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MODIFIED ASCENT SEQUENCES AVOIDING A PATTERN OF LENGTH 4

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Abstract Modified ascent sequences, initially defined as the bijective images of ascent sequences under a certain hat map, have also been characterized as Cayley permutations where each entry is a leftmost copy if and only if it is an ascent top. These sequences play a significant role in the study of Fishburn structures. In this paper, we investigate (primitive) modified ascent sequences avoiding a pattern of length 4 by combining combinatorial and algebraic techniques, including the application of the kernel method. Our results confirm several conjectures posed by Cerbai.

Keywords: ascent sequences; modified ascent sequences; Catalan numbers; pattern avoidance; kernel method.

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1. Introduction

Fishburn structures are combinatorial objects enumerated by the Fishburn numbers, which appear as sequence A022493 in the OEIS [18]. These numbers have an elegant generating function:

$$\sum_{n\geq 0}\prod_{i=1}^{n}(1-(1-x)^i) = 1 + x + 2x^2 + 5x^3 + 15x^4 + 53x^5 + 217x^6 + 1014x^7 + \cdots . \quad (1.1)$$

This generating function was first studied by Zagier [20] in his work concerning bounds on the dimension of the space of Vassiliev's knot invariants.

The first Fishburn structure to be systematically studied was the interval orders by Fishburn [10–12], even though it was not initially identified as a Fishburn structure. It was in honour of him that Claesson and Linusson [7] named the coefficients of Zagier's series as Fishburn numbers. In 2010, Bousquet-Mélou *et al* [1] introduced ascent sequences to build

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Table 1. Enumeration of modified ascent sequences avoiding a single pattern σ.

σ	$\lvert\hat{\mathcal{A}}_n(\sigma)\rvert$	$\lvert\hat{\mathcal{A}}_n^{\mathrm{pr}}(\sigma)\rvert$	Reference
1324	A007317	Catalan	Section 3.1
1342	A007317	Catalan	Section 3.2
1243	New	A082582	Section 4.1
1423	New	New	Section 4.2
1223	A047970	New	Section 5
1123	Catalan	A082582	[5] and Section 6

connections among three other combinatorial structures: interval orders, permutations avoiding a bivincular pattern of length 3 and Stoimenow matchings [19]. They showed that all these four combinatorial objects share the common generating function (1.1) and hence are all Fishburn structures.

Since then, a large number of papers have been devoted to the study of Fishburn structures, and more examples continue to be discovered, including modified ascent sequences [1], Fishburn matrices [6, 8], descent correcting sequences [7], Fishburn diagrams [13] and inversion sequences avoiding a vincular pattern of length 3 [15]. Recently, Cerbai and Claesson [4] introduced two new Fishburn structures, Fishburn trees and Fishburn covers, which simplify some existing bijections. Among them, ascent sequences and modified ascent sequences play central roles in the study of Fishburn structures.

An ascent sequence is a sequence of positive integers in which each entry is bounded by the number of ascents preceding it in the sequence. As previously mentioned, ascent sequences are useful in encoding recursive constructions of some Fishburn structures. However, for some other Fishburn structures, such as interval orders, Fishburn matrices, Fishburn trees, and Fishburn covers, it is easier to establish relationships with modified ascent sequences. Modified ascent sequences are originally defined as the bijective images of ascent sequences under a certain hat map [1]. Recently, Cerbai and Claesson [4] provided an alternative definition of modified ascent sequences as Cayley permutations where each entry is a leftmost copy if and only if it is an ascent top. This enables them to study modified ascent sequences as independent objects, from both geometrical and enumerative perspective [2–5].

Cerbai [2, 3] initiated systematic study of pattern avoidance on modified ascent sequences, paralleling the work of Duncan and Steingrímsson on ascent sequences [9]. He obtained rich enumerative results on modified ascent sequences avoiding a single pattern of length less than or equal to 4 and posed several enumerative conjectures. The main objective of this paper is to further investigate the (primitive) modified ascent sequences avoiding a pattern of length 4. In particular, all of his enumerative conjectures in [2] concerning length-4 patterns are solved. Note that the general and the primitive modified ascent sequences avoiding the same pattern σ are related by the binomial transformation if σ is itself primitive; see proposition 2.1. An overview of our work can be found in table 1.

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The rest of this paper is organized as follows. We start by giving the necessary tools and definitions in section 2.

In section 3, we provide recursive constructions of 1324- and 1342-avoiding primitive modified ascent sequences and show that they are both enumerated by the Catalan numbers, settling two conjectures posed by Cerbai [2]. The enumeration of the general cases then follows immediately from proposition 2.1.

In section 4, we solve the patterns 1243 and 1423 for both primitive and general cases by similar recursive constructions as those used for the patterns 1342 and 1324 in section 3. We obtain the generating functions for them and find that the primitive modified ascent sequences avoiding 1243 are equal to the number of Dyck paths of semilength n with no **uudd** factors.

In section 5, we solve the pattern 1223 for the general case by constructing a bijection with 1232-avoiding modified ascent sequences. For the primitive case, by restricting the bijection to the primitive counterpart and characterizing their images, we obtain the continued fraction expansion of its generating function.

In section 6, by making use of the kernel method, we derive the generating function for primitive modified ascent sequences avoiding 1123 with respect to multiple statistics. Then we slightly tweak proposition 2.1 and reprove that 1123-avoiding modified ascent sequences are enumerated by the Catalan numbers. En passant, we obtain a new Narayana statistic on 1123-avoiding modified ascent sequences.

Section 7 contains some final remarks and suggestions for future work.

2. Preliminaries

Let n be a positive integer and let $[n] = \{1, 2, \ldots, n\}$. An *endofunction* of size n is a map $x : [n] \mapsto [n]$, which is represented as the word $x = x_1x_2\cdots x_n$ with $x_i = x(i)$. In this paper, given a set E with elements equipped with a notion of size, we will denote E_n the set of elements in E that have size n. Conversely, given a definition of E_n (elements of size n), we let $E = \bigcup_{n\geq 0} E_n$. For example, let End_n be the set of endofunctions of size n and let $\mathrm{End} = \bigcup_{n\geq 0} \mathrm{End}_n$.

Given an endofunction $x = x_1x_2\cdots x_n$, we call i $(2 \leq i \leq n)$ an *ascent* (resp., a *descent*) of x if $x_{i-1} < x_i$ (resp., $x_{i-1} > x_i$). We always designate 1 as an ascent of x if x is non-empty. Let $\mathrm{Asc}(x)$ denote the set of ascents and let $\mathrm{asc}(x) = |\mathrm{Asc}(x)|$. We call an endofunction x an *ascent sequence* if $1 \leq x_i \leq \mathrm{asc}(x_1x_2\cdots x_{i-1}) + 1$ for all $1 \leq i \leq n$. Let $\mathcal{A}_n$ denote the set of ascent sequences of length n. For example, we have $\mathcal{A}_3 = \{111, 112, 121, 122, 123\}$.

Let x be an ascent sequence in $\mathcal{A}_n$. Suppose that $\mathrm{Asc}(x) = \{a_1, a_2, \ldots, a_k\}$ with $a_1 < a_2 < \cdots < a_k$. The *modified ascent sequence* of x, denoted by $\hat{x}$, is defined by the following procedure:

for $i = a_1, a_2, \ldots, a_k$ do
for $j = 1, 2, \ldots, i-1$ do
if $x_j \geq x_i$ then $x_j := x_j + 1$.

This map is commonly referred to as the *hat map*. For example, let $x = 123112313$. We have $\mathrm{Asc}(x) = \{1, 2, 3, 6, 7, 9\}$ and $\hat{x} = 156112413$. Let $\hat{\mathcal{A}}_n$ denote the set of modified ascent sequences of length n. That is, $\hat{\mathcal{A}}_n = \{\hat{x} \mid x \in \mathcal{A}_n\}$. Note that the construction

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of the hat map can be easily inverted since $\mathrm{Asc}(x) = \mathrm{Asc}(\hat{x})$. Hence, the hat map is a bijection between $\mathcal{A}_n$ and $\hat{\mathcal{A}}_n$.

Let $x = x_1x_2\cdots x_n$ be an endofunction and let $1 \le \ell \le 1+\max x$. Define $y = \mathrm{Add}(x,\ell)$ to be the endofunction of length $n+1$ depending on whether $\ell > x_n$:

- If $1 \le \ell \le x_n$, we have $y = x\ell$. Here $x\ell$ is the concatenation of x and ℓ.
- If $x_n < \ell \le 1 + \max x$, we have $y(n+1) = \ell$ and for $1 \le i \le n$,

$$y(i) = \begin{cases} x_i, & \text{if } x_i < \ell, \\ x_i + 1, & \text{if } x_i \ge \ell. \end{cases}$$

Note that any prefix of an ascent sequence is also an ascent sequence. This leads to a recursive definition [1] of modified ascent sequences from the hat map as follows. There is exactly one modified ascent sequence of length zero and one, that is, $\hat{\mathcal{A}}_0 = \{\epsilon\}$ and $\hat{\mathcal{A}}_1 = \{1\}$. For $n \ge 1$, every $y = y_1y_2\cdots y_{n+1} \in \hat{\mathcal{A}}_{n+1}$ can be constructed uniquely from some $x \in \hat{\mathcal{A}}_n$ by letting $y = \mathrm{Add}(x, y_{n+1})$. Conversely, given $x \in \hat{\mathcal{A}}_n$ and $1 \le \ell \le 1+\max x$, we can construct a unique modified ascent sequence y by letting $y = \mathrm{Add}(x,\ell)$. Therefore, each modified ascent sequence $x \in \hat{\mathcal{A}}_n$ can generate $1+\max x$ modified ascent sequences of length $n+1$.

A *Cayley permutation* is an endofunction $x : [n] \mapsto [n]$ such that $\mathrm{Im}(x) = [k]$ for some $k \le n$. Here $\mathrm{Im}(x) := \{x_i : 1 \le i \le n\}$ is the set of all images of x. Let Cay_n denote the set of Cayler permutations of length n. Very recently, Claesson and Cerbai [5] showed that

$$\hat{\mathcal{A}}_n = \{x \in \mathrm{Cay}_n \mid \mathrm{Asc}(x) = \mathrm{nub}(x)\},$$

where

$$\mathrm{nub}(x) = \{\min x^{-1}(i) \mid i \in \mathrm{Im}(x)\}.$$

For example, for $x = 156112413$, we have $\mathrm{Asc}(x) = \mathrm{nub}(x) = \{1,2,3,6,7,9\}$, thereby x is a modified ascent sequence. This alternative description takes the mystery out of modified ascent sequences, which we shall use frequently in the paper. From the definition of modified ascent sequence, it is straightforward to verify that each non-empty modified ascent sequence starts with 1.

Let $x = x_1x_2\cdots x_n$ and $\sigma = \sigma_1\sigma_2\cdots\sigma_k$ be two endofunctions with $k \le n$. An *occurrence* of σ in x is a subsequence $x_{i_1}x_{i_2}\cdots x_{i_k}$ which is order isomorphic to σ. Here, *order isomorphic* means $x_{i_s} < x_{i_t}$ if and only if $\sigma_s < \sigma_t$ and $x_{i_s} = x_{i_t}$ if and only if $\sigma_s = \sigma_t$. We say x *contains* the pattern σ if x contains an occurrence of σ. Otherwise, we say x *avoids* the pattern σ or x is σ*-avoiding*. For example, the modified ascent sequence $x = 156112413$ contains three occurrences of the pattern 1342, namely, the subsequences 1562, 1564, and 1563, but avoids the pattern 1234. Let $\mathrm{End}_n(\sigma)$ denote the set of endofunctions in End_n that avoid the pattern σ. We will keep this notation when End_n is replaced by its subsets such as $\mathcal{A}_n$ and $\hat{\mathcal{A}}_n$, etc.

