Proceedings of the Royal Society of Edinburgh: Section A Mathematics, 1–35, *2025*
DOI:10.1017/prm.2025.10038

Variational method for fractional Hamiltonian system in bounded domain

Weimin Zhang
School of Mathematical Sciences, Zhejiang Normal University, Jinhua, Zhejiang Province, P.R. China and
School of Mathematical Sciences, Key Laboratory of Mathematics and Engineering Applications (Ministry of Education) & Shanghai Key Laboratory of PMMP, East China Normal University, Shanghai, P.R. China (zhangweimin2021@gmail.com)

(Received 30 December 2024)

Here we consider the following fractional Hamiltonian system

$$\begin{cases} (-\Delta)^s u = H_v(u,v) & \text{in } \Omega, \\ (-\Delta)^s v = H_u(u,v) & \text{in } \Omega, \\ u = v = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $s \in (0,1)$, $N > 2s$, $H \in C^1(\mathbb{R}^2, \mathbb{R})$, and $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain. To apply the variational method for this problem, the key question is to find a suitable functional setting. Instead of usual fractional Sobolev spaces, we use the solution space of $(-\Delta)^s u = f \in L^r(\Omega)$ for $r \geq 1$, for which we show the (compact) embedding properties. When H has subcritical and superlinear growth, we construct two frameworks, respectively with the interpolation space method and the dual method, to show the existence of nontrivial solution. As byproduct, we revisit the fractional Lane–Emden system, i.e. $H(u,v) = \frac{1}{p+1}|u|^{p+1} + \frac{1}{q+1}|v|^{q+1}$, and consider the existence, uniqueness of (radial) positive solutions under subcritical assumption.

Keywords: fractional Laplacian; Hamiltonian system; variational method; Lane-Emden system; interpolation space; Legendre–Fenchel transform

2020 *Mathematics Subject Classification:* 35A15; 35J50; 58E05

1. Introduction and main results

In present paper, we are interested in the following fractional system of Hamiltonian type:

$$\begin{cases} (-\Delta)^s u = H_v(u,v) & \text{in } \Omega, \\ (-\Delta)^s v = H_u(u,v) & \text{in } \Omega, \\ u = v = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.1}$$

where $s \in (0,1)$, $N > 2s$, $H \in C^1(\mathbb{R}^2, \mathbb{R})$, and $\Omega \subset \mathbb{R}^N$ is a bounded domain. We will not focus on the regularity condition of the domain Ω, and say simply that Ω

© The Author(s), 2025. Published by Cambridge University Press on behalf of The Royal Society of Edinburgh

https://doi.org/10.1017/prm.2025.10038 Published online by Cambridge University Press

is smooth, even most results work with $C^{1,1}$ boundary. The study of system (1.1) is mainly motivated by the following classical Hamiltonian system:

$$\begin{cases} -\Delta u = H_v(u,v) & \text{in } \Omega, \\ -\Delta v = H_u(u,v) & \text{in } \Omega, \\ u = v = 0 & \text{in } \partial\Omega. \end{cases} \tag{1.2}$$

Formally the energy functional of (1.2) is

$$\mathcal{L}(u,v) = \int_\Omega \nabla u \nabla v dx - \int_\Omega H(u,v) dx. \tag{1.3}$$

A crucial question is to decide on which space we will consider the functional $\mathcal{L}$. Let us look at the famous Lane–Emden system where $H(u,v) = \frac{|u|^{p+1}}{p+1} + \frac{|v|^{q+1}}{q+1}$, $p,q>0$, i.e.

$$\begin{cases} -\Delta u = |v|^{q-1}v & \text{in } \Omega, \\ -\Delta v = |u|^{p-1}u & \text{in } \Omega, \\ u = v = 0 & \text{in } \partial\Omega. \end{cases} \tag{1.4}$$

It is well known that the existence of positive solutions to (1.4) on a ball is decided by the position of (p,q) with respect to the critical hyperbola

$$p,q>0, \quad \frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N}. \tag{1.5}$$

A naive choice of the functional space for $\mathcal{L}$ is $H_0^1(\Omega)^2$. However, the Sobolev embedding greatly restricts the growth of H, and we could only work with $\max(p,q) \le \frac{N+2}{N-2}$, hence many other choices of H were eliminated. Another difficulty is to deal with the strong indefiniteness of the quadratic part in $\mathcal{L}$, which is neither bounded from below nor from above on any subspace of $H_0^1(\Omega)^2$ with finite codimension.

To include more choices of H as for (1.4), people thought about functional spaces without symmetry in u and v, such as $W_0^{1,t}(\Omega) \times W_0^{1,\frac{t}{t-1}}(\Omega)$ with $t>1$. But a new difficulty occurs since this choice is not a Hilbert space when $t \neq 2$, which prevents us from applying the linking theory due to Benci and Rabinowitz [3].

