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An entire function with slow growth and simple dynamics

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Abstract. In this paper we prove the existence of an entire transcendental function of arbitrarily slow growth which exhibits simple dynamics on its Fatou set.

The study of the dynamics of transcendental entire functions (TEFs) goes back to Fatou [**7**] who investigated this along with the dynamics of rational functions. While there are many similarities between the dynamics of TEFs and the dynamics of polynomials, there are also important differences, e.g. the existence of wandering domains for TEFs (see [**4**] for a survey). There are certain classes of TEFs which are 'polynomial-like' in their dynamical behavior, for example those with a finite set of singular values [**6**].

Another way to describe a TEF as 'polynomial-like' is to require it to have very slow growth. It is known that if $h\colon [0, \infty) \to [0, \infty)$ is any positive increasing function, there is a TEF f such that the growth of f satisfies $\log M(r, f) = \mathrm{O}(h(r) \log r)$ as $r \to \infty$. In fact, if $h(r) \sim \log \log r$ one can show that $\log M(r, f^n) = \mathrm{O}((\log r)^2)$ for all iterates f^n.

However, the dynamics of such TEFs can display non-polynomial-like behavior. Such functions may have wandering domains and the Julia set may fail to be uniformly perfect as shown by Baker [**2**] and Hinkkanen [**8**]. It is also possible for the Julia set of such functions to equal $\mathbb{C}$, as shown in [**5**].

In this paper we show that the dynamics of TEFs of arbitrarily slow growth can display simple behavior. Namely, we modify the techniques in [**5**] to prove the following.

THEOREM 1. *Given any positive increasing function* $h\colon [0, \infty) \to [0, \infty)$ *there exists a transcendental entire function* f *such that the Fatou set* $F(f)$ *consists entirely of the basin of attraction for an attracting fixed point and such that the growth of* f *satisfies* $\log M(r, f) = \mathrm{O}(h(r) \log r)$.

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We now develop the material needed for the proof of Theorem 1.

Let f be an entire function with attracting fixed point α. We will write $B_f(\alpha) = \{z \in \mathbb{C} : f^n(z) \to \alpha\}$, i.e. $B_f(\alpha)$ is the basin of attraction for the fixed point α. We will also write $O_f^+(z) = \{f^n(z) : n \geq 0\}$ for the forward orbit of a given point z under f. We will write $D(r)$ for $\{z : |z| < r\}$.

PROPOSITION 1. *Let the polynomial p with $d = \deg p \geq 2$ and points $z_1, z_2, \ldots, z_{k-1}$ be given satisfying the following properties:*

(1) $p(0) = 0$, $|p'(0)| < 1$, $p(1) = 1$;
(2) *if a is the leading coefficient of p, then $|a| \leq 1$;*
(3) *there are numbers $r_1 > r_2 > 0$ such that $p(D(r_1)) \subset D(r_2)$;*
(4) *$z_1, \ldots, z_{k-1} \in B_p(0)$; in particular, there are positive integers m_j such that $p^{m_j}(z_j) \in D(r_1)$ for $1 \leq j \leq k-1$;*
(5) *if $c_1, \ldots, c_\ell$ are the distinct critical points of p, then $c_1 \in B_p(0)$ and $c_2, \ldots, c_\ell \in B_p(\infty)$.*

Then, given any point $z_k \in \mathbb{C} \setminus B_p(0)$, $\epsilon \in (0, r_1 - r_2)$ and $R > 0$, there exists a polynomial q, points $z_1', \ldots, z_k'$ and a positive integer m_k satisfying:

(1) $\deg q = d + 1$;
(2) $q(0) = 0$, $|q'(0)| < 1$, $q(1) = 1$;
(3) $q(D(r_1)) \subset D(r_2 + \epsilon)$;
(4) $z_1', \ldots, z_k' \in B_q(0)$; $q^{m_j}(z_j') \in D(r_1)$ for $1 \leq j \leq k$;
(5) $|z_j - z_j'| < \epsilon$ for $1 \leq j \leq k$;
(6) $|p(z) - q(z)| < \epsilon$ for $|z| < R$;
(7) *one critical point of q belongs to $B_q(0)$, the rest belong to $B_q(\infty)$.*

Finally, if $a_1, \ldots, a_d$ are the zeros of p, then there are zeros $a_1', \ldots, a_d'$ of q satisfying $|a_j - a_j'| < \epsilon$ and one zero a_{d+1}' of q satisfying $|a_{d+1}'| > R$.

