b_k^{(1)} \ldots e_k^{(1)} \ldots z_k^{(1)}$ and the corresponding cycle $F_{k'}^{(2)} = f_V(F_k^{(1)})$, $F_{k'}^{(2)} = a_{k'}^{(2)} b_{k'}^{(2)} \ldots e_{k'}^{(2)} \ldots z_{k'}^{(2)}$, $1 \le k, k' \le m$. By the definition of the rotation system, the edge $e_k^{(1)}$ will be met exactly once in a row $F_l^{(1)}$ (the case $k = l$ is not excluded). In this case one draws an edge $\overline{e}_k^{(1)}$ between $F_k^{(1)}$ and $F_l^{(1)}$, which is the edge of the dual graph $X_1^\#$. Proceeding similarly with $F_{k'}^{(2)}$ and $F_{l'}^{(2)}$, one establishes an edge function $f_E : \overline{e}_k^{(1)} \to \overline{e}_{k'}^{(2)}$ between the dual graphs $X_1^\#$ and $X_2^\#$. Since $f = f_V \cup f_E$, an isomorphism f is established and Lemma 1 is proved. □

Let $N_0 = N \setminus O(V_X)$ be a surface with a boundary, obtained from N by removing small open neighborhoods $O(V_X)$ of vertices of X in the given cellular embedding $X \to N$. Let us consider a union of disc regions $D_1 \cup \cdots \cup D_m = N_0$ such that $\overline{D}_i \cap \overline{D}_j \in E_X (i \ne j)$ intersect only at the edges of X.

LEMMA 2. *The dual graph $X^\#$ can be obtained by a homotopy of N_0, in which the D_i are transformed into the vertices of $X^\#$ and $\overline{D}_i \cap \overline{D}_j$ go to the edges of $X^\#$.*

Proof. One easily constructs a homotopy $h_t : N_0 \times [0, 1] \to N_0, t \in [0, 1]$, such that $h_0 = \mathrm{id}_{N_0}, h_1(N_0) = X^\#$. □

To finish the proof of Theorem 6 one must invert the homotopy h_t and deform a homeomorphism $h_0 : X_1^\# \to X_2^\#$, established on the dual graphs, into a homeomorphism $h_1 : N_0 \to N_0$.

Let $D_i^{(1)}$ and $D_i^{(2)}$ $(1 \le i \le m)$ be the disc regions mentioned in Lemma 2 such that $h_0(D_i^{(1)}) = D_i^{(2)}$ (where h_0 is the vertex function $V_{X_1^\#} \mapsto V_{X_2^\#}$ defined in Lemma 1). The boundary ∂D_i of D_i consists of segments $e_k \cup b_k$, where $e_k = \overline{D}_i \cap \overline{D}_j$ and $b_k \subset \partial N_0$.

† Here we prefer to talk about cycles in graphs, always keeping in mind that they are in one-to-one correspondence with the 'faces' of the embedded graph.

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First define a homeomorphism h on $\partial D_i = B \cup E$, where $B = \cup b_k$, $E = \cup e_k$. It may very well happen, that on some e_k, h is already defined (since e_k touches other regions of N_0). In this case we fix $h|_{e_k}$ and extend it to the other points of ∂D_i in an arbitrary but continuous way. It is a trivial exercise to prolong h to the interior of D_i (e.g. it can be done using a level set of some scalar function in D_i).

Now a homeomorphism $h_1 : N_0 \to N_0$, which is a homotopy deformation of a graph isomorphism h_0, is established. Choose $h = h_1$ and extend it to the whole N by 'squeezing' the holes $O(V_X)$ to the vertex points of X. The homeomorphism h is the required one, and it conjugates two embeddings $i_1, i_2 : X \to N$ of the graph X. Lemma 2 is proven. □

Now we can establish a topological conjugacy between Morse–Smale flows (N, ε_1^t) and (N, ε_2^t). Consider the embeddings $i_1(X)$ and $i_2(X)$ of the orgraph X, corresponding to ε_1^t and ε_2^t. It follows from the above arguments that there exists a homeomoprhism $h : N \to N$ bringing $i_1(X)$ to $i_2(X)$.

