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Characteristic classes for permutation representations

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1. *Introduction*

To a finite-dimensional real representation V of a finite group G there are associated its Stiefel–Whitney classes $w_k(V)$ $(k = 1, 2, 3, \ldots)$ in the cohomology groups $H^k(G; \mathbb{F}_2)$. ($\mathbb{F}_2$ is the field with two elements.) The total Stiefel–Whitney class

$$w(V) = 1 + w_1(V) + w_2(V) + \ldots$$

in the ring $H^*(G; \mathbb{F}_2)$ is natural with respect to G in the obvious sense, and, in addition,

(*a*) *exponential*, i.e. $w(V \oplus W) = w(V).w(W)$,

and

(*b*) *stable*, i.e. $w(V) = 1$ when V is a trivial representation.

In particular one can take $V = \mathbb{R}^S$, the permutation representation associated to a finite G-set S. Then $S \mapsto w(S) = w(\mathbb{R}^S)$ is a characteristic class defined for G-sets. It is exponential in the sense that $w(S \amalg T) = w(S).w(T)$, where $\amalg$ denotes disjoint union.

In this note we shall present a general method of constructing stable exponential characteristic classes for G-sets S. In general the classes we construct do not depend solely on the group-representation $\mathbb{R}^S$, and can distinguish between G-sets which give rise to the same representation. Such a class

$$S \mapsto a(S) \in 1 + \prod_{k>0} H^k(G; F),$$

where F is a commutative ring of coefficients and k runs through only even integers unless F has characteristic 2, defines a homomorphism

$$a\colon A(G) \to 1 + \prod_{k>0} H^k(G; F),$$

where $A(G)$ is the *Burnside ring* of S ((4), (8)), i.e. the Grothendieck ring of finite G-sets under the operation $\amalg$, which additively is the free abelian group on the set of isomorphism classes of transitive G-sets.

The simplest case of the construction is

Proposition (1·1). *For any integer $m > 0$ there is a stable exponential class*

$$a^{(m)}(S) = 1 + a_1^{(m)}(S) + a_2^{(m)}(S) + \ldots$$

defined for G-sets S, with $a_k^{(m)}(S) \in H^{k(2^m-1)}(G; \mathbb{F}_2)$. It is characterized by the properties:

(i) $a^{(m)}(S) = 1$ *if* $\mathrm{card}\,(S) < 2^m$,

(ii) $a^{(m)}(S) = 1 + w_{2^m-1}(S)$ *if* $\mathrm{card}\,(S) = 2^m$,

(iii) $a_k^{(m)}(S) = 0$ *if* $\mathrm{card}\,(S) < k.2^m$.

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(Here property (i) is written down simply for emphasis: it follows from either (ii) or (iii). Notice that if $m = 1$ then $a_k^{(m)} \in H^k(G; \mathbb{F}_2)$, like the Stiefel–Whitney class w_k; the w_k satisfy (i) and (ii) but not (iii).)

The corresponding statement for odd primes is

PROPOSITION (1·1)′. *For any prime p and any integer $m > 0$ there is a stable exponential characteristic class*

$$b^{(m)}(S) = 1 + b_1^{(m)}(S) + b_2^{(m)}(S) + \dots$$

defined for G-sets S, with $b_k^{(m)}(S) \in H^{2k(p^m-1)}(G; \mathbb{F}_p)$. It has the properties:

(i) $b^{(m)}(S) = 1$ *if card* $(S) < p^m$,

(ii) $b^{(m)}(S) = 1 + c_{p^m-1}(S)$ *if card* $(S) = p^m$, *where c_{p^m-1} is the $(p^m - 1)$th Chern class reduced modulo p,*

(iii) $b_k^{(m)}(S) = 0$ *if card* $(S) < k.p^m$.

In this case we do not show (although it is true) that the properties characterize the classes. One can certainly characterize them by prescribing their values on the regular representations of the elementary abelian p-groups $A_n = \mathbb{F}_p^n$. The values are:

$$b^{(m)}(A_n) = 1 + c_{p^n-p^{n-m}}(A_n).$$

If we define $b = \Pi_{m \geq 1} b^{(m)}$ we obtain a new exponential class. Let $\tilde{A}(G)$ denote the augmentation ideal in $A(G)$, consisting of all virtual G-sets of cardinal 0. The following result is a simple consequence of (1·1).