As far as we are aware, Hulshof and van der Vorst [24] first used the interpolation space framework to deal with the system with $H(u,v) = F(u) + G(v)$, that is

$$\begin{cases} -\Delta u = g(v) & \text{in } \Omega, \\ -\Delta v = f(u) & \text{in } \Omega, \\ u = v = 0 & \text{in } \partial\Omega, \end{cases} \tag{1.6}$$

where $F(t) = \int_0^t f(\tau)d\tau$, $G(t) = \int_0^t g(\tau)d\tau$. They replaced the first integral in (1.3) by

$$\int_\Omega \widetilde{A}^t u \widetilde{A}^{2-t} v dx \quad \text{with } \ t \in (0,2),$$

https://doi.org/10.1017/prm.2025.10038 Published online by Cambridge University Press

where $\widetilde{A}^t : \widetilde{H}^t(\Omega) \to L^2(\Omega)$ is a family of interpolation operators and $\widetilde{H}^t(\Omega)$ is a family of interpolation spaces between $L^2(\Omega)$ and $H^2(\Omega) \cap H_0^1(\Omega)$. More precisely, for any

$$p, q > 0, \quad \frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2}{N}, \tag{1.7}$$

there exist some $t \in (0, 2)$ such that the embedding

$$\widetilde{H}^t(\Omega) \times \widetilde{H}^{2-t}(\Omega) \subset L^{p+1}(\Omega) \times L^{q+1}(\Omega)$$

is compact, which yields the existence of nontrivial solution to (1.6) under following conditions

- (Subcritical) $f(u) = O(|u|^p), g(v) = O(|v|^q)$ as $|u|, |v| \to \infty$ with $p, q > 1$ satisfying (1.7);
- (Superlinear) $f(u) = o(|u|)$, $g(v) = o(|v|)$ as $|u|, |v| \to 0$;
- (AR condition) $f, g \in C(\mathbb{R})$, $f(0) = g(0) = 0$, and there exist $\gamma > 2$, $R > 0$ such that

$$0 < \gamma F(u) \leq uf(u), \quad 0 < \gamma G(v) \leq vg(v), \quad \forall\, |u|, |v| \geq R.$$

This generalized clearly the study of Lane–Emden system (1.4).

Later, de Figueiredo and Felmer also applied in [19] the interpolation space method to handle (1.2), with more general coupled H where p, q satisfy (1.7),

$$0 \leq H(u, v) \leq C(|u|^{p+1} + |v|^{q+1}),$$

and H is superlinear at 0, that is

$$C(|u|^{p+1} + |v|^{q+1}) \leq H(u, v) \;\text{ with }\; pq > 1, |u| + |v| < r \text{ for some } r > 0.$$

In addition, Ambrosetti–Rabinowitz ((AR) for short) type condition (as $(H2)$ below) was assumed. They proved that (1.2) admits a nontrivial solution if

$$(N-4)\max\{p, q\} < N + 4.$$

Remark that to work with more general coupled H, we need to restrict the upper bound of p, q.

Clément and van der Vorst in [15] proposed another approach to study (1.2), their idea is to take advantage of the dual method developed by Clarke, Ekeland, and Temam [16, 21]. Here the nonlinearity H is assumed strictly convex, subcritical (see (1.7)) and superlinear at 0. This dual method will transform (1.2) into a problem where the energy functional possesses a mountain pass structure. In fact,

https://doi.org/10.1017/prm.2025.10038 Published online by Cambridge University Press

consider the Legendre–Fenchel transform of H (see [21], [37, chapter I, section 6]):

$$H^*(x) = \sup_{w\in\mathbb{R}^2}\big\{\langle w,x\rangle - H(x)\big\} \quad \text{for any } x\in\mathbb{R}^2,$$

one can obtain solutions to (1.2) by critical points of

$$\mathcal{J}(u,v) = \int_\Omega H^*(u,v)dx - \int_\Omega v\mathcal{A}udx, \quad \forall\,(u,v)\in L^{1+\frac{1}{p}}(\Omega)\times L^{1+\frac{1}{q}}(\Omega),$$

where $\mathcal{A}$ is the inverse of

$$-\Delta : W^{2,1+\frac{1}{p}}(\Omega)\cap W_0^{1,1+\frac{1}{p}}(\Omega)\to L^{1+\frac{1}{p}}(\Omega). \tag{1.8}$$

If moreover H satisfies (AR) type condition (similar to $(H5)$ and $(H7)$ below), the existence of nontrivial solution to (1.2) was shown in [15]. This dual method was also used to handle system with critical growth, see Hulshof et al. [25].

Some other approaches exist. de Figueiredo, do Ó and Ruf [18] used Orlicz space to obtain nontrivial solutions of (1.6), they replaced $W_0^{1,t}(\Omega)\times W_0^{1,\frac{t}{t-1}}(\Omega)$ by Sobolev–Orlicz space $W_0^1L_A(\Omega)\times W_0^1L_{\widetilde{A}}(\Omega)$, where A is an N-function and $\widetilde{A}$ is its Young-conjugate. Owing to the fact that this setting is not a Hilbert space, they used a finite-dimensional approximation method. Their models also contain nonlinearites with nearly critical growth. The Lyapunov–Schmidt reduction approach was also applied to problem (1.6), see for instance [32, 33]. For more literature on this topic, we refer to [6] and references therein.

Coming back to the special case (1.4). In [14], the existence of positive solution to (1.4) was firstly considered based on a topological method. When $p,q>0$ and $pq<1$, the uniqueness of positive solution was investigated in [17]. In [5], (1.4) was reduced to the following single equation

$$\begin{cases}\Delta\big(|\Delta u|^{\frac{1}{p}-1}\Delta u\big) = |u|^{q-1}u & \text{in } \Omega,\\ u=\Delta u=0 & \text{in } \partial\Omega.\end{cases} \tag{1.9}$$

If $p,q>0$, $pq\neq 1$ and subcritical as in (1.7), then (1.4) admits a positive classical ground state solution (it has minimal energy among all solutions), see [5]. If Ω is a ball, we can use the Schwartz rearrangement to show that the ground state solution is radially symmetric. Furthermore, (1.4) has no positive solutions, if Ω is star-shaped and $p,q>0$ do not satisfy (1.7), see [29].