This homeomorphism also conjugates every foliation defined on N, and in particular those given by the flows ε_1^t and ε_2^t. Since X is an orgraph, h preserves orientation on leaves of the foliations, establishing the orbital conjugacy of flows: $\varepsilon_2^t = h^{-1} \circ \varepsilon_1^t \circ h$.

Theorem 6 is proven.

3.2. *Proof of Theorem 7.*

Necessary conditions. Let $\pi^t : M \times \mathbb{R} \to M$ be a Morse–Smale flow on an orientable surface M. Then it is a trivial exercise to check that its Peixoto orgraph X^R satisfies all the conditions of Definitions 8 and 9.

Sufficient conditions. Let $X = X_1 \cup \cdots \cup X_m$ be an orgraph consisting of m connected components (we remind the reader that $m > 1$ if and only if the flow has trivial cycles; see Remark 2). The realization theorem will be proved for each X_i $(1 \leq i \leq m)$ independently. Next, we will glue up $M = N_1 \# \cdots \# N_m$, where N_i is a realization manifold for X_i.

(i) Suppose that for some X_i, condition (ii) of Definition 8 is satisfied. Then $M_i = S^2$ and ε^t is a polar flow on a sphere.

(ii) Suppose that X_i is an orgraph with σ-level vertices, endowed with the rotation system R_i. By the Heffter–Edmonds theorem there exists a 2-cell embedding of X_i into an orientable surface N_i. By Definition 9 all cycles of R_i are either the standard or Andronov molecules shown in Figure 1. Therefore, the elementary cells of the embedding $X_i \to M_i$ will be only those shown in Figure 2. Since X_i is an orgraph, the direction on all separatrices is fixed and it remains to foliate the interiors of the cells in a standard way, so as to correspond to Figure 2. By virtue of the decomposition theorem the obtained flow on N_i will be Morse–Smale.

(iii) If all the X_i are realized as Morse–Smale flows ε_i^t on surfaces N_i, then according to condition (i) of Definition 8, one should 'blow-up' those pairs of sinks (sources) which were punctured and glue together the corresponding banks of the cuts. Finally, one obtains a Morse–Smale flow $\pi^t = \varepsilon_1^t \# \cdots \# \varepsilon_m^t$ with stable (unstable) limit cycles on the manifold $M = N_1 \# \cdots \# N_m$.

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Since all cases are considered, Theorem 7 is proven.

3.3. *Proof of Theorem 5.* The proof we give here will be a geometric one—for other (and more extended) expositions of the question, we recommend the monographs of Gross and Tucker [**5**] and Ringel [**16**].

Let X be a finite graph endowed with a rotation system R. As was mentioned earlier, according to the face tracing algorithm (see §1) the rotation R gives rise to a certain (finite) system of faces $F_1, \dots, F_m$ on X, which we denote here as

$$\begin{aligned} F_1 &= a_1 b_1 \dots z_1 \\ F_2 &= a_2 b_2 \dots z_2 \\ &\vdots \\ F_m &= a_m b_m \dots z_m, \end{aligned} \tag{1}$$

where $\{a_i \dots z_i\}_{i=1}^m$ is the set of edges of X. Each face in system (1) is a disc domain bounded by the respective edge sequence and every $e_k \in E_X$ occurs exactly twice there: $e_k = F_i \cap F_j, i \neq j$.

We will prove Theorem 5 by an explicit construction of the fundamental domain of the manifold M. It will be a polygon (namely, a $4g$-gon) Δ, filled with the system of faces (1), $\Delta = F_1 \cup \dots \cup F_m$, and such that the boundary components of $\partial\Delta$ are identified in a natural way. Next we will obtain $M \simeq \Delta/\Gamma$, gluing up Δ by the action of the absolutely discontinuous group Γ, generated by (1).