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Given an endofunction $x = x_1x_2\cdots x_n$, we say x is *primitive* if it contains no flat steps, where a *flat* step means two consecutive equal entries $x_i = x_{i+1}$. For instance, the endofunction $x = 156112413$ is not primitive since it contains a flat step 11 while 15612413 which is obtained from x by removing one copy of 1 in the flat step 11 is primitive. More generally, given any endofunction x, it corresponds to a unique primitive endofunction which is obtained from x by merging each the consecutive flat step of x into one single copy. Conversely, any endofunction can be uniquely obtained from a primitive one by inserting flat steps. If σ is a primitive pattern and x avoids σ, then any insertion of flat steps in x does not create any new occurrence of σ. Let $\mathrm{End}_n^{\mathrm{pr}}$ denote the set of primitive endofunctions of length n. We will also keep this notation when End_n is replaced by its subsets. For example, $\hat{\mathcal{A}}_n^{\mathrm{pr}}$ is the set of primitive modified ascent sequences of length n. Let

$$\hat{\mathcal{A}}_\sigma(t) = \sum_{n\geq 0} |\hat{\mathcal{A}}_n(\sigma)|t^n \quad \text{and} \quad \hat{\mathcal{A}}_\sigma^{\mathrm{pr}}(t) = \sum_{n\geq 0} |\hat{\mathcal{A}}_n^{\mathrm{pr}}(\sigma)|t^n$$

be the generating functions of $\hat{\mathcal{A}}_n(\sigma)$ and $\hat{\mathcal{A}}_n^{\mathrm{pr}}(\sigma)$, respectively. Based on the above reasoning, Cerbai [2] obtained the following relation between $\hat{\mathcal{A}}_n(\sigma)$ and $\hat{\mathcal{A}}_n^{\mathrm{pr}}(\sigma)$.

Proposition 2.1. Cerbai *Let σ be a primitive Cayley permutation. Then for $n \geq 0$,*

$$|\hat{\mathcal{A}}_{n+1}(\sigma)| = \sum_{i=0}^{n} \binom{n}{i} |\hat{\mathcal{A}}_{i+1}^{\mathrm{pr}}(\sigma)|.$$

Equivalently,

$$\hat{\mathcal{A}}_\sigma(t) = \hat{\mathcal{A}}_\sigma^{\mathrm{pr}}\left(\frac{t}{1-t}\right).$$

Let x be a sequence of positive integers. The *standardization* of x, denoted by $\mathrm{st}(x)$, is defined to be the sequence which is obtained from x by relabeling the sequence according to their relative order. More specifically, the smallest integers in x are replaced with 1, the next smallest with 2, and so forth. For example, given $x = 1681981353$, we have $\mathrm{st}(x) = 1451651232$. We end this section with some obvious but useful observations, which we omit the proofs.

Lemma 2.2. *Given a primitive modified ascent sequence $x = x_1x_2\cdots x_n$, if $x_i = x_j$ with $1 \leq i < j \leq n$, then $x_{j-1} > x_j$.*

Lemma 2.3. *Given a sequence x of positive integers, $\mathrm{st}(x)$ is a Cayley permutation. Moreover,*

- *if* x *is primitive, then* $\mathrm{st}(x)$ *is primitive;*
- *if* $\mathrm{Asc}(x) = \mathrm{nub}(x)$, *then* $\mathrm{st}(x)$ *is a modified ascent sequence;*
- *if* x *avoids a pattern σ, then so does* $\mathrm{st}(x)$.

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Lemma 2.4. *Given a pattern* σ *and* $x = x_1x_2\cdots x_n \in \hat{\mathcal{A}}_n^{\mathrm{pr}}(\sigma)$, *then* $\mathrm{st}(x_1x_2\cdots x_k) \in \hat{\mathcal{A}}_k^{\mathrm{pr}}(\sigma)$ *for any* $1 \le k \le n$.

3. Patterns 1324 and 1342

In this section, we concentrate on the enumeration of modified ascent sequences that avoid the pattern 1324 or 1342. Since both the patterns 1324 and 1342 are primitive, owing to proposition 2.1, it is sufficient to consider their primitive cases.

3.1. Pattern 1324

Let $\hat{\mathcal{A}}_{n,=2}^{\mathrm{pr}}(1324)$ and $\hat{\mathcal{A}}_{n,\neq 2}^{\mathrm{pr}}(1324)$ denote the set of primitive modified ascent sequences in $\hat{\mathcal{A}}_n^{\mathrm{pr}}(1324)$ that end in a 2 and that do not end in a 2, respectively. Then we have the following lemma.

Lemma 3.1. *For* $n \ge 1$, *we have*

$$|\hat{\mathcal{A}}_n^{\mathrm{pr}}(1324)| = |\hat{\mathcal{A}}_{n,=2}^{\mathrm{pr}}(1324)| + |\hat{\mathcal{A}}_{n+1,=2}^{\mathrm{pr}}(1324)|.$$

Proof. Each modified ascent sequence either ends in a 2 or does not end in a 2, which leads to $|\hat{\mathcal{A}}_n^{\mathrm{pr}}(1324)| = |\hat{\mathcal{A}}_{n,=2}^{\mathrm{pr}}(1324)| + |\hat{\mathcal{A}}_{n,\neq 2}^{\mathrm{pr}}(1324)|$. To complete the proof, it suffices to show that $|\hat{\mathcal{A}}_{n,\neq 2}^{\mathrm{pr}}(1324)| = |\hat{\mathcal{A}}_{n+1,=2}^{\mathrm{pr}}(1324)|$. Given a modified ascent sequence $x \in \hat{\mathcal{A}}_n$ with $x(n) \neq 2$, It is easily seen that x is primitive if and only if $\mathrm{Add}(x, 2)$ is primitive. On the other hand, by the construction of Add, $\mathrm{Add}(x, 2)$ avoids 1324 if and only if x avoids 1324. Hence the map $\mathrm{Add}(x, 2)$ induces a bijection between $\hat{\mathcal{A}}_{n,\neq 2}^{\mathrm{pr}}(1324)$ and $\hat{\mathcal{A}}_{n+1,=2}^{\mathrm{pr}}(1324)$. It yields that $|\hat{\mathcal{A}}_{n,\neq 2}^{\mathrm{pr}}(1324)| = |\hat{\mathcal{A}}_{n+1,=2}^{\mathrm{pr}}(1324)|$, as desired. □

Let $n \ge 2$ and $x \in \hat{\mathcal{A}}_n^{\mathrm{pr}}(1324)$. It is obvious that x contains at least one copy of 2. Then x can be decomposed uniquely as $x = 1L2R$ according to the rightmost 2 of x.

Lemma 3.2. *Given* $n \ge 2$, *let* $x \in \hat{\mathcal{A}}_n^{\mathrm{pr}}(1324)$ *with the above decomposition as* $x = 1L2R$. *Then for any entry* $\mathrm{a} > 2$ *in* L *and* $\mathrm{b} > 2$ *in* R, *we have* $\mathrm{a} > \mathrm{b}$.

Proof. If there is some $a > 2$ in L and some $b > 2$ in R such that $a \le b$, we consider two cases. If $a < b$, then $1a2b$ is an occurrence of the pattern 1324, a contradiction to the fact that x is 1324-avoiding. If $a = b$, then by lemma 2.2, b is preceded by a larger entry in x, say c. Thus $1a2c$ is an occurrence of the pattern 1324 in x, a contradiction. This completes the proof. □

Given $n \ge 0$, let $f_n = |\hat{\mathcal{A}}_n^{\mathrm{pr}}(1324)|$. We have the following recurrence for f_n.

Lemma 3.3. *For* $n \ge 2$, *we have*

$$f_n = \sum_{k\ge 1}^{n-1} f_k f_{n-k}, \tag{3.1}$$

with the initial value $f_1 = 1$.

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Proof. It is clear that $f_1 = 1$. Given $n \geq 2$, let x be a primitive modified ascent sequence in $\hat{\mathcal{A}}_n^{\mathrm{pr}}(1324)$ with the above decomposition as $x = 1L2R$. Let $L' = \mathrm{st}(1L2)$ and let $R' = \mathrm{st}(R)$ if R starts with the entry 1 or else let $R' = \mathrm{st}(1R)$. Let k be the length of L'. From lemma 2.4 we immediately obtain that $L' \in \hat{\mathcal{A}}_{k,=2}^{\mathrm{pr}}(1324)$.

Next we consider R'. If $k = n$, then $R = \epsilon$ and $R' = 1$. If $k < n$, we claim that $R' \in \hat{\mathcal{A}}^{\mathrm{pr}}(1324)$. Indeed, by lemma 3.2, we have $\mathrm{Asc}(R) = \mathrm{nub}(R)$ if R starts with 1. Then by lemma 2.3, we have $R' \in \hat{\mathcal{A}}_{n-k}^{\mathrm{pr}}(1324)$. If R does not start with 1, then $\mathrm{Asc}(1R) = \mathrm{nub}(1R)$. The insertion of 1 will not create a new 1324 occurrence $x_{i_1}x_{i_2}x_{i_3}x_{i_4}$ in R. Otherwise, replacing x_{i_1} with the leftmost 1 of x in $x_{i_1}x_{i_2}x_{i_3}x_{i_4}$ will form an occurrence of 1324 in x, a contradiction. Hence, We obtain $R' \in \hat{\mathcal{A}}_{n+1-k}^{\mathrm{pr}}(1324)$ from lemma 2.3. This proves the claim.

To conclude, we have $|\hat{\mathcal{A}}_{k,=2}^{\mathrm{pr}}(1324)|$ ways to choose L' and $f_{n-k} + f_{n+1-k}$ ways to choose R' if $k < n$. Conversely, given $L' \in \hat{\mathcal{A}}_{k,=2}^{\mathrm{pr}}(1324)$ and $R' \in \hat{\mathcal{A}}_{n-k}^{\mathrm{pr}}(1324)$ or $R' \in \hat{\mathcal{A}}_{n+1-k}^{\mathrm{pr}}(1324)|$ with $k < n$, one can easily recover the unique primitive modified ascent sequence $x \in \hat{\mathcal{A}}_n^{\mathrm{pr}}(1324)$ from lemma 3.2. Therefore,

$$\begin{aligned}
f_n &= \sum_{k\geq 2}^{n-1} |\hat{\mathcal{A}}_{k,=2}^{\mathrm{pr}}(1324)|(f_{n-k} + f_{n+1-k}) + |\hat{\mathcal{A}}_{n,=2}^{\mathrm{pr}}(1324)| \\
&= \sum_{k\geq 2}^{n-1} |\hat{\mathcal{A}}_{k,=2}^{\mathrm{pr}}(1324)|f_{n-k} + \sum_{k\geq 2}^{n-1} |\hat{\mathcal{A}}_{k,=2}^{\mathrm{pr}}(1324)|f_{n+1-k} + |\hat{\mathcal{A}}_{n,=2}^{\mathrm{pr}}(1324)| \\
&= \sum_{k\geq 1}^{n-1} |\hat{\mathcal{A}}_{k,=2}^{\mathrm{pr}}(1324)|f_{n-k} + \sum_{k\geq 1}^{n-1} |\hat{\mathcal{A}}_{k+1,=2}^{\mathrm{pr}}(1324)|f_{n-k} \\
&\qquad \text{(by the fact } |\hat{\mathcal{A}}_{1,=2}^{\mathrm{pr}}(1324)| = 0 \text{and } f_1 = 1) \\
&= \sum_{k\geq 1}^{n-1} (|\hat{\mathcal{A}}_{k,=2}^{\mathrm{pr}}(1324)| + |\hat{\mathcal{A}}_{k+1,=2}^{\mathrm{pr}}(1324)|)f_{n-k} \\
&= \sum_{k\geq 1}^{n-1} f_k f_{n-k} \quad \text{(by lemma 3.1)},
\end{aligned}$$

as desired. □

The following theorem follows easily from lemma 3.3 and proposition 2.1.

