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Lyapunov spectrum for multimodal maps

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Abstract. We study the dimension spectrum of Lyapunov exponents for multimodal maps of the interval and their generalizations. We also present related results for rational maps on the Riemann sphere.

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1. *Introduction*

This paper is an interval multimodal mapping counterpart of the authors' earlier paper on complex rational maps on the Riemann sphere, [**14**]. Some new ideas also yield progress in the complex setting (see Appendix A).

We consider a C^3 (or C^2 with bounded distortion) multimodal map f on a finite union $\widehat{I}$ of pairwise disjoint closed intervals in $\mathbb{R}$ and the iteration of f on the maximal forward invariant subset K (in an equivalent approach: K is f-forward invariant and $f|_K$ satisfies the Darboux property†). We assume that $f|_K$ is topologically transitive and has positive topological entropy. Given $\alpha \in \mathbb{R}$, we consider the level sets of Lyapunov exponents of f

$$\mathcal{L}(\alpha) := \left\{ x \in K : \chi(x) := \lim_{n\to\infty} \frac{1}{n} \log |(f^n)'(x)| = \alpha \right\}$$

and study the function $\alpha \mapsto \dim_{\mathrm{H}} \mathcal{L}(\alpha)$, which is called Hausdorff dimension spectrum of level sets $\mathcal{L}(\alpha)$ for Lyapunov exponents, or, shortly, the *Lyapunov spectrum*. We prove that this spectrum can be expressed in terms of the Legendre transform of the *geometric pressure function*, that is, of the function $t \mapsto P(f|_K, -t \log |f'|)$. We only allow non-flat critical points in K.

This is a classical result in the uniformly hyperbolic case (see, for example, [**28**, **29**, **45**] and also earlier papers by mathematical physicists, for example, [**6**, **12**, **41**]). But our system is just 'chaotic' in the sense that it is topologically transitive and has positive topological entropy. Observe that our system has a uniformly hyperbolic factor obtained in the following way: identifying endpoints of gaps (intervals complementary to K in the case where K is a Cantor set; it can be *a priori* a finite union of closed intervals $\widehat{I}$), by a continuous change of coordinates we get a piecewise continuous piecewise monotone map of the interval with constant slope $h_{\mathrm{top}}(f|_K)$. In particular, this map is piecewise uniformly hyperbolic, see [**36**, Appendix A]. Unfortunately, the geometric potential

† See [**36**, Appendix A] and references therein.

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$x \mapsto -t \log |f'(x)|$ does not survive this coordinate change. Hence, this approach would not help in studying the geometric properties of the original map.

We are also studying the irregular part of the spectrum, that is, the level sets of irregular points

$$\mathcal{L}(\alpha, \beta) := \{x \in K : \underline{\chi}(x) = \alpha,\ \overline{\chi}(x) = \beta\}, \quad \alpha < \beta$$

for the lower and the upper Lyapunov exponents. In some cases we obtain a precise formula, conjectured in previous papers in both the real and the complex cases. Our main tools are hyperbolic approximation to show lower bounds and conformal measures and their generalizations to show upper bounds. The existence of those conformal measures has been proved via an *inducing scheme* in [**36**, Theorem A].

In particular, we show that certain level sets are empty. Denoting by $\chi_{\inf}$ the infimum of exponents of f-invariant probability measures on K, we obtain $\chi(x) \geq \chi_{\inf}$ for every $x \in K$ for which $\chi(x)$ exists and is different from $-\infty$. This was conjectured before in [**20**, **21**]†.

Moreover, in the case that $K = \widehat{I}$, f is non-exceptional, and f has no indifferent periodic orbits in K, we conclude that $\underline{\chi}(x)^\sharp \geq \chi_{\inf}$ provided $\overline{\chi}(x) > 0$. Here $\underline{\chi}(x)^\sharp$ denotes a certain non-negative average of $\underline{\chi}(x)$ and $\overline{\chi}(x)$, see §6.

The novelty of this paper, compared to methods used in [**14**], is that we investigate not only non-critical pull-backs but also allow pull-backs of bounded criticality (Topological Collet–Eckmann type feature).

Our setting of multimodal maps has been elaborated recently by Przytycki and Rivera-Letelier [**36**]. Substantial earlier contributions to this theory have been made in particular by Hofbauer, Iommi and Todd.

1.1. *Generalized multimodal maps.* In this paper we work in the *multimodal* setting elaborated in [**36**]. We will consider a C^r map f on a certain open subset of the real line into the real line. We assume $r = 2$ if not stated otherwise. A point x is said to be *critical* if $f'(x) = 0$. We denote by $\mathrm{Crit}(f)$ the set of all critical points of f. We denote by $\mathrm{Crit}_T(f) \subset \mathrm{Crit}(f)$ the set of all turning critical points and by $\mathrm{Crit}_I(f) \subset \mathrm{Crit}(f)$ the set of all inflection critical points. We say that f is *non-flat* at a critical point c if $f(x) = \pm|\phi(x)|^d + f(c)$ for some $d \geq 2$ and ϕ a C^r diffeomorphism in a neighborhood of c with $\phi(c) = 0$, see [**25**, Ch. IV]. In this paper, we assume that f has only a finite number of critical points and that all of them are non-flat. We say such a map is *multimodal*.

We denote by $\deg f$ the geometric degree of f, that is,

$$\deg f := \sup_{x \in K} \# f^{-1}|_K(\{x\}).$$

Let $K \subset \mathbb{R}$ be an infinite compact set in the domain of f. We assume that K is *forward invariant* under f, that is, $f(K) = K$. We also assume that $f : K \to K$ is *topologically transitive*, that is, for all $V_1, V_2 \subset K$ non-empty and open in K there is $n \geq 1$ such that $f^n(V_1) \cap V_2 \neq \varnothing$.

† After this paper was written, J. Rivera-Letelier explained to us that $\chi(x) \geq \chi_{\inf}$ follows easily from $-\lim_{n\to\infty} 1/n \log \max |W_n| \geq \chi_{\inf}$, where the maximum is taken over all connected components W_n of preimages $f^{-n}(W)$ with $W \subset \widehat{I}$ a sufficiently small interval. The latter inequality was proved in [**42**] under additional, but not substantial, assumptions on f. See Appendix B for details.

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We will usually assume that $f|_K$ has positive topological entropy.

We assume that there exists a covering of K by a finite family of pairwise disjoint closed intervals $\widehat{I}_j$ in the domain of f, with end points in K, such that K is the maximal compact forward invariant set in

$$\widehat{I}_K := \bigcup_j \widehat{I}_j.$$

We assume that $\widehat{I}_K \setminus K$ does not contain critical points.

Furthermore, we consider a C^r extension of f to some neighborhoods U^j of $\widehat{I}_j$ such that U^j are pairwise disjoint and none of $U^j \setminus K$ contains a critical point. Let $U = \bigcup_j U^j$. Sometimes to distinguish U we write U_K.

Following [**36**], we denote the space of the quadruples $(f, K, \widehat{I}_K, U)$ satisfying the above properties by $\mathscr{A}^r_+$. We simply write $\mathscr{A}_+$ if $r = 2$. As in [**36**], the subscript $+$ indicates that $f|_K$ has positive topological entropy. We will omit it if this assumption is not necessary. Sometimes we abbreviate to $(f, K, \widehat{I}_K)$, (f, K, U), or just (f, K). For such quadruples, triples, couples, we sometimes use the name *generalized $\mathscr{A}$-multimodal maps* or *systems*.

If $(f, K) \in \mathscr{A}$, then K is either the union of a finite collection of compact intervals $K = \widehat{I}_K$ or a Cantor set, see [**36**, Lemma 2.1].

1.2. *Periodic orbits.* We say a point $p \in U$ is *periodic* if there exists $m \geq 1$ such that $f^m(p) = p$, and we call the smallest such number m its *period*. We denote by $\mathcal{O}(p)$ its periodic orbit and define the *basin of attraction* of $\mathcal{O}(p)$ by

$$\mathscr{B}(p) := \operatorname{int}\{x \in U : f^n(x) \to \mathcal{O}(p) \text{ as } n \to \infty\}.$$

The orbit $\mathcal{O}(p)$ is said to be *attracting* if $\mathcal{O}(p) \subset \mathscr{B}(\mathcal{O}(p))$. The orbit is said to be *repelling* if $|(f^m)'(p)| \geq 1$ and some small neighborhood W of $\mathcal{O}(p)$ is forward invariant for $g = f|_W{}^{-1}$ and satisfies $g^n(W) \to \mathcal{O}(p)$ as $n \to \infty$. If $|(f^m)'(p)| > 1$ then p is said to be *hyperbolic repelling*. In the remaining cases we say $\mathcal{O}(p)$ is *indifferent*. We denote the set of indifferent periodic points by $\operatorname{Indiff}(f)$.

The multimodal systems we study in this paper are natural generalizations of the class of multimodal maps of the interval $f : I \to I$ (extended to a neighborhood U of I), where $K = I \setminus \bigcup_p \mathscr{B}(p)$ is called a *Julia set*, see [**36**, Example 1.8] or [**25**, Ch. IV]. A wider class of examples are so-called *basic sets* in the spectral decomposition, see again [**36**, Example 1.9] or [**25**, Ch. III.4].

A special case occurs if $K = \widehat{I}_K$ and, in particular, if $K = I$. This will allow us to prove stronger forms of Theorems 2 and 3.

For $(f, K) \in \mathscr{A}$ there are no periodic orbits in $\widehat{I}_K \setminus K$. Further, periodic orbits in K are either hyperbolic repelling (and there are infinitely many such orbits if $(f, K) \in \mathscr{A}_+$) or non-hyperbolic repelling or indifferent (and of those two types there are at most finitely many orbits). By changing f outside K and shrinking U if necessary, we can also assume that periodic orbits outside K are hyperbolic repelling (see [**36**, Remark 1.6 and Appendix A]).

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Definition 1.1. For $(f, K, U) \in \mathscr{A}$ we say K is *weakly isolated* if there exists an open set $U' \subset U$ such that every f-periodic orbit $\mathcal{O}(p)$ which is contained in U' must be contained in K.

Recall that in the definition of $\mathscr{A}$ we required that K is the maximal invariant set in $\widehat{I}_K$, so the weak isolation condition is not automatically satisfied; *a priori* there can be hyperbolic repelling periodic orbits close to K intersecting $\mathbb{R}\backslash\widehat{I}_K$, see [**36**, Example 2.12].

If K is maximal in an open neighborhood of K, then it is weakly isolated, and even isolated (see the definition in §2.1), but observe that $\widehat{I}_K$ does not contain an open neighborhood of K.

In the main theorems of this paper we need to assume that K is weakly isolated. Notice however that if K is a Julia set (see the definition above), then the weak isolation condition is automatically satisfied.

1.3. *Distortion.* Let us introduce some more notation following the terminology in [**25**]. We define as

$$\operatorname{Dist} g|_Z := \sup_{x,y\in Z} \frac{|g'(x)|}{|g'(y)|}$$

the maximal *distortion* of a differentiable map g on a set Z. Given two intervals $U \subset V$, we call V an *ε-scaled neighborhood* of U if $V\backslash U$ has two components of length $\varepsilon|U|$. In this case, we also define

$$V := (1 + 2\varepsilon) \diamond U. \tag{1.1}$$

Correspondingly, we write

$$U = (1 + 2\varepsilon)^{-1} \diamond V.$$

We say that $(f, K) \in \mathscr{A}$ satisfies a *bounded distortion condition* if there exists $\delta > 0$ such that for every $\varepsilon > 0$ there is $C = C(\varepsilon) > 0$ such that the following holds: for every pair of intervals $I_1, I_2 \subset U$ such that I_1 intersects K and $|I_2| \le \delta$, and every $n > 0$ for which f^n maps an interval I_1' containing I_1 diffeomorphically onto an interval I_2' being the ε-scaled neighborhood of I_2,

$$\operatorname{Dist} f^n|_{I_1} \le C(\varepsilon). \tag{1.2}$$

This easily implies that there exists $\tau(\varepsilon)$ such that I_1' contains a $\tau(\varepsilon)$-neighborhood of I_1. If $(f, K) \in \mathscr{A}$ satisfies the bounded distortion condition, we write $(f, K) \in \mathscr{A}^{\mathrm{BD}}$.

Remark 1.2. For $(f, K) \in \mathscr{A}^{\mathrm{BD}}$, the only periodic orbits in K are either hyperbolic repelling or indifferent one-side repelling (see [**36**, Remark 1.11]).

If $(f, K) \in \mathscr{A}^3$ and all periodic orbits in K are hyperbolic repelling, then f can be changed outside K if necessary, so that $(f, K) \in \mathscr{A}^{\mathrm{BD}}$ (see [**36**, comments after Theorem A, Remark 2.14, and Lemma A.4]). The $\mathscr{A}^{\mathrm{BD}}$ assumption can therefore be considered as the weaker and the appropriate one if only facts concerning $f|_K$ are asserted. Indeed, it assumes only C^2 smoothness and allows indifferent periodic orbits in K.

1.4. *Exceptional sets.* Denote by $NO(f, K)$ the set of points where $f|_K$ is not open. Observe that $NO(f, K) \subset \operatorname{Crit}_T(f) \cup \partial\widehat{I}_K$ (see also [**36**, Lemma 2.2]). We call

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$S := \mathrm{Crit}(f) \cup \partial \widehat{I}_K$ the *singular set* and $S' := \mathrm{Crit}(f) \cup NO(f, K)$ the *restricted singular set*. In fact, S' plays a more substantial role in this paper (and in [**36**]) than S does.

Given an arbitrary finite set $\Sigma \subset K$, we say a non-empty set $E \subset K$ is *weakly* Σ*-exceptional* if E is non-dense in K and satisfies

$$f|_K{}^{-1}(E) \setminus \Sigma \subset E. \tag{1.3}$$

Condition (1.3) can be read as: all backward trajectories for f which start in E and go outside E are 'immediately blocked' by Σ. We say that E is Σ*-exceptional* if it is weakly Σ-exceptional and forward invariant.

Observe that the union of (weakly) Σ-exceptional sets is (weakly) Σ-exceptional. Note that in the case that $(f, K) \in \mathscr{A}_+$, by [**36**, Proposition 2.7] all weakly Σ-exceptional sets E are finite and have a uniformly bounded cardinality. Therefore, there is a finite maximal weakly Σ-exceptional set. Indeed, the union of all weakly Σ-exceptional sets clearly has the property (1.3) and is not dense since there is a uniform bound for its cardinality.

Finally, we say a point $x \in K$ is *(weakly)* Σ*-exceptional* if x belongs to a (weakly) Σ-exceptional set.

We shall apply this for $\Sigma = S'$. If there does not exist any S'-exceptional set, we say f is *non-S'-exceptional* or just *non-exceptional*, otherwise we say f is *exceptional*.

Example 1.3. Chebyshev maps such as, for example, $f(x) = x^2 - 2$ with $K = \widehat{I}_K$ being the interval $[-2.2]$, are S-exceptional, but many other interval maps are not.

The critical value of a unimodal map (for example, a quadratic polynomial) constitutes a (one-element) weakly S-exceptional set.

1.5. *Lyapunov exponents.* We are interested in the spectrum of Lyapunov exponents of the map $f\colon K \to K$. Given x, we denote by $\underline{\chi}(x)$ and $\overline{\chi}(x)$ the *lower* and *upper Lyapunov exponents* at x, respectively, where

$$\underline{\chi}(x) := \liminf_{n\to\infty} \frac{1}{n} \log |(f^n)'(x)|, \quad \overline{\chi}(x) := \limsup_{n\to\infty} \frac{1}{n} \log |(f^n)'(x)|.$$

If both values coincide then we call the common value the *Lyapunov exponent* at x and denote it by $\chi(x)$. We say such x is *Lyapunov regular*. We allow $\chi(x) = -\infty$. Given $\alpha \in \mathbb{R}$, let

$$\mathcal{L}(\alpha) := \{x \in K\colon \chi(x) = \alpha\},$$

that is, the set of Lyapunov regular points in K which have exponent α. For given numbers $0 \le \alpha \le \beta$, we consider also the following level sets

$$\mathcal{L}(\alpha, \beta) := \{x \in K\colon \underline{\chi}(x) = \alpha,\ \overline{\chi}(x) = \beta\}.$$

If $\alpha < \beta$, then $\mathcal{L}(\alpha, \beta)$ is contained in the set of so-called *irregular points*

$$\mathcal{L}_{\mathrm{irr}} := \{x \in K\colon \underline{\chi}(x) < \overline{\chi}(x)\}.$$

Given a compact f-invariant set $X \subset K$, we denote by $\mathcal{M}(X)$ the set of all f-invariant Borel probability measures supported on X. We denote by $\mathcal{M}_{\mathrm{E}}(X)$ the subset of ergodic measures contained in $\mathcal{M}(X)$. We simply write $\mathcal{M} = \mathcal{M}(X)$ and $\mathcal{M}_{\mathrm{E}} = \mathcal{M}_{\mathrm{E}}(X)$ if $X = K$.

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Note that in the following definitions it is irrelevant whether we consider $\mu \in \mathcal{M}_{\mathrm{E}}$ or $\mu \in \mathcal{M}$. Given $\mu \in \mathcal{M}$, we denote by $h_\mu(f)$ the *measure theoretic entropy* of μ and by

$$\chi(\mu) := \int \log |f'| \, d\mu$$

the *Lyapunov exponent* of μ (if $\mu \in \mathcal{M}_{\mathrm{E}}$ then for almost every x, $\chi(\mu)$ is equal to $\chi(x)$ by Birkhoff's Ergodic Theorem). Notice that for $(f, K) \in \mathscr{A}$ we have $\chi(\mu) \geq 0$ for any probability f-invariant μ supported on K (see [**32**, Theorem B], and [**42**, Appendix A] for a simplified proof).

1.6. *Pressure and Legendre-like transform.* Given $t \in \mathbb{R}$, we define the function $\varphi_t \colon K \to [-\infty, +\infty]$ as

$$\varphi_t(x) := -t \log |f'(x)|. \tag{1.4}$$

We define the *variational pressure* of φ_t (with respect to $f|_K$) as

$$P_{\mathrm{var}}(\varphi_t) := \max_{\mu \in \mathcal{M}(f)} \left(h_\mu(f) + \int_\Lambda \varphi_t \, d\mu \right) = \max_{\mu \in \mathcal{M}(f)} (h_\mu(f) - t\chi(\mu)). \tag{1.5}$$

We shall often denote this pressure (and other pressures being equal to it, see (2.1)) simply by $P(t)$ and call it the *geometric pressure*.

The function $t \mapsto P(t)$ is convex as the maximum of convex (even affine) functions

$$t \mapsto h_\mu(f) - t\chi(\mu).$$

It is monotone decreasing, since for all $\mu \in \mathcal{M}(f)$ we have $\chi \geq 0$ (see the references above).

Let

$$\chi_{\inf} := \inf\{\chi(\mu) \colon \mu \in \mathcal{M}\}, \quad \chi_{\sup} := \sup\{\chi(\mu) \colon \mu \in \mathcal{M}\}. \tag{1.6}$$

The definition of the variational pressure (1.5) implies

$$\chi_{\inf} = \lim_{t \to \infty} -\frac{1}{t} P_{\mathrm{var}}(\varphi_t), \quad \chi_{\sup} = \lim_{t \to -\infty} -\frac{1}{t} P_{\mathrm{var}}(\varphi_t).$$

Observe that $\chi_{\inf}$ and $\chi_{\sup}$ coincide with the inclinations of, respectively, the right and left asymptotes of $-P_{\mathrm{var}}(\varphi_t)$, see [**14**].

Given real $\alpha \neq 0$, let us define

$$F(\alpha) := \frac{1}{|\alpha|} \inf_{t \in \mathbb{R}} (P(t) + \alpha t). \tag{1.7}$$

Observe that $\alpha \mapsto \inf_t (P(t) + \alpha t)$ is a Legendre-like transform of $P(t)$; more precisely, $\alpha \mapsto -\inf_t (P(t) - \alpha t)$ is the Legendre transform of the function $P(t)$ (see, for example, [**29**, **40**]). Note that, while the function $\alpha \mapsto \inf_t (P(t) + \alpha t)$ is always concave, the function $\alpha \mapsto F(\alpha)$ is not always concave (see, for example, [**18**]). However, on any compact interval it achieves its minimal value at an endpoint. Some of the possible shapes of the graphs of the function $t \mapsto P(t)$ are shown in Figure 1; the corresponding graphs of $\alpha \mapsto F(\alpha)$ are shown in Figure 2. Also let

$$F(0) := \lim_{\alpha \to 0+} F(\alpha). \tag{1.8}$$

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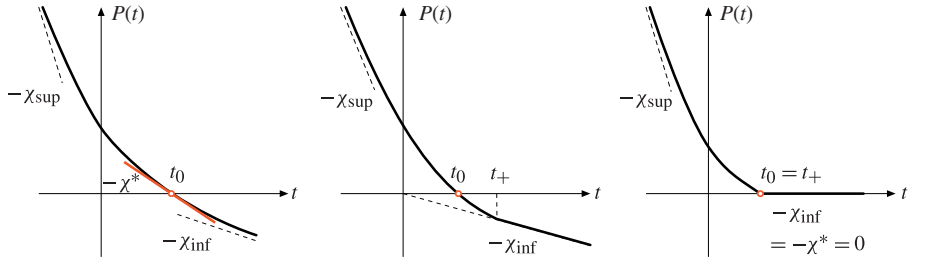


FIGURE 1. The geometric pressure: LyapHyp with $t_+ = \infty$, LyapHyp with $t_+ < \infty$, and non-LyapHyp.

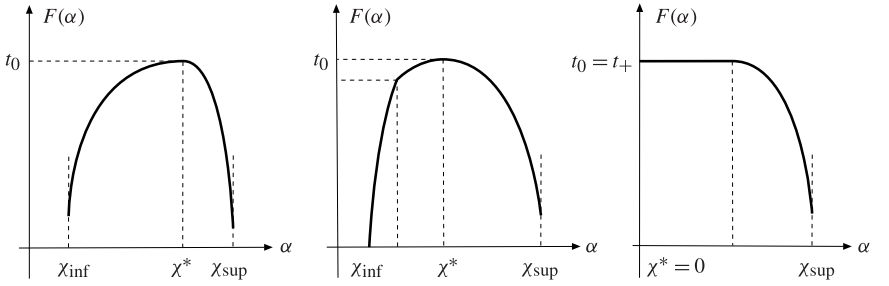


FIGURE 2. $F(\alpha)$: LyapHyp with $t_+ = \infty$, LyapHyp with $t_+ < \infty$, and non-LyapHyp.