PROPOSITION (1·2). *The homomorphism*

$$b: \tilde{A}(G) \to 1 + \prod_{k>0} H^{2k}(G; \mathbb{F}_p)$$

is injective if G is an elementary abelian p-group.

This proposition provides a slightly different proof of a theorem of Laitinen(1), as we shall now explain.

The usual approach to the Stiefel–Whitney classes of group representations is to identify $H^*(G)$ with the cohomology of its classifying space BG. A real representation V of G defines a real vector bundle E_V on BG, and E_V has a Stiefel–Whitney class $w(E_V)$ in the usual sense (2). We define $w(V)$ to be $w(E_V)$.

Even more simply, a finite G-set S defines a finite covering space $Y_S \to BG$. (Indeed the category of finite G-sets is equivalent to the category of finite covering spaces of BG.) The problem of defining stable exponential characteristic classes for G-sets (for all finite groups G) is precisely equivalent to that of defining stable exponential characteristic classes in $H^*(X)$ for finite covering spaces $Y \to X$ (for all para-compact spaces X): for any n-sheeted covering of a paracompact space X can be got by pulling back the standard n-sheeted covering of $B\Sigma_n$, where Σ_n is the symmetric group on n letters, by a map $X \to B\Sigma_n$.

If Cov (X) denotes the monoid of isomorphism classes of finite covering spaces of X under disjoint union then the theorem of Barratt–Priddy–Quillen relating $B\Sigma_\infty$ to the loop-spaces of spheres, in the version given in (5), asserts that any stable exponential characteristic class

$$a: \text{Cov}\,(X) \to h^*(X),$$

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with values in any multiplicative cohomology theory h^*, factorizes canonically through the stable cohomotopy $\sigma^0(X)$ of X. Thus the classes we are constructing can be thought of as exponential transformations $\sigma^0 \to H^*$. In particular the classes a and b of (1·1) and (1·1)′ factorize canonically

$$A(G) \to \sigma^0(BG) \to 1 + \Pi H^k(G).$$

This, together with (1·2), gives Laitinen's result (1):

COROLLARY (1·3). *$A(G) \to \sigma^0(BG)$ is injective when G is an elementary abelian p-group.*

Our method constructs characteristic classes in any multiplicative generalized cohomology theory which admits power operations in the usual sense (which is recalled below). The general result is Proposition (3·1). In (7) it is applied to complex cobordism to obtain a stronger result than (1·3), namely that $A(G) \to \sigma^0(BG)$ is injective whenever G is abelian.

2. *Power operations*

Suppose that h^* is a multiplicative generalized cohomology theory. We shall say that h^* has *power operations* in dimension q if for each n the n-fold external power map $h^q(X) \to h^{nq}(X^n)$ factorizes naturally

$$h_q(X) \xrightarrow{P^n} h^{nq}(E_n X) \xrightarrow{\text{Res}} h^{nq}(X^n),$$

where $E_n X = E\Sigma_n \times_{\Sigma_n} X^n$ is the fibre bundle over $B\Sigma_n$ with fibre X^n, associated to the permutation action of Σ_n on X^n. 'Res' is the restriction map to the fibre X^n in $E_n X$.

The maps P^n, besides being natural, will normally have appropriate behaviour on sums and products. But the only property we shall need in the present work is the following one of compatibility for various n. Notice that if $p+q = n$ the product $E_p X \times E_q X$ is a subspace of $E_n X$. We require

$$(P^n\xi)|E_p X \times E_q X = (P^p\xi) \otimes (P^q\xi).$$

(Here $\otimes$ denotes the external product in cohomology.)

3. *Construction of the classes*

In this section h^* will be any multiplicative cohomology theory which has power operations in dimension r. For any finite group G we shall write $h^*(G)$ for $h^*(BG)$.