Take a face $\Delta_1 = F_1$ of list (1) and place it arbitrarily in the plane, labeling all the boundary edges by $a_1, b_1, \dots, z_1$. Each edge e_k of this sequence will occur once more in a row F_k (the case $k = 1$ is not excluded), so that F_1 has a common edge with every face of the sequence $F_{a_1}, F_{b_1}, \dots, F_{z_1}$. We glue up these faces to the polygon Δ_1 along its boundary $\partial\Delta_1$. Taking the edges lying at the boundary $\partial\Delta_2$ of the polygon $\Delta_2 = F_1 \cup F_{a_1} \cup F_{b_1} \cup \dots \cup F_{z_1}$, we use list (1) to find all adjacent faces of Δ_2 and glue them correspondingly together to obtain a polygon Δ_3. Proceeding inductively, one stops when all faces in list (1) have been used. As a result, we come to a polygon $\Delta_p = \Delta$, pictured in Figure 3.

The boundary $\partial\Delta$ of the polygon consists of the edges which we denote $abcba\dots$ and since each edge on $\partial\Delta$ has its counterpart also in $\partial\Delta$, all the sides of the polygon Δ are pairwise identified.

We call a vertex point p of the graph X an inner point, if $p \in \operatorname{int}\Delta$. Otherwise, it is called a corner point (in Figure 3 the point p is a corner point, while q is an inner point). Clearly, if a corner point p has a pair of its incident edges identified, then it can be made an inner point (for example, in the case $a = b$ the point p in Figure 3 becomes the inner point). In what follows, we suppose that for the polygon Δ all such identifications are made.

Now Δ is a $2n$-gon (a polygon with $2n$ sides) and there are no nearby sides in $\partial\Delta$ which are identified. Let us consider the following cases.

(i) If $n = 2g$, then Δ is a $4g$-gon. In this case Δ is a fundamental domain of a surface $M \simeq \Delta/\Gamma$, where Γ is an absolutely discontinuous group, generated by the identifications on $\partial\Delta$. Clearly, we have a 2-cell embedding $X \to M$, where M is

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an orientable surface of genus g.

(ii) If $n = 2g - 1$, then Δ is a $(4g - 2)$-gon. To arrive at the 'standard' case (i), one must add two extra edges to $\partial\Delta$. It can be done as follows.

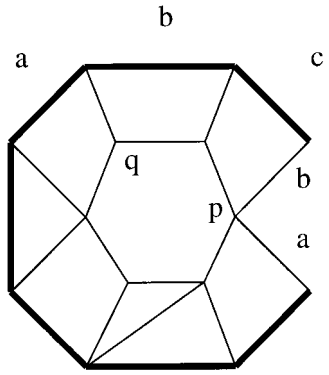


FIGURE 3. Fundamental polygon.

Take an edge $e_0 \in E_X$ incident to a corner point $p \in \partial\Delta$ and split p to p_1 and p_2, moving them aside. The edge e_0 will also be truncated into the edges $e_0^{(1)}$ and $e_0^{(2)}$. On the boundary $\partial\Delta$ there will appear an extra edge p_1p_2. Similarly, we add an extra edge q_1q_2, where $q_1 = \Gamma(p_1), q_2 = \Gamma(p_2)$. Now, one arrives at case (i), identifying p_1p_2 with q_1q_2. Thus far, we have obtained an embedding $\overline{X} \to M$, where $\overline{X}$ has two extra triangle faces in M. One squeezes these faces bringing the point p_1 to the point p_2 and q_1 to q_2. We will now obtain a 2-cell embedding $X \to M$, where M is an orientable surface of genus g.

Theorem 5 is proved. □

4. *Example*

To illustrate how the rotation systems work, here we will consider a classical example, taken from [**15**]. Let π_1^t and π_2^t be two Morse–Smale flows on a sphere, as sketched in Figure 4. It was shown in [**15**] that the orgraphs X of these flows coincide; see Figure 5. If one takes the corresponding rotation systems, they also coincide in all vertices of X except in α_3 and ω. Namely, for the flows π_1^t and π_2^t the cyclic order in each vertex is given in Table 1. (The edges of X are labeled as is indicated in Figure 5.) Clearly, $R_1 \not\simeq R_2$ and, therefore, their invariants X^{R_1} and X^{R_2} are distinct.