Theorem 3.4. *Let* $c(t) := \sum_{n\geq 0} c_n t^n = \frac{1-\sqrt{1-4t}}{2t}$ *be the generating function for the Catalan numbers* $\mathrm{c_n}$. *Then,*

$$\hat{\mathcal{A}}_{1324}^{\mathrm{pr}}(t) = \sum_{n\geq 0} f_n t^n = 1 + tc(t).$$

Therefore, $|\hat{\mathcal{A}}_n^{\mathrm{pr}}(1324)| = c_{n-1}$ *and*

$$|\hat{\mathcal{A}}_{n+1}(1324)| = \sum_{i=0}^{n} \binom{n}{i} c_i,$$

which is the binomial transformation of Catalan numbers.

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Theorem 3.4 confirms a conjecture posed by Cerbai [2]. Let

$$\hat{\mathcal{A}}^{\mathrm{pr}}_{1324,=2}(t) = \sum_{n\geq 1} |\hat{\mathcal{A}}^{\mathrm{pr}}_{n,=2}(1324)| t^n.$$

By lemma 3.1, we obtain

$$\hat{\mathcal{A}}^{\mathrm{pr}}_{1324}(t) = 1 + (1 + \frac{1}{t})\hat{\mathcal{A}}^{\mathrm{pr}}_{1324,=2}(t),$$

which implies

$$\hat{\mathcal{A}}^{\mathrm{pr}}_{1324,=2}(t) = \frac{t^2}{1+t} c(t) = t^2 + 2t^4 + 3t^5 + 11t^6 + 31t^7 + 101t^8 + \cdots.$$

This indeed represents the absolute value of the *alternating sum of the Catalan numbers*; see [18, A032357].

3.2. Pattern 1342

Given a primitive modified ascent sequence x with $\max x = m$, we assert that m occurs exactly once in x. Otherwise, if m appears more than once, then by lemma 2.2, the second copy of m would be preceded by a number greater than m in x, thereby conflicting with the assertion that m is the maximum value. This enables us to decompose x uniquely as $x = 1LmR$.

Lemma 3.5. *Given $n \geq 2$, let $x \in \hat{\mathcal{A}}^{\mathrm{pr}}_n(1342)$ with the above decomposition as $x = 1LmR$. Then for any entry $a \geq 2$ in L and $b \geq 2$ in R, we have $a \leq b$.*

Proof. If there is some $a \geq 2$ in L and some $b \geq 2$ in R such that $a > b$, we have $a < m$ and $b < m$ since m is the unique maximum value in x. Hence $1amb$ is an occurrence of 1342 in x, a contradiction to the fact that x is 1342-avoiding. □

Let $g_n = |\hat{\mathcal{A}}^{\mathrm{pr}}_n(1342)|$. The following lemma shows that g_n shares the same recurrence (3.1) as $|\hat{\mathcal{A}}^{\mathrm{pr}}_n(1324)|$.

Lemma 3.6. *For $n \geq 2$, we have*

$$g_n = \sum_{k\geq 1}^{n-1} g_k g_{n-k}, \tag{3.2}$$

with the initial value $g_1 = 1$.

Proof. Given $n \geq 3$, let $x \in \hat{\mathcal{A}}^{\mathrm{pr}}_n(1342)$ with the decomposition $x = 1LmR$, where m is the unique maximum value in x. Let k be the length of L. From lemma 3.5 we see that $1L \in \hat{\mathcal{A}}^{\mathrm{pr}}_{k+1}(1342)$. Now, there are exactly three possibilities to be considered.

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Case 1. $k=0$. In this case, L is empty and R is non-empty as $n \geq 3$. Recall that $\mathrm{Asc}(x) = \mathrm{nub}(x)$ since x is a modified ascent sequence. This implies that the first entry of R is 1. Moreover, one can check that $\mathrm{Asc}(R) = \mathrm{nub}(R)$. Then by lemma 2.3, we have $R \in \hat{\mathcal{A}}^{\mathrm{pr}}_{n-2}(1342)$, namely, there are g_{n-2} such primitive modified ascent sequences in $\hat{\mathcal{A}}^{\mathrm{pr}}_{n}(1342)$ satisfying that L is empty.
Case 2. $k = n-2$. In this case, we have R is empty and L is non-empty. Notice that $1L \in \hat{\mathcal{A}}^{\mathrm{pr}}_{n-1}(1342)$. There are g_{n-1} such primitive modified ascent sequences in $\hat{\mathcal{A}}^{\mathrm{pr}}_{n}(1342)$ satisfying tht R is empty.
Case 3. $1 \leq k \leq n-3$. Both L and R are non-empty in this case. It is known that there are g_{k+1} choices for L. Now we consider R. There are two cases depending on whether R contains the number m', where $m' = \max L$.

- If R does not contain m', let $R' = \mathrm{st}(R)$. By lemma 3.5 and the fact $\mathrm{Asc}(x) = \mathrm{nub}(x)$, we deduce that R starts with 1 and furthermore $\mathrm{Asc}(R) = \mathrm{nub}(R)$. Then $R' \in \hat{\mathcal{A}}^{\mathrm{pr}}_{n-2-k}(1342)$ follows directly from lemma 2.3. Conversely, given $1L \in \hat{\mathcal{A}}^{\mathrm{pr}}_{k+1}(1342)$ and $R' \in \hat{\mathcal{A}}^{\mathrm{pr}}_{n-2-k}(1342)$, by lemma 3.5, one can recover a unique $x \in \hat{\mathcal{A}}^{\mathrm{pr}}_{n}(1342)$. Hence, there are $g_{k+1}g_{n-2-k}$ such $x \in \hat{\mathcal{A}}^{\mathrm{pr}}_{n}(1342)$ in this case.
- If R contains at least one copy of m', a 1 is inserted immediately before the first occurrence of m' in R, followed by standardization to derive a new sequence, denoted by R''. We proceed to show that $R'' \in \hat{\mathcal{A}}^{\mathrm{pr}}_{n-1-k}(1342)$. First, again by lemma 3.5 and the fact $\mathrm{Asc}(x) = \mathrm{nub}(x)$, we deduce that R starts with 1 or m'. By the construction of R'', in either case, we have R'' starts with 1 and furthermore $\mathrm{Asc}(R'') = \mathrm{nub}(R'')$. Since m' is preceded by a number larger that m' in R, the insertion of 1 will not result in two consecutive occurrences of 1. Thus, R'' is primitive from lemma 2.3. Note that the insertion of 1 and standardization will not create any occurrence of 1342. In conclusion, $R'' \in \hat{\mathcal{A}}^{\mathrm{pr}}_{n-1-k}(1342)$. Conversely, given $1L \in \hat{\mathcal{A}}^{\mathrm{pr}}_{k+1}(1342)$ and $R'' \in \hat{\mathcal{A}}^{\mathrm{pr}}_{n-1-k}(1342)$, by lemma 3.5, one can recover a unique $x \in \hat{\mathcal{A}}^{\mathrm{pr}}_{n}(1342)$. Hence there are $g_{k+1}g_{n-1-k}$ such x in this case.

Combining the above three cases, we derive that for $n \geq 3$,

$$\begin{aligned} g_n &= g_{n-2} + g_{n-1} + \sum_{k\geq 1}^{n-3} g_{k+1}(g_{n-2-k} + g_{n-1-k}) \\ &= g_{n-2} + g_{n-1} + \sum_{k\geq 2}^{n-2} g_k(g_{n-1-k} + g_{n-k}) \\ &= \sum_{k\geq 1}^{n-2} g_k(g_{n-1-k} + g_{n-k}), \end{aligned} \tag{3.3}$$

where the last step is obtained by using the simple fact $g_1 = 1$. Let $g'_1 = 1$ and for $n \geq 2$, let $g'_n = \sum_{k\geq 1}^{n-1} g_k g_{n-k}$. Eq. (3.3) tells us that $g_n + g_{n-1} = g'_n + g'_{n-1}$ for $n \geq 3$. It is routine

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to check that $g_2 = g'_2 = 1$, which yields that $g_n = g'_n$ for $n \geq 0$. Hence, $g_n = \sum_{k\geq 1}^{n-1} g_k g_{n-k}$ for $n \geq 2$, as desired. □

Analogous to theorem 3.4, the following theorem can be derived directly from lemma 3.6 and proposition 2.1.

Theorem 3.7. *We have*

$$\hat{\mathcal{A}}^{\mathrm{pr}}_{1342}(t) = \sum_{n\geq 0} g_n t^n = 1 + tc(t).$$

Therefore, $|\hat{\mathcal{A}}^{\mathrm{pr}}_n(1342)| = c_{n-1}$ *and*

$$|\hat{\mathcal{A}}_{n+1}(1342)| = \sum_{i=0}^{n} \binom{n}{i} c_i.$$

Theorem 3.7 confirms a conjecture of Cerbai [2] concerning the pattern 1342.

4. Patterns 1243 and 1423

This section is devoted to the enumeration for patterns 1243 and 1423, employing the same decompositions as those used for patterns 1342 and 1324, respectively. Based on proposition 2.1, we only need to consider the primitive cases.

4.1. Pattern 1243

Let $x \in \hat{\mathcal{A}}^{\mathrm{pr}}_n(1243)$ with $\max x = m$. Recall that every primitive modified ascent sequence contains exactly one copy of the maximum value. We can decompose x uniquely as $x = 1LmR$. The following lemma is analogous to lemma 3.5, and hence we omit the proof.

Lemma 4.1. *Given $n \geq 2$, let $x \in \hat{\mathcal{A}}^{\mathrm{pr}}_n(1243)$ with the above decomposition as $x = 1LmR$. Then for any entry $a \geq 2$ in L and $b \geq 2$ in R, we have $a \geq b$.*

Let $h_n = |\hat{\mathcal{A}}^{\mathrm{pr}}_n(1243)|$. We have the following recurrence for h_n.

Lemma 4.2. *For $n \geq 1$, we have*

$$h_n = h_{n-1} + \sum_{\mathrm{k}=2}^{n-1} h_k h_{n-1-k}, \tag{4.1}$$

with the initial value $h_0 = 1$.

Proof. Let $n \geq 1$ and $x \in \hat{\mathcal{A}}^{\mathrm{pr}}_{n+1}(1243)$ with the above decomposition $x = 1LmR$, where m is the unique maximum value in x. Let k be the length of $1L$ and let m' be the

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minimum value of L such that $m' \geq 2$. We distinguish two cases depending on whether L is empty.

Case 1. $k=1$. That is, L is empty. By a similar argument as Case 1 of lemma 3.6, we have $R \in \hat{\mathcal{A}}^{\mathrm{pr}}_{n-1}(1243)$. Hence, there are h_{n-1} such sequences in $\hat{\mathcal{A}}^{\mathrm{pr}}_{n+1}(1243)$ satisfying that L is empty.

Case 2. $2 \leq k \leq n$. In this case, L is non-empty. Let $L' = \mathrm{st}(1L)$. By lemma 2.4, $L' \in \hat{\mathcal{A}}^{\mathrm{pr}}_{k}(1243)$. Now we consider R. By lemma 4.1 and the fact that $\mathrm{Asc}(x) = \mathrm{nub}(x)$, R either starts with 1 or m'.