Note that $F \neq -\infty$ precisely on the interval $[\chi_{\inf}, \chi_{\sup}]$, where F is non-negative.

Define

$$t_0 := \inf\{t \in \mathbb{R} \colon P_{\mathrm{var}}(\varphi_t) = 0\}.$$

This number is well defined since the latter set is non-empty, see [**36**, Proposition 1.19]. It coincides with the so-called hyperbolic dimension, that is, the supremum of the Hausdorff dimension of expanding Cantor repellers (see §3.4 and [**36**, **40**]).

There are two significantly different behaviors of f:

The case $\chi_{\inf} > 0$*, Lyapunov hyperbolic condition (LyapHyp)*†. This property implies that $P(t) \to -\infty$ as $t \to \infty$, with the slope bounded away from zero. Note that for every $(f, K) \in \mathscr{A}_+^{\mathrm{BD}}$ satisfying the weak isolation condition, LyapHyp is equivalent to TCE together with the absence of indifferent periodic points (see §2.4 for the definition of TCE and [**36**, Theorem C]).

The case $\chi_{\inf} = 0$. In this case we have $P(t) \equiv 0$ for all $t \geq t_0$. Indeed, by contradiction, assume that $P(t) < 0$. Since for all $\mu \in \mathcal{M}$ by the variational principle we have $A(\mu, t) := h_\mu(f) - t\chi(\mu) \leq P(t)$ and $A(\mu, 0) = h_\mu(f) \geq 0$, we would obtain $\chi(\mu) = (A(\mu, 0) - A(\mu, t))/t \geq -P(t) > 0$, a contradiction.

This distinction is illustrated in Figure 1. For the definition of t_+, see §2.2.

Denote by χ^* the right derivative of $P(t)$ at $t = t_0$. Notice that $F(\alpha)$ attains its maximum at χ^* and note that this maximum is equal to t_0. Observe that $F(\alpha)$ is increasing

† In [**36**] this property is called *Lyapunov*.

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on $[\chi_{\inf}, \chi^*]$ and non-increasing on $[\chi^*, \chi_{\sup}]$. Hence, for every $\alpha \le \beta$ we have

$$\min\{F(q) : \alpha \le q \le \beta\} = \min\{F(\alpha), F(\beta)\}. \tag{1.9}$$

In the case $\chi_{\inf} = 0$, the maximum of F is attained at 0, so we have $F(0) = t_0$. If $P(t)$ is not differentiable at t_0, then we have $F(\alpha) \equiv t_0$ on the whole interval between the right and the left derivatives of $-P(t)$ at t_0.

Let us remark that usually, for example for each C^3 multimodal map f of the interval with all periodic points being hyperbolic repelling and $|(f^n)'(f(c))| \to \infty$ as $n \to \infty$ for all f-critical points c, we have $t_0 = 1$ and there exists an f-invariant probability measure absolutely continuous with respect to the Lebesgue measure (abbreviated as pacim), see [**4**]. However, even in the special case of f being S-unimodal, pacim need not exist, see [**1**] and references therein. Furthermore, $t_0 < 1$ can happen, for example for unimodal Fibonacci combinatorics polynomials $z^\ell + c$ with integer ℓ large enough, see [**3**, Theorem 10.5]. For other examples see [**5**].

1.7. *Main results.*

THEOREM 1. *Let $(f, K) \in \mathscr{A}_+^{\mathrm{BD}}$ satisfy the weak isolation condition and be non-exceptional. For any $\alpha \le \beta \le \chi_{\sup}$ with $\beta > 0$, and additionally assuming $\alpha > 0$ if $\chi_{\inf} = 0$, we have*

$$\min\{F(\alpha), F(\beta)\} \le \dim_{\mathrm{H}} \mathcal{L}(\alpha, \beta) \le \max\Big\{0, \max_{\alpha \le q \le \beta} F(q)\Big\}. \tag{1.10}$$

In particular, for any $\alpha \in [\chi_{\inf}, \chi_{\sup}]\backslash\{0\}$ we have

$$\dim_{\mathrm{H}} \mathcal{L}(\alpha) = F(\alpha).$$

We also have

$$\dim_{\mathrm{H}} \mathcal{L}(0) \ge F(0).$$

Moreover,

$$\{x \in K : -\infty < \chi(x) < \chi_{\inf}\} = \varnothing, \tag{1.11}$$

$$\{x \in K : \overline{\chi}(x) > \chi_{\sup}\} = \varnothing, \tag{1.12}$$

and

$$\dim_{\mathrm{H}}\{x \in K : 0 < \overline{\chi}(x) < \chi_{\inf}\} = 0. \tag{1.13}$$

The assumptions of the above theorem are satisfied if, for example, $(f, K) \in \mathscr{A}_+^3$ and there are no indifferent periodic orbits (see Remark 1.2).

Remark 1.4. Theorem 1 has the same assertion as [**14**, Theorem 2] (except that we do not assume $\alpha > 0$ which is, in fact, needed neither here nor there). The strategy of the proof is also the same. However, there are technical differences, and the part which has an entirely new proof, since the holomorphic one does not work, is to show that $\{x \in K : 0 \le \chi(x) < \chi_{\inf}\} = \varnothing$ in (1.11). This answers a conjecture in [**21**, p. 4] and [**20**, Remark 4.1]. In Theorem 3 below, we prove an even stronger fact under the additional assumptions $K = \widehat{I}_K$ and K contains no indifferent periodic orbits.

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Remark 1.5. After this paper was written, J. Rivera-Letelier explained to us that (1.11) in Theorem 1 follows from his paper [**42**]. See Appendix B for the discussion.

Let us mention that the above results extend [**20**] in the following sense: our results enable us to also cover the boundary points of the spectrum $\mathcal{L}(\alpha)$ for $\alpha = \chi_{\inf}$ and $\chi_{\sup}$, as well as the irregular points $\mathcal{L}(\alpha, \beta)$ for $\alpha < \beta$. Moreover, we are additionally able to consider maps that have indifferent periodic orbits.

Hofbauer [**16**] proves results about the Lyapunov spectrum in a situation which is, in some aspects, considerably more general than the one discussed in the present paper (he considers piecewise monotone piecewise continuous interval maps), but which does not allow singularities $\pm\infty$ for the geometric potential.

As in the complex case, we do not know whether $\{x \in K : \chi(x) = -\infty\} = \bigcup_{n\geq 0} f^{-n}(\mathrm{Crit}(f))$, unless there is at most one critical point in K. In this case the proof of the equality is the same as in the complex situation.

Under additional assumptions we can improve (1.10), the right-hand side inequality, sometimes even getting the equality.

THEOREM 2. *Let $(f, K) \in \mathscr{A}_+^{\mathrm{BD}}$ (or $\mathscr{A}_+^3$) satisfy the weak isolation condition and be non-exceptional and without indifferent periodic orbits. Then, given numbers $\alpha \leq \beta \leq \chi_{\sup}$ with $\beta > 0$, we have*

$$\dim_{\mathrm{H}} \mathcal{L}(\alpha, \beta) \leq \max\{0, F(\beta)\}. \tag{1.14}$$

If we assume $K = \widehat{I}_K$ (that is, K is the finite union of intervals), then

$$\dim_{\mathrm{H}} \mathcal{L}(\alpha, \beta) \leq \max\{0, \min\{F(\alpha^\sharp), F(\beta)\}\}, \tag{1.15}$$

where

$$\alpha^\sharp := \frac{\beta}{1 + (\beta - \alpha)/\chi_{\sup}}.$$

Note that $\alpha^\sharp$ is positive and $\alpha < \alpha^\sharp < \beta$ (except for the case $\alpha = \beta$, when $\alpha^\sharp = \alpha = \beta$).

Remark 1.6. If $F(\alpha) \geq F(\beta)$ then, together with Theorem 1, we obtain the equality

$$\dim_{\mathrm{H}} \mathcal{L}(\alpha, \beta) = F(\beta). \tag{1.16}$$

In particular, we obtain the equality (1.16) for non-LyapHyp systems (f, K) (systems where $\chi_{\inf} = 0$).

For uniformly hyperbolic systems, for all $\chi_{\inf} \leq \alpha \leq \beta \leq \chi_{\sup}$, the equality

$$\dim_{\mathrm{H}} \mathcal{L}(\alpha, \beta) = \min\{F(\alpha), F(\beta)\}$$

holds. The question of whether this equality holds in general remains open.

The trouble with improving the maximum by the minimum in the right-hand estimate (1.10), is that the times one can go to large scale avoiding S' for a conical point x (see §3.8) are not related to the times of an *a priori* given Lyapunov exponent between α and β. The new idea is to consider a larger set of (restricted Pliss hyperbolic) times where one can go to large scale with finite criticality, thus allowing us to control the exponent.

We deduce also the following strong version of a part of the property (1.11) of Theorem 1.

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THEOREM 3. *Let* $(f, K) \in \mathscr{A}_+^{\mathrm{BD}}$ *(or* $\mathscr{A}_+^3$*) satisfying* $K = \widehat{I}_K$ *be non-exceptional and without indifferent periodic orbits. Then, given numbers* $\alpha \le \beta$ *with* $\alpha^\sharp < \chi_{\inf}$ *and* $\beta > 0$*, the set* $\mathcal{L}(\alpha, \beta)$ *is empty.*

Remark 1.7. The proof of the fact that $\{x \in K : -\infty < \chi(x) < \chi_{\inf}\} = \varnothing$ (observe that this set concerns Lyapunov regular points only) in Theorem 1, provided in §5, is conceptually simpler: it does not use the existence of conformal measures, see the end of §2.3 for this notation.

The existence of true conformal measures is the only reason for which we have to assume no indifferent periodic orbits in Theorems 2 and 3.

Theorem 1 is proven in §4, except for relation (1.11) which is shown in §5. Theorems 2 and 3 are proven in §6.

2. *Tools*

2.1. *Pressure functions.* Recall that f is said to be *uniformly expanding* or *uniformly hyperbolic* on a set $X \subset \mathbb{R}$ if there exists $\lambda > 1$ such that for every $n \ge 1$ we have $|(f^n)'(x)| \ge \lambda^n$ for every $x \in X$. A compact set $X \subset \mathbb{R}$ is said to be *isolated* if there is an open neighborhood U of X in the domain of f such that $f^n(x) \in U$ for all $n \ge 0$ implies $x \in X$. A set $X \subset \mathbb{R}$ is said to be *invariant* if $f(X) \subset X$. A compact f-invariant isolated uniformly expanding set $X \subset \mathbb{R}$ is called an *expanding repeller*.

Given $X \subset \mathbb{R}$ invariant and $\varphi : X \to \mathbb{R}$ continuous, we denote by $P_{f|_X}(\varphi)$ the standard *topological pressure* of φ with respect to $f|_X$ (see, for example, [**44**] or [**40**]). Sometimes we write $P(f|_X, \varphi)$. In the case $X = K$ we simply write $P(\varphi)$. Furthermore, we denote by $P_{\mathrm{hyp}}(\varphi)$ the *hyperbolic pressure* defined by

$$P_{\mathrm{hyp}}(\varphi) := \sup_{X \subset K} P_{f|_X}(\varphi),$$

where the supremum is taken over all expanding repellers $X \subset K$. The *variational hyperbolic pressure* of φ with respect to $f|_K$ is defined by almost the same formula as the variational pressure:

$$P_{\mathrm{varhyp}}(\varphi) := \sup_{\mu \in \mathcal{M}_E(f)} \left(h_\mu(f) + \int_\Lambda \varphi \, d\mu \right),$$

where the supremum is taken over ergodic f-invariant measures of positive Lyapunov exponent.

By [**36**, Theorem B], for every $(f, K) \in \mathscr{A}_+^{\mathrm{BD}}$ and $t \in \mathbb{R}$ we have

$$P(t) = P(\varphi_t) := P_{\mathrm{var}}(\varphi_t) = P_{\mathrm{varhyp}}(\varphi_t) = P_{\mathrm{hyp}}(\varphi_t). \tag{2.1}$$

The proof of the equality $P_{\mathrm{var}}(\varphi_t) = P_{\mathrm{varhyp}}(\varphi_t)$ is not easy in the interval case and follows from the equality of various definitions of Topological Collet–Eckmann maps, see [**42**] or [**36**, Lemma 4.6 and Theorem C]. The inequality $P_{\mathrm{varhyp}}(\varphi_t) \ge P_{\mathrm{hyp}}(\varphi_t)$ follows immediately from the variational principle. The proof of $P_{\mathrm{varhyp}}(\varphi_t) \le P_{\mathrm{hyp}}(\varphi_t)$ will be outlined in §3 for completeness, and to introduce another variant of $P_{\mathrm{hyp}}(\varphi_t)$ (on Cantor sets).

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2.2. *Phase transitions.* Let $(f, K) \in \mathscr{A}_+^{\mathrm{BD}}$. Let

$$t_+ := \sup\{t \in \mathbb{R}\colon\ P(\varphi_t) + t\chi_{\inf} > 0\}.$$

Observe that since $P(\varphi_0) = h_{\mathrm{top}}(f|_K) > 0$, we have $t_+ > 0$. Note that cases $t_+ = \infty$ and $t_+ < \infty$ are both possible. The latter case occurs, for example, in the non-LyapHyp case, that is, if $\chi_{\inf} = 0$, see §2.4 and references therein. Observe that the case $t_+ < \infty$ can happen even if $\chi_{\inf} > 0$ (and P can be differentiable or non-differentiable at t_+), see [**7**] and [**8**].

Also let

$$t_- := \inf\{t \in \mathbb{R}\colon\ P(\varphi_t) + t\chi_{\sup} > 0\}.$$

Observe that $t_- < 0$. There are examples in which $t_- > -\infty$ (for example, $f(x) = x^2 - 2$), but this can happen only for f exceptional. This was proved in the interval setting under some restrictions, see [**19**, Theorem B].

That in our situation $t_- = -\infty$ for non-exceptional f can be seen from the following argument. Suppose to the contrary that $P(\varphi_t) = -t\chi_{\sup}$ for some $t < 0$. Then this holds for every $t' < t$ by the definition of P. By the definition of P, there exists a sequence $\mu_n \in \mathcal{M}$ weakly* convergent to some $\mu_{t'} \in \mathcal{M}$ such that $-t'\chi(\mu_n) + h_{\mu_n}(f) \to P(\varphi_{t'})$, hence $-t'\chi(\mu_n) \to P(\varphi_{t'})$, as the entropies must be equal to zero. Then by the upper semi-continuity of the function $\mu \mapsto \chi(\mu)$ on $\mathcal{M}(f)$ we get $-t'\chi(\mu_{t'}) = P(t')$ and hence $\chi(\mu_{t'}) = \chi_{\sup}$ (in particular, $\mu_{t'}$ is an equilibrium state). Now, by the Key Lemma in [**17**]† adapted to $(f, K) \in \mathscr{A}_+^{\mathrm{BD}}$, since we assume (f, K) non-exceptional, we get $P(\varphi_{t'}) > -t'\chi(\mu_{t'})$, a contradiction.

Figure 1 illustrates various cases. In all three cases the asymptotes $-\chi_{\sup}$ are depicted. In the first case (LyapHyp with $t_+ = \infty$) the asymptote $-\chi_{\inf}$ does not always pass through 0; it could intersect the vertical axis at a positive number. The simplest example is provided by a piecewise linear expanding map with two branches with slope $\exp(\chi_{\inf})$ and two branches with slope $\exp(\chi_{\sup})$. This corresponds to the case where $F(\chi_{\sup})$ and $F(\chi_{\inf})$ are positive. See also [**43**] for uniformly hyperbolic examples with general Hölder potentials.

2.3. *Conformal measures.* Given a function $\phi\colon K \to \mathbb{R}$, we call a finite Borel measure μ supported on K a *ϕ-conformal measure* if it is forward quasi-invariant for f, that is, sets of measure zero are sent by f to sets of measure zero, and if it satisfies

$$\mu(f(A)) = \int_A \phi\, d\mu \tag{2.2}$$

for every $A \subset K$ on which f is injective. Conformal measures do not always exist for $t < 0$, in particular in the presence of critical points in K. It is unclear at the moment whether conformal measures exist for $t > 0$ or $t < 0$ in the exceptional case. However, the following weaker version will be sometimes sufficient for our setting.

We say μ is *conformal away from the restricted singular set S'* or a *CaS measure* if (2.2) is satisfied under the additional assumption that A is disjoint from S' and μ is forward quasi-invariant for f on $K \setminus S'$. As usual, we call ϕ the *Jacobian* of f for μ.

† $t > 0$ is assumed there, but J. Rivera-Letelier has informed us that under small modifications the proof works also for $t < 0$.

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PROPOSITION 2.1. *If f is non-exceptional, then for every $t \in \mathbb{R}$ there exists a CaS measure with Jacobian $\lambda(t)|f'|^t$ for $\log \lambda(t) = P(t)$ which is positive on open sets. We will denote this measure by μ_t^*.*

Proof. We construct μ_t^* by Patterson–Sullivan's method, as in [**40**, pp. 333–334] (see also [**38**] for the $t < 0$ complex case).

For $t > 0$, Patterson–Sullivan's construction gives a conformal measure in the complex case, but (2.2) can fail at points in S' in the interval case, see [**36**, Appendix C] for a more precise description. So we get only a CaS measure.

The measure μ_t^* is positive on open sets. Indeed, it is proved in [**36**, Lemma C.3] that, for each $t \in \mathbb{R}$, if μ_t^* vanishes on an open set then it is supported on a weakly S'- exceptional set E (which is finite, as mentioned in §1.4).

In the case where f is non-exceptional this leads to a contradiction. Below is an explanation (repeated for completeness after [**36**, Lemma C.3]).

By Patterson–Sullivan's construction, for every $x \in K$ and every $z \in f^{-1}(x) \cap K$ we get (see [**36**, (C.2), (C.3)])

$$\mu_t^*(z)|f'(z)|^t \le \mu_t^*(x) \le \sum_{y \in f^{-1}(x) \cap K} \mu_t^*(y)|f'(y)|^t, \tag{2.3}$$

provided no expression of the form $0 \cdot \infty$ is involved.

We consider three cases:

Case $t < 0$. If $\mu_t^*(x) > 0$ at $x \in E$, then we have atoms of μ_t^* at all points $f^n(x)$ for $n \ge 0$ due to the left-hand side inequality (2.3). In particular, since f is non-exceptional, μ_t^* has an atom at some $w = f^n(x)$ not weakly S'-exceptional (this will be proved in Lemma 3.6(ii) below). This contradicts the condition that μ_t^* is supported on E.

Case $t > 0$. If μ_t^* has an atom at x, then it has an atom at a point $y \in f^{-1}(x)$ by the right-hand side inequality (2.3). Therefore, it has atoms at all f^k-preimages of x which constitute an infinite set which is therefore not contained in E. We have a contradiction again.

Case $t = 0$. We can consider a measure with maximal entropy as μ_0^*. □

In the proof of Theorem 2, in §6 we rely on a part of [**36**, Theorem A], asserting that for $(f, K) \in \mathscr{A}_+^{\mathrm{BD}}$ without indifferent periodic orbits there exists a probability measure μ_t which is truly conformal for the function $\varphi = e^{P(t)}|f'|^t$ for every $t < t_+$, non-atomic, positive on open sets, and supported on the set of conical points in K, see §3.8.

2.4. *Topological Collet–Eckmann and related notions.* In this section we will discuss some (partially equivalent) conditions for $(f, K) \in \mathscr{A}$.

Given $y = f^n(x)$, denote by $\mathrm{Comp}_x f^{-n}(B)$ the component of the preimage of an interval $B \ni y$ which contains x. We sometimes call this component a *pull-back* of B by f^n. For $(f, K) \in \mathscr{A}$ we consider only pull-backs intersecting K. This component is an interval; to denote its length we use either $|\cdot|$ or diam (to avoid confusion with other meanings of $|\cdot|$).

Lyapunov Hyperbolic Condition (LyapHyp): It holds that $\chi_{\inf} > 0$.

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Topological Collet–Eckmann Condition (TCE): There exist numbers $M \geq 0$, $P \geq 1$, and $r > 0$ such that, for every $x \in K$, there exists a strictly increasing sequence of positive integers $(n_j)_{j\geq 1}$ such that for every $j \geq 1$ we have $n_j \leq P \cdot j$ and

$$\#\{k \in \{0, \ldots, n_j - 1\} \colon \operatorname{Comp}_{f^k(x)} f^{-(n_j-k)} B(f^{n_j}(x), r) \cap \operatorname{Crit}(f) \neq \varnothing\} \leq M. \quad (2.4)$$

We only consider this condition in the absence of indifferent periodic orbits; in this situation all components above are well inside U_K for r sufficiently small (see BaShrink below). Of course, the number r can be taken arbitrarily small and P can be taken arbitrarily close to one (at the cost of increasing M).

Exponential shrinking of components (ExpShrink): There exist numbers $\lambda > 1$ and $r > 0$ such that, for every $x \in K$, every $n > 0$, and every connected component W of $f^{-n}(B(x, r))$ which intersects K, we have

$$|W| \leq \lambda^{-n}.$$

See [**36**] for the usage of the condition LyapHyp in the multimodal case and, for example, [**37**] in the complex holomorphic setting. In the interval case the proof of the fact that, assuming weak isolation, LyapHyp implies TCE is different from the holomorphic case and is given in the multimodal case in [**42**] and later in [**36**, Theorem C] (in the more general case of $f \in \mathscr{A}_+^{\mathrm{BD}}$).

By [**36**, Theorem C], for every $(f, K) \in \mathscr{A}_+^{\mathrm{BD}}$ satisfying the weak isolation condition, LyapHyp is equivalent to TCE together with the absence of indifferent periodic points which in turn is equivalent to ExpShrink. For more conditions and their equivalences in the interval case, see [**36**, Theorem C] or [**42**].

In what follows, instead of ExpShrink, we will use only the following weaker condition which is satisfied for every $(f, K) \in \mathscr{A}_+$ (see [**36**, Lemma 2.10]).

Backward Shrinking (BaShrink): For every $\varepsilon > 0$ there exists $\delta > 0$ such that if T is an open interval in $\mathbb{R}$ intersecting K, disjoint from $\operatorname{Indiff}(f)$, and satisfying $|T| \leq \delta$, then for every $n \geq 0$ and every component T' of $f^{-n}(T)$ intersecting K we have $|T'| \leq \varepsilon$. Moreover, the lengths of all components of $f^{-n}(T)$ intersecting K converge to zero uniformly as $n \to \infty$.