To the best of our knowledge, for the fractional Laplacian case, although some special cases such as fractional Lane–Emden systems were studied (see [26] and references therein), there exists no study of (1.1) for general coupled subcritical nonlinearities so far. As mentioned above, a key step to handle (1.1) with the variational approach is to establish a suitable functional framework. Furthermore, in the Laplacian case, regardless of interpolation method, dual method, or reduced into a single equation, one needs the isomorphism given by (1.8). In the fractional Laplacian case, we need to find suitable functional space which plays the role of $W^{2,1+\frac{1}{p}}(\Omega)\cap W_0^{1,1+\frac{1}{p}}(\Omega)$.

https://doi.org/10.1017/prm.2025.10038 Published online by Cambridge University Press

1.1. Weak solution and fractional spaces

Let Ω be a smooth bounded domain and $s \in (0,1)$, we denote

$$X_0^s(\Omega) := \left\{ u \in L^2(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \text{ and } \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy < \infty \right\},$$

endowed with the norm

$$\|u\|_{X_0^s(\Omega)} := \left(\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy \right)^{1/2}.$$

The embedding $X_0^s(\Omega) \hookrightarrow L^r(\Omega)$ is continuous for $r \in [1, \frac{2N}{N-2s}]$ and compact for $r \in [1, \frac{2N}{N-2s})$, see [20, Theorems 6.5, 7.1]. $(-\Delta)^s$ is an operator from $X_0^s(\Omega)$ into its dual space, namely

$$\langle (-\Delta)^s u, v \rangle = \int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}} dxdy, \quad \forall\, u, v \in X_0^s(\Omega).$$

Formally the energy functional associated to (1.1) is

$$\mathcal{K}(u,v) = \int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}} dxdy - \int_\Omega H(u,v) dx. \tag{1.10}$$

As mentioned before, the crucial question is to find a suitable functional space to work with $\mathcal{K}$, and $X_0^s(\Omega)^2$ would not be the right one if we hope to handle more general function H.

Next we recall the regularity result for the linear equation

$$(-\Delta)^s u = f \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^N \setminus \Omega. \tag{1.11}$$

We denote

$$\delta(x) := \operatorname{dist}(x, \partial\Omega), \quad x \in \Omega,$$

and

$$C_\delta^\alpha(\overline{\Omega}) := \left\{ u \in C(\overline{\Omega}) : \frac{u}{\delta^s} \text{admits a continuous extension belonging to } C^\alpha(\overline{\Omega}) \right\}.$$

Ros-Oton and Serra in [35, Proposition 1.1, Theorem 1.2] proved that

LEMMA 1.1. *Let Ω be a bounded $C^{1,1}$ domain and $f \in L^\infty(\Omega)$. There exists a unique $u \in X_0^s(\Omega)$ solving (1.11). Moreover $u \in C^s(\overline{\Omega}) \cap C_\delta^\alpha(\overline{\Omega})$, and*

$$\|u\|_{C^s(\overline{\Omega})} + \left\| \frac{u}{\delta^s} \right\|_{C^\alpha(\overline{\Omega})} \le C \|f\|_\infty$$

for some $0 < \alpha < \min\{s, 1-s\}$. The constants α, C depend only on Ω and s.

https://doi.org/10.1017/prm.2025.10038 Published online by Cambridge University Press

Here we shall consider a weaker solution (see definition below) and choose the test function space as

$$\mathcal{T}_s(\Omega) := \{u \in X_0^s(\Omega) : (-\Delta)^s u \in C_c^\infty(\Omega)\}.$$

Hence $\mathcal{T}_s(\Omega) \subset C^s(\overline{\Omega}) \cap C_\delta^\alpha(\overline{\Omega})$. Let

$$L^1(\Omega; \delta^s dx) := \left\{u \in L^1_{loc}(\Omega) : \int_\Omega |u|\delta^s dx < \infty\right\}$$

be endowed with the norm $\|u\|_{L^1(\Omega;\delta^s dx)} = |u\delta^s|_1$. In this paper, $|\cdot|_r$ denotes always the norm of $L^r(\Omega)$.

Definition 1.2. *Let $s \in (0,1)$, $N > 2s$ and $f \in L^1(\Omega; \delta^s dx)$. We say that u is a L^1-weak solution to (1.11) if $u \in L^1(\Omega)$, and*

$$\int_\Omega u(-\Delta)^s \varphi dx = \int_\Omega f\varphi dx, \quad \forall\, \varphi \in \mathcal{T}_s(\Omega). \tag{1.12}$$

Note that similar notions were given in [27, 34]. We will prove in proposition 2.3 that for any $f \in L^1(\Omega; \delta^s dx)$, there exists a unique L^1-weak solution to (1.11). We also give the comparison principle and maximum principle for fractional Laplacian in L^1-weak sense, see lemmas 2.1 and 2.2. Accordingly, we define

Definition 1.3. *For $s \in (0,1)$ and $N > 2s$, (u,v) is said a L^1-weak solution to system (1.1) if $u, v \in L^1(\Omega)$, $H_u(u,v), H_v(u,v) \in L^1(\Omega; \delta^s dx)$ and for any $(\varphi, \psi) \in \mathcal{T}_s(\Omega) \times \mathcal{T}_s(\Omega)$,*

$$\int_\Omega u(-\Delta)^s \varphi dx = \int_\Omega H_v(u,v)\varphi dx, \quad \int_\Omega v(-\Delta)^s \psi dx = \int_\Omega H_u(u,v)\psi dx.$$

If in addition $u, v \in L^\infty(\Omega)$, we call (u,v) a classical solution.