- If R starts with 1, R cannot contain m'. Otherwise, the m' contained in R will be preceded by a larger number, a contradiction to lemma 4.1. Since x is a Cayley permutation, again by lemma 4.1, we derive that R contains all the numbers between 1 and $m'-1$. It is easily seen that $\mathrm{Asc}(R) = \mathrm{nub}(R)$. By lemma 2.3, $R \in \hat{\mathcal{A}}^{\mathrm{pr}}_{n-k}(1243)$. Conversely, given any $L' \in \hat{\mathcal{A}}^{\mathrm{pr}}_{k}(1243)$ and $R \in \hat{\mathcal{A}}^{\mathrm{pr}}_{n-k}(1243)$, by lemma 4.1, we can recover a unique primitive modified ascent sequence $x \in \hat{\mathcal{A}}^{\mathrm{pr}}_{n+1}(1243)$. Hence there are $h_k h_{n-k}$ such $x \in \hat{\mathcal{A}}^{\mathrm{pr}}_{n+1}(1243)$ in this case.
- If R starts with m', let R' be the sequence obtained from R by deleting the first entry m'. By a similar reasoning as the case R starts with 1, we obtain that $R' \in \hat{\mathcal{A}}^{\mathrm{pr}}_{n-1-k}(1243)$. Conversely, given any $L' \in \hat{\mathcal{A}}^{\mathrm{pr}}_{k}(1243)$ and $R' \in \hat{\mathcal{A}}^{\mathrm{pr}}_{n-1-k}(1243)$, we can recover a unique sequence $x \in \hat{\mathcal{A}}^{\mathrm{pr}}_{n+1}(1243)$. Hence there are $h_k h_{n-1-k}$ such $x \in \hat{\mathcal{A}}^{\mathrm{pr}}_{n+1}(1243)$ in this case.

Combining the above two cases gives

$$h_{n+1} = h_{n-1} + \sum_{k=2}^{n} h_k (h_{n-k} + h_{n-1-k}), \tag{4.2}$$

for $n \geq 1$. Let $h'_0 = 1$ and $h'_n = h_{n-1} + \sum_{k=2}^{n-1} h_k h_{n-1-k}$ for $n \geq 1$. Then by adding h_n to both sides of equation (4.2), we have $h_{n+1} + h_n = h'_{n+1} + h'_n$ for $n \geq 1$. It is routine to check that $h_1 = h'_1 = 1$, which yields that $h_n = h'_n$ for $n \geq 0$. Therefore, $h_n = h_{n-1} + \sum_{k=2}^{n-1} h_k h_{n-1-k}$ for $n \geq 1$, as desired. □

Theorem 4.3. *We have*

$$\hat{\mathcal{A}}^{\mathrm{pr}}_{1243}(t) = \frac{1 + t^2 - \sqrt{1 - 4t + 2t^2 + t^4}}{2t} \tag{4.3}$$

and

$$\hat{\mathcal{A}}_{1243}(t) = \frac{1 + 2t^2 - 2t - \sqrt{1 - 8t + 20t^2 - 20t^3 + 8t^4}}{2t - 2t^2}. \tag{4.4}$$

Proof. Turning the recurrence relation (4.1) into generating function yields

$$\hat{\mathcal{A}}^{\mathrm{pr}}_{1243}(t) - 1 = t\hat{\mathcal{A}}^{\mathrm{pr}}_{1243}(t)^2 - t^2 \hat{\mathcal{A}}^{\mathrm{pr}}_{1243}(t).$$

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Solving the above equation, we obtain (4.3). It then follows from proposition 2.1 that (4.4) holds. □

Eq. (4.3) gives the sequence

$$(|\hat{\mathcal{A}}_n^{\mathrm{pr}}(1243)|)_{n\geq 0} = (1, 1, 1, 2, 5, 13, 35, 97, 275, 794, 2327, \cdots),$$

which appears as A082582 in OEIS [18]. This sequence also counts the Dyck paths of semilength n with no **uudd** (or with no **dduu**).

4.2. Pattern 1423

For $n \geq 1$, let $d_n = |\hat{\mathcal{A}}_n^{\mathrm{pr}}(1423)|$. Let $e_n = |\{x \in \hat{\mathcal{A}}_n^{\mathrm{pr}}(1423) \mid x_n = 2\}|$. With a similar proof as lemma 3.1, we have the following relation for d_n and e_n.

Lemma 4.4. *For $n \geq 1$, we have*

$$d_n = e_n + e_{n+1}. \tag{4.5}$$

For $n \geq 2$, let

$$r_n = |\{x \in \hat{\mathcal{A}}_n^{\mathrm{pr}}(1423) \mid x_2 = 2\}| \quad \text{and} \quad s_n = |\{x \in \hat{\mathcal{A}}_n^{\mathrm{pr}}(1423) \mid x_2 > 2\}|.$$

For convenience, we set $d_0 = 0$ and $e_n = r_n = s_n = 0$ if $n < 2$. Obviously, for $n \geq 2$, we have

$$d_n = r_n + s_n. \tag{4.6}$$

Lemma 4.5. *For $n \geq 2$, we have*

$$r_n = d_{n-1} + d_{n-2} + s_n + s_{n-1}. \tag{4.7}$$

Proof. It is routine to check the conclusion for $n \leq 4$. Now we assume that $n > 4$. Let $x \in \hat{\mathcal{A}}_n^{\mathrm{pr}}(1423)$ with $x_2 = 2$. We consider the following four cases.

Case 1. If $x_3 = 1$ and there is exactly one copy of 2 in x, deleting the first two entries in x and standardizing, gives a sequence in $\hat{\mathcal{A}}_{n-2}^{\mathrm{pr}}(1423)$. Hence there are d_{n-2} such x in this case.

Case 2. If $x_3 = 1$ and there are at least two copies of 2 in x, let x' be the sequence obtained from x by deleting the first two entries of x and inserting a 1 directly before the second 2 of x. One can check that $x' \in \hat{\mathcal{A}}_{n-1}^{\mathrm{pr}}(1423)$. Note that $x_4 > 2$ as $x_2 = 2$ and $x_3 = 1$. It follows that $x'_2 > 2$. Hence there are s_{n-1} such x in this case.

Case 3. If $x_3 > 1$ and there is exactly one copy of 2 in x, deleting the unique 2 in x and standardizing, gives a sequence in $\hat{\mathcal{A}}_{n-1}^{\mathrm{pr}}(1423)$, Thus, there are d_{n-1} such x in this case.

Case 4. If $x_3 > 1$ and there are at least two copies of 2 in x, let x' be the sequence obtained from x by deleting the first 2 in x and inserting a 1 directly before the second 2 in x. By a similar argument as in Case 2, we deduce that $x' \in \hat{\mathcal{A}}_n^{\mathrm{pr}}(1423)$ with $x'_2 > 2$. Thus, there are s_n such x in this case.

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Combining the above four cases gives (4.7), as desired. □

Given $n \geq 2$, let $x \in \hat{\mathcal{A}}_n^{\mathrm{pr}}(1423)$. There is at least one copy of 2 in x. It yields that x can be decomposed uniquely as $x = 1L2R$ according to the rightmost 2 of x. The following lemma is analogous to lemma 3.2, which we omit the proof here.

Lemma 4.6. *Given $n \geq 2$, let $x \in \hat{\mathcal{A}}_n^{\mathrm{pr}}(1423)$ with the above decomposition as $x = 1L2R$. Then for any entry $a \geq 2$ in L and $b \geq 2$ in R, we have $a \leq b$.*

Lemma 4.7. *For $n \geq 2$, we have*

$$d_n = d_{n-1} + d_{n-2} - r_n + \sum_{2 \leq k \leq n} e_k r_{n+2-k}. \tag{4.8}$$

Proof. It is routine to verify (4.8) for $2 \leq n \leq 4$. Now we assume that $n > 4$. Let $x \in \hat{\mathcal{A}}_n^{\mathrm{pr}}(1423)$ with the above decomposition as $x = 1L2R$. Let k be the length of $L' = 1L2$. There are three cases.

Case 1. $k = 2$. That is L is empty. Note that x_2 is the unique 2 in x. From Case 1 and Case 3 of lemma 4.5, we have $d_{n-1} + d_{n-2}$ such x.

Case 2. $k = n$. That is R is empty. Then x is a sequence in $\hat{\mathcal{A}}_n^{\mathrm{pr}}(1423)$ that ends with 2, and so there are e_n such x.

Case 3. $3 \leq k \leq n-1$. From lemma 4.6, L' is a Cayley permutation. Furthermore, we have $L' \in \hat{\mathcal{A}}_k^{\mathrm{pr}}(1423)$ which ends in a 2. Hence, there are e_k choices for L'. Let $m = \max L'$. We need to consider four cases for R.

(i) If R does not contain m and starts with 1, let $R' = \mathrm{st}(R)$. Then $R' \in \hat{\mathcal{A}}_{n-k}^{\mathrm{pr}}(1423)$ and so there are d_{n-k} such R'.

(ii) If R contains no m and does not start with 1, let $R' = \mathrm{st}(1R)$. Then $R' \in \hat{\mathcal{A}}_{n+1-k}^{\mathrm{pr}}(1423)$ and so there are d_{n+1-k} such R'.

(iii) If R contains m and starts with 1, let R' be the sequence obtained from R by inserting a 1 directly before the leftmost m of R and then standardizing. Note that m is preceded by a larger number in R and the number m in R corresponds to 2 in R'. Thus, the second entry of R' is greater than 2, thereby there are s_{n+1-k} such R' in this case.

(iv) If R contains m and does not start with 1, let R' be the sequence obtained from R by inserting a 1 before R and inserting a 1 directly before the leftmost m in R and then standardizing. Similarly, one can verify that R' is a primitive modified ascent sequence in $\hat{\mathcal{A}}_{n+2-k}^{\mathrm{pr}}(1423)$ with $R'(2) > 2$. It follows that there are s_{n+2-k} such R'.

To conclude, there are $d_{n-k} + d_{n+1-k} + s_{n+1-k} + s_{n+2-k}$ choices for R'.

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Combining the above three cases gives

$$\begin{aligned} d_n &= d_{n-1} + d_{n-2} + e_n + \sum_{3\le k\le n-1} e_k(d_{n-k} + d_{n+1-k} + s_{n+1-k} + s_{n+2-k}) \\ &= d_{n-1} + d_{n-2} + e_n + \sum_{3\le k\le n-1} e_k r_{n+2-k} \quad \text{(by (4.7))} \\ &= d_{n-1} + d_{n-2} - r_n + \sum_{2\le k\le n} e_k r_{n+2-k}, \quad \text{(by the fact } e_2 = r_2 = 1), \end{aligned}$$

as desired. □

Theorem 4.8. *We have*

$$\hat{\mathcal{A}}^{\mathrm{pr}}_{1423}(t) = \frac{-t^4 - 2t^3 + t^2 + 6t + 2 - (t^2+2t)\sqrt{t^4 - 2t^2 - 4t + 1}}{2(t+1)}$$

and

$$\hat{\mathcal{A}}_{1423}(t) = \frac{2t^4 - 6t^3 + 5t^2 + 2t - 2 - (t^2-2t)\sqrt{4t^4 - 12t^3 + 16t^2 - 8t + 1}}{2(t^3 - 3t^2 + 3t - 1)}.$$

Proof. Let $D(t), E(t), R(t)$, and $S(t)$ be the generating functions of d_n, e_n, r_n, and s_n, respectively. Note that $D(t) = \hat{\mathcal{A}}^{\mathrm{pr}}_{1423}(t) - 1$. Multiplying both sides of equalities (4.5)–(4.8) by t^n and summing over all possible n yields

$$\begin{cases} D(t) = E(t) + \dfrac{E(t)}{t}, \\ D(t) = t + R(t) + S(t), \\ R(t) = tD(t) + t^2 D(t) + S(t) + tS(t), \\ D(t) = t + tD(t) + t^2 D(t) - R(t) + \dfrac{E(t)R(t)}{t^2}. \end{cases}$$

Solving the above equations for $D(t)$ gives the desired expression for $\hat{\mathcal{A}}^{\mathrm{pr}}_{1423}(t)$. The expression for $\hat{\mathcal{A}}_{1423}(t)$ then follows from proposition 2.1. □

5. Pattern 1223

This section focuses on the enumeration of the pattern 1223, including both general and primitive cases, and solves a conjecture posed by Cerbai [2]. For the general case, we construct a bijection ϕ between $\hat{\mathcal{A}}_n(1223)$ and $\hat{\mathcal{A}}_n(1232)$. For the primitive case, by restricting the map ϕ to $\hat{\mathcal{A}}^{\mathrm{pr}}_n(1223)$ and characterizing their images, we obtain the generating function of $\hat{\mathcal{A}}^{\mathrm{pr}}_n(1223)$. The following lemma is needed for the construction of ϕ.