In particular, for δ sufficiently small, for every $\varepsilon' > 0$ there exists a positive integer $\varkappa = \varkappa(\delta, \varepsilon')$ such that for every T as above, satisfying $|T| \leq \delta$, and for every component T' of $f^{-n}(T)$ intersecting K with $n \geq \varkappa$, we have $|T'| \leq \varepsilon'$.

3. *Preliminary constructions*

3.1. *Outside singular set.*

LEMMA 3.1. *For $(f, K) \in \mathscr{A}$ there exists $\delta > 0$ such that for every interval $T \subset U_K$ intersecting K and disjoint from S', if $|T| < \delta$, then $f \colon T \to f(T)$ is a homeomorphism and*

$$f(T) \cap K = f(T \cap K). \quad (3.1)$$

In [**36**] such f on T is called a *K-homeomorphism*. For completeness we provide the proof of this lemma.

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Proof. For $x \in \partial \widehat{I}_K$ (there are only finitely many such points) consider the point $f(x)$. There are two possibilities: either $f(x)$ is accumulated by points in K from both sides, or $f(x)$ is a boundary point of a component of $\mathbb{R}\backslash K$. In the latter case we denote by g_x the length of the component of $\mathbb{R}\backslash K$ adjacent to $f(x)$. Let U' be a neighborhood of K with closure in U_K. Let L be Lipschitz constant for $f|_{U'}$. Choose

$$\delta := \min\left\{\operatorname{dist}(K, \partial U'), \frac{1}{L}\min_x g_x\right\}.$$

Consider now an open interval T of length less than δ which intersects K and is disjoint from S'. Since K is forward invariant, the only possibility of (3.1) failing is when $f(T) \cap K \backslash f(T \cap K) \neq \varnothing$. In such a situation T must contain a point $x \in \partial \widehat{I}_K$. Indeed, T intersects $\widehat{I}_K$ because it intersects K, and if $T \subset \widehat{I}_K$, then (3.1) holds, since $f^{-1}(K \cap f(T)) \subset K$, by maximality of K. Let T' be the component of $T\backslash\{x\}$ which is disjoint from $\widehat{I}_K$. If $f(x)$ is accumulated by points in $K \cap f(T')$, then $x \in NO(f, K)$ and hence $x \in S'$, a contradiction. Otherwise, $f(x)$ is a boundary point of a component of $\mathbb{R}\backslash K$ and, by our choice of δ, we have $f(T') \cap K = \varnothing$. Since $T'' := T\backslash T' \subset \widehat{I}_K$, maximality of K implies the property (3.1) for T''. Hence (3.1) holds for T. □

3.2. *More on K-homeomorphisms.*

LEMMA 3.2. *Let $(f, K) \in \mathscr{A}_+$ be non-exceptional, satisfying the weak isolation condition. Consider any $x \in K$ not weakly S'-exceptional. Then, for a constant $r > 0$ small enough depending on (f, K) and every integer n large enough, the following holds. Assume that $B(f^n(x), r/2)$ does not contain any indifferent periodic points. Let W_n be the pull-back of $B(f^n(x), r/2)$ for f^n which contains x. For all $j = 0, \ldots, n$ define $W_n^j := f^j(W_n)$. Then for all $j = 0, \ldots, n-1$ the interior of W_n^j is disjoint from $NO(f, K)\backslash \mathrm{Crit}_T(f)$.*

Proof. Let $r > 0$ be small enough that every T intersecting K, containing no indifferent periodic points, with $|T| \leq r$, satisfies BaShrink.

Choose an arbitrary periodic point p whose periodic orbit $\mathcal{O}(p)$ consists of points not weakly S'-exceptional. This is possible since the set of weakly S'-exceptional points is finite, see §1.4, and the set of all periodic points in K is infinite, see [**36**, Proposition 2.5].

Suppose there exists $j_0 \in \{0, \ldots, n-1\}$ and $z \in \operatorname{int} W_n^{j_0}$, not a turning critical point, at which $f|_K$ is not open. We assume that r, and hence $|W_n^{j_0}|$, is small enough (smaller than the minimal mutual distance between the points in $NO(f, K)$) that there are no turning critical points in $W_n^{j_0}$. Then there exists $z' \in W_n^{j_0+1}$ arbitrarily close to $f(z)$ whose f-preimage z'' in $W_n^{j_0}$ is not in K. Let $z_0 = x, z_1, \ldots$ be a backward trajectory of x in K, (that is, $f(z_j) = z_{j-1}$), converging to $\mathcal{O}(p)$ omitting S' (possible since we have assumed x is not weakly S'-exceptional) and let $p = y_0, y_1, \ldots$ be a backward trajectory of p in K such that z' is its accumulation point. Choose n_0 such that y_{n_0} is close enough to z' to be in $W_n^{j_0+1}$. In fact, we want $|y_{n_0} - z'| < \varepsilon$ for an arbitrarily small $\varepsilon > 0$.

Choose $\delta > 0$ small enough that all consecutive pull-backs V_j for f of $B(p, \delta)$ along y_j, $j = 0, 1, 2, \ldots, n_0$ are disjoint from S' and V_{n_0} is also close enough to z' to be in $W_n^{j_0+1}$, and even within an arbitrarily chosen neighborhood, say in $B(z', \varepsilon)$.

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If n is large enough, then $|W_n|$ is so small that all consecutive pull-backs Z_j of W_n for f along z_j, $j = 0, 1, 2, \ldots$ are disjoint from S'. Choose n_1 such that $Z_{n_1} \subset B(p, \delta)$.

Let us summarize: $f^{j_0+1}(W_n) = W_n^{j_0+1}$, $V_{n_0} \subset W_n^{j_0+1}$, $f^{n_0}(V_{n_0}) = B(p, \delta)$, $Z_{n_1} \subset B(p, \delta)$, and $f^{n_1}(Z_{n_1}) = W_n$.

Hence there exists a periodic orbit $\widetilde{\mathcal{O}}$ of period $j_0 + 1 + n_0 + n_1$ close to K and passing arbitrarily close to z'' for ε small. Hence, by the weak isolation assumption, $\widetilde{\mathcal{O}} \subset K$, and hence $z'' \in K$, a contradiction. □

The considerations on shadowing as above can be found for other aims in [**36**, §7.3, item 1a]. It is also related to the 'bridges' technique in §3.4.

COROLLARY 3.3. *In the situation of Lemma 3.2, in particular for x not weakly S'-exceptional, for r small enough, and all n large enough, f is a K-homeomorphism on each* int W_n^j *containing no turning critical points, onto* int W_n^{j+1}. *If there is a turning critical point $c \in$* int W_n^j *then f is a K-homeomorphism on both components of* int $W_n^j \backslash \{c\}$*; one of them is mapped onto* int W_n^{j+1} *and the other one is mapped into* int W_n^{j+1}.

Proof. By Lemma 3.2, if int W_n^j contains $z \in NO(f, K)$, then $z \in \mathrm{Crit}_T(f)$. So if there is no turning critical point in int W_n^j, there is no point belonging to $NO(f, K)$ either. Hence, f is a homeomorphism on int W_n^j, and also a K-homeomorphism, since int W_n^j contains $f^j(x) \in K$, by Lemma 3.1.

If there is a turning critical point $c \in$ int W_n^j, then there are no other turning critical points in int W_n^j since the interval is short enough, and $f(c)$ is an end of $f(W_n^j)$, and at least one of the ends of W_n^j is mapped to the other end of $f(W_n^j)$. Since there are no points belonging to $NO(f, K)\backslash\{c\}$ in W_n^j if this interval is short enough, the K-homeomorphism property on each component of int $W_n^j \backslash \{c\}$, asserted in the corollary, follows. □

3.3. *More on topological recurrence and exceptional sets.* The fact that $f\colon K \to K$ is topologically transitive and has positive topological entropy implies that it is weakly topologically exact (see [**36**, Lemma A.7]). However, we will sometimes need a weaker conclusion, that $f|_K$ is strongly topologically transitive, which immediately follows from weak topological exactness and, less easily, from the topological transitivity, that is, it is automatically true for $(f, K) \in \mathscr{A}$, see [**22**] or [**36**, Proposition 2.4].

We say that $f|_K$ is *weakly topologically exact* (see also [**36**, Definition 1.15]) if there exists a positive integer N such that, for every non-empty open set $V \subset K$, there exists a positive integer $n = n(V)$ such that $f^n(V) \cup \cdots \cup f^{n+N-1}(V) = K$. Notice that this automatically implies the same for all $n \geq n(V)$, by applying $f^{n-n(V)}$ on both sides of the equality.

Definition 3.4. The map $f|_K$ is *strongly topologically transitive*† if for every non-empty open $V \subset K$ there is a positive integer $m = m(V)$ such that $V \cup f(V) \cup \cdots \cup f^m(V) = K$. Note that $f|_K$ is strongly topologically transitive if, and only if, the union of backward trajectories of every point in K is dense in K.

† This name was used in [**22**]. See also the references therein. In [**36**, Definition 2.3] it is called the *density of preimages property* (dp).

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Notice that for a given $\Delta > 0$ one can choose $m(V)$ which works for all $V = B(x, \Delta)$, $x \in K$. We will denote this number by $m_c(\Delta)$.

Later we will need the following notation. For a measure μ and $\Delta > 0$ let

$$\Upsilon(\mu, \Delta) := \inf_{x \in K} \mu(B(x, \Delta)). \tag{3.2}$$

In our case, for μ being ϕ-conformal, this number is positive, which follows easily from the strong topological transitivity of (f, K) due to the existence of $m = m_c(\Delta)$. In the case of μ being a CaS measure, this positivity also holds provided this measure has infinite support. Indeed, let $S_1 = \bigcup_{j=0}^{m-1} f^{-j}(S')$ and $S_2 = \bigcup_{j=0}^{m} f^j(S_1)$. If $\mu(B(x, \Delta))$ is small, then forward quasi-invariance implies that $\mu(\bigcup_{j=0}^{m} f^j(B(x, \Delta)\backslash S_1))$ is small: smaller than $1 - \mu(S_2)$. This contradicts the fact that $\bigcup_{j=0}^{m}(f^j(B(x, \Delta)\backslash S_1)$ contains $K\backslash S_2$.

In §6 we shall need the following very easy general fact. We will apply it to $X = K$.

LEMMA 3.5. *Let (X, ρ) be a compact infinite metric space without isolated points. Then for every $r > 0$ and integer $m > 0$ there exists $\varepsilon = \varepsilon(r, m) > 0$ such that, for every $x \in X$, there exists $Z \subset X \cap B(x, r)$ with $\#Z \geq m$ such that for all $z_1, z_2 \in Z$ with $z_1 \neq z_2$ we have $\rho(z_1, z_2) \geq \varepsilon$.*

Proof. For every point $x \in X$ for every r and m there exists $\varepsilon(x)$ satisfying the assertion because x is not isolated; the problem in this lemma is to prove that one can choose one ε which works for all x. Assume that this is not the case and that there exists a sequence $(x_n)_n$ such that for $\varepsilon(x_n)$ maximal possible, we have $\varepsilon(x_n) \to 0$. Let x be an accumulation point of sequence (x_n); passing to a subsequence we can assume $x = \lim x_n$. Then $B(x, r/2) \subset B(x_n, r)$ for n large enough, and hence for the same m and for $r' = r/2$ we would have $\varepsilon(x) = 0$, a contradiction. □

The following lemma provides a justification of the notions of 'exceptional' and 'non-exceptional'. Given $x \in K$, let

$$\mathcal{O}^-_{\mathrm{reg}}(x) := \{y = (f|_K)^{-n}(x) \colon n = 0, 1, \ldots, f^k(y) \notin S' \text{ for every } k = 0, \ldots, n-1\}$$

be the union of all backward trajectories (finite or infinite) for $f|_K$ starting at x that do not hit S'. Recall the notion of being weakly S'-exceptional in §1.4.

LEMMA 3.6. *Let $(f, K) \in \mathscr{A}_+$. Then the following hold.*

(i) *For every $x \in K$ not weakly S'-exceptional (we do allow $x \in S'$) the set $\mathcal{O}^-_{\mathrm{reg}}(x)$ is dense in K.*

(ii) *For every $x \in K$ not S'-exceptional there exists $n \geq 0$ such that $f^n(x)$ is not weakly S'-exceptional.*

(iii) *For every invariant measure μ which gives measure 0 to all S'-exceptional sets, the set of points $x \in K$ for which the set $\mathcal{O}^-_{\mathrm{reg}}(x)$ is dense in K has positive measure μ.*

Proof. By its definition, the set $\mathcal{O}^-_{\mathrm{reg}}(x)$ satisfies (1.3) for $\Sigma = S'$. Hence, as x does not belong to any weakly S'-exceptional set, $\mathcal{O}^-_{\mathrm{reg}}(x)$ must be dense in K which proves item (i).

Suppose that each $y = f^n(x)$, $n \geq 0$ belongs to a weakly S'-exceptional set $\Sigma(y)$. Then, by definition, $\mathcal{O}^-_{\mathrm{reg}}(y) \subset \Sigma(y)$, and hence it is finite and weakly S'-exceptional. Then

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$\Sigma := \bigcup_{y\in\mathcal{O}^+(x)} \mathcal{O}^-_{\mathrm{reg}}(y)$, for $\mathcal{O}^+(x)$ denoting the forward orbit of x, is finite, in particular not dense, and forward invariant. Therefore Σ is S'-exceptional, and hence x belongs to an S'-exceptional set. This proves item (ii).

Item (iii) follows immediately from item (ii). □

The weak exactness property has not been used directly in the proof of Lemma 3.6. However, it is needed to prove the finiteness of the union of all weakly S'-exceptional sets, see [**36**, Proposition 2.7, Part 2], used in Proof of Lemma 3.6.

Have in mind the example of irrational rotation of the circle (though formally it is not in $\mathbb{R}$) or a 'solenoid' $K = \omega(c)$ for an infinitely renormalizable unimodal map of the interval and c its critical value, where all finite blocks $\{c, f(c), \ldots, f^n(c)\}$ are weakly S'-exceptional sets. These examples are strongly topologically transitive, but not weakly exact, and assertions 2 and 3 of Lemma 3.6 fail.

3.4. *Cantor repellers and 'bridges'.* We describe a construction to 'connect' two given uniformly expanding repellers by building 'bridges' between them.

Let $f: U \to \mathbb{R}$ be a $C^{1+\varepsilon}$ map on an open domain $U \subset \mathbb{R}$. We call a set $X \subset U$ an (f-uniformly) *expanding Cantor repeller (ECR)* if $f|_X$ is a uniformly expanding repeller being a topological Cantor set and being a limit set of a finite graph directed system (GDS) satisfying the strong separation condition (SSC) with respect to f. Recall that a GDS satisfying the SSC with respect to f is a family of domains and maps satisfying the following conditions (compare [**24**, pp. 3, 58]).

(i) There exists a finite family $\mathcal{U} = \{U_k\}$ of open intervals with pairwise disjoint closures.

(ii) There exists a family $G = \{g_{k\ell}\}$ of branches of f^{-1} mapping $\overline{U_\ell}$ into U_k with bounded distortion (not all pairs k, ℓ must appear here).
Note that a general definition of GDS allows many maps g from each $\overline{U_\ell}$ to each U_k. Here however there can be at most one, since we assume that f-critical points are far away from X, the intervals U_k are short, and the maps g are branches of f^{-1}.

(iii) We have
$$X = \bigcap_{n=1}^{\infty} \bigcup_{k_1,\ldots,k_n} g_{k_1k_2} \circ g_{k_2k_3} \circ \cdots \circ g_{k_{n-1}k_n}(U_{k_n}).$$
We assume that we have $f(X) = X$ and hence that for each k there exists ℓ, and for each ℓ there exists k, such that $g_{k\ell} \in G$.

We can view k's as vertices and $g_{k\ell}$ as edges from ℓ to k of a directed graph $\Gamma = \Gamma(\mathcal{U}, G)$.

Notice that the limit set X of any GDS as above is either a Cantor set or, provided $f|_X$ is topologically transitive, a periodic orbit.

Observe that, even if X contains points in $NO(f, K)$, the resulting restricted map $f|_X$ is open as X is a limit set of a GDS satisfying the SSC.

We import the following result from Pesin–Katok theory (see, for example, [**40**, Theorem 11.6.1], [**36**, Theorem 4.1] (concerning our interval case) [**13**], and also [**26**] for earlier related results). Recall again the definition of weakly isolated (Definition 1.1).

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LEMMA 3.7. *Consider $f: U \to \mathbb{R}$ for an open set $U \subset \mathbb{R}$, being a C^2 map with all critical points non-flat. Consider an arbitrary compact f-invariant $X \subset U$. Let $\mu \in \mathcal{M}_E$ be supported on X and have positive Lyapunov exponent. Let $\varphi: U \to \mathbb{R}$ be a continuous function. Then there exists a sequence $(X_k)_{k\geq 1} \subset U$ of topologically transitive ECRs or periodic orbits such that, for every k,*

$$\liminf_{k\to\infty} P_{f|X_k}(\varphi) \geq h_\mu(f) + \int \varphi \, d\mu, \tag{3.3}$$

and that every sequence $(\mu_k)_{k\geq 1}$ of measures $\mu_k \in \mathcal{M}_E(X_k)$ converges to μ in the weak-topology. Moreover, we have*

$$\lim_{k\to\infty} \chi(\mu_k) \to \chi(\mu). \tag{3.4}$$

If X is weakly isolated, then one can choose $X_k \subset X$.

For $X = K$ for $(f, K) \in \mathscr{A}_+^{\mathrm{BD}}$ satisfying the weak isolation condition, Lemma 3.7 easily implies $P_{\mathrm{varhyp}}(\varphi_t) \leq P_{\mathrm{hyp}}(\varphi_t)$, see (2.1). Indeed, by definition we can consider μ such that $h_\mu(f) + \int \varphi_t \, d\mu$ is arbitrarily close to $P_{\mathrm{varhyp}}(\varphi_t)$ and next find X_k as in Lemma 3.7 applied to $\varphi \equiv 0$. From (3.3), $h_{\mathrm{top}}(f|_{X_k})$ is at least $h_\mu(f)$ (up to an arbitrarily small positive number) and, applying (3.4) for μ_k being measures of maximal entropy on X_k, we obtain

$$P_{\mathrm{varhyp}}(\varphi_t) \leq P_{\mathrm{ECR}}(\varphi|_X) := \sup_{X\subset K} P_{f|_X}(\varphi) \leq P_{\mathrm{hyp}}(\varphi_t), \tag{3.5}$$

where the supremum is taken over all ECRs and periodic repelling orbits $X \subset K$. This uses an inequality in the classical variational principle for the continuous potentials $\varphi_t|_{X_k}$, namely: $h_{\mu_k}(f|_{X_k}) + \int \varphi_t \, d\mu_k \leq P_{f|_{X_k}}(\varphi_t)$.

Compare also [**36**, Lemma 4.2].

Definition 3.8. Consider $(f, K) \in \mathscr{A}_+^{\mathrm{BD}}$ satisfying the weak isolation condition, and set $X = K$. Consider two ECRs or periodic repelling orbits X_1 and X_2 contained in K, resulting from families of domains $\mathcal{U}_1 = \{U_{1,k}\}$ and $\mathcal{U}_2 = \{U_{2,k}\}$ respectively. A *bridge* connecting X_1 to X_2 is a backward trajectory that starts in X_1, goes close to X_2, and in between avoids the singular set S'. More precisely, it is a sequence of points $x_n \in K$ for $n = 0, -1, \ldots, -N$ such that $f(x_n) = x_{n-1}$, $x = x_0 \in U_{1,k}$ for an integer k, and the components W_n of $f^{-n}(U_{1,k})$ containing x_n are disjoint from S' for $n = 0, \ldots, N$, disjoint from $\bigcup_k U_{i,k}$ for $n = 1, \ldots, N-1$, and $W_N \subset U_{2,k'}$ for an index k'.

If (f, K) is non-exceptional, then for any two X_1, X_2 a bridge exists. In fact, by Lemma 3.6(iii), in X_2, for every invariant measure μ on X_2 there is a positive measure set of points x whose backward trajectories omitting S' are dense in K. This allows us to prove that for every X_1 a bridge from X_1 to X_2 exists (for appropriate $\mathcal{U}_i$).

Based on this idea of bridges connecting X_1 to X_2 and X_2 to X_1, one can prove the following lemma (compare [**14**, Lemma 2]).

LEMMA 3.9. *For any two disjoint ECRs or repelling periodic orbits $X_1, X_2 \subset K$ there exists an ECR set $X \subset K$ containing the set $X_1 \cup X_2$. If f is topologically transitive on each X_i, $i = 1, 2$, then $f|_X$ is topologically transitive.*

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In case two different ECRs or repelling periodic orbits intersect, in a preliminary step we first have to modify one of them by the following lemma. For that we also have to take into consideration ECRs with respect to f^n.

In the following two results we use the standard notation $S_\ell\phi = \phi + \phi \circ f + \cdots + \phi \circ f^{\ell-1}$ for a function $\phi\colon X \to \mathbb{R}$ and a natural number $\ell > 0$.

LEMMA 3.10. *Let X be a topologically transitive f^n-ECR. Let $\phi_i\colon X \to \mathbb{R}$ be a finite number of Hölder continuous functions. Then, for any open sufficiently small interval D intersecting X, and for any $\varepsilon > 0$, one can find a set $X' \subset D \cap X$ and a natural number $\ell > 0$ such that X' is an f^ℓ-ECR and for every ϕ_i satisfies*

$$\frac{1}{\ell} P_{f^\ell|X'}(S_\ell\phi_i) \geq \frac{1}{n} P_{f^n|X}(S_n\phi_i) - \varepsilon.$$

Notice that for every repelling periodic orbit $X_1 \subset K$ there exists an ECR $X \subset K$ containing X. Indeed, take another repelling periodic orbit $X_2 \subset K$ existing due to $(f, K) \in \mathscr{A}_+$ and find X as in Lemma 3.9. Thus, combining Lemmas 3.7, 3.9, and 3.10, we obtain the following key result.