Another important functional space for us is the set of u such that $(-\Delta)^s u \in L^r(\Omega)$, namely for $r \geq 1$,

$$\mathcal{W}^{2s,r}(\Omega) := \{u \in L^1(\Omega) : \exists\, f \in L^r(\Omega) \text{ such that (1.12) holds true}\} \tag{1.13}$$

endowed with the norm

$$\|u\|_{\mathcal{W}^{2s,r}(\Omega)} = |(-\Delta)^s u|_r.$$

We shall use $\mathcal{W}^{2s,r}(\Omega)$ to play the role of $W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega)$ in the Laplacian case. In particular, we denote

$$\mathcal{H}^{2s}(\Omega) = \mathcal{W}^{2s,2}(\Omega).$$

Some (compact) embedding properties of $\mathcal{W}^{2s,r}(\Omega)$ will be shown in proposition 2.6.

https://doi.org/10.1017/prm.2025.10038 Published online by Cambridge University Press

1.2. Fractional Hamiltonian system

Motivated by [19, 24], we apply firstly the interpolation space method to study the system (1.1). For $0 \leq \alpha \leq 2s$, consider the interpolation space

$$E^\alpha := \Big\{ u = \sum_{j\geq 1} a_j \varphi_j \in L^2(\Omega) : \sum_{j\geq 1} \lambda_j^{\frac{\alpha}{s}} a_j^2 < \infty \Big\}, \tag{1.14}$$

where λ_j is the j-th eigenvalue of $(-\Delta)^s$ with corresponding eigenfunction φ_j, and $\{\varphi_j\}$ forms an orthonormal basis of $L^2(\Omega)$. For $\alpha \in (0, 2s)$, let $A^\alpha : E^\alpha \to L^2(\Omega)$ be given by

$$A^\alpha u := \sum_{j\geq 1} \lambda_j^{\frac{\alpha}{2s}} a_j \varphi_j \in L^2(\Omega), \quad \forall\, u = \sum_{j\geq 1} a_j \varphi_j \in E^\alpha. \tag{1.15}$$

E^α is clearly a Hilbert space with the scalar product

$$(u, v)_{E^\alpha} = \int_\Omega A^\alpha u A^\alpha v dx = \sum_{j\geq 1} \lambda_j^{\frac{\alpha}{s}} \langle u, \varphi_j \rangle_{L^2} \langle v, \varphi_j \rangle_{L^2}.$$

To handle the Hamiltonian system (1.1), we define

$$\mathbf{E}_\alpha := E^\alpha \times E^{2s-\alpha}, \quad \alpha \in (0, 2s).$$

Instead of considering $\mathcal{K}$ over $X_0^s(\Omega)^2$, we consider the energy functional

$$\mathcal{E}(u, v) = \int_\Omega A^\alpha u A^{2s-\alpha} v dx - \int_\Omega H(u, v) dx, \quad \forall\, (u, v) \in \mathbf{E}_\alpha. \tag{1.16}$$

The choice of $\mathbf{E}_\alpha$ originates from three observations.

- $\mathcal{H}^{2s}(\Omega) = E^{2s}$, seeing proposition 3.1;
- E^α can be embedded compactly into some $L^r(\Omega)$, seeing proposition 2.6 and remark 3.2;
- Every critical point of $\mathcal{E}$ is a L^1-weak solution to (1.1), seeing proposition 3.4.

Applying a linking theorem in [22], we show the following existence result, which extends the study for the Laplacian case in [19].

THEOREM 1.4. *Let $N > 2s$ and $p, q > 0$ satisfy*

$$1 > \frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2s}{N}, \tag{1.17}$$

and

$$(N - 4s) \max\{p, q\} < N + 4s. \tag{1.18}$$

Assume that $H \in C^1(\mathbb{R}^2, \mathbb{R})$ satisfies

(H 1) $H \geq 0$ in $\mathbb{R}^2$;

https://doi.org/10.1017/prm.2025.10038 Published online by Cambridge University Press

(H 2) There exists $R > 0$ such that

$$\frac{1}{p+1}H_u(u,v)u+\frac{1}{q+1}H_v(u,v)v \geq H(u,v) > 0, \quad \forall\, |u|+|v| \geq R; \tag{1.19}$$

(H 3) There exist $r > 0$, $C > 0$ such that

$$|H(u,v)| \leq C(|u|^{p+1} + |v|^{q+1}), \quad \forall\, |u| + |v| \leq r; \tag{1.20}$$

(H 4) There exists $C > 0$ such that for any $(u,v) \in \mathbb{R}^2$,

$$|H_u(u,v)| \leq C\Big(|u|^p+|v|^{\frac{p(q+1)}{p+1}}+1\Big), \quad |H_v(u,v)| \leq C\Big(|v|^q+|u|^{\frac{q(p+1)}{q+1}}+1\Big). \tag{1.21}$$

Then, there exists a nontrivial classical solution to (1.1).

Note that theorem 1.4 requires the assumption (1.18). For getting the existence of solutions to (1.1) in a more broad range of p, q, we will also apply the dual method. Three major difficulties subsist.