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Lemma 5.1. *Let $x \in \hat{\mathcal{A}}_n(1223)$ and let $a \geq 2$ be an integer that occurs at least twice in x. Then either all copies of* a *are consecutive in x or excluding the leftmost* a*, all remaining copies of* a *are consecutive in x.*

Proof. We proceed by contradiction to prove this conclusion. Without loss of generality, assume that the second a and the third a are not consecutive in x. Here we always count all copies of a from left to right. Since x is a modified ascent sequence, we have $\mathrm{Asc}(x) = \mathrm{nub}(x)$. It follows that the third a is preceded by a larger entry, say b. Then $1aab$ is an occurrence of the pattern 1223, a contradiction. This completes the proof. □

Given $x \in \hat{\mathcal{A}}_n$, let $\alpha(x) = (a_1, a_2, \ldots, a_k)$ be the list of integers that are greater than 1 and occur at least twice in x. We list them based on the positions of their leftmost occurrences in x, proceeding from left to right. That is, the leftmost a_i is to the left of the leftmost a_{i+1} in x for $1 \leq i \leq k-1$. Let x be a modified ascent sequence in $\hat{\mathcal{A}}_n(1223)$ with $\alpha(x) = (a_1, a_2, \ldots, a_k)$. Now we proceed to define $\phi(x)$. For $1 \leq i \leq k$, from lemma 5.1, we have two cases for a_i:

- If all occurrences of a_i are consecutive in x, do nothing.
- If there are other numbers between the first and the second occurrence of a_i, move all consecutive occurrences of a_i from the second to the last one behind the first a_i, while preserving the relative positions of all other entries.

Let $\phi(x)$ be the resulting sequence.

Example 5.2. Let $x = 13124155443332211 \in \hat{\mathcal{A}}_n(1223)$. Then we have $\alpha(x) = (3, 2, 4, 5)$ and $\phi(x) = 13333122244415511$ with the following detailed procedure, where the entries coloured in red indicate the entries that are moved at current stage.

$$
\begin{aligned}
x = {} & 13124155443332211 \\
& \downarrow \\
& 13\color{red}{333}124155442211 \\
& \downarrow \\
& 1333312\color{red}{22}41554411 \\
& \downarrow \\
& 133331222\color{red}{44}15511 = \phi(x).
\end{aligned}
$$

It should be mentioned that the definition of ϕ is actually independent of the choice of the order of $a_1, a_2, \ldots, a_k$. We define it this way primarily to facilitate the construction of its inverse map.

Lemma 5.3. *Given $x \in \hat{\mathcal{A}}_n(1223)$ with $\alpha(x) = (a_1, a_2, \ldots, a_k)$, we have $\phi(x) \in \hat{\mathcal{A}}_n(1232)$ with $\alpha(\phi(x)) = (a_1, a_2, \ldots, a_k)$.*

Proof. We first show that $\phi(x)$ is a modified ascent sequence. Without loss of generality, we assume that there are other numbers between the first and the second occurrence of a_1. Let b be the entry preceding the second occurrence of a_1 in x. Let y be the sequence

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obtained from x after the movement of a_1. It is sufficient to show that $\mathrm{Asc}(y) = \mathrm{nub}(y)$, meaning each entry is a leftmost copy of y if and only if it sits at an ascent top of y. Notice that x is a modified ascent sequence, thus $\mathrm{Asc}(x) = \mathrm{nub}(x)$. By the construction of ϕ, each entry is a leftmost copy of y if it is a leftmost copy of x. We need to show each entry is an ascent top of y if and only if it is an ascent top of x, which is easily verified except for the entry, say c, after the rightmost a_1. Since x avoids 1223, we deduce that the entries to the right of the rightmost a_1 in x are smaller than a_1. It follows that c is not an ascent top in x. After movement of a_1, the entry preceding c is b in y. It is easily seen that $b > a_1 > c$ since x is a modified ascent sequence. Hence, c is not an ascent top in y. To conclude, we have $\mathrm{Asc}(y) = \mathrm{nub}(y)$, that is, y is a modified ascent sequence.

By the construction of ϕ, all copies of a_i $(1 \leq i \leq k)$ appear consecutively in $\phi(x)$. This yields that $\phi(x)$ avoids 1232, thereby $\phi(x) \in \hat{\mathcal{A}}_n(1232)$. On the other hand, The order of the leftmost copy of each number in $(a_1, a_2, \ldots, a_k)$ is unchanged under the mapping ϕ. Hence we have $\alpha(\phi(x)) = \alpha(x) = (a_1, a_2, \ldots, a_k)$, as desired. □

From the proof of lemma 5.3, we see that there are some entries larger than a_1 to the right of the rightmost a_1 in y. Furthermore, the entry b is the last entry larger than a_1 in y. Hence we can recover x from y by moving all consecutive occurrences of a_1 from the second to the last one behind the entry b. This enables us to define the inverse map ψ of ϕ. Let y be a modified ascent sequence in $\hat{\mathcal{A}}_n(1232)$ with $\alpha(y) = (a_1, a_2, \ldots, a_k)$. Since y avoids 1232, it is obvious that all occurrences of a_i are consecutive in y. We define $\psi(y)$ as follows. For $i = k, k-1, \ldots, 1$, we consider two cases in each step.

- If there is no entry larger than a_i to the right of a_i, do nothing.
- If there is some entry larger than a_i to the right of a_i, let b be the rightmost such entry. Move all consecutive occurrences of a_i from the second to the last one behind the entry b, while preserving the relative positions of all other entries.

For example, let $y = 13333122244415511 \in \hat{\mathcal{A}}_n(1232)$. Then we have $\psi(y) = 13124155443332211$. See the procedure in example 5.2 backward for an illustration.

Theorem 5.4. *The map ϕ is a bijection between $\hat{\mathcal{A}}_n(1223)$ and $\hat{\mathcal{A}}_n(1232)$.*

Proof. Let y be a modified ascent sequence in $\hat{\mathcal{A}}_n(1232)$ with $\alpha(y) = (a_1, a_2, \ldots, a_k)$ and let $x = \psi(y)$. Since ψ also preserves the order of the leftmost copy of each entry, we have $\alpha(x) = \alpha(y) = (a_1, a_2, \ldots, a_k)$. By the constructions of ϕ and ψ, it is obvious that they are inverses of each other. According to lemma 5.3, to prove the conclusion, it suffices to show that $x = \psi(y) \in \hat{\mathcal{A}}_n(1223)$. By a similar argument as in the proof of lemma 5.3, we have x is a modified ascent sequence. We proceed to show that x avoids 1223. This will be verified by showing that there is no entry larger than a_i to the right of the second a_i in x. We only check this for a_k as the other a_i can be verified similarly. By the construction of ψ, there exists no entry larger than a_k to the right of the second a_k after the movement of a_k. We need to prove that the movement of a_i $(1 \leq i \leq k-1)$ will not affect this property of a_k. It is sufficient to show that the movement of a_{k-1} will not affect this property of a_k. Let y' be the sequence obtained from y after the movement of a_k. Assume that there exist some entries larger than a_{k-1} to the right of the rightmost a_{k-1} in y'. Let b be the rightmost such entry. We consider two cases. If b is to the left of

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the second occurrence of a_k in y', it is obvious that the movement of a_{k-1} will not affect the property of a_k. If b is to the right of the second occurrence of a_k in y' or $b = a_k$, then $b \leq a_k$. After the movement of a_{k-1}, there is still no larger entry to the right of the second occurrence of a_k. This completes the proof. □

The following theorem follows directly from [2, proposition 5.2] and theorem 5.4.

Theorem 5.5. *For $n \geq 1$,*

$$|\hat{\mathcal{A}}_n(1223)| = \sum_{k=1}^{n} \binom{n-1}{k-1} \sum_{j=1}^{k} (j-1)! S(k-j+1, j),$$

where $S(n,k)$ denotes the (n, k)-th Stirling number of the second kind. Furthermore,

$$\hat{\mathcal{A}}_{1223}(t) = \sum_{i \geq 0} \frac{t^i(1-t)}{1-(i+1)t}.$$

The rest of this section is devoted to the enumeration of $\hat{\mathcal{A}}_n^{\mathrm{pr}}(1223)$. We will accomplish this by restricting the map ϕ to $\hat{\mathcal{A}}_n^{\mathrm{pr}}(1223)$ and characterizing their images.

Lemma 5.6. *Let $y \in \hat{\mathcal{A}}_n(1232)$ with $\alpha(y) = (a_1, a_2, \ldots, a_k)$. Then $y \in \phi(\hat{\mathcal{A}}_n^{pr}(1223))$ if and only if for $1 \leq i \leq k$,*

- *there are no two consecutive occurrences of 1 in y;*
- *each a_{i} appears exactly twice and the two occurrences of a_{i} are consecutive in y;*
- *there is some entry larger than a_{i} to the right of the second a_{i} in y.*

Proof. Let $x = \psi(y)$. Since ϕ and ψ are inverses of each other, we have $y = \phi(x)$. From lemma 5.3, $\alpha(x) = (a_1, a_2, \ldots, a_k)$. Assume that $y \in \phi(\hat{\mathcal{A}}_n^{\mathrm{pr}}(1223))$, that is, $x \in \hat{\mathcal{A}}_n^{\mathrm{pr}}(1223)$. Recall that there are no two equal entries which appear consecutively in a primitive sequence. Hence, there are no two consecutive occurrences of 1 in x. Since each 1 is followed by an ascent top which is equivalently a leftmost copy of some number, such ascent top will not be moved under the mapping ϕ. Thus, there are no two consecutive occurrences of 1 in y. Again since x is primitive, by lemma 5.1, each a_i appears exactly twice and the two occurrences of a_i are not consecutive in x. By the construction of ϕ, each a_i appears exactly twice and the two occurrences of a_i are consecutive in y. If there is no entry larger than a_i to the right of the second a_i in y, then by the construction of ψ, these two occurrences of a_i appear consecutively in x, a contradiction. Hence, there is some entry larger than a_i to the right of the second a_i in y.