PROPOSITION 3.11. *For every non-exceptional $(f, K) \in \mathscr{A}_+^{\mathrm{BD}}$ satisfying the weak isolation condition, there exists a sequence $(a_m)_m$ of positive integers and a sequence $X_m \subset K$ of f^{a_m}-invariant uniformly expanding topologically transitive ECRs such that, for every $t \in \mathbb{R}$, we have*

$$P(\varphi_t) = \lim_{m\to\infty} \frac{1}{a_m} P_{f^{a_m}|X_m}(S_{a_m}\varphi_t) = \sup_{m\geq 1} \frac{1}{a_m} P_{f^{a_m}|X_m}(S_{a_m}\varphi_t). \tag{3.6}$$

For every $\alpha \in (\chi_{\inf}, \chi_{\sup})$ we have

$$F(\alpha) = \lim_{m\to\infty} F_m(\alpha) = \sup_{m\geq 1} F_m(\alpha) \tag{3.7}$$

and

$$\lim_{m\to\infty} \chi^m_{\inf} = \inf_{m\geq 1} \chi^m_{\inf} = \chi_{\inf}, \quad \lim_{m\to\infty} \chi^m_{\sup} = \sup_{m\geq 1} \chi^m_{\sup} = \chi_{\sup}, \tag{3.8}$$

where F_m and $\chi^m_{\inf/\sup}$ are defined as in (1.7) and (1.6) but for the pressure $(1/a_m)P_{f^{a_m}|X_m}(S_{a_m}\phi)$ instead of $P(\phi)$.

The proofs of Lemmas 3.7, 3.9, and 3.10 are, almost verbatim, as the proofs of [**14**, Lemmas 2–4]. We would like to remark that to make the proof of Lemma 3.10 work (and this should also have been mentioned in the proof of [**14**, Lemma 4]) we need to choose D of sufficiently small size that the considered pull-backs (see §3.6 below) of D are all univalent.

Finally, notice that our sets X_m are in K, since periodic points are dense in them and if they are sufficiently close to K they are in K by the weak isolation property.

In fact one can assume all $a_m = 1$. This follows from the fact that one can replace X_m by $Y := \bigcup_{j=0}^{a_m-1} X_m$ and find Y' containing Y also f-invariant uniformly expanding topologically transitive and, moreover, isolated (the property we might have lost for Y), see [**40**, Proposition 4.5.6].

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3.5. *Local properties at critical points.* Let c be a non-flat critical point of a C^1 map f. There exist constants $R_0 > 0$ and $A_0 > 1$ such that for every x with $|x - c| \le R_0$ there is $d(c) \ge 2$ so that

$$A_0^{-1} \le \frac{|f'(x)|}{|x - c|^{d(c)-1}} \le A_0. \tag{3.9}$$

In our situation there are only finitely many critical points, hence A_0 can be chosen uniformly.

3.6. *Telescope constructions.* A number $n \in \mathbb{N}$ is called a *Pliss hyperbolic time* for a point x with exponent $\sigma > 0$ if

$$|(f^k)'(f^{n-k}(x))| \ge e^{k\sigma} \quad \text{for every } 1 \le k \le n. \tag{3.10}$$

The following is an immediate consequence of the Pliss lemma (see, for example, [**30**] or [**2**, Lemma 3.1]. In Lemma 6.1 below we will provide a more refined abstract result.

LEMMA 3.12. (Pliss lemma) *Let f be a C^1 map of a closed interval. For every point x with $\overline{\chi}(x) > \sigma > 0$ there exist infinitely many Pliss hyperbolic times for x with exponent σ. Moreover, if $H \subset \mathbb{N}$ denotes the set of hyperbolic times, then we have*

$$\overline{d}(H) := \limsup_{n\to\infty} \frac{\#(H \cap [0, n])}{n} > 0.$$

We will call $\overline{d}(H)$ the *upper density* of H.

PROPOSITION 3.13. (Telescope) *Let $f : I \to I$ be a C^1 multimodal map on a closed interval I. Given $\varepsilon > 0$ and $\sigma > 0$, there exist constants $A_1 = A_1(\varepsilon) > 0$ and $R_1 = R_1(\varepsilon, \sigma) > 0$ such that the following is true. Given a point $x \in I$ with $\overline{\chi}(x) > \sigma > 0$, for every number $r \in (0, R_1)$, for every $n \ge 1$ which is a Pliss hyperbolic time for x with exponent σ, and for every $k = 1, \ldots, n$ we have*

$$\begin{aligned}\operatorname{diam} \operatorname{Comp}_{f^{n-k}(x)} f^{-k}(B(f^n(x), r)) &\le r\, A_1\, e^{k\varepsilon}\, |(f^k)'(f^{n-k}(x))|^{-1} \\ &\le r\, A_1\, e^{-k(\sigma-\varepsilon)}.\end{aligned}$$

This proposition was first stated and proved, with various variants, in the complex setting in [**31**]. Here we provide a slightly simplified proof for interval maps. Notice that only C^1 regularity is required and that no advanced distortion properties are needed.

We first prove a preliminary result. Recall the choice of constants A_0, R_0 in §3.5.

LEMMA 3.14. *Given a point $c \in \operatorname{Crit}(f)$, let $T \subset B(c, R_0)$ be an interval and $x \in T$. Then*

$$\operatorname{diam} f(T) \ge \frac{1}{2d(c)A_0^4}\, |f'(x)|\, \operatorname{diam} T.$$

Proof. Let $d = d(c)$. Let y be the endpoint of T furthest away from c and z the endpoint closest to c. We can assume that y is to the left of c.

Suppose first that T is on one side of c (that is, its interior does not contain c). Hence $\operatorname{diam} T = |y - z|$. By (3.9) we have

$$|f'(x)| \ge A_0^{-1}\, |x - c|^{d-1},$$

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which implies

$$\operatorname{diam} f(T) \geq \int_{|z-c|}^{|y-c|} A_0^{-1} s^{d-1}\, ds.$$

Combining this inequality with

$$|f'(y)| \leq A_0\, |y-c|^{d-1},$$

we obtain

$$\begin{aligned}\operatorname{diam} f(T) &\geq \frac{1}{A_0 d}(|y-c|^d - |z-c|^d)\\ &\geq \frac{1}{A_0 d}\, |y-c|^{d-1} \cdot |y-z|\\ &\geq \frac{1}{A_0^2 d}|f'(x)| \operatorname{diam} T.\end{aligned}$$

Suppose now that y and z are on the opposite sides of c. Suppose first that $x \in [y, c]$. Then we apply the above to T replaced by $[y, c]$ and obtain

$$\operatorname{diam} f(T) \geq \operatorname{diam} f([y, c]) \geq \frac{1}{A_0^2 d}|f'(x)| \cdot |y-c| \geq \frac{1}{2A_0^2 d}|f'(x)| \cdot \operatorname{diam} T.$$

Suppose now that $x \in [c, z]$. Then we apply (3.9) to the point $\widehat{x} = c - (x-c)$. Then $\widehat{x} \in [y, c]$ and by the previous estimate and applying (3.9) we obtain

$$\operatorname{diam} f(T) \geq \frac{1}{2A_0^2 d}|f'(\widehat{x})| \cdot \operatorname{diam} T \geq \frac{1}{2A_0^4 d}|f'(x)| \cdot \operatorname{diam} T.$$

The proof is complete. □

Remark 3.15. If the bounded distortion condition holds, the following unifying version of Lemma 3.14 holds. There exists $C > 0$ depending only on f such that for any interval T short enough and $x \in T$

$$\operatorname{diam} f(T) \geq C|f'(x)| \operatorname{diam} T.$$

Indeed, if T is close to a critical point, this follows from Lemma 3.14. Otherwise, if, say, $2 \diamond T \cap \operatorname{Crit}(f) = \varnothing$, it follows from the bounded distortion condition.

Let $\lambda(r)$ be the minimum of $|f'|$ outside the r-neighborhood of Crit. Let $\gamma(\cdot)$ be the modulus of continuity of the function $|f'|$. We have the following simple estimation.

LEMMA 3.16. *Given an interval T of length R' that has distance at least R from* $\operatorname{Crit}(f)$*, we have*

$$\sup_{x,y \in T} \frac{|f'(x)|}{|f'(y)|} \leq \frac{\lambda(R) + \gamma(R')}{\lambda(R)}.$$

Proof of Proposition 3.13. Fix σ and $\varepsilon \in (0, \min\{\sigma, 1\})$. Let n be a Pliss hyperbolic time for x with exponent σ. For $k = 0, \ldots, n$ and $r > 0$ consider the sets

$$W_k(r) := \operatorname{Comp}_{f^{n-k}(x)} f^{-k}(B(f^n(x), r)).$$

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Let $R_1 < R_*$ be positive numbers to be defined below. Let

$$M := \#\,\mathrm{Crit}, \quad d := \max d(c).$$

Let $A(N)$ be the minimum distance ℓ such that there exists a critical point c and a point $y \in B(c, \ell)$ such that $f^k(y) \in B(c, \ell)$ for some $k \le N$. For a critical point c we say that a number $k \le n$ is a c-critical time if $\mathrm{dist}(c, W_k(R_1)) < R_*$. Let C^{crit} be the set of times which are c-critical for some $c \in \mathrm{Crit}$.

The following claim will immediately prove the proposition.

CLAIM. *There exist positive numbers $R_* \in (0, R_0/2)$, $R_1 \in (0, R_*)$, A_1, and a positive integer N such that for all $k = 0, \ldots, n$ and $r \in (0, R_1)$ we have:*

(a) $\mathrm{diam}\, W_k(R_1) \le R_*$;

(b) *for every $i \in C^{\mathrm{crit}}$, $i \le k$, we have*

$$\frac{\mathrm{diam}\, W_{i-1}(r)}{\mathrm{diam}\, W_i(r)} \ge \frac{1}{2dA_0^4}\,|f'(f^{n-i}(x))|$$

while

$$\prod_{i \notin C^{\mathrm{crit}}, i \le k} \mathrm{Dist}_{W_i(R_1)} f \le \left(1 + \frac{\varepsilon}{2}\right)^k;$$

(c) *given $c \in \mathrm{Crit}$, any two c-critical times $\le k$ differ at least by N; and*

(d) $\mathrm{diam}\, W_k(r) \le r\, A_1\, e^{k\varepsilon}\, |(f^k)'(f^{n-k}(x))|^{-1}$.

Proof. Let us first fix some constants. Let

$$A_1 > (2dA_0{}^4)^{2M}. \tag{3.11}$$

Choose a positive integer N satisfying

$$A_1 < e^{N\varepsilon/2}. \tag{3.12}$$

Choose a positive number R_* satisfying

$$R_* < \min\left\{\frac{R_0}{2}, \frac{A(N)}{2}\right\}. \tag{3.13}$$

Choose a positive number R_1 satisfying

$$R_1 < \min\left\{\lambda(R_0)\frac{R_*}{2}A_1, \frac{R_*{}^d}{2}A_0A_1d\right\} \tag{3.14}$$

and

$$\gamma(R_1) < \varepsilon\frac{\lambda(R_*)}{2}. \tag{3.15}$$

The proof is by induction on k. Note that all statements (a)–(d) hold for $k = 0$. Suppose that the claim holds for $k = \ell$.

Observe that we have chosen A_1 and N sufficiently large such that we can conclude (d) from (a)–(c). We have chosen R_* sufficiently small such that (a) implies (c). Finally, we have chosen R_1 small enough that the previous lemmas and (a) imply (b). We are now going to conclude that properties (a)–(d) hold for $\ell + 1$.

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Property (a). Since (d) holds at time ℓ and since n is a Pliss hyperbolic time, by the choice of ε we have

$$\operatorname{diam} W_\ell(R_1) \le R_1 A_1 e^{\ell\varepsilon} |(f^\ell)'(f^{n-\ell}(x))|^{-1} \le R_1 A_1 e^{\ell(\varepsilon-\sigma)} < R_1 A_1.$$

Thus, together with (3.14), we can conclude

$$\operatorname{diam} W_{\ell+1}(R_1) \le \frac{R_1 A_1}{\lambda(R_0)} + (A_0\, d\, R_1 A_1)^{1/d} \le R_*$$

and therefore, we have (a) at time $\ell + 1$.

Property (b). We will distinguish two cases.

If $\ell + 1 \in C^{\mathrm{crit}}$ then $\operatorname{dist}(c, W_{\ell+1}(R_1)) < R_* < R_0/2$ for some $c \in \mathrm{Crit}$ and by Lemma 3.14 we have

$$\frac{\operatorname{diam} W_\ell(r)}{\operatorname{diam} W_{\ell+1}(r)} \ge \frac{1}{2dA_0^4} |f'(f^{n-(\ell+1)}(x))|$$

while the estimate of the product remains unaltered. Thus, (b) holds for $\ell + 1$ in this case.

If $\ell + 1 \notin C^{\mathrm{crit}}$, then $\operatorname{dist}(c, W_{\ell+1}(R_1)) \ge R_*$ for all $c \in \mathrm{Crit}$ and hence, by Lemma 3.16 and by (3.15), we obtain

$$\operatorname{Dist}_{W_{\ell+1}(R_1)} f = \sup_{x,y \in W_{\ell+1}(R_1)} \frac{|f'(x)|}{|f'(y)|} \le \frac{\lambda(R_*) + \gamma(R_1)}{\lambda(R_*)} \le 1 + \frac{\varepsilon}{2}.$$

Thus, we obtain

$$\prod_{i \notin C^{\mathrm{crit}}, i \le \ell+1} \operatorname{Dist}_{W_i(R_1)} f \le \left(1 + \frac{\varepsilon}{2}\right)^{\ell+1}.$$

On the other hand, we have

$$\begin{aligned} \frac{\operatorname{diam} W_\ell(r)}{\operatorname{diam} W_{\ell+1}(r)} &\ge |f'(f^{n-(\ell+1)}(x))| \cdot (\operatorname{Dist}_{W_{\ell+1}(R_1)} f)^{-1} \\ &\ge \frac{1}{2dA_0^4} |f'(f^{n-(\ell+1)}(x))|, \end{aligned}$$

that is, (b) holds for $\ell + 1$ in this case as well.

Property (c). If (a) holds at times $0, \ldots, \ell$, then it follows from (3.13) that (c) holds at time $\ell + 1$. Indeed, by (a), $W_i(R_1) \subset B(c, 2R_*)$ holds for all c-critical times $i \le \ell$. The definition of $A(N)$ implies that no point $y \in W_i(R_1)$ has a preimage $f^{-(\ell+1)}(y) \in B(c, 2R_*)$ for $\ell + 1 < N$. Hence, times $i + \ell + 1$ are not c-critical for $\ell + 1 < N$.

Property (d). It remains to show that (d) is true at time $\ell + 1$. For all non-critical times $i \le \ell + 1$ we have

$$\frac{\operatorname{diam} W_{i-1}(r)}{\operatorname{diam} W_i(r)} \ge |f'(f^{n-i}(x))| \cdot (\operatorname{Dist}_{W_i(R_1)} f)^{-1},$$

while for all c-critical times $i \le \ell + 1$, using (b), we have

$$\frac{\operatorname{diam} W_{i-1}(r)}{\operatorname{diam} W_i(r)} \ge |f'(f^{n-i}(x))| \cdot \frac{1}{2dA_0^4}.$$

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Thus, together with (3.11), we obtain

$$\begin{aligned}\operatorname{diam} W_{\ell+1}(r) &= r\frac{\operatorname{diam} W_1(r)}{\operatorname{diam} W_0(r)}\cdots\frac{\operatorname{diam} W_{\ell+1}(r)}{\operatorname{diam} W_\ell(r)}\\ &\le r\,|(f^{\ell+1})'(f^{n-(\ell+1)}(x))|^{-1}\cdot\prod_{i\notin C^{\mathrm{crit}},\, i\le \ell+1}\operatorname{Dist}_{W_i(R_1)} f\cdot(2dA_0^4)^{2M}\\ &\le r\,|(f^{\ell+1})'(f^{n-(\ell+1)}(x))|^{-1}e^{(\ell+1)\varepsilon}A_1\end{aligned}$$

Here we also used the fact that there are only M critical points and that by (c) the critical times $\le \ell+1$ for each of them happen at most once every N iteration. Thus, we obtain (d) for $\ell+1$. This finishes the proof. □

This completes the proof of the proposition. □

Remark 3.17. The proof is simpler when one assumes the bounded distortion condition. In this case it is not necessary to go step by step; one can take the whole non-critical stretch of the trajectory of x and use the bounded distortion there. One proceeds as in the pull-back construction, see §3.7.

3.7. *Pull-back construction.* To formulate our second preliminary technical result we use the following construction, see [**15**, **37**, **38**] and [**36**, Definition 5.1]. Some complications arise in the presence of indifferent periodic orbits.

Recall that S' denotes the restricted singular set

$$S' = \operatorname{Crit}(f)\cup NO(f, K).$$

Pull-back construction: Let $(f, K, U)\in\mathscr{A}_+$. Given $n>0$, $R>0$, and $y\in K$, consider a backward trajectory $(y_i)_{i=1}^n$ of y, that is, $y_0=y$ and $y_{i+1}\in f^{-1}(y_i)$ for every $i=1,\dots,n-1$. Let k_1 be the smallest integer for which $\operatorname{Comp}_{y_{k_1}} f^{-k_1}(B(y,R))$ contains a point in S'. For every $\ell\ge 1$ let $k_{\ell+1}>k_\ell$ be the smallest integer such that $\operatorname{Comp}_{y_{k_{\ell+1}}} f^{-(k_{\ell+1}-k_\ell)}(B(y_{k_\ell},R))$ contains a point in S' and so on.

In this way, with each backward branch $(y_i)_i$ considered we associate a sequence $(k_\ell)_\ell$ of numbers $1\le k_\ell\le n$ that has a maximal element which we denote by k. We consider the set of all pairs (y_k, k) built from all the backward branches that start from y, and denote by $N(y,n,R)$ the cardinality of this set.

In order to make this construction work, we need to know that every considered preimage is well defined. If there are no indifferent periodic orbits in K then this is easy to achieve provided that R is small enough that all pull-backs of intervals of length at most $2R$ intersecting K are well inside U_K (it is sufficient if their closures are in U_K), so that they do not become truncated. This is possible in the absence of indifferent periodic orbits by [**36**, Lemma 2.10]; compare the definition of BaShrink in §2.4†.

In the presence of indifferent periodic orbits this construction works if we *a priori* know or assume the following:

(∗) Let

$$R' := \tfrac{1}{2}\operatorname{dist}(S', \operatorname{Indiff}(f)).$$

† Notice that this difficulty does not appear for rational mappings on the Riemann sphere.

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Assume that all pull-backs $\mathrm{Comp}_{y_i} f^{-i}(B(y, R))$ for $i = 0, \ldots, k_1$ have diameters smaller than R' and are well inside U_K.

Additionally, we assume that R is small enough that for every $x \in S'$ the pull-backs $\mathrm{Comp}_z f^{-k}(B(x, R))$, for all $k = 0, 1, \ldots$ and $f^k(z) = x$, have diameters smaller than R' (in particular, with $k = 0$ we have $2R < R'$). This is possible since $S' \cap \mathrm{Indiff}(f) = \varnothing$ (S' has no periodic points) and S' is finite (refer again to [**36**, Lemma 2.10], by BaShrink). We also assume, as before, that R is small enough that all pull-backs of intervals of length at most $2R$ intersecting K and disjoint from $\mathrm{Indiff}(f)$ are well inside U_K.

Under the above assumptions, the pull-back construction works. Indeed, the pull-backs $\mathrm{Comp}_{y_i} f^{-i}(B(y, R))$ for $i = 0, \ldots, k_1$ are short by assumption. Then for $c \in S' \cap \mathrm{Comp}_{y_{k_1}} f^{-k_1}(B(y, R))$ we have $|c - y_{k_1}| < R'$; hence $B(y_{k_1}, R')$, and therefore $B(y_{k_1}, R)$ are disjoint from $\mathrm{Indiff}(f)$ and we can continue the construction.

Remark 3.18. As mentioned before, the condition $(*)$ holds for R small enough if $B(y, R)$ does not contain any indifferent periodic points. Notice that it also holds if k_1 is a Pliss hyperbolic time for y_{k_1}, by Proposition 3.13.

In fact, due to Proposition 3.13, the backward shrinking as in, say, $(*)$ holds for any piece of a trajectory $x, f(x), \ldots, f^n(x)$ if n is a Pliss hyperbolic time for $x \in K$.

PROPOSITION 3.19. (Pull-back) *Given $\varepsilon > 0$, there exist numbers $A_2 = A_2(\varepsilon) > 0$ and $R_2 = R_2(\varepsilon) > 0$ such that for every $r \in (0, R_2)$ and every $y \in K$ we have*

$$N(y, n, r) < A_2\, e^{n\varepsilon} \quad \text{for all } n \geq 1.$$

Proof. Let

$$M_1 := \#S', \quad M_2 := \#S' + \sum_{c \in \mathrm{Crit}(f)} (d(c) - 1).$$

Given $\varepsilon > 0$, let us choose a positive integer N large enough such

$$\frac{M_1 \ln((N+1)\, M_2)}{N} \leq \varepsilon. \tag{3.16}$$

Observe that there are no cycles in S'; that is, $f^k(x) \neq x$ for every $k \geq 1$ and every $x \in S'$, see [**36**, Lemma 2.2]. Observe, though, that a point in S' may be mapped to another point in this set. So, given N, there is

$$\delta'(N) := \min_{x \in S'} \min_{1 \leq k \leq N} |f^k(x) - x| > 0.$$

Moreover, let $\delta(N)$ be sufficiently small that for any interval I of length $\delta(N)$ and for any $1 \leq k \leq N$ the size of each component of $f^{-k}(I)$ is smaller than $\delta'(N)/2$.

Let us consider an integer $n > N\varepsilon^{-1}$.

Let $\delta \in (0, \delta(N))$. Given a point $y \in K$ and $r \in (0, \delta)$, consider the ball $B = B(y, r)$. By definition of the number $N(y, n, \delta)$, every backward branch of y determines a sequence of numbers $(k_\ell)_\ell$. The definition of $\delta(N)$ implies that if in our construction along some backward trajectory $(y_i)_{i=1}^n$ of y there are two 'singular times' k_ℓ, k_m, $1 \leq k_\ell < k_m \leq n$, corresponding to the *same* singular point $x \in S'$, then $k_m - k_\ell > N$. As a consequence, given a stretch of this backward trajectory with times in $\{j + 1, \ldots, j + N\}$, there are at most M_1 'singular times' k_ℓ corresponding to a point in S'.

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Finally, given $1 \le i \le n$, considering all possible backward trajectories starting in y, at most M_2 of them have a *smallest* 'singular time' $k_1 = i$, that is, the corresponding component of $f^{-i}(B)$ will intersect S' for the first time after exactly i backward iterations. Observe that all of those components will be disjoint. This implies, in particular, that at most M_2 of them can intersect S'. The same argument will apply not only for backward branches starting at y, but also the branches starting at $f^{-k_i}(y)$ that we use in the construction.