- The first problem is still to construct a suitable functional framework. We choose (see §4) the energy functional as follows

$$\mathcal{J}(f,g) = \mathcal{H}^*(f,g) - \int_\Omega g\mathcal{A}f dx, \quad \forall\, (f,g) \in X := L^{1+\frac{1}{p}}(\Omega) \times L^{1+\frac{1}{q}}(\Omega)$$

  where $\mathcal{A}$ is the inverse of $(-\Delta)^s : \mathcal{W}^{2s,1+\frac{1}{p}}(\Omega) \to L^{1+\frac{1}{p}}(\Omega)$ and $\mathcal{H}^*$ denotes the Legendre–Fenchel transform of

$$\mathcal{H}(u,v) = \int_\Omega H(u,v)dx, \quad \forall\, (u,v) \in X^* = L^{p+1}(\Omega) \times L^{q+1}(\Omega).$$

- Secondly, we need to verify the differentiability of $\mathcal{J}$, and the correspondence between critical points of $\mathcal{J}$ and solutions to (1.1). It should be mentioned that these arguments were not proven explicitly in [15] for the Laplacian case.
- Finally, we need to check the compact embedding properties of $\mathcal{W}^{2s,1+\frac{1}{p}}(\Omega)$ in order to check the Palais–Smale condition.
- Finally, we need to check the compact embedding properties of $\mathcal{W}^{2s,1+\frac{1}{p}}(\Omega)$ in order to check the Palais–Smale condition.

To deal with these difficulties, we use properties of Legendre–Fenchel transform (see lemma 2.9) to ensure the well-definedness of $\mathcal{J}$. For the differentiability of $\mathcal{J}$, we will prove in lemma 4.5 that

$$\mathcal{H}^*(f,g) = \int_\Omega H^*(f,g)dx, \quad \forall\, (f,g) \in X,$$

where H^* is the Legendre–Fenchel transform of $H : \mathbb{R}^2 \to \mathbb{R}$. We use the fact $(\nabla H)^{-1} = \nabla H^*$ (see lemma 4.3) which guarantees that $\mathcal{H}^*$ is well defined and of

https://doi.org/10.1017/prm.2025.10038 Published online by Cambridge University Press

class C^1 over X. Moreover, under the superlinear growth assumption of H, $\mathcal{J}$ has a mountain pass geometry, and $H_z^*(f,g) := \nabla H^*(f,g)$ provides a weak solution to (1.1) given any critical point $(f,g) \in X$ for $\mathcal{J}$, see proposition 4.7. The compact embeddings of $\mathcal{W}^{2s,1+\frac{1}{p}}(\Omega)$ are given in proposition 2.6.

THEOREM 1.5. *Let $N > 2s$, $p, q > 0$ satisfy (1.17). Assume that $H \in C^1(\mathbb{R}^2, \mathbb{R})$ satisfies*

(H 5) There exist positive numbers C_1, C_2 such that

$$C_1|u|^{p+1} \le H_u(u,v)u \le C_2\left(|u|^{p+1} + |u|^{\alpha}|v|^{\beta}\right),$$
$$C_1|v|^{q+1} \le H_v(u,v)v \le C_2\left(|v|^{q+1} + |u|^{\alpha}|v|^{\beta}\right),$$

with

$$\frac{\alpha}{p+1} + \frac{\beta}{q+1} = 1, \quad \alpha, \beta > 1; \tag{1.22}$$

(H 6) ∇H is strictly monotone, i.e.

$$\langle \nabla H(u_1,v_1) - \nabla H(u_2,v_2),\ (u_1,v_1) - (u_2,v_2)\rangle > 0, \quad \text{for any disjoint } (u_1,v_1),\ (u_2,v_2) \in \mathbb{R}^2;$$

(H 7) There are $\theta \in (0,1)$ and positive numbers C_3, C_4 such that

$$\theta H_u(u,v)u + (1-\theta)H_v(u,v)v - H(u,v) \ge C_3\left(|u|^{p+1} + |v|^{q+1}\right) - C_4.$$

Then there exists a nontrivial classical solution to (1.1).

REMARK 1.6. $(H5)$ implies indeed $(H1)$ and $(H4)$. $(H7)$ is useful to prove that every Palais–Smale sequence of $\mathcal{J}$ is bounded, see lemma 4.8. Without loss of generality, we can assume $H(0,0) = 0$. Otherwise, we replace $H(u,v)$ by $H(u,v) - H(0,0)$.

REMARK 1.7. The nonlinearities H in theorems 1.4 and 1.5 both have subcritical and superlinear growth. But some differences exist between the two families of assumptions. For example, let H have the form

$$H_\varepsilon(u,v) = |u|^{p+1} + |v|^{q+1} + \varepsilon|u|^{\alpha}|v|^{\beta}, \tag{1.23}$$

where p, q satisfy (1.17), and α, β satisfy (1.22). When (1.18) holds true, H_ε satisfies $(H1)$–$(H4)$ for all $\varepsilon > 0$. On the other hand, we need not (1.18) in theorem 1.5, but the strictly convexity assumption $(H6)$ fails for H_ε when ε is sufficiently large. In other words, theorem 1.4 holds for more broad coupling nonlinearities but requires narrow choices of p, q; theorem 1.5 can work for all subcritical and superlinear p, q, meanwhile the strict convexity is more restrictive for the coupling term.