Our main result is

PROPOSITION 3·1. *Let α be an element of $h^r(\Sigma_d)$ whose restriction to $\Sigma_i \times \Sigma_{d-i}$ vanishes when $0 < i < d$. Then there is canonically associated to α a stable exponential characteristic class*

$$a = 1 + a_1 + a_2 + \ldots,$$

with $a_k \in h^{kr}$, whose value on the standard permutation representation of Σ_d is $1 + \alpha \in h^*(\Sigma_d)$. Furthermore

(i) *$a(S) = 1$ if card $(S) < d$, and*

(ii) *$a_k(S) = 0$ if card $(S) < kd$.*

A more explicit restatement of this is

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PROPOSITION (3·2). *Beginning with* $\alpha \in h^r(\Sigma_d)$ *as above one can construct classes* $a_k^{(n)} \in h^{kr}(\Sigma_n)$ *such that*

(i) $a_k^{(n)} | \Sigma_m = a_k^{(m)}$ *if* $m \leqslant n$,

(ii) $a_k^{(n)} | \Sigma_p \times \Sigma_q = \sum_{i=0}^{k} a_i^{(p)} \otimes a_{k-i}^{(q)}$ *if* $p+q = n$,

(iii) $a_1^{(d)} = \alpha$,

(iv) $a_k^{(n)} = 0$ *if* $kd > n$.

In general we do not know that the properties listed characterize the classes.

The first step of the construction is as follows. Beginning with $\alpha \in h^r(\Sigma_d)$ such that $\alpha | \Sigma_i \times \Sigma_{d-i} = 0$ when $0 < i < d$, we define $\alpha_n \in h^{nr}(\Sigma_{nd})$ by $\alpha_n = \operatorname{tr} P^n \alpha$, where

$$P^n : h^r(\Sigma_d) \to h^{nr}(\Sigma_n \wr \Sigma_d)$$

is the nth power, and tr is the transfer associated to $\Sigma_n \wr \Sigma_d \to \Sigma_{nd}$. Then we have

PROPOSITION (3·3).

$$\alpha_n | \Sigma_i \times \Sigma_{nd-i} = 0 \quad \text{if } i \text{ is not a multiple of } d,$$

$$= \alpha_s \otimes \alpha_{n-s} \quad \text{if} \quad i = sd.$$

Proof. We use the 'Mackey formula', which asserts that if H and K are subgroups of G and $\alpha \in h^*(H)$ then

$$(\operatorname{tr}_{H,G} \alpha) | K = \sum_{\sigma} \operatorname{tr}_{H_\sigma, K}(\alpha | H_\sigma),$$

where $\operatorname{tr}_{A,B}$ denotes the transfer corresponding to an injection $A \to B$, and

$$H_\sigma = (g_\sigma^{-1} K g_\sigma) \cap H,$$

and $\{g_\sigma\}$ is a set of representatives for the double coset space $K \backslash G / H$.

In the present case $K = \Sigma_i \times \Sigma_{nd-i}$ and $H = \Sigma_n \wr \Sigma_d$, and $K \backslash G / H$ can be thought of as the set of ways of putting i red balls and $nd-i$ blue balls into n indistinguishable baskets each containing d balls. Thus $K \backslash G / H$ is the set of sequences $\mathbf{j} = (j_0, \ldots, j_d)$ such that $\Sigma j_k = n$ and $\Sigma k j_k = i$, where j_k is the number of baskets containing k red balls. The corresponding $H_{\mathbf{j}}$ is $\Pi_k \Sigma_{j_k} \wr (\Sigma_k \times \Sigma_{d-k})$. Now

$$(P^n \alpha) | H_{\mathbf{j}} = \prod_k P^{j_k} (\alpha | \Sigma_k \times \Sigma_{d-k}).$$

This vanishes unless $j_k = 0$ when $0 < k < d$. In that case $i = ds$, where $s = j_d$, and $H_{\mathbf{j}} = (\Sigma_s \wr \Sigma_d) \times (\Sigma_{n-s} \wr \Sigma_d)$, and $(P^n \alpha) | H_{\mathbf{j}} = (P^s \alpha) \otimes (P^{n-s} \alpha)$. Thus

$$(\operatorname{tr} P^n \alpha) | \Sigma_{sd} \times \Sigma_{(n-s)d} = \operatorname{tr}((P^s \alpha) \otimes (P^{n-s} \alpha)) = \alpha_s \otimes \alpha_{n-s}.$$

Let us now define $a_n^{(m)} \in h^{nr}(\Sigma_m)$ for $m \geqslant nd$ by $a_n^{(m)} = \operatorname{tr}(\alpha_n \otimes 1)$, where the transfer is from $\Sigma_{nd} \times \Sigma_{m-nd}$ to Σ_m. If $m < nd$ we define $a_n^{(m)} = 0$.