Let us summarize the construction in the statement of the lemma. When we divide the time interval $\{1, \ldots, n\}$ into intervals $I_k = \{kN + 1, kN + 2, \ldots, (k + 1)N\}$, for any backward trajectory of y at most M_1 singular times can be in I_k. Hence, there are at most $(N + 1)^{M_1}$ possibilities for $\{k_j\} \cap I_k$. Moreover, given the sequence $\{k_i\} \cap \{1, \ldots, kN\}$ each of the possible subsets $\{k_j\} \cap I_k$ can be realized for at most $M_2^{M_1}$ subbranches. This lets us estimate the number of backward branches for which there is a singular time in I_k by

$$[(N + 1)M_2]^{(k+1)M_1},$$

and the number of all singular branches up to a time n by

$$N(y, n, r) \le \sum_{k=0}^{\lceil n/N \rceil} [(N + 1)M_2]^{(k+1)M_1} \le A_2(\varepsilon)e^{n\varepsilon}.$$

This proves the proposition. □

3.8. *Conical points.* After [**9**], for $(f, K, U_K) \in \mathscr{A}$, a point $x \in K$ is said to be *conical* if there exist a number $r > 0$, a sequence of positive integers $n_i \nearrow \infty$, and a sequence $(U_i)_i$ of neighborhoods $U_i \subset U_K$ of x, such that f^{n_i} is defined on U_i, $f^{n_i}(U_i) = B(f^{n_i}(x), r)$ and Dist $f^{n_i}|_{U_i}$ is bounded uniformly in i. We say that x is S'*-conical* if, additionally, for all i and $0 \le k < n_i$ we have $f^k(U_i) \cap S' = \varnothing$.

Notice that $|U_i| \to 0$ as $n_i \to \infty$, otherwise there would exist a wandering interval or an attracting (or indifferent attracting from the side of K) periodic orbit in K, which is not possible; see [**25**, Ch. 4, Theorem A], and compare [**36**, §2.3].

Remark 3.20. We observe that the concept of conical points is in fact very much related to the concept of points that are 'seen' by an inducing scheme (see, for example, [**20**] for an application of this technique to questions similar to those discussed here). Similarly to Proposition 3.21 below, in [**20**, Theorem 3.3(c)] the authors show that the inducing scheme captures all but a set of points of zero Hausdorff dimension. See also [**35**] in the holomorphic case.

PROPOSITION 3.21. (Non-conical hyperbolic points) *The set of points $x \in K$ that are not S'-conical and satisfy $\overline{\chi}(x) > 0$ has Hausdorff dimension zero.*

Proof. Let us choose some numbers $\sigma > 0$ and $\varepsilon > 0$. Let

$$r := \frac{1}{2(A_1 + 2)} \min\{R_1, R_2\},$$

where A_1 and R_1 are constants given by Proposition 3.13 and where R_2 is given by Proposition 3.19. Let $R := 2^{-1} \min\{R_1, R_2\}$.

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We fix a finite family of balls $\{B_i\}_{i=1}^N$ of radius $3r$ such that any ball of radius $2r$ which intersects K must be contained in one of the balls in this family.

Given $m \geq 1$ and $\sigma > 0$, let $G(m, \sigma)$ be the set of points $x \in K$ with $\overline{\chi}(x) > \sigma$ for which, for all $\ell > m$ being Pliss hyperbolic times with exponent σ, the pull-back of $B(f^\ell(x), 2r)$ for f^ℓ containing x meets a point in S'; that is, we have

$$\mathrm{Comp}_{f^k(x)}\, f^{-(\ell-k)}(B(f^n(x), 2r)) \cap S' \neq \varnothing \tag{3.17}$$

for some $k \in \{0, \ldots, \ell - 1\}$. Since ℓ is a Pliss hyperbolic time, by Proposition 3.13 all the above pull-backs are small, and in particular well inside U_K. Hence the assumption $(*)$ enabling the pull-back construction holds. Compare Remark 3.18.

We claim that $\dim_{\mathrm{H}} G(m, \sigma) = 0$ for any $m \geq 1$ and $\sigma > 0$. Indeed, given $n > m$, let us denote by $G(m, \sigma, n)$ the set of points in $G(m, \sigma)$ for which n is a Pliss hyperbolic time with exponent σ. Recall that $\overline{\chi}(x) > \sigma > 0$ implies that there exist infinitely many Pliss hyperbolic times for x with exponent σ. Hence, we have

$$G(m, \sigma) = \bigcap_{k \geq m} \bigcup_{n > k} G(m, \sigma, n). \tag{3.18}$$

Let $x \in G(m, \sigma)$. Then $x \in G(m, \sigma, n)$ for some $n > m$ arbitrarily large. Let B_j be a ball from the above chosen family which contains $B(f^n(x), 2r)$ and pick y such that $B_j = B(y, 3r)$. We will apply the pull-back construction of Proposition 3.19 to the point y, the number n, the radius R, and f_x^{-n}. Let (y_k, k) be given by the pull-back construction.

We first note that Proposition 3.13 and $|f^n(x) - y| \leq r \leq R_1$ imply that

$$|(f^{n-k}(x)) - y_k| \leq r\ A_1 e^{-k(\sigma-\varepsilon)} < r\ A_1.$$

By our choice of constants we have $r\ A_1 + 2r = R$, which implies

$$B(f^{n-k}(x), 2r) \subset B(y_k, R).$$

Since $x \in G(m, \sigma)$, the pull-back of $B(f^\ell(x), 2r)$ must meet a point in S' whenever $\ell > m$. On the other hand, k is the maximal element and hence the pull-back of $B(y_k, R)$ does not meet any critical point. Thus, we conclude $m \geq n - k$. Thus, fixing n and B_j, by Proposition 3.19 the point x must belong to one of at most $A_2 e^{n\varepsilon} (\deg f)^m$ preimages of B_j.

Also observe that $B_j = B(y, 3r) \subset B(f^n(x), 4r)$ and $4r < R_1$. Since n is a Pliss hyperbolic time for x with exponent σ, by Proposition 3.13 this preimage of B_j has diameter not greater than $4r\ A_1 e^{-n(\sigma-\varepsilon)}$.

Hence, we showed that every point in $G(m, \sigma, n)$ belongs to the nth preimage of some ball B_j along a backward branch of y and the maximal element (y_k, k) as defined in the pull-back construction satisfies $k \geq m - n$. Thus, by Proposition 3.13, the set $G(m, \sigma, n)$ is contained in a union of at most $A_2 e^{n\varepsilon} (\deg f)^m$ sets of diameter not greater than $4r\ A_1 e^{-n(\sigma-\varepsilon)}$. We apply $G(m, \sigma) \subset \bigcup_{n>m} G(m, \sigma, n)$ to obtain

$$\dim_{\mathrm{H}} G(m, \sigma) \leq \frac{\varepsilon}{\sigma - \varepsilon}$$

(recall (3.18)). As ε can be chosen arbitrarily small, this proves $\dim_{\mathrm{H}} G(m, \sigma) = 0$.

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Finally, let us consider the set

$$G := \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} G\left(m, \frac{1}{n}\right).$$

By the above we have $\dim_{\mathrm{H}} G = 0$. Let $x \in K \setminus G$ be a point with $\overline{\chi}(x) > 0$. Let us now show that x must be S'-conical. By the above, there is a sequence of Pliss hyperbolic times $(n_i)_i$ such that $\mathrm{Comp}_{f^k(x)} f^{-(n_i-k)}(B(f^{n_i}(x), 2r))$ does not intersect S' for any $k = 0, \ldots, n_i - 1$. Hence, in particular, f^{n_i} maps $I_1^i = \mathrm{Comp}_{f^k(x)} f^{-(n_i-k)}(B(f^{n_i}(x), 2r))$ diffeomorphically onto $I_2^i = B(f^{n_i}(x), 2r)$. Note that $(1-\varepsilon) I_2^i \subset B(f^{n_i}(x), r)$. Hence, as f satisfies bounded distortion near K, there exists $C(\varepsilon) > 1$ such that for every $i \geq 1$

$$\mathrm{Dist}\, g|_{B(f^{n_i}(x), r)} \leq C(\varepsilon) \quad \text{where } g = (f^{n_i}|_{I_1^i})^{-1}.$$

Thus, x is S'-conical with r, $(n_i)_i$, and $U_i = I_1^i$. This finishes the proof. □

4. *Proof of Theorem 1*

In this section we shall prove Theorem 1, except for (1.11) which is postponed to §5.

4.1. *No conical points with small exponent.* We start with a proposition, which, together with Proposition 3.21 in the previous section, proves in particular (1.13) in Theorem 1 (compare also [**20**, Remark 4.1]). In the following sections we shall provide improvements, skipping the assumption that the points are conical, in particular proving (1.11) and its generalizations.

PROPOSITION 4.1. *Let $(f, K) \in \mathscr{A}_+^{\mathrm{BD}}$ be non-exceptional and satisfy the weak isolation condition. Then the set $\{x : x \text{ is conical}, \ \overline{\chi}(x) < \chi_{\inf}\}$ is empty.*

In particular, above we can write S'-conical, which together with Proposition 3.21 is also sufficient to prove (1.13). Moreover, we point out that the assumption of weak isolation is not needed in this proof.

Let us summarize that we get a decomposition of K into three sets:

- points x being non-S'-conical with $\overline{\chi}(x) \leq 0$;
- points x being non-S'-conical with $\overline{\chi}(x) > 0$; and
- x being S'-conical.

We do not know much about the first set. By Proposition 3.21, the second one has Hausdorff dimension 0. Any x in the third set satisfies $\overline{\chi}(x) \geq \chi_{\inf}$ by Proposition 4.1.

Sketch of Proof of Proposition 4.1. For x conical we find an arbitrarily small neighborhood U of x, and n such that f^n maps U onto $B(f^n(x), r)$ with bounded distortion. By strong transitivity, $f^{-m}(x)$ is $r/2$-dense for sufficiently large m. As $|U|$ is small, its respective pull-back U' is in $B(f^n(x), r)$. Then one finds a shadowing periodic orbit and distributes an invariant measure μ on it. We obtain $\chi(\mu) \leq \overline{\chi}(x) + \varepsilon$ with ε arbitrarily close to 0. Observe that capturing critical points when going forward from U' to U does not hurt as the small derivative of f close to them works to our advantage. □

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4.2. *Upper bound.* The content of this subsection, like §4.1 and 4.3, is very similar to the complex case in [**14**]. However, the modified versions of the proof of Proposition 4.2 will be considered in §5 and 6, so we include the details.

Thus, the first part of Theorem 1 consists of the following upper bound.

PROPOSITION 4.2. *Given numbers $\alpha \le \beta \le \chi_{\sup}$ with $\beta > 0$, and additionally assuming $\alpha > 0$ if $\chi_{\inf} = 0$, we have*

$$\dim_{\mathrm{H}} \mathcal{L}(\alpha, \beta) \le \max\Big\{0, \max_{\alpha \le q \le \beta} F(q)\Big\}.$$

The possibility of restricting to S'-conical points due to Proposition 3.21 makes the proof easier than the proofs in §5 and §6, where one deals with finite criticality (TCE). It turns out that considering 'non-critical conical times' is not sufficient and that one has to deal with true conformal measures.

Proof. Let α and β be as in the assumption and $x \in \mathcal{L}(\alpha, \beta)$. By Proposition 3.21, it is enough to assume that x is S'-conical with a corresponding number $r > 0$, a sequence $n_i \nearrow \infty$, and a sequence $(U_i)_i$ of neighborhoods of x such that $f^{n_i}(U_i) = B(f^{n_i}(x), r)$ (U_i's are pull-backs in another terminology) and that Dist $f^{n_i}|_{U_i}$ is bounded uniformly in i by some constant C. By passing to a subsequence we can assume that $1/n_i \log |(f^{n_i})'(x)|$ has a limit $q \in [\alpha, \beta]$. If $\chi_{\inf} > 0$ then as in the proof of Proposition 4.1 we can find a periodic point x with $\chi(x)$ less than $q + \varepsilon$ for ε arbitrarily close to zero, for n_i sufficiently large. Hence $q \ge \chi_{\inf} > 0$. If $\chi_{\inf} = 0$ we use the assumption $\alpha > 0$ to ensure that $q > 0$.

Every $f^k(U_i)$ and thus $f^{k-n_i}(B(f^{n_i}(x), r))$, for every $k = 0, \ldots, n_i - 1$, is disjoint from S'. So when we consider μ_t^* (the measure defined in §2.3 with Jacobian $\lambda |f'|^t$ for $\lambda = e^{P(t)}$), applying the conformality relation for f repeated n_i times, we have, due to Lemma 3.1,

$$\int_{f_x^{-n_i}(B(f^{n_i}(x),r))} |(f^{n_i})'|^t \, d\mu_t^* = \mu_t^*(B(f^{n_i}(x), r)) \, \lambda^{-n_i}.$$

Given r, there is a constant $c = c(r, t) > 1$ such that $c^{-1} \le \mu_t^*(B(y, r)) \le c$ for every $y \in K$, see §3.3 (where the notation Υ was used). This, together with the distortion results, implies

$$\mu_t^*(f_x^{-n_i}(B(f^{n_i}(x), r))) \ge |(f^{n_i})'(x)|^{-t} C^{-1} c^{-1} e^{-n_i(P(t)+\varepsilon)}. \tag{4.1}$$

Now we conclude as in the proof of [**14**, Proposition 2] (see also the end of §6) that the lower local dimension of μ_t^* is bounded from above by $t + (P(t)/q)$ at x. Then, continuing as in this proof, and applying the Frostman lemma, we obtain

$$\dim_{\mathrm{H}} \mathcal{L}(\alpha, \beta) \le \max\Big\{0, \max_{\alpha \le q \le \beta} F(q)\Big\}. \tag{4.2}$$

In the case $\chi_{\inf} > 0$ we have got a formally better result, with $\max\{\alpha, \chi_{\inf}\} \le q \le \beta$ in the above estimate, but this does not make a difference since $F(\alpha) = -\infty$ for $\alpha < \chi_{\inf}$. This finishes the proof. □

Notice that, in particular for $0 < q \le \beta < \chi_{\inf}$, for t large enough we have $P(t) < -q\,t$. Hence, for ε small enough the measures in (4.1) diverge. So such S'-conical points x

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cannot exist. Observe that in this way we have obtained an alternative proof of the partial result in Proposition 4.1 stating that the set of S'-conical points is empty (the emptiness of the conical set does not follow, since to use the CaS measure we need to omit S').

Summarizing, Proposition 4.1 together with Proposition 3.21 prove (1.13). This approach will be used again in §6.

4.3. *Lower bound—interior of the spectrum.* The proof of the lower bound for the regular spectrum in Theorem 1 goes exactly like in [**14**, §5], with the only difference being that we use S' instead of the set of critical points. The main idea is to construct ECRs and use bridges, as recalled in §3.4, and to apply Proposition 3.11.

We use the formula for the Lyapunov spectrum on a uniformly hyperbolic repeller X_m (see, for example, [**29**]) which gives in particular

$$\dim_{\mathrm{H}}\{x \in X_m : \chi(x) = \alpha\} \geq F_m(\alpha).$$

Hence, using (3.7), for any $\alpha \in (\chi_{\inf}, \chi_{\sup})$ we get

$$\dim_{\mathrm{H}}\{x \in K : \chi(x) = \alpha\} \geq \sup_m F_m(\alpha) = F(\alpha),$$

which is the lower bound asserted in Theorem 1.

4.4. *Lower bound—boundary and irregular part.* It is sufficient to have the following result.

THEOREM 4. *Let $(X_i)_i$ be a sequence of ECRs invariant for f^{a_i} in K (Cantor sets; we do not allow individual periodic orbits here), disjoint from $\partial\widehat{I}_K$. Let $(\phi_i)_i$ be a sequence of Hölder continuous potentials and let $(\mu_i)_i$ be a sequence of equilibrium states for ϕ_i with respect to $f^{a_i}|_{X_i}$. Then we get*

$$\dim_{\mathrm{H}}\left\{x \in K : \underline{\chi}(x) = \liminf_{i\to\infty} \chi(\mu_i),\ \overline{\chi}(x) = \limsup_{i\to\infty} \chi(\mu_i)\right\} \geq \liminf_{i\to\infty} \dim_{\mathrm{H}} \mu_i$$

and for the packing dimension

$$\dim_{\mathrm{P}}\left\{x \in K : \underline{\chi}(x) = \liminf_{i\to\infty} \chi(\mu_i),\ \overline{\chi}(x) = \limsup_{i\to\infty} \chi(\mu_i)\right\} \geq \limsup_{i\to\infty} \dim_{\mathrm{H}} \mu_i.$$

Proof. This proof goes exactly like the proof of [**14**, Theorem 3]. However, in our setting, even assuming the weak isolation condition, it is not *a priori* clear that the Cantor set C of points x constructed in [**14**] is in K. So, we should care about this in the construction. The points in C are the points whose forward trajectory spends time n_1 close to a trajectory in X_1 Birkhoff for ϕ_1 (that is, with finite time Birkhoff average close to $\int \phi_1\, d\mu_1$); next it goes along a bridge which joins X_2 to X_1 (in the forward direction), then it spends time n_2 close to a trajectory in X_2 Birkhoff for ϕ_2; next it goes to X_3, etc. The sequence $(n_i)_i$ grows sufficiently fast to guarantee the required properties.

Since the measures μ_i are non-atomic, the (countable) set $\bigcup_{m=0}^{\infty} f^m(\partial\widehat{I}_K)$ has measure zero. Hence, we can choose any point outside this set and construct (backward) bridges from such a point. These bridges will be disjoint from $\partial\widehat{I}_K$. If $\mathcal{U}_i$ consist of components of length small enough, then $\bigcup_{m=0}^{\infty} f^m(C) \subset \widehat{I}_K$. Hence, by the maximality of K in $\widehat{I}_K$, we have $C \subset K$, see §1.1. □

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To prove lower bounds for, say, $\dim_{\mathrm{H}} \mathcal{L}(\alpha, \beta)$ with $\alpha, \beta \in (\chi_{\inf}, \chi_{\sup})$, we consider ECRs X_i as in Proposition 3.11 (indexed X_m there). Consider also

$$\phi_i := \phi_{t_i}|_{X_i} \quad \text{where } t_i \text{ is such that } \left.\frac{\partial P(f|_{X_i}, \phi_t)}{\partial t}\right|_{t_i} = \begin{cases} \alpha & i \text{ odd}, \\ \beta & i \text{ even}, \end{cases}$$

and let μ_i be the equilibrium for ϕ_i with respect to $f|_{X_i}$.

We can assume X_i to be disjoint from $\partial \widehat{I}_K$. Otherwise we can subtract from X_i an open set W covering $\partial \widehat{I}_K$ which, for example, can be taken to be small 'cylinders' of the form $g_{k_1 k_2} \circ g_{k_2 k_3} \circ \cdots \circ g_{k_{n-1} k_n}(U_{k_n})$, see §3.4. This does not change the topological pressures on X_i much. Indeed, this continuity follows from the following.

LEMMA 4.3. *Let $f: X \to X$ be an expanding repeller in $\mathbb{R}$ (or, more generally, an open distance-expanding map, see* [**40**]*) which is topologically transitive and of positive topological entropy. Let $\phi: X \to \mathbb{R}$ be a continuous function. Take an arbitrary finite set $Z \subset X$. Denote $X(r) := X \setminus Y_r$ where $Y_r = \bigcup_{j=0}^{\infty} f^{-j}(B(Z, r))$. Then $P(f|_{X(r)}, \phi|_{X(r)}) \nearrow P(f, \phi)$ as $r \to 0$.*

For completeness we provide an elementary proof. Another proof follows from the use of conformal measures, see [**11**]. Compare also [**36**, Proof of Theorem 4.1, Case 2] from which yet another proof can be extracted. Remark also that using a Markov partition one can approximate ϕ by a function constant on small 'cylinders', and reduce the consideration to the continuity of the leading eigenvalue of the corresponding Markov matrices.

Proof of Lemma 4.3. We replace Z by a finite subset Z' of its forward trajectory, so that no point in Z' is pre-periodic but non-periodic and such that the forward trajectories of any pair of distinct points in Z' are disjoint. Then for an arbitrary $r' > 0$ and $W = B(Z', r')$ we get $\bigcup_{j=0}^{\infty} f^{-j}(W) \supset Y_r$ for all r small enough.

Let ℓ_0 be such that every point $x \in X$ has at least two distinct preimages of level ℓ_0 in X. Fix n and take $r' > 0$ small enough that all the sets $f^j(B(z, r'))$, $z \in Z'$, $j = 0, \ldots, n + \ell_0$ are pairwise disjoint, with the possible exception of $f^{j_1}(B(z, r'))$, $f^{j_2}(B(z, r'))$ for $z \in Z'$ periodic with $j_2 - j_1$ being a multiplicity of the period (such sets, naturally, must intersect).

Consider

$$X(W) := X \setminus \bigcup_{j=-\ell_0}^{\infty} f^{-j}(W).$$

Take $x \in X(W)$. Consider two distinct points $y, y' \in f^{-\ell_0}(x)$. Then by the definition of W at least one of them, say y, has the property that all the preimages $f^{-j}(\{y\})$, $j = 0, \ldots, n$ are disjoint from W. By definition of $X(W)$ all $f^j(y)$, $j = 1, \ldots, \ell_0$ are also outside W. Thus, all the backward branches of x of length $n + \ell_0$ passing through y are disjoint from W.

Let ψ be a non-negative potential obtained by adding a constant to ϕ. We choose n large enough that for all $w \in X$ we have

$$\frac{1}{n - \ell_0} \log \sum_{z \in f^{-n+\ell_0}(w)} \exp\left(\sum_{\ell=0}^{n-1-\ell_0} \psi(f^\ell(z)) \right) \geq P(f, \psi) - \varepsilon.$$

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Indeed, this follows by [**40**, Proposition 4.4.3]. This proposition formally assumes Hölder continuity. However, the uniform convergence of pressure holds also in the general case of a merely continuous potential, since every continuous function can be uniformly approximated by a Hölder continuous one.