Conversely, assume that y satisfies the three conditions given in the lemma. We need to show that x is primitive. By a similar reasoning as above, all occurrences of 1 are not consecutive in x. Now it is sufficient to show that all the two occurrences of a_i are not consecutive in x for $1 \leq i \leq k$, which can be easily verified by the construction of ψ. □

Let $y \in \phi(\hat{\mathcal{A}}_n^{\mathrm{pr}}(1223))$ with $\alpha(y) = (a_1, a_2, \ldots, a_k)$. We associate y with a permutation by removing all occurrences of 1 in y, removing one copy of each a_i for $1 \leq i \leq k$ in y

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and decreasing each remaining entry by 1. Let $\theta(y)$ denote the resulting permutation. For example, let $y = 13312244151 \in \phi(\hat{\mathcal{A}}_{12}^{\mathrm{pr}}(1223))$. We have $\theta(y) = 2134$.

Let $\mathcal{S}_n$ denote the set of permutations of $[n]$. Given a permutation $\pi = \pi_1\pi_2\cdots\pi_n \in \mathcal{S}_n$, we say π_i is a *right-to-left maximum* if $\pi_i > \pi_j$ for all $i < j \le n$. Let $\mathrm{rlmax}(\pi)$ denote the number of right-to-left maxima in π and let $p_{n,a,r}$ denote the number of permutations in $\mathcal{S}_n$ with $a+1$ ascents and r right-to-left maxima. We need the following continued fraction expression due to Zeng [21, theorem 3].

Theorem 5.7 (Zeng) *The generating function for $p_{n,a,r}$ is*

$$P(t,u,v) := \sum_{n\ge 0}\sum_{0\le a\le n-1}\sum_{0\le r\le n-a} p_{n,a,r}t^n u^a v^r = \cfrac{1}{1-vt-\cfrac{vut^2}{1-(v+u+1)t-\cfrac{2(v+1)ut^2}{\ddots\cfrac{}{1-(v+n(u+1))t-\cfrac{(n+1)(v+n)ut^2}{\ddots}}}}}.$$

In fact, Zeng [21] obtained a stronger version of theorem 5.7 which contains five classic permutation statistics.

Let $q_{n,a,r}$ denote the number of modified ascent sequences y in $\phi(\hat{\mathcal{A}}_n^{\mathrm{pr}}(1223))$ such that $\theta(y)$ have $a+1$ ascents and r right-to-left maxima.

Theorem 5.8. *We have*

$$Q(t,u,v) := \sum_{n\ge 0}\sum_{0\le a\le n-1}\sum_{0\le r\le n-a} q_{n,a,r}t^n u^a v^r = (1+t)P\left((1+t)t^2, \frac{u(1+t)}{t}, \frac{v}{1+t}\right). \tag{5.1}$$

Proof. Let π be a permutation in $\mathcal{S}_n$ with $a+1$ ascents and r right-to-left maxima. To obtain a modified ascent sequence y in $\phi(\hat{\mathcal{A}}_n^{\mathrm{pr}}(1223))$ such that $\theta(y) = \pi$, by the definition of θ, we first increase each entry in π by one, resulting in a permutation π'. By lemma 5.6, we can choose to replace each entry a in π', where a is not a right-to-left maximum, by aa. It is obvious that there are $n-r$ entries which are not right-to-left maxima in π'. Let π'' be the resulting permutation. Then, add one copy of 1 before each descent bottom of π'' and add one copy of 1 before π''. This operation ensures that the leftmost copy of each entry in the resulting sequence is an ascent top. There are $n-1-a$ descents in π''. Hence, we must add $n-a$ copies of 1 to π''. As for the positions before each ascent top (excluding the position before π'') and the position after π'', we can choose to add one copy of 1 or not to add. There are $a+1$ such positions. To conclude, we have

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$$\begin{aligned} Q(t,u,v) &= \sum_{n\geq 0}\sum_{0\leq a\leq n-1}\sum_{0\leq r\leq n-a} p_{n,a,r}t^n u^a v^r (1+t)^{n-r}t^{n-a}(1+t)^{a+1} \\ &= (1+t)\sum_{n\geq 0}\sum_{0\leq a\leq n-1}\sum_{0\leq r\leq n-a} p_{n,a,r}((1+t)t^2)^n \left(\frac{u(1+t)}{t}\right)^a \left(\frac{v}{1+t}\right)^r \\ &= (1+t)P\left((1+t)t^2, \frac{u(1+t)}{t}, \frac{v}{1+t}\right), \end{aligned}$$

as desired. □

Setting $u = v = 1$ in equation (5.1), we immediately obtain the generating function for $\hat{\mathcal{A}}_n^{\mathrm{pr}}(1223)$.

Theorem 5.9. *We have*

$$\hat{\mathcal{A}}_{1223}^{\mathrm{pr}}(t) = (1+t)P\left((1+t)t^2, \frac{1+t}{t}, \frac{1}{1+t}\right).$$

6. Pattern 1123

In this section, we prove that $\hat{\mathcal{A}}_n^{\mathrm{pr}}(1123)$ and $\hat{\mathcal{A}}_n^{\mathrm{pr}}(1243)$ are equinumerous by showing that they share the same generating function as shown in theorem 4.3. In fact, we obtain the generating function of $\hat{\mathcal{A}}_n^{\mathrm{pr}}(1123)$ with respect to several statistics by using kernel method. Incidentally, we reprove that $\hat{\mathcal{A}}_n(1123)$ is counted by the Catalan numbers and find a new Narayana statistic on $\hat{\mathcal{A}}_n(1123)$.

Cerbai [5] showed that the hat map is a bijection between $\mathcal{A}_n(1123)$ and $\hat{\mathcal{A}}_n(1123)$. By the definition of the hat map, it is easily seen that an ascent sequence is primitive if and only if $\hat{x}$ is primitive. Hence, the hat map also induces a bijection between $\mathcal{A}_n^{\mathrm{pr}}(1123)$ and $\hat{\mathcal{A}}_n^{\mathrm{pr}}(1123)$. This motivates us to consider ascent sequences and primitive ascent sequences that avoid the pattern 1123 in this section. Mansour and Shattuck [16] proved that $|\mathcal{A}_n(1123)| = c_n$, confirming a conjecture of Duncan and Steingrímsson [9]. We will adapt their method to the enumeration of primitive counterpart.

Let us introduce some statistics first. Given an endofunction x, let $\mathrm{fwd}(x)$ be the length of the maximal final weakly decreasing sequence in x and let $\mathrm{one}(x)$ be the number of copies of 1 in x. If x is primitive, let $\mathrm{NS}(x)$ be the set of entries which cannot play the role "1" in any pattern 123 and let $\mathrm{ns}(x) = |\mathrm{NS}(x)|$. For general x, define $\mathrm{ns}(x) = \mathrm{ns}(\tilde{x})$, where $\tilde{x}$ is the primitive sequence obtained from x by merging each the consecutive flat step of x into one single copy. For example, let $x = 135224155126643321$. We have $\mathrm{fwd}(x) = 7$ since 6643321 has length 7 and $\mathrm{ns}(x) = 8$ since the entries colored by red in its corresponding primitive sequence 13**5**24**1**5**1**26**4**3**21** cannot play the role "1" in any pattern 123.

Given $n \geq 1$ and $1 \leq r, l, k \leq n$, let

$$A_{n,r,l,k} := \{x \in \mathcal{A}_n^{\mathrm{pr}}(1123) : \mathrm{one}(x) = r, \mathrm{fwd}(x) = l, \mathrm{ns}(x) = k\}.$$

and let $a_{n,r,l,k} = |A_{n,r,l,k}|$. Let $B_{n,r,l,k} := \{x \in A_{n,r,l,k} : x_n \neq 1\}$ and let $b_{n,r,l,k} = |B_{n,r,l,k}|$.

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Lemma 6.1. *For $n \geq 2$ and $1 \leq r, l, k \leq n$, we have*

$$a_{n,r,l,k} = b_{n,r,l,k} + b_{n-1,r-1,l-1,k-1}.$$

Proof. This follows from the the fact that $|A_{n,r,l,k}| = |B_{n,r,l,k}| + |\{x \in A_{n,r,l,k} : x_n = 1\}|$ and the one-to-one correspondence between $\{x \in A_{n,r,l,k} : x_n = 1\}$ and $B_{n-1,r-1,l-1,k-1}$ by simply removing the last entry 1. □

The array $b_{n,r,l,k}$ is determined by the recurrences as showed in the following lemma.

Lemma 6.2. *The array $b_{n,r,l,k}$ may be non-zero only when $n \geq 2$, $1 \leq r \leq \lfloor \frac{n}{2} \rfloor$, $1 \leq l \leq n+1-2r$, and $1 \leq k \leq n$. Furthermore, $b_{n,r,l,k}$ is determined by the following recurrences:*

$$b_{n,1,l,k} = \sum_{i \geq 1} (b_{n-1,i,l,k} + b_{n-2,i-1,l-1,k-1}), \quad n \geq 3, \tag{6.1}$$

$$b_{n,2,l,k} = \sum_{i \geq 1} \sum_{j \geq l} (b_{n-2,i,j,k-1} + b_{n-3,i-1,j-1,k-2} + b_{n-3,i,j,k-2} + b_{n-4,i-1,j-1,k-3}), \quad n \geq 5, \tag{6.2}$$

$$b_{n,r,l,k} = \sum_{j \geq l+1} b_{n-1,r-1,j,k-1} + \sum_{j \geq l} b_{n-2,r-1,j,k-2}, \quad n \geq 2, r \geq 3, \tag{6.3}$$

and the initial values

$$b_{2,1,l,k} = \begin{cases} 1, & \text{if } l = 1 \text{and } k = 2; \\ 0, & \text{otherwise,} \end{cases}$$

and

$$b_{4,2,l,k} = \begin{cases} 1, & \text{if } l = 1, k = 3 \text{or } l = 1, k = 4; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Given $n \geq 2$, let x be a sequence in $B_{n,r,l,k}$. Since x is primitive and it does not end in a 1, x contains at most $\lfloor \frac{n}{2} \rfloor$ copies of 1. Moreover, the rightmost 1 must occur at the position $2r-1$ or to its right. It follows that the first $2r-1$ entries of x do not contribute to the statistic "fwd". Therefore, we have $\mathrm{fwd}(x) \leq n - (2r-1) = n+1-2r$.

It is routine to check the initial values. We proceed to prove the recurrences of $b_{n,r,l,k}$. If $n \geq 3$ and $r = 1$, by deleting the first entry 1 and decreasing each remaining entry by 1, we obtain a sequence in $A_{n-1,i,l,k}$ for some $i \geq 1$. Hence, $b_{n,1,l,k} = \sum_{i \geq 1} a_{n-1,i,l,k}$. By lemma 6.1, we obtain (6.1).

If $n \geq 5$ and $r = 2$, then the second occurrence of 1 in x must directly precede the first letter of the maximal final weakly decreasing sequence. For if not, there is some ascent to

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the right of the second 1. Then there would be an occurrence of 1123 in x, with the two copies of 1 play the roles "11" and the letters of the rightmost ascent play the roles "23". Now we consider two cases. If the entries immediately before and after the second 1 are not equal in x, deleting the two copies of 1 and decreasing each remaining entry by 1, we obtain a primitive ascent sequence $x' \in A_{n-2,i,j,s}$ for some $i \geq 1$. Since the last l entries are weakly decreasing, we have $j \geq l$. Note that the first occurrence of 1 is not in $\mathrm{NS}(x)$ and the second occurrence of 1 is in $\mathrm{NS}(x)$, which implies that $s = k-1$. Hence there are $\sum_{i\geq 1}\sum_{j\geq l} a_{n-2,i,j,k-1}$ such primitive ascent sequences in this case. If the entries immediately before and after the second 1 are equal in x, deleting one copy of these two corresponding entries in x', we obtain a primitive ascent sequence $x'' \in A_{n-3,i,j,s}$ for some $i \geq 1$. We also have $j \geq l$. Note that the entry we newly delete is in $\mathrm{NS}(x)$, which implies that $s = k-2$. Hence there are $\sum_{i\geq 1}\sum_{j\geq l} a_{n-3,i,j,k-2}$ such primitive ascent sequences in this case. Combining the above two cases and lemma 6.1 results in (6.2).