https://doi.org/10.1017/prm.2025.10038 Published online by Cambridge University Press

1.3. Fractional Lane–Emden system

As a special example, we revisit the fractional Lane–Emden system

$$\begin{cases} (-\Delta)^s u = |v|^{q-1}v & \text{in } \Omega, \\ (-\Delta)^s v = |u|^{p-1}u & \text{in } \Omega, \\ u = v = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.24}$$

where $s \in (0,1)$, $p, q \in (0,\infty)$, $N > 2s$. As for the classical Laplacian case, we consider subcritical exponents p, q, that is

$$p, q > 0, \quad \frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2s}{N}. \tag{1.25}$$

Leite and Montenegro [26] showed the existence of positive viscosity solutions to (1.24) under the subcritical condition (1.25). They reduced (1.24) to a single equation and considered energy functional on $W_0^{1,1+\frac{1}{q}}(\Omega) \cap W^{2s,1+\frac{1}{q}}(\Omega)$. Choi and Kim [12, 13] studied the related problems with respect to spectral fractional Laplacian.

Different from [26], we work with the functional space $\mathcal{W}^{2s,1+\frac{1}{q}}(\Omega)$. Comparing with [26], our setting avoid many regularity problems since every $\mathcal{W}^{2s,1+\frac{1}{q}}(\Omega)$ solution is naturally a L^1-weak solution. More precisely, we consider

$$\begin{cases} (-\Delta)^s \left(|(-\Delta)^s u|^{\frac{1}{q}-1}(-\Delta)^s u \right) = |u|^{p-1}u & \text{in } \Omega, \\ u = (-\Delta)^s u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.26}$$

Definition 1.8. *We call* u *an energy solution of* (1.26)*, if* $u \in \mathcal{W}^{2s,1+\frac{1}{q}}(\Omega)$ *and*

$$\int_\Omega |(-\Delta)^s u|^{\frac{1}{q}-1}(-\Delta)^s u(-\Delta)^s \varphi dx = \int_\Omega |u|^{p-1}u\varphi dx, \quad \forall\, \varphi \in \mathcal{W}^{2s,1+\frac{1}{q}}(\Omega). \tag{1.27}$$

Note that the above terms are well defined since $(-\Delta)^s u \in L^{1+\frac{1}{q}}(\Omega)$ for any $u \in \mathcal{W}^{2s,1+\frac{1}{q}}(\Omega)$. Putting $v = |(-\Delta)^s u|^{\frac{1}{q}-1}(-\Delta)^s u$, we can prove that (u, v) is a classical solution to (1.24) if and only if u is an energy solution to (1.26), see proposition 5.1.

Moreover, solutions to (1.26) coincide with critical points of the following C^1 functional in $\mathcal{W}^{2s,1+\frac{1}{q}}(\Omega)$,

$$\mathcal{I}(u) := \frac{q}{q+1} |(-\Delta)^s u|_{1+\frac{1}{q}}^{1+\frac{1}{q}} - \frac{1}{p+1}|u|_{p+1}^{p+1}.$$

Consider the Nehari manifold associated to $\mathcal{I}$,

$$\mathcal{N}_\mathcal{I} := \{u \in \mathcal{W}^{2s,1+\frac{1}{q}}(\Omega) \setminus \{0\} : \langle \mathcal{I}'(u), u\rangle = 0\}, \tag{1.28}$$

and the ground state level is defined as

$$c_\mathcal{I} := \inf_{u \in \mathcal{N}_\mathcal{I}} \mathcal{I}(u). \tag{1.29}$$

https://doi.org/10.1017/prm.2025.10038 Published online by Cambridge University Press

We establish the existence of positive solutions to (1.24) by showing that $c_{\mathcal{I}}$ can be attained.

THEOREM 1.9. *Assume that $s \in (0,1)$, $N > 2s$, p, q satisfy (1.25) and $pq \neq 1$. Then*

(i) (1.24) admits a positive classical ground state solution;
(ii) If $pq < 1$, the positive classical solution of (1.24) is unique;
(iii) If Ω is a ball, then (1.24) has a positive radially symmetric classical solution.

When Ω is a ball and $pq<1$, the uniqueness of positive classical solution to (1.24) ensures that the solution is radially symmetric. For $pq>1$, we can not claim directly as in [5] that the ground state solution is radially symmetric, but we can work simply with $\mathcal{W}^{2s,r}_{\mathrm{rad}}(B_R)$, the subset of radial functions in $\mathcal{W}^{2s,r}(B_R)$. In fact, for $f \in C(\overline{B_R})$, $f \geq 0$, if u, w are respectively solutions of

$$\begin{cases} (-\Delta)^s u = f & \text{in } B_R, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \overline{B_R}, \end{cases} \qquad \begin{cases} (-\Delta)^s w = f^{\#} & \text{in } B_R, \\ w = 0 & \text{in } \mathbb{R}^N \setminus \overline{B_R}, \end{cases}$$

where $f^{\#}$ denotes the radial decreasing rearrangement of f, we do not have always $u^{\#} \leq w$, see [23], while it is true for $s=1$.

REMARK 1.10. For any classical solution (u, v) of (1.24), we have

$$\int_\Omega |v|^{q+1} dx = \langle (-\Delta)^s u, v\rangle = \langle (-\Delta)^s v, u\rangle = \int_\Omega |u|^{p+1} dx.$$

Combining with $\langle \mathcal{I}'(u), u\rangle = 0$, there holds $\mathcal{K}(u,v) = \mathcal{I}(u)$. Hence $\mathcal{K}(u,v)$ reaches the minimal energy among all classical solutions if u is a ground state of (1.26).