Given $x \in X(W)$ and y as above, using the fact that ψ is non-negative, noting also that $z \in f^{-n+\ell_0}(y)$ implies $z \in X(W)$, we obtain

$$\sum_{z \in f^{-n}(x) \cap X(W)} \exp\left(\sum_{\ell=0}^{n-1} \psi(f^\ell(z))\right) \geq \sum_{z \in f^{-n+\ell_0}(y)} \exp\left(\sum_{\ell=0}^{n-1-\ell_0} \psi(f^\ell(z))\right). \tag{4.3}$$

Composing these two inequalities we obtain a uniform lower bound for the left-hand side of (4.3) valid for every $x \in X(W)$. In particular we can apply it for each z in place of x in (4.3). Thus, using this lower bound repeatedly we obtain for every positive integer m

$$\sum_{z \in f^{-mn}(x)} \exp\left(\sum_{\ell=0}^{mn-1} \psi(f^\ell(z))\right) \geq \exp(m(n-\ell_0)(P(f, \psi) - \varepsilon))$$

and hence

$$P(f|_{X(r)}, \psi) \geq \frac{n-\ell_0}{n}(P(f, \psi) - \varepsilon).$$

This proves the lemma. □

One can correct X_i, defined as $X(r)$ above, to become topologically transitive: first, by slightly extending it so that $f|_{X_i}$ becomes an open map (see [**40**, Proposition 4.5.6]) but still $X_i \subset \operatorname{int} \widehat{I}_K$; next, replacing it by its subset of all non-wandering points of $f|_{X_i}$ (by the variational principle it does not change pressures); and finally, restricting it to an appropriate basic set in the spectral decomposition, see [**40**, Theorem 4.3.8].

If at least one of the numbers α, β is in the boundary of the interval $[\chi_{\inf}, \chi_{\sup}]$, then we consider μ_i for a sequence of numbers t_i corresponding to $\gamma_i \in (\chi_{\inf}, \chi_{\sup})$ which accumulates at β and at α. This proves in particular $\dim_{\mathrm{H}} \mathcal{L}(0) \geq F(0)$ in Theorem 1.

5. *On the completeness of the spectrum*

Analogously to [**14**, §3] we investigate which numbers can occur at all as upper/lower Lyapunov exponent. The proof of (1.12) in Theorem 1 that

$$\{x \in K : \overline{\chi}(x) > \chi_{\sup}\} = \varnothing$$

is the same as in [**14**, Proof of Lemma 5]. The proof of (1.11) in Theorem 1 that

$$\{x \in K : -\infty < \chi(x) < \chi_{\inf}\} = \varnothing$$

is more difficult and in part different from the one in [**14**]. This section is devoted to this proof.

5.1. *No points with exponent less than* $\chi_{\inf}$.

We prove the completeness part of Theorem 1, promised at the beginning of this section.

THEOREM 5. *Let* $(f, K) \in \mathscr{A}_+^{\mathrm{BD}}$ *and satisfy the weak isolation condition. Then for every* $x \in K$ *such that finite* $\chi(x)$ *exists, we have* $\chi(x) \geq \chi_{\inf}$.

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Proof. The proof of $\chi(x) \geq 0$ under the hypothesis $\lim_{n\to\infty} \log |f'(f^n(x))| = 0$ is the same as in [**14**, Lemma 6]. However, to prove $\chi(x) \notin [0, \chi_{\inf})$ we cannot use complex methods as in [**14**], so we provide an alternative proof.

Let us suppose that $\chi(x) \geq 0$. We can also freely assume that x is not eventually periodic. Indeed, otherwise, $\chi(x) = \chi(p)$ for some periodic p and $\chi(p) \geq \chi_{\inf}$ because we can distribute an invariant measure on the orbit of p. If $\chi_{\inf} = 0$ then automatically $\chi(x) \geq \chi_{\inf}$ and the assertion holds. Thus, it remains to consider the case $\chi_{\inf} > 0$. By the properties listed in §2.4, we know that (f, K) satisfies TCE and lacks indifferent periodic points in K.

Fix any $n = n_j$ from the sequence $(n_j)_j$ in TCE and consider the intervals

$$\widehat{W} = B(f^n(x), r) \quad \text{and} \quad W := \tfrac{1}{2} \diamond \widehat{W} = B(f^n(x), r/2),$$

that is, $\widehat{W}$ is the $1/2$-scaled neighborhood of W (recall (1.1)). Given $k = 0, \ldots, n$, denote by $\widehat{W}_k$ the pull-back of $\widehat{W}$ for f^k which contains $f^{n-k}(x)$,

$$\widehat{W}_k := \mathrm{Comp}_{f^{n-k}(x)}\, f^{-k}(\widehat{W}),$$

and denote by W_k the corresponding pull-back of W. Denote by

$$\widehat{W}_k \setminus W_k = L_k \cup R_k$$

the union of the left and right interval (margin) in this difference.

We shall now prove the following fact.

CLAIM. *There exists a number $\tau > 0$ (which depends only on f, not on x, n) such that each $\widehat{W}_k$ contains an τ-scaled neighborhood of W_k.*

Compare, for example, [**33**, Lemma 1.4] in the complex case.

Proof. Recall the choice of δ in (1.2). Assume that in TCE the number r is small enough that in BaShrink all pull-backs of $\widehat{W}$ for iterates of f have lengths less than δ.

For $k \in \{0, \ldots, n\}$ denote by τ_k the maximal number for which the claim still holds. By construction, $\tau_0 \geq 1/2$.

Let $0 \leq k_1 < \cdots < k_N \leq n-1$ be the subsequence of all consecutive integers k such that $\widehat{W}_k \cap \mathrm{Crit}(f) \neq \varnothing$. By TCE, we have $N \leq M$. Note that M depends neither on x nor on n.

Let us divide the interval $\{0, \ldots, n-1\}$ into critical times $\{k_i\}$ and intercritical intervals $\{k_i + 1, \ldots, k_{i+1} - 1\}$, $i = 1, \ldots, N-1$. By (3.9), at any critical time the number τ_j can decrease at most by a constant factor. By (1.2), in an intercritical interval the number τ_j can decrease at most by a constant factor. As the number of critical times and intercritical intervals are bounded by M and $M+1$, respectively, the claim follows. □

Notice that $f^n(\partial W_n) \subset \partial W$ (it can happen that both ends of W_n are mapped to one end of W). Since $f^n(W_n)$ contains also the point $f^n(x)$, we get $|f^n(W_n)| \geq r/2$.

We will show in the following that there is a constant $a \in (0, 1)$ such that

$$a \diamond (f^n(W_n)) \cap K \neq \varnothing. \tag{5.1}$$

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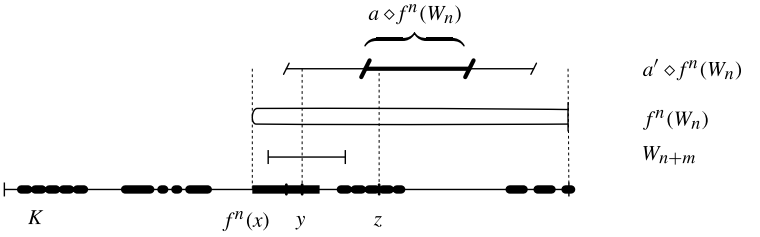


FIGURE 3. Capturing a periodic orbit.

Observe that this condition says that there is a point belonging to K which is 'well inside' $f^n(W_n)$. We will show this fact not for all n_j but for some subsequence, slightly corrected if necessary, satisfying (2.4) (with the same r, but the constant M may be altered), or else assuming that x is not pre-periodic. But let us postpone the proof for a while.

(i) Proof of $\chi(x) \geq \chi_{\inf}$ *assuming (5.1).*

(i)(a) Capturing a shadowing periodic orbit. Let $a' = (a+1)/2$. Pick some $z \in a \diamond (f^n(W_n)) \cap K$. Recall the function m_c in Definition 3.4. There exists some $m \in \{0, \ldots, m_c((a'-a)r/4))\}$ and $y \in f^{-m}(x)$ such that $|y - z| < (a'-a)r/4$. Hence, as $|f^n(W_n)| \geq r/2$, we have $y \in a' \diamond (f^n(W_n))$.

Hence, due to BaShrink, the pull-back interval W_{n+m} of W_n for f^m which contains y has length smaller than $(1-a')r/4$, for n large enough. Hence, it is strictly contained in $f^n(W_n)$. Therefore, there exists a periodic point $p \in W_{n+m}$ of period $n+m$. By the weak isolation condition, $p \in K$. See Figure 3.

(i)(b) Comparison of derivatives. Let us now estimate $|(f^{n+m})'(p)|$ from above. Denote by Lip the Lipschitz constant of $f|_K$. First observe that existence and finiteness of $\chi(x)$ imply $\lim_{n\to\infty} 1/n \log |f'(f^n(x))| = 0$. Hence, given $\xi > 0$, for large enough n we have $|(f^n)'(x)| \leq \exp(\xi N n)$. For n large enough, we have

$$|(f^{n+m})'(p)| = |(f^m)'(p)| \cdot \prod_{i=1}^{N} |(f^{k_i - k_{i-1}-1})'(f^{m+n-k_i+1}(p))| \cdot \prod_{i=1}^{N} |f'(f^{m+n-k_i}(p))|.$$

Now let τ be as in the Claim and $C(\tau)$ as in (1.2). We can further estimate

$$\begin{aligned} |(f^{n+m})'(p)| &\leq \mathrm{Lip}^{m+N}\, C(\tau)^N \cdot \prod_{i=1}^{N} |(f^{k_i - k_{i-1}-1})'(f^{n-k_i+1}(x))| \\ &\leq \mathrm{Lip}^{m+N}\, C(\tau)^N \cdot |(f^n)'(x)| \exp(\xi N n). \end{aligned} \tag{5.2}$$

We conclude

$$\frac{1}{n+m} \log |(f^{n+m})'(p)| \leq N\xi + \lim_{n\to\infty} \frac{1}{n} \log |(f^n)'(x)| + O\left(\frac{1}{n}\right).$$

Considering the f-invariant probability measure

$$\mu_{n,\xi} := \frac{1}{n+m} \sum_{k=0}^{n+m-1} \delta_{f^k(p)}, \tag{5.3}$$

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its Lyapunov exponent is $\chi(p)$. Hence, letting $\xi \to 0$ and $n \to \infty$, we obtain a sequence of measures $\mu_{n,\xi}$ such that $\inf_{n,\xi} \chi(\mu_{n,\xi}) \le \chi(x)$. This proves the assertion assuming that (5.1) holds.

What remains to prove is (5.1).

(ii) Proof of (5.1), changing n and M if necessary. Now let us prove the postponed (5.1). We shall specify the constant a later on. For any $\rho > 0$ define

$$\partial_\rho := \{z \in K : (z, z+\rho) \cap K = \varnothing \text{ or } (z-\rho, z) \cap K = \varnothing\}.$$

Considering $a > 1/2$, set $b = 2a - 1$ (hence $1 - b = 2(1-a)$) and $\rho = br/2$. Assume first that

$$f^n(x) \notin B(\partial_\rho, (1-b)r/4). \tag{5.4}$$

In words this means that $f^n(x) \in K$ is not closer than $(1-b)r/4$ to a 'big hole in K' which is of length at least ρ. Then K intersects both intervals $R = b \diamond (f^n(x),\ f^n(x) + r/2)$ and $L = b \diamond (f^n(x) - r/2,\ f^n(x))$. Since $f^n(W_n)$ covers either $(f^n(x),\ f^n(x) + r/2)$ or $(f^n(x) - r/2,\ f^n(x))$, then (5.1) follows. Indeed, notice that $(f^n(x),\ f^n(x) + r/2)$ is a $(1-b)r/4$-scaled neighborhood of R (analogously for L), so $f^n(W_n)$ contains a $(1-a)r/4$-scaled neighborhood of R (or L).

Notice that if $0 < r_1 < r_2$ then $\partial_{r_2} \subset \partial_{r_1}$. If $r/3 \le \rho$, and thus $b \ge 2/3$, then (5.4) and thus (5.1) hold for all n such that

$$f^n(x) \notin B(\partial_{r/3}, (1-b)r/4) =: B. \tag{5.5}$$

Notice that our subscript at ∂ is independent of b now. Notice that the set $\partial_{r/3}$ is finite, with the number of elements depending only on r. This set can contain periodic points, which must be hyperbolic repelling. Let $\theta > 0$ be a constant such that for each periodic point $q \in \partial_{r/3}$ with period $m(q)$, for every $y \in B(q, \theta)$ we have $|(f^{m(q)})'(y)| > 1$.

Let us consider the upper density

$$\overline{d} := \limsup_{m\to\infty} \frac{1}{m} \#\{0 < k \le m : f^k(x) \notin B\}.$$

(ii)(a) The case $\overline{d} > 0$. Taking P in the definition of TCE small enough that $1/P + \overline{d} > 1$, possibly passing to a subsequence, we can assume that there is a strictly increasing sequence $(n_j)_j$ such that TCE and (5.5) hold. Hence, this shows (5.1) and finishes the proof of Theorem 5 in this case.

(ii)(b) The case $\overline{d} = 0$. Set

$$\alpha := \operatorname{dist}(f(\partial_{r/3}) \setminus \partial_{r/3},\ \partial_{r/3}) > 0.$$

Then, for each $z \in \partial_{r/3} \setminus f^{-1}(\partial_{r/3})$,

$$f(B(z, \operatorname{Lip}^{-1} \alpha/2)) \cap B(\partial_{r/3}, \alpha/2) = \varnothing. \tag{5.6}$$

Recall that Lip denotes the Lipschitz constant of $f|_K$. Define

$$\Theta := \min\{|w - w'| : w \ne w',\ w, w' \in \partial_{r/3}\}.$$

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We set

$$\alpha' := \min\left\{\frac{\alpha}{2\,\mathrm{Lip}}, \frac{\Theta}{2}, \frac{\theta}{2}\right\}.$$

Let b be defined by $(1-b)r/4 = \alpha'$. This, in turn, eventually specifies a. Finally, recalling the definition of $\varkappa(\cdot)$ in BaShrink, define

$$M' := M + \#\partial_{r/3} + \varkappa(2r, \alpha'). \tag{5.7}$$

Recall again that $f^k(x) \notin B$ implies (5.5) and hence (5.1). As we have $\overline{d} = 0$, for all m large enough, an arbitrarily large proportion of $\{1, 2, \ldots, m\}$ is covered by blocks A_j of consecutive integers k so that $f^k(x) \in B(z_k, \alpha')$ for an appropriate $z_k \in \partial_{r/3}$; such z_k is unique by the choice of α'. Moreover, it also follows from the choice of α' that for integers $k, k+1$ belonging to one block A_j we have

$$f(z_k) = z_{k+1}. \tag{5.8}$$

Hence, for each block $A_j = \{\ell_j, \ell_j + 1, \ldots, \ell'_j\}$, if $\#A_j > \#\partial_{r/3}$, then the point z_{ℓ_j} is eventually periodic; namely, there exists a periodic point $q \in \partial_{r/3}$ with period not greater than $\#A_j - 1$, which we denote by t_j†.

Note that we can exclude the case $f^{\ell_j}(x) = z_{\ell_j}$, because then x would be eventually periodic (we considered this case in the beginning of the proof of Theorem 5). Increasing m if necessary, we can assume that the last block ends at m and that all the blocks A_j have been chosen to be maximal, that is, $f^{\ell'_j+1}(x) \notin B$ for all j. Observe that each block must indeed be finite by definition of α'.

So (5.5) and hence (5.1) hold for $n_j = \ell'_j + 1$ for each j. What remains to be checked is that this number also satisfies (2.4) in TCE.

We will prove that (2.4) holds for $n_j = \ell'_j + 1$ with the constant M' provided it holds for some $m_j \in A_j$ with the constant M. Indeed, the pull-backs $\mathrm{Comp}_{f^k(x)} f^{-(\ell'_j+1-k)} B(f^{\ell'_j+1}(x), r)$ have lengths smaller than α' for all $k \geq \varkappa(2r, \alpha')$ by the definition of the latter constant in BaShrink, and hence their consecutive pull-backs do not capture critical points for $\ell_j + t_j \leq k \leq \ell'_j - \varkappa(2r, \alpha')$; they can capture critical points only for $k \in \{m_j, \ldots, m_j + t_j - 1\}$ and the number of such k's is bounded by $\#\partial_{r/3}$. In conclusion, when passing from m_j to $\ell'_j + 1$, TCE still holds with M replaced by M' defined in (5.7).

Finally, recall that the union of the blocks A_j covers an arbitrarily large proportion of $\{0, \ldots, m\}$ for m large enough. Hence there exist arbitrarily large m_j belonging to some A_j also satisfying TCE, and hence there exist arbitrarily large $n_j = \ell'_j + 1$ satisfying TCE with M', and simultaneously satisfying (5.1).

This finishes the proof of the theorem. □

6. *Strong upper bound for irregular part*

We now shall prove Theorems 2 and 3 from §1.7. We start with a technical lemma, to be applied later for $\Phi(n) = \log|(f^n)'(x)|$. It will allow us to strengthen Pliss lemma 3.12

† Notice that we can assume that $z_{\ell_j} \in f^{\ell_j}(W_{\ell_j})$. Otherwise, $f^{\ell_j}(W_{\ell_j})$ would cover the side of $f^{\ell_j}(x)$ not containing z_{ℓ_j} and hence (5.1) would hold immediately. However, we will not use this observation and assumption later on.

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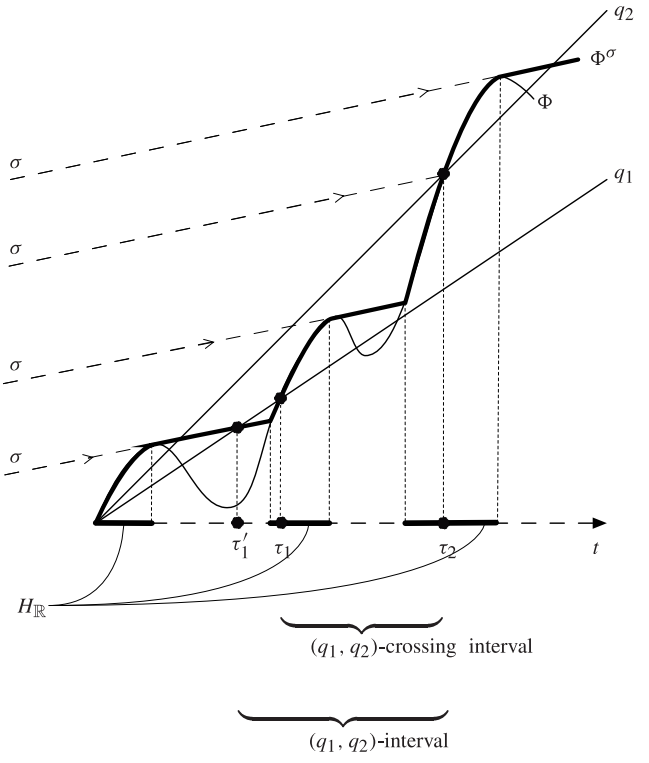


FIGURE 4. (q_1, q_2)- and crossing intervals.

which states that for x with $\overline{\chi}(x) > \sigma > 0$ there is an abundance of hyperbolic times with exponent σ for x (that is, 'illuminated' from an angle σ, see Figure 4). We will conclude that, in fact, there are times with exponent $(1/n)\log|(f^n)'(x)|$ within an arbitrary *a priori* fixed reasonable interval $[q_1, q_2]$.

LEMMA 6.1. *Let $\Phi\colon \mathbb{R}_+ \to \mathbb{R}$ be a continuous function, and $L > 0$ be such that for each $0 \le t_1 < t_2$ we have $\Phi(t_2) - \Phi(t_1) \le L(t_2 - t_1)$. Let $0 \le \sigma \le q_1 < q_2$ be real numbers. Let $C := L^{-1}(q_2 - q_1)$ and define*

$$\Phi^\sigma(t) := \sup_{s<t}(\Phi(s) + \sigma(t-s))$$

and

$$H_{\mathbb{R}} := \{x \in \mathbb{R}_+\colon \Phi(x) = \Phi^\sigma(x)\} \quad \text{and} \quad H_{\mathbb{N}} := \{n \in \mathbb{N}\colon \Phi(n) = \Phi^\sigma(n)\}.$$

Let $\tau_1 < \tau_2$ be positive numbers satisfying $\Phi^\sigma(\tau_1) = q_1\tau_1$, $\Phi^\sigma(\tau_2) = q_2\tau_2$. Then, with ℓ denoting the length measure, we have

$$\frac{1}{\tau_2}\ell(H_{\mathbb{R}} \cap [\tau_1, \tau_2]) \ge C. \tag{6.1}$$

If Φ is affine between each pair of consecutive integers and τ_2 is an integer then

$$\frac{1}{\tau_2}\#(H_{\mathbb{N}} \cap [\tau_1, \tau_2]) \ge C. \tag{6.2}$$

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Proof. Notice that the function Φ^σ is affine on certain open intervals, on each such interval its graph has slope σ, and these intervals are all in the complement of $H_{\mathbb{R}}$. Since by our assumption $\sigma < L$, the function Φ^σ is monotone increasing Lipschitz continuous with Lipschitz constant L. Hence

$$\Phi^\sigma(\tau_2) - \Phi^\sigma(\tau_1) \le L \cdot \ell(H_{\mathbb{R}} \cap [\tau_1, \tau_2]) + \sigma \cdot \ell([\tau_1, \tau_2] \setminus H_{\mathbb{R}}).$$

On the other hand,

$$\Phi^\sigma(\tau_2) - \Phi^\sigma(\tau_1) = q_2\tau_2 - q_1\tau_1 = (q_2 - q_1)\tau_2 + q_1(\tau_2 - \tau_1).$$

From $\sigma \le q_1$ we get

$$(q_2 - q_1)\tau_2 \le L \cdot \ell(H_{\mathbb{R}} \cap [\tau_1, \tau_2])$$

and (6.1) follows.

Finally, notice that if Φ is affine between each two consecutive integers, then $t \in H_{\mathbb{R}}$ implies $\lceil t \rceil \in H_{\mathbb{R}}$ where $\lceil t \rceil$ denotes the smallest integer $\ge t$. Hence $\#(H_{\mathbb{N}} \cap [\tau_1, \tau_2]$ for integer τ_2 is not less than the number of intervals of the form $(m, m+1]$ intersecting $H_{\mathbb{R}} \cap [\tau_1, \tau_2]$. Hence

$$\#(H_{\mathbb{N}} \cap [\tau_1, \tau_2]) \ge \ell(H_{\mathbb{R}} \cap (\tau_1, \tau_2)) \ge L^{-1}(q_2 - q_1)\tau_2,$$

which proves (6.2). □

Below we will apply this lemma to numbers $0 \le \sigma \le q_1 < q_2 < L$ so that

$$\Phi^\sigma(\tau_1) = q_1\tau_1, \quad q_1 t \le \Phi^\sigma(t) \le q_2 t \quad \text{for all } t \in [\tau_1, \tau_2], \quad \Phi^\sigma(\tau_2) = q_2\tau_2.$$

We call such an interval $[\tau_1, \tau_2]$ a (q_1, q_2)-*interval* for Φ^σ. If it is the minimal interval with this property, we call it a (q_1, q_2)-*crossing interval* (compare Figure 4).