PROPOSITION (3·4). $a_n^{(m)} | \Sigma_l = a_n^{(l)}$ *when* $l \leqslant n$.

Proof. It follows from (3·3) that $\alpha_n | \Sigma_l = 0$ when $l < nd$, so it is enough to consider the case $m \geqslant l \geqslant nd$. Using the Mackey formula again, we have to consider

$$\Sigma_l \backslash \Sigma_m / \Sigma_{nd} \times \Sigma_{m-nd}.$$

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This is the set of ways of choosing nd things from m when l of the m are indistinguishable. If we pick s of the indistinguishables the corresponding H_σ is

$$\Sigma_s \times \Sigma_{l-s} \subset \Sigma_{nd} \times \Sigma_{m-nd}.$$

Thus $(\alpha_n \otimes 1)|H_\sigma = 0$ unless $s = nd$, in which case

$$H_\sigma = \Sigma_{nd} \times \Sigma_{l-nd} \quad \text{and} \quad (\alpha_n \otimes 1)|H_\sigma = \alpha_n \otimes 1.$$

So
$$a_n^{(m)}|\Sigma_l = \operatorname{tr}(\alpha_n \otimes 1) = a_n^{(l)},$$

as we want.

The proof of (3·2) will now be complete when we prove

PROPOSITION (3·5).

$$a_n^{(m)}|\Sigma_p \times \Sigma_q = \sum_{s=0}^{n} a_s^{(p)} \otimes a_{n-s}^{(q)},$$

when $p+q = m$.

Proof. For this we have to consider the intersections of $\Sigma_{nd} \times \Sigma_{m-nd}$ with the conjugates of $\Sigma_p \times \Sigma_q$. These are indexed by r such that $0 \leqslant r \leqslant \inf(nd, p, q-nd)$, and the corresponding H_r is $(\Sigma_r \times \Sigma_{nd-r}) \times (\Sigma_{p-r} \times \Sigma_{m-nd-p+r})$. The only relevant ones are those where $r = sd$: then $(\alpha_n \otimes 1)|H_{sd} = (\alpha_s \otimes 1) \otimes (\alpha_{n-s} \otimes 1)$. Thus

$$a_n^{(m)}|(\Sigma_p \times \Sigma_q) = \sum_s \operatorname{tr}((\alpha_s \otimes 1) \otimes (\alpha_{n-s} \otimes 1)) = \sum_s a_s^{(p)} \otimes a_{n-s}^{(q)}.$$

4. *Euler classes*

If E is a real n-dimensional vector bundle on a space X we shall denote the unit disk and unit sphere bundles contained in E (for some choice of metric) by D_E and S_E. An *orientation* (or *Thom class*) of E for a cohomology theory h^* is an element $\eta_E \in h^n(D_E, S_E)$ which is a free generator of $h^*(D_E, S_E)$ as an $h^*(X)$-module. The restriction of η_E to the zero-section of E is called the *Euler class* $e(E) \in h^n(X)$ of E. From the exact sequence

$$h^n(D_E, S_E) \to h^n(X) \to h^n(S_E)$$

(arising because $D_E \simeq X$) we see that $e(E) = 0$ if E has a non-vanishing cross-section.

Suppose now that all real vector bundles have a canonical orientation for the theory h^*. Then we can define an Euler class $e(V) \in h^n(G)$ for any representation of a group G on an n-dimensional real vector space V. The standard representation of Σ_n on $\mathbb{R}^n$ contains the trivial representation $\mathbb{R}$: let the complementary $(n-1)$-dimensional representation be V_n. It is immediate that $e(V_n) \in h^{n-1}(\Sigma_n)$ has the property of proposition (3·1) that $e(V_n)|(\Sigma_k \times \Sigma_{n-k}) = 0$ when $0 < k < n$, for $V_n|(\Sigma_k \times \Sigma_{n-k}) \cong V_k \oplus V_{n-k} \oplus \mathbb{R}$.