If $n \geq 2$ and $r \geq 3$, since x avoids 1123, all the entries larger than 1 and to the right of the second 1 are weakly decreasing. And the maximal final weakly decreasing sequence of x is preceded by the rightmost 1. We also have two cases. Suppose the entries immediately before and after the rightmost 1 are not equal in x. Deleting the rightmost 1 of x, we obtain a primitive ascent sequence in $B_{n-1,r-1,j,k-1}$ for some $j \geq l+1$. Suppose the entries immediately before and after the rightmost 1 are equal in x. Deleting the rightmost 1 of x and the entry immediately after it, we obtain a primitive ascent sequence in $B_{n-2,r-1,j,k-2}$ for some $j \geq l$. Combining these two cases gives (6.3). □

Let $B(t,u,v,w) = \sum_{n,r,l,k} b_{n,r,l,k} t^n u^r v^l w^k$, where n, r, l, k range over all possible values such that $b_{n,r,l,k} > 0$. In what follows, for convenience, we always assume the summation variables take over all possible values unless specifically stated otherwise. Given $r \geq 1$, let $B_r(t,v,w) = \sum_{n,l,k} b_{n,r,l,k} t^n v^l w^k$. The polynomial $B_r(t,v,w)$ satisfies the following relations.

Lemma 6.3. *We have*

$$B_1(t,v,w) = t^2vw^2 + (t + t^2vw)B(t,1,v,w), \tag{6.4}$$

$$\begin{aligned} B_2(t,v,w) = {} & t^4vw^4 + \frac{t^2vw}{1-v}(1 + 2tw + t^2w^2)B(t,1,1,w) \\ & - \frac{t^2vw}{1-v}(1 + tvw + tw + t^2vw^2)B(t,1,v,w), \end{aligned} \tag{6.5}$$

and for $r \geq 3$,

$$B_r(t,v,w) = \frac{tvw + t^2vw^2}{1-v}B_{r-1}(t,1,w) - \frac{tw + t^2vw^2}{1-v}B_{r-1}(t,v,w). \tag{6.6}$$

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Proof. If $r=1$, we have

$$\begin{aligned}
B_1(t,v,w) &= \sum_{n,l,k} b_{n,1,l,k} t^n v^l w^k \\
&= t^2vw^2 + \sum_{n\geq 3,l,k} b_{n,1,l,k} t^n v^l w^k \quad \text{(by the initial values in lemma 6.2)} \\
&= t^2vw^2 + \sum_{n\geq 3,i,l,k} (b_{n-1,i,l,k} + b_{n-2,i-1,l-1,k-1}) t^n v^l w^k \qquad \text{(by (6.1))} \\
&= t^2vw^2 + (t+t^2vw)B(t,1,v,w),
\end{aligned}$$

which gives (6.4).

If $r=2$, we have

$$\begin{aligned}
B_2(t,v,w) &= \sum_{n,l,k} b_{n,2,l,k} t^n v^l w^k \\
&= t^4vw^3 + t^4vw^4 + \sum_{n\geq 5,l,k} b_{n,2,l,k} t^n v^l w^k \quad \text{(by the initial values in lemma 6.2)} \\
&= t^4vw^3(1+w) + \sum_{n\geq 5,i,l,k} \sum_{j\geq l} \Big(b_{n-2,i,j,k-1} + b_{n-3,i-1,j-1,k-2} \\
&\qquad + b_{n-3,i,j,k-2} + b_{n-4,i-1,j-1,k-3} \Big) t^n v^l w^k \quad \text{(by (6.2))} \\
&= t^4vw^3(1+w) + \sum_{n\geq 5,i,j,k} \Big(b_{n-2,i,j,k-1} + b_{n-3,i-1,j-1,k-2} \\
&\qquad + b_{n-3,i,j,k-2} + b_{n-4,i-1,j-1,k-3} \Big) t^n w^k \sum_{1\leq l\leq j} v^l \\
&= t^4vw^3(1+w) + \frac{v}{1-v} \sum_{n\geq 5,i,j,k} \Big(b_{n-2,i,j,k-1} + b_{n-3,i-1,j-1,k-2} \\
&\qquad + b_{n-3,i,j,k-2} + b_{n-4,i-1,j-1,k-3} \Big) t^n (1-v^j) w^k \\
&= t^4vw^3(1+w) + \frac{v}{1-v} \Big(t^2w\big[B(t,1,1,w) - t^2w^2 - B(t,1,v,w) + t^2vw^2\big] \\
&\qquad + t^3w^2\big[B(t,1,1,w) - vB(t,1,v,w)\big] \\
&\qquad + t^3w^2\big[B(t,1,1,w) - B(t,1,v,w)\big] \\
&\qquad + t^4w^3\big[B(t,1,1,w) - vB(t,1,v,w)\big] \Big) \\
&= t^4vw^4 + \frac{t^2vw}{1-v}\big(1+2tw+t^2w^2\big)B(t,1,1,w) \\
&\qquad - \frac{t^2vw}{1-v}\big(1+tvw+tw+t^2vw^2\big)B(t,1,v,w),
\end{aligned}$$

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where the last step is obtained by careful calculation and simplification. Hence (6.5) follows.

If $r \geq 3$, we have

$$\begin{aligned}
B_r(t,v,w) &= \sum_{n,l,k} b_{n,r,l,k} t^n v^l w^k \\
&= \sum_{n,l,k} \left(\sum_{j \geq l+1} b_{n-1,r-1,j,k-1} + \sum_{j \geq l} b_{n-2,r-1,j,k-2} \right) t^n v^l w^k \\
&= \sum_{n,j \geq 2,k} b_{n-1,r-1,j,k-1} t^n w^k \sum_{1 \leq l \leq j-1} v^l + \sum_{n,j,k} b_{n-2,r-1,j,k-2} t^n w^k \sum_{1 \leq l \leq j} v^l \\
&= \frac{v}{1-v} \sum_{n,j,k} b_{n-1,r-1,j,k-1} t^n w^k (1 - v^{j-1}) + \frac{v}{1-v} \sum_{n,j,k} b_{n-2,r-1,j,k-2} t^n w^k \\
&\quad \times (1 - v^j) \\
&= \frac{tw}{1-v} \big(v B_{r-1}(t,1,w) - B_{r-1}(t,v,w) \big) + \frac{t^2 v w^2}{1-v} \big(B_{r-1}(t,1,w) \\
&\quad - B_{r-1}(t,v,w) \big) \\
&= \frac{tvw + t^2 v w^2}{1-v} B_{r-1}(t,1,w) - \frac{tw + t^2 v w^2}{1-v} B_{r-1}(t,v,w),
\end{aligned}$$

which gives (6.6). □

The polynomial $B(t,u,v,w)$ satisfies the following functional equation.

Lemma 6.4. *We have*

$$\begin{aligned}
&(1 - v + tuw + t^2 uvw^2) B(t,u,v,w) \\
=&(1-v) t^2 uvw^2 + tu(1-v)(tuw+1)(tvw+1) B(t,1,v,w) \\
&+ tuvw(1+tw) B(t,u,1,w).
\end{aligned} \tag{6.7}$$

Proof. Note that

$$B(t,u,v,w) = \sum_{r \geq 1} B_r(t,v,w) u^r.$$

Then by lemma 6.3, we have

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$$\begin{aligned}B(t,u,v,w) &= t^2uvw^2 + tu(1+tvw)B(t,1,v,w)\\ &\quad + t^4u^2vw^4 + \frac{t^2u^2vw}{1-v}\big(1+2tw+t^2w^2\big)B(t,1,1,w)\\ &\quad - \frac{t^2u^2vw}{1-v}\big(1+tvw+tw+t^2vw^2\big)B(t,1,v,w)\\ &\quad + \frac{tvw+t^2vw^2}{1-v}\sum_{r\geq 3}B_{r-1}(t,1,w)u^r\\ &\quad - \frac{tw+t^2vw^2}{1-v}\sum_{r\geq 3}B_{r-1}(t,v,w)u^r\\ &= t^2uvw^2 + tu(1+tvw)B(t,1,v,w)\\ &\quad + t^4u^2vw^4 + \frac{t^2u^2vw}{1-v}\big(1+2tw+t^2w^2\big)B(t,1,1,w)\\ &\quad - \frac{t^2u^2vw}{1-v}\big(1+tvw+tw+t^2vw^2\big)B(t,1,v,w)\\ &\quad + \frac{tuvw+t^2uvw^2}{1-v}\big(B(t,u,1,w)-uB_1(t,1,w)\big)\\ &\quad - \frac{tuw+t^2uvw^2}{1-v}\big(B(t,u,v,w)-uB_1(t,v,w)\big). \end{aligned} \tag{6.8}$$

By (6.4), we have

$$B_1(t,1,w) = t^2w^2 + (t+t^2w)B(t,1,1,w). \tag{6.9}$$

Substituting (6.4) and (6.9) into (6.8) we obtain (6.7) after simplification. □

Lemma 6.5. *We have*

$$B(t,1,1,w) = \frac{-\big(t^2+t\big)w - t + 1 - \sqrt{(t^4-2\,t^3+t^2)w^2+t^2+2\,(t^3-t)w-2\,t+1}}{2\,(t^2w+t)}. \tag{6.10}$$

Proof. Letting $u=1$ in (6.7) implies

$$\begin{aligned}&(t^3v^2w^2 - t^3vw^2 + t^2v^2w + t^2vw^2 - t^2w + tv + tw - t - v + 1)B(t,1,v,w)\\ &\qquad = (1-v)t^2vw^2 + tvw(1+tw)B(t,1,1,w). \end{aligned} \tag{6.11}$$

We proceed to solve equation (6.11) by using kernel method. Let $v_0 = v_0(t,w)$ satisfy

$$t^3v_0^2w^2 - t^3v_0w^2 + t^2v_0^2w + t^2v_0w^2 - t^2w + tv_0 + tw - t - v_0 + 1 = 0,$$

namely,

$$v_0 = \frac{\big(t^2-t\big)w - t + 1 - \sqrt{(t^4-2\,t^3+t^2)w^2+t^2+2\,(t^3-t)w-2\,t+1}}{2\,t^2w}. \tag{6.12}$$

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(Note that there are two solutions for v_0, we choose the one by the condition $B(0,1,1,1) = 0$.) Letting $v = v_0$ in (6.11), and solving for $B(t,1,1,w)$, gives

$$B(t,1,1,w) = \frac{tw(v_0-1)}{tw+1}.$$

Substituting (6.12) into the above expression and simplifying implies (6.10), as desired. □

To obtain the expression for $B(t,u,v,w)$, we still need to find expressions for $B(t,1,v,w)$ and $B(t,u,1,w)$. By (6.11), we have

$$B(t,1,v,w) = \frac{(1-v)t^2vw^2 + tvw(1+tw)B(t,1,1,w)}{t^3v^2w^2 - t^3vw^2 + t^2v^2w + t^2vw^2 - t^2w + tv + tw - t - v + 1}. \tag{6.13}$$

To find an expression for $B(t,u,1,w)$, we again use kernel method. Let $v_1 = v_1(t,u,w)$ satisfy

$$1 - v_1 + tuw + t^2uv_1w^2 = 0,$$

namely,

$$v_1 = \frac{tuw+1}{1-t^2uw^2}.$$

Letting $v = v_1$ in (6.7), and solving for $B(t,u,1,w)$ gives

$$B(t,u,1,w) = \frac{(v_1-1)tv_1w^2 + (v_1-1)(tuw+1)(tv_1w+1)B(t,1,v_1,w)}{v_1w(1+tw)}. \tag{6.14}$$

It then follows from (6.7) that

$$B(t,u,v,w) = \frac{(1-v)t^2uvw^2 + tu(1-v)(tuw+1)(tvw+1)B(t,1,v,w)}{(1-v+tuw+t^2uvw^2)} \tag{6.15}$$

where $B(t,1,v,w)$ and $B(t,u,1,w)$ are given by (6.13) and (6.14), respectively.