LEMMA 6.2. *Let Φ, L be as in Lemma 6.1, $\beta = \limsup_{t\to\infty} \Phi(t)/t$, and $\alpha = \liminf_{t\to\infty} \Phi(t)/t$ (we allow α to be negative). Let*

$$L_0 := \limsup_{N\to\infty} \sup_k \frac{1}{N} |\Phi(k+N) - \Phi(k)|.$$

Then for $\sigma \in [0, \beta)$ we have

$$\liminf_{t\to\infty} \frac{\Phi^\sigma(t)}{t} \le \frac{\beta + \sigma(\beta - \alpha)/(L_0 - \sigma)}{1 + (\beta - \alpha)/(L_0 - \sigma)}. \tag{6.3}$$

In particular, for $\sigma = 0$, see Figure 5, we have

$$\liminf_{t\to\infty} \frac{\Phi^0(t)}{t} \le \frac{\beta}{1 + (\beta - \alpha)/L_0}.$$

Applying this lemma we shall consider $\sigma > 0$ arbitrarily close to zero, so we will be allowed to use the latter formula.

Proof. We always have

$$\alpha < \alpha^\sharp(\sigma) := \frac{\beta + \sigma(\beta - \alpha)/(L_0 - \sigma)}{1 + (\beta - \alpha)/(L_0 - \sigma)}. \tag{6.4}$$

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Fix a small $\varepsilon < \min\{\beta - \sigma, (\alpha^\sharp(\sigma) - \alpha)/2\}$. Let $(n_i)_i$ be a sequence of times for which $\Phi(n_i) \le (\alpha + \varepsilon)n_i$. We can freely assume (passing to a subsequence if necessarily) that $\Phi^\sigma(n_i) \ge \Phi(n_i)$ for every i, otherwise the assertion follows immediately. Hence, each n_i lies on a plateau of Φ^σ; denote by k_i and ℓ_i the beginning and the end of this plateau. We have

$$\Phi(k_i) + (\ell_i - k_i)\sigma = \Phi(\ell_i)$$

and, for i large enough,

$$\Phi(k_i) \le (\beta + \varepsilon)k_i.$$

We can freely assume that $\Phi^\sigma(\ell_i) \ge (\alpha^\sharp(\sigma) - \varepsilon)\ell_i$, otherwise the assertion follows. As

$$\Phi(\ell_i) \le \Phi(n_i) + (\ell_i - n_i)L,$$

this implies that

$$\frac{\ell_i}{n_i} \ge \frac{L - \alpha - \varepsilon}{L - \alpha^\sharp(\sigma) + \varepsilon} > 1,$$

and hence, for i large enough,

$$\Phi(\ell_i) \le \Phi(n_i) + (L_0 + \varepsilon)(\ell_i - n_i) \le (\alpha + \varepsilon)k_i + L_0(\ell_i - k_i).$$

Hence,

$$\frac{\ell_i}{k_i} \ge 1 + \frac{1}{L_0 + \varepsilon - \sigma}\left(\frac{\Phi(k_i)}{k_i} - (\alpha + \varepsilon)\right)$$

and

$$\frac{\Phi(\ell_i)}{k_i} - \sigma\frac{\ell_i}{k_i} = \frac{\Phi(k_i)}{k_i} - \sigma.$$

Dividing side by side we get

$$\frac{\Phi(\ell_i)}{\ell_i} - \sigma \le \frac{((\Phi(k_i)/k_i) - \sigma)(L_0 + \varepsilon - \sigma)}{(\Phi(k_i)/k_i) + L_0 - \sigma - \alpha}.$$

The function $x \to (x - \sigma)/(x - \sigma + L_0 - \alpha)$ is increasing. As $\Phi(k_i)/k_i \le \beta + \varepsilon$,

$$\frac{\Phi(\ell_i)}{\ell_i} \le \sigma + \frac{(\beta + \varepsilon - \sigma)(L_0 + \varepsilon - \sigma)}{L_0 - \sigma + \beta - \alpha + \varepsilon}$$

Passing with ε to zero we get the assertion. □

In the rest of the section we consider the following piecewise affine function

$$\Phi(t) := \begin{cases} \log|(f^\ell)'(x)| & \text{if } t = \ell \in \mathbb{N}, \\ (\ell + 1 - t)\cdot\Phi(\ell) + (t - \ell)\cdot\Phi(\ell + 1) & \text{if } \ell < t < \ell + 1. \end{cases} \tag{6.5}$$

Observe that, by [**36**, Proposition 4.7], when considering the orbit of a point x and the above defined function we have $L_0 \le \chi_{\sup}$. Hence, in the following, for this function we let

$$\alpha^\sharp := \frac{\beta}{1 + (\beta - \alpha)/\chi_{\sup}}.$$

LEMMA 6.3. *For every $d \in (0, 1]$ and every positive integer k there exist an integer $m = m(d, k)$ and a number n_0 such that, for every $n \ge n_0$ and every $J \subset \{1, \dots, n\}$ such that $\#J \ge dn$, there exist numbers $a_1 < a_2 < \cdots < a_k$ in J such that $a_k - a_1 \le m$.*

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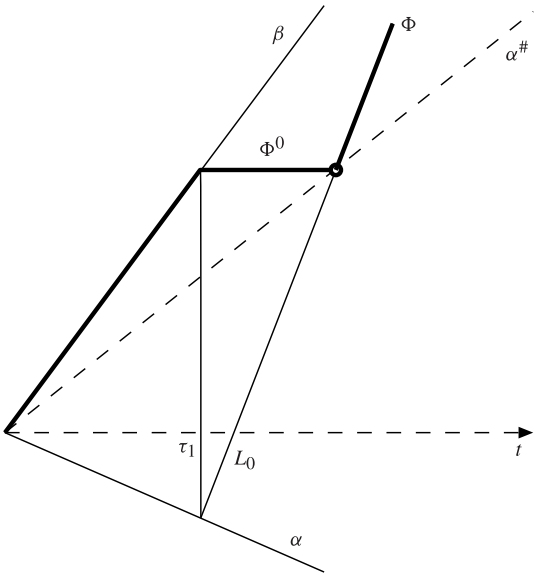


FIGURE 5. Sketch of the proof of Lemma 6.2 ($\sigma = 0$).

Proof. Let $m = \lfloor 2k/d \rfloor + 1$, where $\lfloor \cdot \rfloor$ denotes the integer part. Then for n large enough

$$\sum_{j \in J} |[j,\, j+m] \cap [1,\, n]| \geq m(\#J - m) \geq mdn - m^2 \geq \frac{1}{2} mnd.$$

Hence, there exists $i \in \{0, \ldots, n\}$ such that

$$A(i) := \{j \in J : i \in [j,\, j+m] \cap [1,\, n]\} \quad \text{satisfies } \#A(i) \geq \tfrac{1}{2} md.$$

We now let $a_1 < a_2 < \cdots < a_k$ be the consecutive indices j in $A(i)$. □

The following lemma will provide a substantial technical ingredient to prove Theorem 2.

LEMMA 6.4. *Let $(f, K) \in \mathscr{A}$. Given $\alpha \leq \beta$ with $\beta > 0$, for every $q > \sigma > 0$ such that $\alpha^\sharp(\sigma) \leq q \leq \beta$ and every $\varepsilon > 0$, there exist $M \geq 0$, $r > 0$, $\varepsilon' > 0$ such that, for every $x \in \mathcal{L}(\alpha, \beta)$, there exists a subset H of $\mathbb{N}$ of upper density*

$$\overline{d}(H) := \limsup_{n \to 0} \frac{\#(H \cap [1, n])}{n} > 0$$

such that for every $n \in H$ and every $k = 0, \ldots, n$ for the pull-back W_k of $B(f^n(x), r)$ for f^k which contains $f^{n-k}(x)$, the following properties hold:

$$|(f^k)'(f^{n-k}(x))|^{-1} \leq e^{-k\sigma}, \tag{6.6}$$

$$|W_k| \leq e^{-k(\sigma - \varepsilon)}, \tag{6.7}$$

$$n(q - \varepsilon) \leq \log |(f^n)'(x)| \leq n(q + \varepsilon), \tag{6.8}$$

$$\#\{k : 0 \leq k < n,\, W_k \cap \mathrm{Crit}(f) \neq \varnothing\} \leq M. \tag{6.9}$$

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Moreover, for Φ defined in (6.5), if $\alpha < \beta$ then there exists $n^{\mathrm{up}} > n$ such that

$$\frac{\Phi(n^{\mathrm{up}})}{n^{\mathrm{up}}} \geq \frac{\Phi(n)}{n} + \varepsilon' \quad \text{and} \quad \frac{\Phi(t)}{t} \geq \frac{\Phi(n)}{n} - \varepsilon \quad \text{for all } n < t < n^{\mathrm{up}}. \tag{6.10}$$

If $\alpha = \beta$ then only the second inequality holds for all $t > n$.

Observe that (6.9) is as in (2.4) in TCE, however we conclude this property for the subset $\mathcal{L}(\alpha, \beta) \subset K$ only.

The property (6.10) is technical, and is not needed in the case $K = \widehat{I}_K$. The possibility of finding a positive upper density set H of integers for which Pliss's type properties (6.6), (6.7) and (6.9) simultaneously hold has been recently proved in [**23**], using 'shadows' from [**39**].

Proof.

Step 1. Let H_1 be the set of all Pliss hyperbolic times for x with exponent σ. As $\sigma < \overline{\chi}(x) = \beta$, by the Pliss Lemma 3.12 (see §3.6) H_1 has positive upper density $\overline{d}(H_1)$. Now the property (6.6) follows by definition of the Pliss hyperbolic times and (6.7) holds by the Telescope Lemma 3.13.

To prove (6.8) and (6.10) we can assume $\alpha < \beta$, since for $\alpha = q = \beta$ these properties hold immediately.

Let $q_1 < q_2'$ and $[q_1, q_2'] \subset (q - \varepsilon, q + \varepsilon) \cap (\alpha^\sharp(\sigma), \beta)$. Assume also $\sigma < q_1$. Let $q_2 = (q_1 + q_2')/2$ (hence $q_2 - q_1 \leq \varepsilon$). We define $\varepsilon' := q_2' - q_2$.

To apply Lemma 6.1, let us consider the piecewise affine function Φ, defined in (6.5). Now let $H_{\mathbb{N}}$ be the set from Lemma 6.1 applied to this function. Recall that every integer $n \in H_{\mathbb{N}}$ satisfies $\Phi^\sigma(n) = \Phi(n)$ and is a Pliss hyperbolic time for x with exponent σ, that is, $H_{\mathbb{N}} \subset H_1$. Continue to consider the set H_2 being the set $H_{\mathbb{N}}$ intersected with the union of all the (q_1, q_2)-crossing intervals (recall the definition right after the proof of Lemma 6.1). If $[\tau_1, \tau_2]$ is now such a crossing interval then, for $n = \tau_2$ and for C as in Lemma 6.1, we have

$$\frac{\#(H_{\mathbb{N}} \cap [\tau_1, n])}{n} \geq C > 0.$$

Moreover, by

$$\liminf_{n\to\infty} \frac{1}{n} \log |(f^n)'(x)| \leq \alpha^\sharp(\sigma) < q_1 \quad \text{and} \quad q_2' < \limsup_{n\to\infty} \frac{1}{n} \log |(f^n)'(x)| = \beta$$

there are infinitely many crossing intervals, and hence infinitely many such n's. Thus, the set H_2 has positive upper density and also satisfies (6.6) and (6.8). The crossing property easily implies (6.10).

Step 2. Now we shall concentrate on proving (6.9), further restricting H_2. By [**36**, Lemma 3.1] (see also [**27**], or [**10**] in the holomorphic case) there is a constant $Q > 0$ such that for every $x \in K$ for every n and an arbitrary $c \in \mathrm{Crit}(f)$ we have

$$\sum_{j=0}^{n}{}' - \log |f^j(x) - c| \leq Qn, \tag{6.11}$$

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where Σ' means that in the sum we omit an index j of smallest distance $|f^j(x) - c|$ (only one if there is more than one such j). For normalization (to have only non-negative numbers in (6.11)) we assume $K \subset [0, 1]$; this influences only the constant Q.

We continue, using the ideas from [**39**], following the proof of the fact that ExpShrink implies TCE (see also [**27**]). Given $j \geq 1$, let

$$a(j) := -\log \operatorname{dist}(f^j(x), \operatorname{Crit}).$$

Fix a number $\varkappa > 1/(\sigma - \varepsilon)$. For each j consider the interval ('shadow') $S_j := (j, j + \varkappa a(j)]$. Denote by $\mathbb{1}_{S_j}$ the characteristic function of S_j. Then, by (6.11), we have

$$\sum_{k=1}^{n} \sum_{j=0}^{n} \mathbb{1}_{S_j}(k) \leq \sum_{j=0}^{n} \varkappa a(j) \leq \#\operatorname{Crit} \cdot n + \varkappa Q n = n(\#\operatorname{Crit} + \varkappa Q) =: n Q_\varkappa,$$

where the term $\#\operatorname{Crit} \cdot n$ takes care of the indices which have been omitted from the sum in (6.11). So, given an arbitrary $M > 0$ the cardinality n_M of the set of $k \in \{1, \ldots, n\}$ covered by more than M shadows is less than $n(\#\operatorname{Crit} + \varkappa Q)/M$. Hence, there is $\eta = \eta(M, \varkappa, Q)$ such that

$$1 - n_M/n \geq 1 - (\#\operatorname{Crit} + \varkappa Q)/M = \eta.$$

In particular, the lower density satisfies

$$\liminf_{n\to\infty} \left(1 - \frac{n_M}{n}\right) \geq \eta,$$

and, choosing M sufficiently large, η is arbitrarily close to one.

Now let H be the set of numbers in H_2 which do not belong to more than M shadows. Thus, its upper density is still positive for sufficiently large M.

It remains to show that every $n \in H$ satisfies (6.9). If $n \in H$ and $n \notin S_j$ for $j < n$ then by definition $n - j > \varkappa a_j$, and hence

$$\exp(-(n - j)/\varkappa) < \operatorname{dist}(f^j(x), \operatorname{Crit}).$$

Thus, by (6.7) we obtain $|W_j| \leq e^{-(n-j)(\sigma-\varepsilon)}$. Since $\varkappa > 1/(\sigma - \varepsilon)$ we conclude that W_j is disjoint from Crit. Thus the set H defined above satisfies (6.9). □

We remark that our term 'shadow' originates in its analogy to the 'shadows' in the proof of existence of Pliss hyperbolic times. Indeed, compare Figure 4 where the shadows correspond to the lines with slope σ to catch hyperbolic times with such exponent.

The following proposition is the key step in the proof of Theorem 2. In the remainder of the section we will assume that $(f, K) \in \mathscr{A}_+^{\mathrm{BD}}$ is non-exceptional and we will consider the following family of measures μ_t. In the case that there are no indifferent periodic orbits in K, then for every $t < t_+$ we denote by μ_t the unique (non-atomic) $e^{P(t)}|f'|^t$-conformal measure which is well defined by [**36**, Theorem A]. For $t < 0$, if such a measure does not exist we consider a CaS measure μ_t^* instead, positive on open sets (see §§2.3 and 4.2) and denote it also by μ_t.

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PROPOSITION 6.5. *Let $(f, K) \in \mathscr{A}_+^{\mathrm{BD}}$ satisfy the weak isolation condition and be non-exceptional and without indifferent periodic orbits. Given numbers $\alpha \le \beta$ and q such that $\alpha^\sharp \le q \le \beta$, for every $x \in \mathcal{L}(\alpha, \beta)$ not weakly S'-exceptional there exist a sequence of integers $n_i \to \infty$, a sequence $(q_i)_i$ of positive numbers $q_i \in [q, \beta]$, and $r > 0$ such that for every $\varepsilon > 0$ and $i \ge 1$ we have*

$$\operatorname{diam} \operatorname{Comp}_x f^{-n_i}(B(f^{n_i}(x), r)) \le e^{-n_i(q_i-\varepsilon)}. \tag{6.12}$$

For every $t < t_+$, the measure μ_t satisfies

$$\mu_t(\operatorname{Comp}_x f^{-n_i}(B(f^{n_i}(x), r))) \ge e^{-n_i P(t)} e^{-n_i q_i - n_i \varepsilon |t|} \Upsilon(t), \tag{6.13}$$

where $\Upsilon(t) := \Upsilon(\mu_t, \Delta)$ defined in (3.2)*, for Δ depending only on r, f, and* $\operatorname{sgn}(t)$*. In the case $K = \widehat{I}_K$, one can take $q_i = q$ for all i.*

Proof. The property (6.12) for $n = n_i \in H$ and $q_i \equiv q$ follows immediately from (6.6), (6.8), and Proposition 3.13. (Though later on there will occur a case where we need to work with different choice of q_i.) Therefore the rest of the proof will be devoted to (6.13).

We start by repeating the construction in the proof of Theorem 5. Consider $\widehat{W}$ and $W = 1/2 \diamond \widehat{W} = B(f^n(x), r/2)$ and denote by $\widehat{W}_k$ the pull-back of $\widehat{W}$ for f^k which contains $f^{n-k}(x)$.

The general strategy to prove (6.13) is, as for Proposition 4.2, using the conformal measure $\mu = \mu_t$. We will start by considering a particular case.

(i) Assume $f^n(W_n) = W$†.

(i)(a) The case $t \ge 0$. We will pay particular attention to the encounter of critical points. As in the proof of Theorem 5, let $0 \le k_1 < \cdots < k_{M'} \le n-1$ be the subsequence of all consecutive integers k such that $\widehat{W}_k \cap \operatorname{Crit} \ne \varnothing$. Here, by Lemma 6.4 property (6.9), we capture critical points only $M' \le M$ times.

We pull back $W = B(f^n(x), r/2)$ in place of $B(f^n(x), r)$, and use bounded distortion for $f^{k_i - k_{i-1} - 1}$ on the pull-back $W_{k_i - 1} \ni f^{n-(k_i-1)}(x)$ for $f^{k_i - 1}$ as in the proof of Theorem 5 (the formula (5.2)).

In the proof of Theorem 5, in part (i)(b) we only needed to compare derivatives. Here however we pull back μ_t on K so we must be careful. Let us explain it in detail. As in Lemma 3.2 we use the notation $W_n^j = f^j(W_n)$. By Corollary 3.3 the map $f^{k_i - k_{i-1} - 1}$ is a K-homeomorphism of $W_n^{n-k_i+1}$ onto $W_n^{n-k_{i-1}}$. These are in fact diffeomorphisms with bounded distortion, since we also take into account, while defining k_i, the capture of inflection critical points. Therefore, there exists some constant C coming from the distortion relation (1.2) so that

$$\frac{\mu_t(W_n^{n-k_i+1})}{\mu_t(W_n^{n-k_{i-1}})} \ge C^t e^{-(k_i - k_{i-1} - 1)P(t)} \Big(\frac{|W_n^{n-k_i+1}|}{|W_n^{n-k_{i-1}}|} \Big)^t. \tag{6.14}$$

If $W_n^{n-k_i}$ contains an inflection critical point c, f is a K-homeomorphism onto $W_n^{n-k_i+1}$ again by Corollary 3.3 and, denoting $(f|_{W_n^{n-k_i}})^{-1}$ by g, we have (as μ_t is non-atomic,

† This is so in the iteration of a rational function case. Hence for rational maps the same proof will give us (6.13), see Appendix A.

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in particular not having an atom at c, we can even write equality in place of the first inequality)

$$\mu_t(W_n^{n-k_i}) \geq e^{-P(t)} \int_{W_n^{n-k_i+1}} |g'(y)|^t \, d\mu_t(y)$$
$$\geq C^t e^{-P(t)} \Big(\frac{|W_n^{n-k_i}|}{|W_n^{n-k_i+1}|} \Big)^t \mu_t(W_n^{n-k_i+1}),$$

with a constant C arising from Lemma 3.14, applied for every y under the integral. We again get (6.14) (for $n = 1$).

If $W_n^{n-k_i}$ contains a turning critical point c, then we apply the above estimate by choosing as g the branch of f^{-1} on $W_n^{n-k_i+1}$ mapping it onto $W_n^{n-k_i}$ according to Corollary 3.3.

Notice finally that the case where $W_n^{n-k_i}$ contains no critical points can happen, since this set can be strictly contained in W_{k_i}. Then f is a K-homeomorphism on its image and we also get the above estimate, referring to Remark 3.15.

Composing those inequalities we obtain

$$\mu_t(W_n) \geq C^{t(2M'+1)} e^{-nP(t)} \Big(\frac{|W_n|}{|f^n(W_n)|} \Big)^t \mu_t(f^n(W_n)). \tag{6.15}$$

This ends the proof as we assumed $f^n(W_n) = W$ and hence

$$\mu_t(B(f^n(x), r/2)) \geq \Upsilon(\mu_t, r/2) > 0$$

(see (3.2) and (6.12)).

(i)(b) The case $t < 0$. By Lemma 3.5 applied to $X = K$ there are some $r' = \varepsilon(r/2, M + 1)/2$ and $z \in K \cap B(f^n(x), r/2)$ such that $B(z, r')$ is disjoint with $\bigcup_{j=1}^n f^j(\mathrm{Crit} \cap W_j)$. Then f^n has no critical points and has bounded distortion on the pull-back W_n' of $W' = B(z, r'/2)$ contained in W_n. Hence, by (1.2) and the telescope estimate in Proposition 3.13, since $B(z, r')$ is $1/2$-scaled neighborhood of $B(z, r'/2)$, for every $y \in W_n'$,

$$|(f^n)'(y)| \geq \frac{1}{C(1/2)} \frac{r'}{|W_n'|} \geq \frac{C_1}{|W_n|} \geq C_2 |(f^n)'(x)| e^{-n\varepsilon}$$

for arbitrarily small $\varepsilon > 0$ and some constants C_1, C_2. As f^n is a K-homeomorphism on W_n', by Corollary 3.3, for $t < 0$ we get

$$\mu_t\big(\mathrm{Comp}_x \, f^{-n}(B(f^n(x), r)) \big) \geq \mu_t(W_n')$$
$$\geq C_2^{|t|} e^{-nP(t)} |(f^n)'(x)|^{-t} e^{-n|t|\varepsilon} \cdot \Upsilon(\mu_t, r'/2)$$

giving again (6.13), with $\Upsilon(t) = \Upsilon(\mu_t, r'/2)$.

In steps (i)(a) and (i)(b) we showed the assertion under the assumption that $f^n(W_n) = W$. However, this equality, which is always true for rational maps, is not true in general. To prove the general case we need to modify the construction.