For the proof of Proposition 1, we will need the following lemmas, whose proofs can be found in [**5**].

LEMMA 1. *For every $\delta > 0$ and $\hat{R} > 0$ there exists $\eta > 0$ such that every quasiconformal homeomorphism $\phi : \mathbb{C} \to \mathbb{C}$ fixing 0 and 1 with the Beltrami coefficient $\|\mu\|_\infty < \eta$ satisfies $|\phi(z) - z| < \delta$, for $z \in D(\hat{R})$.*

LEMMA 2. *For every positive integer d, $\eta > 0$ and $a \in \mathbb{C} \setminus \{0\}$ there exists $\gamma \in (0, 1/2)$ with the following property.*

Let h_1 and h_2 be holomorphic functions in the annulus $A(r/2, 4r)$ such that $\|h_i\|_\infty < \gamma$, $i = 1, 2$. Then there exists a quasiregular local homeomorphism $\phi : A(r, 2r) \to \mathbb{C}$ with boundary values

$$\phi(z) = z^d(a + h_1(z)), \quad |z| = r$$

and

$$\phi(z) = z^d(a + h_2(z)), \quad |z| = 2r$$

such that the Beltrami coefficient μ of ϕ satisfies $\|\mu\|_\infty < \eta$.

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Proof of Proposition 1. We first note that the assumptions on the critical points of p imply that $F(p) = B_p(0) \cup B_p(\infty)$. Also, the filled-in Julia set of p satisfies $K(p) = J(p) \cup B_p(0)$.

We recall that the Green's function for $B_p(\infty)$ is given by

$$u(z) = \lim_{n\to\infty} \frac{\log|p^n(z)|}{d^n} \tag{1}$$

which is continuous on $\mathbb{C}$, positive on $B_p(\infty)$ and zero on $K(p)$ (see [**3**] for details).

By shifting the point z_k a small amount, we can assume that $u(z_k) > 0$. Since the level curves $u(z) \equiv k > 0$ have an empty interior, we can further assume that there is $\kappa > 0$ such that

$$|d^n u(z_k) - d^j u(c)| > \kappa d^n \tag{2}$$

holds for all $c \in \mathrm{crit}'(p)$, $n \in \mathbb{N}$ and $j \in \mathbb{Z}$ where $\mathrm{crit}'(p)$ denotes the critical points of p lying in $B_p(\infty)$.

Using this and (1) we get

$$\min_{j\in\mathbb{N}} \left|\log \frac{|p^n(z_k)|}{|p^j(c)|}\right| \to \infty \quad \text{as } n \to \infty \text{ and } c \in \mathrm{crit}'(p) \tag{3}$$

and

$$\min_{0\le j<n} \frac{|p^n(z_k)|}{|p^j(z_k)|} \to \infty \quad \text{as } n \to \infty. \tag{4}$$

Choose $\hat{R}$ satisfying

$$\hat{R} \ge 1 + \max\left\{R, r_1, |c_1|, \ldots, |c_\ell|, |a_1|, \ldots, |a_d|, |z_1|, \ldots, |z_k|, \max_{|z|=R+1} |p(z)|\right\}$$

and apply Lemma 1 to a fixed $\delta > 0$ and this $\hat{R}$ to obtain a number η. Then apply Lemma 2 to η, a and d to obtain γ.

Choose the integer n to be large enough so that with

$$r := \frac{\gamma|p^n(z_k)|}{4}, \tag{5}$$

the following conditions are satisfied:

$$|p^n(z_k)| \ge \max\left\{\frac{4}{\gamma}(\hat{R}+1), \frac{2e(d+1)}{|a|}\right\}; \tag{6}$$

$$|z^{-d}p(z) - a| < \gamma, \quad \text{for } |z| > r/2; \tag{7}$$

$$\min_{j\in\mathbb{N}} \left|\log \frac{|p^n(z_k)|}{|p^j(c)|}\right| > \log\frac{4}{\gamma}, \quad c \in \mathrm{crit}'(p); \tag{8}$$

$$\min_{0\le j<n} \frac{|p^n(z_k)|}{|p^j(z_k)|} > \frac{4}{\gamma}; \tag{9}$$

and

$$O_p^+(c_1) \subset D(r) \quad \text{and} \quad O_p^+(z_j) \subset D(r) \quad \text{for } 1 \le j \le k-1. \tag{10}$$

Condition (6) can be satisfied since $p^n(z_k) \to \infty$ as $n \to \infty$. The inequality (7) follows since $p(z)/z^d \to a$ as $z \to \infty$. Conditions (8) and (9) follow from (3) and (4),

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respectively. Finally, (10) can be satisfied since $c_1, z_j \in B_p(0)$ for $j = 1, \dots, k-1$. Note that $r > \hat{R} + 1$.