The paper is organized as follows. In §2, we introduce some preliminary results. In §3, we construct an interpolation space setting and prove theorem 1.4. In §4, we construct the framework of dual method to show theorem 1.5. In §5, we revisit the fractional Lane–Emden system (1.24).

2. Notations and preliminary

Throughout this paper, we use the following notations.

- We denote by ∇H the gradient of H in $\mathbb{R}^2$, and we write $H_z(u,v) := \nabla H(u(x), v(x))$ for the functional variable case where $z = (u, v)$. The same convention is used for the Legendre–Fenchel transform H^*.
- $C, C', C_1, C_2, \ldots$ denote always generic positive constants.
- For $r \in [1, \infty]$, we denote by $|u|_r$ the usual $L^r(\Omega)$ or $L^r(\mathbb{R}^N)$ norm.
- For any $p, q > 0$, we set $X := L^{1+\frac{1}{p}}(\Omega) \times L^{1+\frac{1}{q}}(\Omega)$ with norm $\|(f,g)\|_X = |f|_{1+\frac{1}{p}} + |g|_{1+\frac{1}{q}}$, hence its dual space is $X^* = L^{p+1}(\Omega) \times L^{q+1}(\Omega)$.

https://doi.org/10.1017/prm.2025.10038 Published online by Cambridge University Press

- The Hamiltonian functional is denoted by

$$\mathcal{H}(u,v)=\int_\Omega H(u,v)dx,\quad (u,v)\in X^*.$$

2.1. Basic properties for L^1-weak solutions

Here we show some elementary facts for L^1-weak solutions. Many of them were established for solutions in $X_0^s(\Omega)$, however it's worthy to check carefully for L^1-weak solutions.

Let us begin with the comparison principle. We say that $(-\Delta)^s u\le(-\Delta)^s v$ in the L^1-weak sense, if $u,v\in L^1(\Omega)$ and

$$\int_\Omega (u-v)(-\Delta)^s\varphi dx\le 0,\quad \forall\,\varphi\in\mathcal{T}_s(\Omega)\text{with }(-\Delta)^s\varphi\ge 0. \tag{2.1}$$

By the definition of $\mathcal{T}_s(\Omega)$, as $(-\Delta)^s\varphi$ fulfilled $C_c^\infty(\Omega)$, we get immediately

LEMMA 2.1. *If $(-\Delta)^s u\le(-\Delta)^s v$ in the L^1-weak sense, then $u\le v$ a.e. in Ω.*

The next one is a Hopf lemma result for L^1-weak supersolutions.

LEMMA 2.2. *Assume that $f\in L^1(\Omega;\delta^s dx)$ is nonnegative and $f\neq 0$. Then there exists $\mathrm{C}>0$ depending only on f, Ω, N and s such that for any $u\in L^1(\Omega)$ satisfying $(-\Delta)^s u\ge f$, we have $u>C\delta^s$ a.e. in Ω.*

Proof. Fix $f_k:=\min\{k,f\}$ with k large enough such that $f_k\neq 0$. Let $u_k\subset X_0^s(\Omega)$ solve (ensured by lemma 1.1)

$$(-\Delta)^s u_k=f_k\ \text{ in }\Omega,\quad u_k=0\ \text{ in }\mathbb{R}^N\setminus\Omega. \tag{2.2}$$

As $f_k\in L^\infty(\Omega)$, by maximum principle (see [36, Proposition 2.2.8], [8, Theorem A.1]) and Hopf's lemma (see [11, Proposition 2.7], [35, Lemma 3.2] and [31, Theorem 1.5]) for bounded source, there holds $u_k>C\delta^s$ for some $C>0$. Using the previous lemma, we get $u\ge u_k$, and finish the proof. □

Inspired by [10], we show the following existence and uniqueness result for L^1-weak solution.

PROPOSITION 2.3. *Given any $f\in L^1(\Omega;\delta^s dx)$, there exists a unique L^1-weak solution u to (1.11). Moreover, there exists $C=C(N,s,\Omega)>0$ such that*

$$|u|_1\le C\|f\|_{L^1(\Omega;\delta^s dx)}. \tag{2.3}$$

Proof. Assume first $f\ge 0$, otherwise we decompose $f=f_+-f_-$. Let $\xi\in X_0^s(\Omega)$ be the solution of

$$(-\Delta)^s\xi=1\ \text{ in }\Omega,\quad \xi=0\ \text{ in }\mathbb{R}^N\setminus\Omega. \tag{2.4}$$

For any $k\in\mathbb{N}$, set $f_k:=\min\{k,f\}$, and $u_k\ge 0$ the solution of (2.2). As $u_k\in X_0^s(\Omega)$, we can use it as test function to (2.4). Applying the estimates in lemma 1.1 for ξ,

$$|u_k|_1=\langle(-\Delta)^s\xi,u_k\rangle=\langle(-\Delta)^s u_k,\xi\rangle=\int_\Omega f_k\xi dx\le C\int_\Omega f_k\delta^s dx\le C\|f\|_{L^1(\Omega;\delta^s dx)}.$$

https://doi.org/10.1017/prm.2025.10038 Published online by Cambridge University Press

Similarly, considering $u_k - u_l$, we have $|u_k - u_l|_{L^1} \le C\|f_k - f_l\|_{L^1(\Omega;\delta^s dx)}$, so $\{u_k\}$ is a Cauchy sequence, hence a convergent sequence in $L^1(\Omega)$. We get easily a L^1-weak solution to (1.11) by taking u, the limit of u_k in $L^1(\Omega)$. The uniqueness is ensured by the comparison principle. □

REMARK 2.4. If $\Omega = B_R$ is a ball and $f \in L^1(B_R, \delta^s dx)$ is radial, then there exists a unique radially symmetric solution to (1.11).