The problem is that, when $f^n \colon W_n \to W$ is not onto, during the pull-back we can lose part of the conformal measure. To have f^n onto, in this construction we replace W with $f^n(W_n)$. Then, however, we lose the uniform bound $\mu_t(f^n(W_n)) \geq c > 0$. (Notice that in §4.2 we did not have this trouble.)

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A priori $\mu_t(f^n(W_n))$ (or its piece if $t<0$ as in (i)(b)) could be arbitrarily close to zero. Notice, however, that if there is a constant $\delta>0$ such that there is $y\in K$ with $B(y,\delta)\subset f^n(W_n)$ then, as before, $\mu_t(f^n(W_n))\geq \Upsilon(\mu_t,\delta)$ and we are done. (Compare (5.5) to be used also later on.)

(ii) Assume $f^n(W_n)\neq W$, but $K=\widehat{I}_K$. In this case the above holds. Indeed, we assume that r is smaller than the lengths of all gaps in $\mathbb{R}\backslash\widehat{I}_K$ and all components of $\widehat{I}_K$. For n large enough, $|W_n|<r$ by BaShrink, see §2.4. Hence, as $x\in K$, at least one end, say z, of W_n is in K. Hence $f^n(z)\in K$. Since $|f^n(x)-f^n(z)|=r/2$ and $f^n(x)\in K$, letting y be the middle point between $f^n(x)$ and $f^n(z)$ and $\delta:=r/4$ we have $\mu_t(f^n(W_n))\geq\Upsilon(\mu_t,\delta)$ and we are done.

(iii) The general $(f,K)\in\mathscr{A}_+^{\mathrm{BD}}$. We shall overcome the difficulty with estimating $\mu_t(f^n(W_n))$ from below by acting similarly as in the last part of the proof of Theorem 5. Unfortunately it can happen that we need to replace q with some $q'>q$.

We first restrict our considerations to a particular subset of the Pliss hyperbolic times H, see Lemma 3.12. Fix a number $\widehat{n}\in\mathbb{N}$ sufficiently large that $\#(H\cap\{1,\dots,\widehat{n}\})\geq\overline{d}(H)/2$. Then for $d=\overline{d}(H)/4$ we have $\#(H\cap\{\lfloor\widehat{n}\,d\rfloor,\dots,\widehat{n}\})\geq d\widehat{n}$. Applying Lemma 6.3 to d, $k=\#\partial_{r/3}+1$ (see the notation in the proof of Theorem 5), $\widehat{n}$, and the set $H_{\widehat{n}}:=H\cap\{1,\dots,\widehat{n}\}$, we find a number $m=m(d,k)>0$ and numbers $a_1<a_2<\cdots<a_k$ all in $H_{\widehat{n}}$ such that $a_k-a_1\leq m$.

Consider now an arbitrary constant $b\in(0,1)$ (to be specified later on), close to one. Define

$$B:=B(\delta_{r/3},(1-b)r/4).$$

We consider two cases.

(iii)(a) Assume there exists $n\in H_{\widehat{n}}$ with $f^n(x)\notin B$. Then, comparing (5.5) and the notation in the proof of Theorem 5, there is some $a\in(0,1)$ such that some point $x'\in K$ is well inside $f^n(W_n)$, that is, $x'\in a\diamond(f^n(W_n))\cap K$, and thus $f^n(W_n)$ contains the ball $B(x',(1-b)r/4)$. So, as above, we conclude

$$\mu_t(f^n(W_n))\geq\Upsilon(\mu_t,(1-b)r/4)>0.$$

This proves the assertion in the case $t\geq 0$.

For $t<0$ we proved already that there is some point $x''\in K$ such that we have $B(x'',(1-b)r/4)\subset f^n(W_n)$. As in step (i)(b), applying Lemma 3.5 to $X=K$ there is $r''=\varepsilon((1-b)r/4,M+1)/2$ and $z\in K\cap B(x'',(1-b)r/4)$ such that the pull-back of $B(z,r'')$ along the trajectory of x does not encounter critical points. As before, we obtain

$$\mu_t(B(x'',(1-b)r/4))\geq\Upsilon(\mu_t,r'')>0.$$

This proves the assertion in the case $t<0$.

(iii)(b) Assume $f^n(x)\in B$ for all $n\in H_{\widehat{n}}$. In particular, this holds for all $n=a_1<a_2<\cdots<a_k$ in $H_{\widehat{n}}$ as chosen above. For any integer $N>0$ define

$$\alpha_N:=\operatorname{dist}\Bigg(\bigcup_{j=1}^{N}f^j(\partial_{r/3})\backslash\partial_{r/3},\,\partial_{r/3}\Bigg)>0.$$

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Then for each $j = 1, \ldots, N$ we have, for each $z \in \partial_{r/3} \setminus \bigcup_{j=1}^{N} f^{-j}(\partial_{r/3})$,

$$f^j(B(z, \mathrm{Lip}^{-N}\, \alpha_N/2)) \cap B(\partial_{r/3}, \alpha_N/2)) = \varnothing, \tag{6.16}$$

where Lip denotes the Lipschitz constant of $f|_K$. Hence, comparing the definition of α' preceding (5.7), for

$$\alpha'_N := \min\left\{\frac{\alpha_N}{2\,\mathrm{Lip}^N}, \frac{\Theta}{2}, \frac{\theta}{2}\right\},$$

for each $0 < s_1 < s_2$, if $f^{s_1}(x) \in B(z_1, \alpha'_N)$ and $f^{s_2}(x) \in B(z_2, \alpha'_N)$ for $z_1, z_2 \in \partial_{r/3}$, then either $s_2 - s_1 > N$ or

$$f^{s_2-s_1}(z_1) = z_2,$$

compare (5.8).

Now we fix $N := m$ and let s_1, s_2 be any a_i, a_j for $i < j$. We fix b such that $(1-b)r/4 = \alpha'_m$. Hence $f^{a_i}(x) \in B(\partial_{r/3}, \alpha'_m) = B$ for all $i = 1, \ldots, k$.

Therefore, there is an index i such that $y := f^{a_i}(x)$ is within the distance at most α'_m from a periodic point $p \in \partial_{r/3}$, of period at most m. Let us take $n := a_i$. Note that p is hyperbolic repelling†.

If $y = p$ then $\chi(x) = \chi(p)$, and in particular $\chi(x)$ exists, contrary to our assumption that $x \in \mathcal{L}(\alpha, \beta)$ for $\alpha \neq \beta$. So we can assume $y \neq p$. Let $\ell \geq n$ be the least integer such that $f^\ell(x) \notin B$. It exists by the definition of θ.

Considering the pull-back W_ℓ of $B(f^\ell(x), r/2)$ containing x, we repeat the reasoning of step (iii) case (a). This provides us lower bounds for μ_t.

Unfortunately we do not know now whether $q(\ell) := \log|(f^\ell)'(x)|/\ell$ is in the interval $[q-\varepsilon, q+\varepsilon]$. This depends on $\chi(p)$. We deal with this difficulty using the item (6.10) in Lemma 6.4. For simplicity, we assume the period of p to be equal to one.

Case 1. $\chi(p) \geq q - 2\varepsilon$. Then $q(\ell) \geq \min\{q-\varepsilon, \chi(p)-\varepsilon\} \geq q - 3\varepsilon$. We obtain (6.12) and (6.13) with ℓ in place of n, with 3ε in place of ε and with $q(\ell) \geq q(n)$.

Case 2. $\chi(p) < q - 2\varepsilon$. In this case

$$q(t) \leq \frac{1}{t}(nq(n) + (t-n)(\chi(p)+\varepsilon)) < q(n)$$

for $t \in (n, l]$. In particular, $n^{\mathrm{up}} > \ell$. Hence, by (6.10), $q(\ell) \geq q(n) - \varepsilon \geq q - 3\varepsilon$. Hence we obtain (6.12) and (6.13) with ℓ in place of n, 3ε in place of ε and with $q(\ell) \geq q - 3\varepsilon$.

To define the sequence $(n_i)_i$ in the proposition, we repeat the above steps infinitely many times, for some increasing sequence of restricted Pliss hyperbolic times obtaining (6.12) and (6.13) for some numbers n_i and $q_i := q(n_i) \in [q, \beta]$. □

Now we can finally prove Theorems 2 and 3.

Proof of Theorem 2.

(i) Special case. Suppose that in Proposition 6.5 all q_i's are equal to q, which occurs, for example, if $K = \widehat{I}_K$ (and also in the case of a rational function on the Riemann sphere as in Appendix A).

† The remainder of the proof would also work if p were indifferent, repelling to one side, but in this proof we had to exclude indifferent periodic orbits to refer to [**36**] for the existence of μ_t.

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Let $x \in \mathcal{L}(\alpha, \beta)$. We can assume x is not weakly S'-exceptional by replacing it with $f^n(x)$ for some $n > 0$, see Lemma 3.6. Indeed, for each n we consider $X_n := \{x \in \mathcal{L}(\alpha, \beta)\colon f^n(x) \text{ is weakly regular}\}$ and any upper estimate of $\dim_{\mathrm{H}}(f^n(X_n))$ holds automatically for X_n. Notice that $x \in \mathcal{L}(\alpha, \beta)$ implies that x is not precritical, hence $f^n(x) \in \mathcal{L}(\alpha, \beta)$. For every $q \in [\alpha^\sharp, \beta]$, for every $t < t_+$ for the measure μ_t as in the proposition, the lower local dimension at x satisfies

$$\begin{aligned}\underline{d}_{\mu_t}(x) &\le \liminf_{i\to\infty} \frac{\log \mu_t(\mathrm{Comp}_x\ f^{-n_i}(B(f^{n_i}(x), r)))}{\log \operatorname{diam} \mathrm{Comp}_x\ f^{-n_i}(B(f^{n_i}(x), r))} \\ &\le \liminf_{i\to\infty} \frac{-n_i(P(t) + (q+\varepsilon)t) + \log \Upsilon(t)}{-n_i(q-\varepsilon)} \le \frac{P(t)+qt}{q} + \varepsilon',\end{aligned}$$

with $\varepsilon' > 0$ arbitrarily close to zero for ε appropriately small. We conclude that

$$\underline{d}_{\mu_t}(x) \le \frac{P(t)+qt}{q} = \frac{P(t)}{q} + t. \tag{6.17}$$

(ii) General case. In general, Proposition 6.5 provides us only with a sequence of $q_i \in [q, \beta]$ instead of $q_i \equiv q$. This leads to the estimate

$$\underline{d}_{\mu_t}(x) \le \max_{q_i \in [q,\beta]} \frac{P(t)}{q_i} + t. \tag{6.18}$$

(iii) Conclusion. To obtain the estimates (1.14) and (1.15) we will use the Frostman lemma. To obtain (1.14) we consider $q = \beta$ and then (6.17) and (6.18) coincide. We can choose $t < t_+$ such that the right-hand side in (1.14) takes a value arbitrarily close to $F(\beta)$ (recall (1.7)) and the assertion follows from Frostman lemma.

To obtain the additional upper bound for (1.15) (note that our assumptions for this inequality guarantee that we are in the special case here) we will use $q = \alpha^\sharp$. As we are in the special case, (6.17) holds and we can find $t < t_+$ such that the right-hand side of (6.17) is arbitrarily close to $F(\alpha^\sharp)$, and then apply the Frostman lemma.

This proves Theorem 2. □

Proof of Theorem 3. We are in the special case in the proof of Theorem 2. Let α and β be such that $\alpha^\sharp < \chi_{\inf}$. Let $x \in \mathcal{L}(\alpha, \beta)$. We have (6.17) for any $t < t_+$. If $\alpha^\sharp < \chi_{\inf}$ then $F(\alpha^\sharp) = -\infty$, and hence there exists $t < t_+$ such that

$$\frac{P(t)}{t} + t < 0.$$

By (6.17), this implies that the corresponding measure μ_t has negative local dimension at x (that is, the measure $\mu_t(B(x, r))$ escapes to infinity as $r \to 0$), which is impossible. Hence, no such point x can exist. □

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A. *Appendix. Strong upper bound. Holomorphic case*

We consider a rational map $f: \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ of degree at least two. Denote by J its Julia set (it corresponds to the set K in our interval maps notation). The sets considered below are subsets of J.

The following results provide the complex counterparts of our main results. The first one is a slightly strengthened version of [**14**, Theorem 2] (observe that there we only assumed that $\alpha > 0$). The other two are new. For their proofs we refer to the remarks and footnotes in §6.

THEOREM A.1. *Let f be a non-exceptional rational map of degree ≥ 2. For any $\alpha \leq \beta \leq \chi_{\sup}$ with $\beta > 0$, and additionally with $\alpha > 0$ if $\chi_{\inf} = 0$, we have*

$$\min\{F(\alpha), F(\beta)\} \leq \dim_{\mathrm{H}} \mathcal{L}(\alpha, \beta) \leq \max\{0, \max_{\alpha \leq q \leq \beta} F(q)\}. \tag{A.1}$$

In particular, for any $\alpha \in [\chi_{\inf}, \chi_{\sup}]\backslash\{0\}$ we have

$$\dim_{\mathrm{H}} \mathcal{L}(\alpha) = F(\alpha) \quad \textit{and} \quad \dim_{\mathrm{H}} \mathcal{L}(0) \geq F(0).$$

Moreover,

$$\{x \in K : -\infty < \chi(x) < \chi_{\inf}\} = \{x \in K : \overline{\chi}(x) > \chi_{\sup}\} = \varnothing$$

and

$$\dim_{\mathrm{H}}\{x \in K : 0 < \overline{\chi}(x) < \chi_{\inf}\} = 0.$$

THEOREM A.2. *Let f be a non-exceptional rational map of degree ≥ 2 without indifferent periodic orbits. For any $\alpha \leq \beta \leq \chi_{\sup}$ with $\beta > 0$, we have*

$$\dim_{\mathrm{H}} \mathcal{L}(\alpha, \beta) = \max\{0, \min\{F(\alpha^\sharp), F(\beta)\}\}, \tag{A.2}$$

where

$$\alpha^\sharp := \frac{\beta}{1 + (\beta - \alpha)/\chi_{\sup}}.$$

THEOREM A.3. *Let f be a non-exceptional rational map without indifferent periodic orbits. Assume $\chi_{\inf} > 0$ (f is Topological Collet–Eckmann). Then, for any $\alpha \leq \beta \leq \chi_{\sup}$ with $\beta > 0$, if $\alpha^\sharp < \chi_{\inf}$, then $\mathcal{L}(\alpha, \beta) = \varnothing$.*

B. *Appendix. On Rivera-Letelier's proof of $\chi(x) \geq \chi_{\inf}$*

After this paper was written, J. Rivera-Letelier explained to us that $\chi(x) \geq \chi_{\inf}$, or, more precisely, (1.11) in Theorem 1, follows easily from his paper [**42**]. We devote this Appendix to a brief explanation and comparison to our proof.

In [**42**], multimodal maps of the unit interval I are considered with $K = J(f)$ being the Julia set (see §1.2: periodic orbits), of class C^3, with all periodic orbits in K hyperbolic repelling and $f|_K$ topologically exact (which is formally stronger than having positive entropy; see [**36**] for a discussion concerning a comparison of these notions). However, this theory is working in our more general setting of $(f, K) \in \mathscr{A}^{\mathrm{BD}}$.

Define

$$\chi_{\mathrm{ExpShrink}} := -\limsup_{n \to \infty} \frac{1}{n} \log\big(\max\{|T_n|\}\big), \tag{B.1}$$

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where the maximum is taken over all connected components T_n of $f^{-n}(T)$ intersecting K, over all open intervals $T \subset \mathbb{R}$ of length at most r, not intersecting G (G being the union of all components of $\mathbb{R}\setminus K$ longer than some constant r' ('large gaps')). The number $r > 0$ is also a constant small enough, in particular not bigger than δ in the definition of Backward Shrinking in §2.4. (Then $\chi_{\text{ExpShrink}}$ occurs independent of r by a variant of the Telescope Proposition 3.13.)

Then a part of the Main Theorem in [**42**, §4] asserts (in our setting) that

$$\chi_{\text{ExpShrink}} \geq \chi_{\text{inf}}. \tag{B.2}$$

COROLLARY B.1. *Let $(f, K) \in \mathscr{A}_+^{\text{BD}}$ satisfy the weak isolation condition. For every regular $x \in K$ (that is, such that $\chi(x)$ exists), if $\chi(x) > -\infty$, then*

$$\chi(x) \geq \chi_{\text{inf}}. \tag{B.3}$$

Proof. First notice, as in Proof of Theorem 5 which refers to [**14**], that $\chi(x) \geq 0$. Notice also that by the existence of a finite $\chi(x)$ we have

$$\liminf_{n\to\infty} \operatorname{dist}(f^n(x), \operatorname{Crit}(f)) = 0. \tag{B.4}$$

Using this, one can prove (compare [**14**, Claim in §3]) that there exists $r > 0$ (maybe smaller than the previous one) such that for any $\varepsilon > 0$ arbitrarily close to zero, for every n large enough the map f^n is a diffeomorphism of $\operatorname{Comp}_x f^{-n}(B(f^n(x), r\exp(-\varepsilon n)/2))$ onto $B(f^n(x), r\exp(-\varepsilon n)/2)$ with distortion bounded by, say, $1/2$. Therefore $\chi(x) \geq \chi_{\text{ExpShrink}} - \varepsilon$, and hence, taking $\varepsilon \to 0$,

$$\chi(x) \geq \chi_{\text{ExpShrink}}. \tag{B.5}$$

Hence, by (B.2) applied to $T = B(f^n(x), r/2)$, we get (B.3), provided there is a sequence n_j such that all $f^{n_j}(x)$ are within a distance at least $r/2$ from ∂G. This is only not the case if x is pre-periodic, where $\chi(x) \geq \chi_{\text{inf}}$ holds directly. □

Remark B.2. In fact, we use above only an easy case of (B.2), where the distance of T from ∂G is bounded away from zero. This is explained in more detail below.

The strategy above, together with the proof of (B.2), is in fact similar to ours. Here is a comparison of both.

• Our proof is direct. We shadow $x, \ldots, f^n(x)$, extended by a block of a backward trajectory of x of length bounded by a constant independent of n, ending close to $f^n(x)$ (we call it 'closing the loop'), by a periodic trajectory of a point p_n close to K, and thus in K by the weak isolation assumption. TCE allows to easily compare the derivatives, though on the other hand we can consider only $n = n_j$ (see (2.4)) which causes (fortunately minor) difficulties. Hence $\chi(x) \geq \liminf_{n\to\infty} \chi(p_n) \geq \chi_{\text{inf}}$.

We can 'close the loop' whenever we can choose n such that $f^n(x)$ is not too close to ∂G, see §5, (5.1). If we cannot, then x is pre-periodic as above and we are done.

• To follow [**42**], one considers an auxiliary number $\chi_{\text{CE2}}(z)$ and proves that for z being safe and hyperbolic (see [**36**, Definitions 1.22 and 1.23] and [**40**, Definition 12.5.7]) one has

$$\chi(x) \geq \chi_{\text{ExpShrink}} \geq \chi_{\text{CE2}}(z) \geq \chi_{\text{inf}}. \tag{B.6}$$

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(1) The first inequality has been already proven above. It is sufficient to consider x not pre-periodic, so we can consider in the definition of $\chi_{\text{ExpShrink}}$ only T within the distance at least r from G, getting the same inequality.

(2) Define $\chi_{\text{CE2}}(z)$ for a point $z \in K$ as logarithm of $\lambda_{\text{CE2}}(z)$ being the supremum of such λ that there exists $C = C(z, \lambda) > 0$ such that for every $n \geq 1$ and every $w \in f^{-n}(z)$,

$$|(f^n)'(w)| \geq C\lambda^n. \tag{B.7}$$

Here w is not necessarily in K but we consider all w such that $\text{Comp}_w\, f^{-n}(B(z, r_1))$ intersects K, for a constant r_1.

Safe and hyperbolic points exist, see [**36**, Lemma 4.4].

The conditions 'safe' and 'hyperbolic' allow us to find via shadowing (as in [**37**, Lemma 3.1] or, for example, [**36**, §3]; compare also Proposition 4.1, although there we do not care about the times of going to large scale) periodic points p_n such that $\chi_{\text{CE2}}(z) \geq \liminf_{n\to\infty} \chi(p_n) \geq \chi_{\text{inf}}$, thus proving the last inequality in (B.6).

(The property $\chi_{\text{CE2}}(z) > 0$ is called CE2$^*(z)$ (or backward Collet–Eckmann condition at z for preimages close to K), see [**36**, §3]. So the last inequality yields LyapHyp $\Rightarrow$ CE2$^*(z)$ for safe hyperbolic z.)

(3) The novelty in [**42**] is proving the missing inequality

$$\chi_{\text{ExpShrink}} \geq \chi_{\text{CE2}}(z). \tag{B.8}$$

As remarked above, we need here only its easy case where T is not close to ∂G. Then also the proof in [**36**, Proof of Theorem C. Case 1] works. We consider a finite set Y of f^j-preimages $y \in K$ of z, $j = 0, 1, \ldots$ which constitute a δ-dense set for K for δ small enough. By finiteness for every λ as in the definition of $\chi_{\text{CE2}}(z)$ we can find a common C. If T is not close to ∂ we can assume it is between two consecutive points y, y' of V. Then pulling $[y, y']$ back we obtain (B.8) for z being y or y'.

Note the following.

The pull-backs of T become short after a bounded time by Backward Shrinking, see §2.4. Note that we can assume $\chi_{\text{CE2}}(z) > 0$, otherwise there is nothing to prove, since $\chi_{\text{ExpShrink}}$ is obviously non-negative. Then we deal with short intervals and rarely capture critical points. One proves by induction that it is increasingly rarely. This yields at most subexponential growth of $|T_n| \cdot |(f^n)'(x_n)|$. The proof resembles the proof of the Telescope Proposition 3.13. One need not use TCE, as one can capture critical points unbounded number of times.

In the complex situation this proof does not work. One does not have the Backward Shrinking property and already at the beginning the pull-backs can become uncontrollably large. Fortunately, one has a different method: joining $f^n(x)$ to z satisfying CE2 with a curve γ in sufficient distance from critical values (of subexponential quasi-hyperbolic length, see [**14**] and [**34**]).

Surprisingly, our proof works in the complex case too, since Backward Shrinking (even ExpShrink) holds due to TCE. Since f is open in the complex case, only the easy part of the proof of Theorem 5.1 is needed ((5.1) holds automatically). So we get a new proof of $\chi(x) \geq \chi_{\text{inf}}$, easier than the one in [**14**], just by shadowing by periodic orbits, without joining $f^n(x)$ with γ to a remote safe hyperbolic z.

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