The following theorem follows directly from lemma 6.1.

Theorem 6.6. *Let* $A(t,u,v,w) := \sum_{n,r,l,k} a_{n,r,l,k}t^nu^rv^lw^k$. *Then,*

$$A(t,u,v,w) = 1 + tuvw + (1+tuvw)B(t,u,v,w),$$

where $B(t,u,v,w)$ *is given by (*6.15*).*

The following result shows that the sequence $\{|\hat{\mathcal{A}}_n^{\mathrm{pr}}(1123)|\}_{n\geq 0}$ matches A082582 in the OEIS [18], as was conjectured by Cerbai [2].

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Theorem 6.7. *We have*

$$A(t,1,1,w) = \frac{(t^2-t)w + t + 1 - \sqrt{(t^4-2t^3+t^2)w^2 + t^2 + 2(t^3-t)w - 2t + 1}}{2t}.$$

In particular,

$$\mathcal{A}^{\mathrm{pr}}_{1123}(t) = \hat{\mathcal{A}}^{\mathrm{pr}}_{1123}(t) = \frac{1 + t^2 - \sqrt{1 - 4t + 2t^2 + t^4}}{2t} \tag{6.16}$$

and so

$$|\hat{\mathcal{A}}^{\mathrm{pr}}_n(1243)| = |\mathcal{A}^{\mathrm{pr}}_n(1123)| = |\hat{\mathcal{A}}^{\mathrm{pr}}_n(1123)|. \tag{6.17}$$

Proof. The first statement follows from lemma 6.5 and theorem 6.6. Letting $w = 1$ in $A(t,1,1,w)$ gives (6.16). Comparing (4.3) with (6.16) yields (6.17). □

Theorem 6.8. *Let*

$$N(t,w) := \sum_{n\geq 0} t^n \sum_{x\in\mathcal{A}_n(1123)} w^{\mathrm{ns}(x)}.$$

Then,

$$N(t,w) = \frac{1 + t - tw - \sqrt{t^2w^2 + t^2 - 2(t^2+t)w - 2t + 1}}{2t}. \tag{6.18}$$

Proof. Any ascent sequence in $\mathcal{A}(1123)$ can be obtained uniquely from some primitive ascent sequence $y \in \mathcal{A}^{\mathrm{pr}}(1123)$ by inserting proper flat steps immediately after the entries in $\mathrm{NS}(y)$. In terms of generating function, this gives

$$N(t,w) = A\left(t, 1, 1, \frac{w}{1-t}\right).$$

The expression (6.18) for $N(t,w)$ then follows from theorem 6.7. □

Remark 6.9. Mansour and Shattuck [16] proved that the statistic "asc" on $\mathcal{A}_n(1123)$ is a Narayana distribution, that is

$$|\{x \in \mathcal{A}_n(1123) : \mathrm{asc}(x) = k+1\}| = \frac{1}{n}\binom{n}{k+1}\binom{n}{k}.$$

Theorem 6.8 shows that the statistic "ns" on $\mathcal{A}_n(1123)$ is a new Narayana distribution.

Theorem 6.10. *For any $x \in \mathcal{A}_n(1123)$, we have*

$$(\mathrm{asc}, \mathrm{one}, \mathrm{fwd}, \mathrm{ns})(x) = (\mathrm{asc}, \mathrm{one}, \mathrm{fwd}, \mathrm{ns})(\hat{x}).$$

Consequently, all results stated in this section for 1123*-avoiding (primitive) ascent sequences are also true for* 1123*-avoiding (primitive) modified ascent sequences.*

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Proof. It can be easily checked that the hat map preserves the statistics "asc", "one" and "fwd". We need to show that it also preserves the statistic "ns". By the definition of the hat map and the definition of statistic "ns", it suffices to prove the result for primitive case. Now we assume that x is primitive. For $1 \leq i < j \leq n$, By the definition of hat map, if $x_i \geq x_j$, then $\hat{x}_i \geq \hat{x}_j$. It yields that if $\hat{x}_i\hat{x}_j\hat{x}_k$ is an occurrence of the pattern 123 in $\hat{x}$, then $x_ix_jx_k$ is also an occurrence of 123 in x. Thus, if $x_i \in \mathrm{NS}(x)$, then $\hat{x}_i \in \mathrm{NS}(\hat{x})$

Conversely, if $x_i \notin \mathrm{NS}(x)$, then there are $i < j < k$ such that $x_ix_jx_k$ forms an occurrence of the pattern 123 in x. We claim that $x_1x_2\cdots x_{i+1} = 12\cdots i(i+1)$. If not, there is some $1 \leq s \leq i+1$ such that $x_s \neq s$. We choose the smallest such s. If $s = i+1$, then $x_1x_2\cdots x_i = 12\cdots i$ and $x_s \neq i+1$. Since x is an ascent sequence, we derive $x_s \leq i$. Then $x_{x_s}x_sx_jx_k$ forms an occurrence of 1123 in x, a contradiction. If $s = i$, with a similar argument as above, we obtain that $x_{x_s}x_sx_jx_k$ forms an occurrence of 1123 in x, a contradiction. If $2 \leq s < i$, then $x_1x_2\cdots x_{s-1} = 12\cdots(s-1)$ and $x_s < s$. Since x avoids 1123, we have $s-1 < x_i < x_j$. For if $x_i \leq s-1$, then $x_{x_i}x_ix_jx_k$ forms an occurrence of the pattern 1123 in x, a contradiction. Hence, we have $x_s \leq s-1 < x_i$. It follows that $x_{x_s}x_sx_jx_k$ is an occurrence of the pattern 1123 in x, a contradiction. This proves the claim.

Choose $j = i+1$ and choose k to be the smallest integer such that $x_ix_jx_k$ forms an occurrence of the pattern 123 in x. For $i+1 \leq l < k-1$, we have $x_l \geq x_{l+1}$. For if not, $x_{x_l}x_lx_{l+1}x_k$ forms an occurrence of the pattern 1123 in x, a contradiction. Then by the definition of the hat map, $\hat{x}_i\hat{x}_j\hat{x}_k$ is also an occurrence of the pattern 123 in $\hat{x}$, which implies $\hat{x}_i \notin \mathrm{NS}(\hat{x})$. To conclude, we have $\mathrm{ns}(x) = \mathrm{ns}(\hat{x})$, as desired. □

7. Final remarks and future directions

In section 3, we proved that both $\hat{\mathcal{A}}_n^{\mathrm{pr}}(1324)$ and $\hat{\mathcal{A}}_n^{\mathrm{pr}}(1342)$ are counted by the Catalan numbers by using generating functions. It is interesting to find a bijection between $\hat{\mathcal{A}}_n^{\mathrm{pr}}(1324)$ and $\hat{\mathcal{A}}_n^{\mathrm{pr}}(1342)$ or bijections between these two sets and other known Catalan objects. We failed to find any statistic on $\hat{\mathcal{A}}_n^{\mathrm{pr}}(1324)$ or $\hat{\mathcal{A}}_n^{\mathrm{pr}}(1342)$ that interprets the Narayana numbers. Thus, we pose the following challenging open problem for further research.

Problem 7.1. *Find some statistics on* $\hat{\mathcal{A}}_n^{\mathrm{pr}}(1324)$ *and* $\hat{\mathcal{A}}_n^{\mathrm{pr}}(1342)$ *which have Narayana distribution.*

In sections 4.1 and 6, we proved that both $\hat{\mathcal{A}}_n^{\mathrm{pr}}(1243)$ and $\hat{\mathcal{A}}_n^{\mathrm{pr}}(1123)$ are equinumerous with the set of Dyck paths of semilength n with no **uudd**. Are there bijections among these three sets?

Given two patterns σ and τ, they are said to be *strong-Wilf-equivalent* if both $|\hat{\mathcal{A}}_n^{\mathrm{pr}}(\sigma)| = |\hat{\mathcal{A}}_n^{\mathrm{pr}}(\tau)|$ and $|\hat{\mathcal{A}}_n(\sigma)| = |\hat{\mathcal{A}}_n(\tau)|$ hold for any $n \geq 1$. We denote the strong-Wilf-equivalence of σ and τ by $\sigma \sim \tau$.

Conjecture 7.2. *For* $k \geq 1$, *we have*

$$2\underbrace{1\ldots1}_{k} \sim 1\underbrace{2\ldots2}_{k}3,$$

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where $\underbrace{a\ldots a}_{k}$ *denotes the sequence that contains* k *copies of* a.

The case $k=1$ of conjecture 7.2 have been verified by Cerbai [2]. In the same paper, Cerbai conjectured that $211 \sim 1223$ which corresponds $k=2$ in conjecture 7.2. Note that we obtain the enumeration of $\hat{\mathcal{A}}_n^{\mathrm{pr}}(1223)$ and $\hat{\mathcal{A}}_n(1223)$ in section 5. Hence, confirming of conjecture 7.2 will lead to the enumeration of $\hat{\mathcal{A}}_n^{\mathrm{pr}}(211)$ and $\hat{\mathcal{A}}_n(211)$, which is one of the two unsolved patterns of length 3. The other unsolved pattern of length 3 is 111.

Conjecture 7.3. *For $k \geq 1$, we have*

$$\underbrace{2\ldots2}_{k}1 \sim 12\underbrace{3\ldots3}_{k}.$$

When $k=1$, conjectures 7.2 and 7.3 are the same, which yields that conjecture 7.3 is valid for $k=1$. When $k=2$, Cerbai [2] proved that $\hat{\mathcal{A}}_n^{\mathrm{pr}}(221)$ is counted by the Bell numbers. Hence, confirming of conjecture 7.3 will yield that $\hat{\mathcal{A}}_n^{\mathrm{pr}}(1233)$ is also counted by the Bell numbers. Cerbai [3] obtained that (primitive) modified ascent sequence avoiding a large number of patterns are counted by the Bell numbers. Numerical evidence shows that 1233 is the last pattern of length less than or equal to 4 that could potentially be counted by the Bell numbers.

Conjecture 7.4. *For $k \geq 2$, all the patterns in the set* $\{\underbrace{2\ldots2}_{i}1\underbrace{2\ldots2}_{k-i} \mid 1 \leq i \leq k-1\}$ *are strong-Wilf-equivalent.*

Conjecture 7.5. *For $k \geq 2$, all the patterns in the set* $\{\underbrace{1\ldots1}_{i}2\underbrace{1\ldots1}_{k-i} \mid 1 \leq i \leq k-1\}$ *are strong-Wilf-equivalent.*

All the conjectures above have been verified up to $k=5$ and $n=10$. Are there any other families of strong-Wilf-equivalence classes?

Parallel to the works in [14, 15, 17] for patterns in inversion sequences or ascent sequences, it would be interesting to investigate systematically (primitive) modified ascent sequences avoiding triples of relations or avoiding vincular patterns.

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