Define a quasiregular map $q_1 \colon \mathbb{C} \to \mathbb{C}$ by

$$q_1(z) = \begin{cases} p(z), & |z| \le r, \\ \phi(z), & r < |z| < 2r, \\ z^d(a - az/p^n(z_k)), & |z| \ge 2r, \end{cases} \tag{11}$$

where ϕ is obtained from Lemma 2 using

$$h_1(z) = z^{-d}p(z) - a \quad \text{and} \quad h_2(z) = -az/p^n(z_k).$$

The required estimates on $\|h_i\|_\infty$ hold by (5) and (7), recalling that $|a| \le 1$.

Properties of q_1. We see that the domain $U := \{z \colon |z| > 2|p^n(z_k)|\}$ lies in $\{z \colon |z| > 2r\}$ and is invariant under q_1. In fact, the q_1-orbits of all points in U tend to infinity.

Since $r_1 < r$, we have that $q_1(D(r_1)) = p(D(r_1)) \subset D(r_2)$ and so all q_1-orbits of points in $D(r_1)$ tend to zero.

The critical points of q_1 are the critical points $c_1, \dots, c_\ell$ of p plus the point $c_{\ell+1} = dp^n(z_k)/(d+1)$. Since the p-orbit of c_1 remains in $D(r)$, we have that the q_1-orbit of c_1 tends to zero. The q_1-orbit of $c_{\ell+1}$ tends to infinity since $|c_{\ell+1}| > 2r$ and so $q_1(c_{\ell+1}) = a(c_{\ell+1})^d/(d+1)$, which we see belongs to U using (6). Condition (8) guarantees that the p-orbits of $c_2, \dots, c_\ell$ do not intersect $A(r, 2|p^n(z_k)|)$ and so their q_1-orbits tend to infinity.

The q_1-orbits of $z_1, \dots, z_{k-1}$ tend to zero by (10) since $q_1(z) = p(z)$ for $|z| < r$. The q_1-orbit of z_k also tends to zero. In fact, $q_1^{n+1}(z_k) = 0$. Condition (9) gives that $|p^j(z_k)| < r$ for $j < n$ so

$$q_1^{n+1}(z_k) = q_1^2(q_1^{n-1}(z_k)) = q_1^2(p^{n-1}(z_k)) = q_1(p^n(z_k)) = 0.$$

Since $q_1(A(r, 2r)) \subset U$ (increase r if necessary), standard quasiconformal surgery arguments applied to q_1 can be used to construct the required polynomial q (see [**5**, Lemma 9.6.2], for example). Specifically, there is a quasiconformal map $\psi \colon \mathbb{C} \to \mathbb{C}$ fixing zero and one such that $q = \psi^{-1} \circ q_1 \circ \psi$ is a polynomial of degree $d+1$. We define $z_j' = \psi^{-1}(z_j)$, $a_j' = \psi^{-1}(a_j)$ and $a_{d+1}' = \psi^{-1}(p^n(z_k))$. The Beltrami coefficients of ψ and ψ^{-1} have the same bound as the Beltrami coefficient for q_1 (namely η) and so

$$|\psi(z) - z| < \delta \quad \text{and} \quad |\psi^{-1}(z) - z| < \delta \quad \text{for } |z| < \hat{R}$$

by Lemma 1.

Thus, by choosing δ small enough, we can guarantee that q and the points $z_1', \dots, z_k', a_1', \dots, a_{d+1}'$ satisfy the conditions of the proposition. □

We note that for a polynomial p of degree d satisfying $p(0) = 0$ and $p(1) = 1$, we may write

$$p(z+1) = \prod_{k=1}^{d} \left(1 - \frac{z}{c_k}\right) \tag{12}$$

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where $c_k = a_k - 1$ and $a_1, \ldots, a_d$ are the zeros of p. In particular,

$$(-1)^d \prod_{k=1}^{d} \frac{1}{c_k}$$

is the leading coefficient of p. Using this observation, if p satisfies the requirements of Proposition 1, R and ϵ may be chosen so that the leading coefficient of q also has modulus at most 1.