For classical cohomology modulo 2 all real bundles have a unique orientation, and it is well known (8) that $e(V)$ is the Stiefel–Whitney class $w_n(V)$ when $\dim(V) = n$. This gives us the classes of Proposition (1·1).

A more frequent situation is for all complex vector bundles to be oriented. If $V_n^{\mathbb{C}}$ is the complexification of V_n then $e(V_n^{\mathbb{C}})$ also has the property of (3·1). For classical cohomology modulo p the class $e(W)$ is the Chern class $c_n(W)$ for a complex n-dimensional representation. This gives us the classes of (1·1)′.

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There is no point in considering Euler classes of degrees other than those of (1·1) and (1·1)′ because of the general fact

PROPOSITION (4·1). *If the theory h^* is p-local then $e(V_n)$ and $e(V_n^{\mathbb{C}})$ are zero unless $n = p^k$ for some k. (A p-local theory is one such that $h^*(X)$ is a $\mathbb{Z}_{(p)}$-module, where $\mathbb{Z}_{(p)}$ is $\mathbb{Z}$ localized at p.)*

Proof. Let m be the largest power of p less than n. Then the index of $\Sigma_m \times \Sigma_{n-m}$ in Σ_n is $\binom{n}{m}$, which is not divisible by p. So $h^*(\Sigma_n) \to h^*(\Sigma_m \times \Sigma_{n-m})$ is injective. But $e(V_n)$ and $e(V_n^{\mathbb{C}})$ belong to the kernel unless $m = n$.

5. *Uniqueness of the classes in classical cohomology*

In this section H^* denotes classical cohomology with coefficients in the field $\mathbb{F}_p$.

An exponential characteristic class is a sequence of elements $a^{(n)} \in H^*(\Sigma_n)$. We can think of $a^{(n)}$ as an additive homomorphism $a^{(n)}\colon H_*(\Sigma_n) \to \mathbb{F}_p$. If we introduce the ring $H_*(\Sigma) = \bigotimes_{n \geqslant 0} H_*(\Sigma_n)$ in which the product is induced by the juxtaposition maps $\Sigma_m \times \Sigma_n \to \Sigma_{m+n}$ then the $a^{(n)}$ fit together to define a map $H_*(\Sigma) \to \mathbb{F}_p$. We have

PROPOSITION (5·1). *An exponential characteristic class in H^* is precisely a ring homomorphism $a\colon H_*(\Sigma) \to \mathbb{F}_p$ which is the identity on $H_0(\Sigma_1) = \mathbb{F}_p$.*

The elementary abelian p-group $A_n = \mathbb{F}_p^n$ is naturally a subgroup of Σ_{p^n} by its regular representation. The following result is well known (cf. (3), (6), (3a)).

PROPOSITION (5·2). *The ring $H_*(\Sigma)$ is generated by the images of $H_*(A_n) \to H_*(\Sigma_{p^n})$.*

COROLLARY (5·3). *An exponential characteristic class a in H^* is determined by the sequence $\{a(A_n) \in H^*(A_n)\}$ of its values on the regular representations of the A_n.*

In Σ_{p^n} the subgroup A_n is normalized by $G_n = GL(n, \mathbb{F}_p)$, so $a(A_n)$ must lie in the invariant part $H^*(A_n)^{G_n}$. Furthermore, under the restriction map $H^*(A_n) \to H^*(A_{n-1})$ the class $a(A_n)$ maps to $a(A_{n-1})^p$. When $p = 2$ it can be shown conversely that any sequence of elements $f_n \in H^*(A_n)^{G_n}$ such that $f_n | A_{n-1} = f_{n-1}^p$ arises from a characteristic class, but we do not need that here.

If $p = 2$ it is known (6) that $H^*(A_n)^{G_n}$ is a polynomial algebra $\mathbb{F}_2[\xi_{n,1}, \ldots, \xi_{n,n}]$ on generators $\xi_{n,k}$ of dimensions $2^n - 2^{n-k}$. The restriction $H^*(A_n)^{G_n} \to H^*(A_{n-1})^{G_{n-1}}$ takes $\xi_{n,k}$ to $(\xi_{n-1,k})^2$ if $k < n$, and takes $\xi_{n,n}$ to 0. This means that the kernel of the restriction is generated by $\xi_{n,n}$, and is injective in dimensions $< 2^n - 1$.