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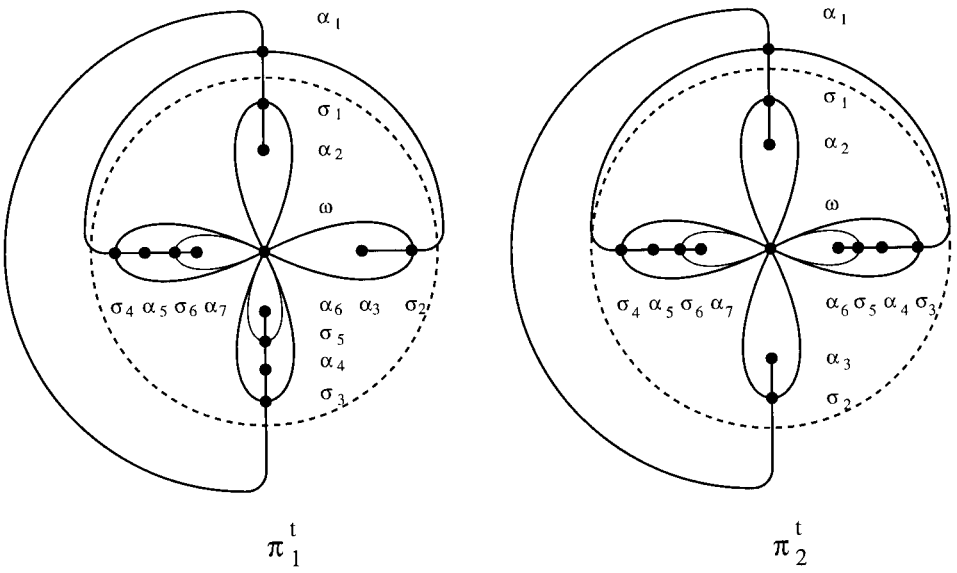


FIGURE 4. Peixoto's counterexample.

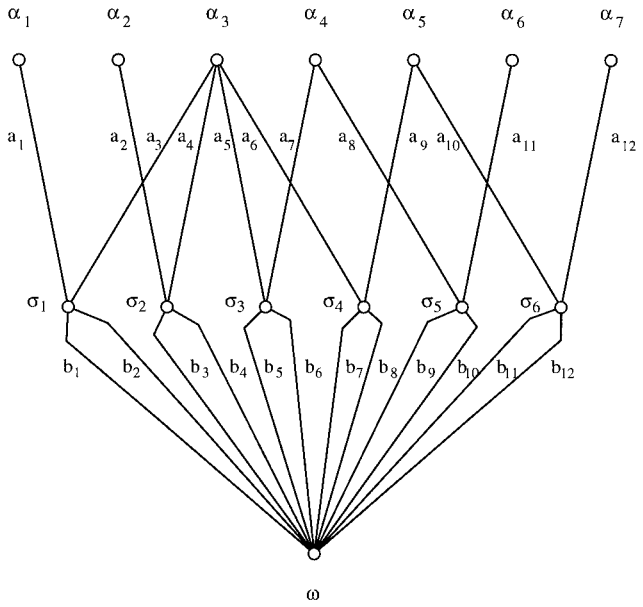


FIGURE 5. Peixoto's graph.

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TABLE 1.

$v \in V_X$	Local rotation for the flow π_1^t	Local rotation for the flow π_2^t
α_1	trivial	—
α_2	trivial	—
α_3	$a_3a_6a_5a_4$	$a_3a_6a_4a_5$
α_4	trivial	—
α_5	trivial	—
α_6	trivial	—
α_7	trivial	—
σ_1	$a_1b_1a_3b_2$	—
σ_2	$a_2b_3a_4b_4$	—
σ_3	$a_5b_6a_7b_5$	—
σ_4	$a_6b_8a_9b_7$	—
σ_5	$a_8b_{10}a_{11}b_9$	—
σ_6	$a_{10}b_{12}a_{12}b_{11}$	—
ω	$b_1b_2b_3b_4b_5b_9b_{10}b_6b_7b_{11}b_{12}b_8$	$b_1b_2b_5b_9b_{10}b_6b_3b_4b_7b_{11}b_{12}b_8$

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