Proof of Theorem 1. Let (z_n) be a dense sequence in $\mathbb{C}$ with $z_1 = 0$. Let (ϵ_n) be a sequence of positive numbers such that

$$\sum_{n=1}^{\infty} \epsilon_n < \frac{1}{8}$$

and (R_n) be an increasing sequence of positive numbers such that $R_1 > 2$ and

$$\sum_{n=1}^{\infty} \frac{1}{R_n} < \infty.$$

Start with $p_2(z) = z^2$ and set $k_2 = 1$. Assume that $z_1, \ldots, z_{k_3-1} \in B_{p_2}(0)$ and $z_{k_3} \notin B_{p_2}(0)$ where $k_3 > 1$. Define $z_{2,j} = z_j$ for $k_2 = 1 \leq j < k_3$. We apply Proposition 1 to the polynomial p_2, points $z_{2,1}, \ldots, z_{2,k_3-1}$, together with z_{k_3}, ϵ_3 and R_3. We obtain a polynomial p_3 and points which we will denote $z_{3,1}, \ldots, z_{3,k_3}$ satisfying the conclusion of Proposition 1. Let $k_4 > k_3$ be the first index such that $z_{k_4} \notin B_{p_3}(0)$. Set $z_{3,j} = z_j$ for $k_3 < j < k_4$. In this way, we repeatedly apply Proposition 1 to obtain a sequence of polynomials p_n, a sequence of integers m_j and points $z_{n,j}$ for $j = 1, \ldots, k_n$ such that:

(1) $\deg p_n = n$ and the leading coefficient of p_n has modulus at most one;
(2) $p_n(0) = 0$, $|p_n'(0)| < 1$, $p_n(1) = 1$;
(3) $p_n(D(1/2)) \subset D(3/8)$;
(4) $p_n^{m_j}(z_{n,j}) \in D(1/2)$ for $j \leq k_n$;
(5) $|z_{k_n} - z_{n,k_n}| < \epsilon_n$;
(6) $|z_{n-1,j} - z_{n,j}| < \epsilon_n$ for $j < k_n$;
(7) $|p_{n-1}(z) - p_n(z)| < \epsilon_n$ for $z \in D(R_n)$.

Further, if $a_{n,1}, \ldots, a_{n,n}$ are the zeros of p_n, $n \geq 3$, then $|a_{n-1,j} - a_{n,j}| < \epsilon_n$ for $1 \leq j < d_n$ and $|a_{n,n}| > R_n$.

Thus $w_j = \lim_{n\to\infty} z_{n,j}$ exists for all j and $|w_j - z_j| \to 0$. Hence $\{w_j\}$ is also dense in $\mathbb{C}$.

The sequence p_n converges locally uniformly on $\mathbb{C}$ to an entire function f. This function satisfies $f(D(1/2)) \subset D(3/8)$ and $f^{m_j}(w_j) \in D(1/2)$. Thus, zero is an attracting fixed point for f whose basin $B_f(0)$ is dense in $\mathbb{C}$ and so $F(f) = B_f(0)$.

To estimate the growth of f, we apply (12) to p_n to obtain

$$p_n(z+1) = \prod_{k=1}^{n} \left(1 - \frac{z}{c_{n,k}}\right)$$

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where $c_{n,k} = a_{n,k} - 1$. Our assumptions give that $c_k = \lim_{n\to\infty} c_{n,k}$ exists in $\mathbb{C}$ and that $|c_k| > R_k - 2$. Passing to the limit we have

$$f(z+1) = \prod_{k=1}^{\infty} \left(1 - \frac{z}{c_k}\right)$$

and it is known that (R_k) can be chosen increasing rapidly enough so that f has arbitrarily slow growth. □

We remark that by the result of [**9**], since $p_n \to f$ locally uniformly on $\mathbb{C}$ and the Fatou set of f consists only of the basin for an attracting fixed point, we have $J(p_n) \to J(f)$ in the Hausdorff metric.

We further note that by a result of Baker [**1**], if the maximum modulus $M(r, f)$ satisfies $\log M(r, f) = \mathrm{O}((\log r)^p)$ as $r \to \infty$ where $0 < p < 3$, then all components of the Fatou set $F(f)$ are bounded. Thus, if the growth of h in Theorem 1 is chosen to be small enough, then all components of $F(f)$ are bounded. In particular, $F(f)$ cannot be connected.

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