Now suppose that $p = 2$ and that $\tilde{a}$ is a characteristic class satisfying the conditions of (1·1). To prove $\tilde{a} = a^{(m)}$ we must show that $\tilde{a}(A_n) = a^{(m)}(A_n)$ for each n. When $n \leqslant m$ this holds by condition (ii). But when $n > m$ the class $\tilde{a}(A_n)$ vanishes above dimensions $2^n - 2^{n-m}$ by condition (iii), so it cannot involve $\xi_{n,n}$, and it is completely determined by $\tilde{a}(A_{n-1})$. Thus $\tilde{a}(A_n) = a^{(m)}(A_n)$ for all n by induction.

6. *Proof of proposition* (1·2)

Let us denote the multiplicative group $1 + \Pi_{k>0} H^{2k}(G; \mathbb{F}_p)$ by $M(G)$.

The proof is by induction on the rank n on the elementary abelian p-group $G \cong \mathbb{F}_p^n$.

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Suppose that $b\colon A(H) \to M(H)$ is injective when H is a proper subgroup of G. Laitinen has pointed out ((1) (3·7)) that the kernel of the restriction $A(G) \to \Pi_H A(H)$, where H runs through the proper subgroups of G, consists of the multiples of a certain element

$$\xi_G = \sum_{k=0}^{n} (-1)^k p^{\frac{1}{2}(k-1)(k-2)} \sigma_k,$$

where σ_k is the sum of the distinct G-sets with p^k elements. From the diagram

$$\begin{array}{ccc} \mathbb{Z}\xi_G & \xrightarrow{b} & M_0 \\ \downarrow & & \downarrow \\ A(G) & \xrightarrow{b} & M(G) \\ \downarrow & & \downarrow \\ \Pi A(H) & \xrightarrow{b} & \Pi M(H), \end{array}$$

where M_0 is the kernel of $M(G) \to \Pi M(H)$, we see that it is enough to show that $b(\xi_G)^n \neq 0$ for all n.

The cohomology ring $H^*(G)$ is the tensor product of a polynomial algebra

$$\mathbb{F}_p[x_1, \ldots, x_n]$$

on 2-dimensional classes and an exterior algebra. The image of b is contained in the part of $M(G)$ consisting of formal power series in $x_1, \ldots, x_n$. As this is torsion-free it is enough to show $b(\xi_G) \neq 0$. If we write $\xi_G = \Sigma(-1)^k n_k \sigma_k$ then we must show

$$\Pi_{k \text{ odd}} b(\sigma_k)^{n_k} \neq \Pi_{k \text{ even}} b(\sigma_k)^{n_k} \qquad (*)$$

in $\mathbb{F}_p[x_1, \ldots, x_n]$.

Now $\mathbb{F}_p[x_1, \ldots, x_n]$ is a unique factorization domain, and one side of (*) is divisible by

$$b^{(n)}(\sigma_n) = 1 + c_{p^n-1}(G) = 1 + \Pi\alpha,$$

where α runs through the non-zero linear expressions $\lambda_1 x_1 + \ldots + \lambda_n x_n$ with $\lambda_i \in \mathbb{F}_p$. Because $(p-1)! = -1$ in $\mathbb{F}_p$, and

$$1 - X^{p-1} = (1-X)(1-2X)\ldots(1-(p-1)X),$$

the last expression is

$$\prod_{\lambda=1}^{p-1} (1 - \lambda \Pi' \alpha),$$

where $\Pi'\alpha$ means the product of one α from each equivalence class under scalar multiplication (i.e. α runs through the projective space of the dual of G). But $1 - \lambda\Pi'\alpha$ is clearly irreducible in $\mathbb{F}_p[x_1, \ldots, x_n]$, and does not divide any element of

$$\mathbb{F}_p[x_1, \ldots, x_{n-1}],$$

or more generally any element in the image of $H^*(\tilde{G}) \to H^*(G)$, where $\tilde{G}$ is a proper quotient of G. So $1 - \lambda\Pi'\alpha$ does not divide $b(S)$ when card $(S) < p^n$, and so does not divide $b(\sigma_k)$ when $k < n$. This establishes the inequality (*), and completes the proof.

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