Next, we state the regularity results for the unique L^1-weak solution when f admits further integrability. Here we summarize the results in [27] and [4, Lemma 2.5].

PROPOSITION 2.5. *Assume that $s \in (0,1)$, $N > 2s$, Ω is a smooth bounded domain. For any $f \in L^r(\Omega)$ with $r \ge 1$, the unique L^1-weak solution u to (1.11) satisfies*

(i) *If $1 \le r < \frac{N}{2s}$, $u \in L^\gamma(\Omega)$ and $|u|_\gamma \le C(\Omega, r, s)|f|_r$, where*

$$\gamma = \frac{Nr}{N-2rs} \text{ if } r > 1, \quad 1 \le \gamma < \frac{N}{N-2s} \text{ if } r = 1;$$

(ii) *If $r \ge \frac{2N}{N+2s}$, we have $u \in X_0^s(\Omega)$ and $\|u\|_{X_0^s(\Omega)} \le C(\Omega, r, s)|f|_r$;*
(iii) *If $r > \frac{N}{2s}$, there holds $u \in L^\infty(\Omega)$ and $|u|_\infty \le C(\Omega, r, s)|f|_r$;*
(iv) *If $r = \frac{N}{2s}$, there is a constant $\alpha(\Omega, f, s) > 0$ such that*

$$\int_\Omega e^{\alpha|u|} dx < \infty.$$

In particular, $u \in L^\gamma(\Omega)$ for all $1 \le \gamma < \infty$.

2.2. Embedding and density

Here we expose some basic properties of $\mathcal{W}^{2s,r}(\Omega)$, including (compact) embedding results and density property, which will be important for constructing the variational framework.

PROPOSITION 2.6. *Assume that $s \in (0,1)$, $N > 2s$, $r \ge 1$, Ω is a smooth bounded domain. Then $\mathcal{W}^{2s,r}(\Omega)$ is a Banach space and $\mathcal{T}_s(\Omega)$ is dense in $\mathcal{W}^{2s,r}(\Omega)$. On the other hand, we have continuous embedding $\mathcal{W}^{2s,r}(\Omega) \subset L^\gamma(\Omega)$ for*

$$\begin{cases} 1 \le \gamma < \dfrac{N}{N-2s}, & \text{if } r = 1; \\ 1 \le \gamma \le \dfrac{Nr}{N-2sr}, & \text{if } 1 < r < \dfrac{N}{2s}; \\ 1 \le \gamma < \infty, & \text{if } 2sr = N; \\ \gamma = \infty, & \text{if } 2sr > N. \end{cases} \tag{2.5}$$

Moreover, the above embeddings are compact provided

$$1 \le \gamma < \frac{Nr}{N-2sr} \text{ if } N > 2sr; \quad \text{or } 1 \le \gamma < \infty, \text{ if } N \le 2sr. \tag{2.6}$$

https://doi.org/10.1017/prm.2025.10038 Published online by Cambridge University Press

Proof. The density of $\mathcal{T}_s(\Omega)$ in $\mathcal{W}^{2s,r}(\Omega)$ is obvious by definition and lemma 1.1.

The completeness of $\mathcal{W}^{2s,r}(\Omega)$ is a simple fact. Let $\{u_j\} \subset \mathcal{W}^{2s,r}(\Omega)$ be a Cauchy sequence, by definition, $\{(-\Delta)^s u_j\}$ is a Cauchy sequence in $L^r(\Omega)$, hence converges to some $v \in L^r(\Omega)$. Using proposition 2.3, there is a unique L^1-weak solution u satisfying (1.11) with $f = v$. Clearly $u \in \mathcal{W}^{2s,r}(\Omega)$, and u_j converges to u in $\mathcal{W}^{2s,r}(\Omega)$.

On the other hand, in virtue of proposition 2.5, the embedding of $\mathcal{W}^{2s,r}(\Omega) \subset L^\gamma(\Omega)$ is continuous if (2.5) holds true, we shall only prove that the embedding is compact provided (2.6).

Without loss of generality, we only consider the case $N > 2sr$, $1 < r$, $1 \le \gamma < \frac{Nr}{N-2sr}$. If $r \ge \frac{2N}{N+2s}$, then $\mathcal{W}^{2s,r}(\Omega) \subset X_0^s(\Omega)$ is continuous by proposition 2.5. By Hölder inequality, fix any $1 < \gamma < \frac{Nr}{N-2sr}$,

$$|u|_\gamma \le |u|_1^{1-\theta} |u|^\theta_{\frac{Nr}{N-2sr}} \quad \text{with } \theta = \frac{Nr - \gamma(N-2sr)}{Nr - (N-2sr)}. \tag{2.7}$$

Since $\mathcal{W}^{2s,r}(\Omega)$ is continuously embedded in $L^{\frac{Nr}{N-2sr}}(\Omega)$ and compactly embedded in $L^1(\Omega)$, $\mathcal{W}^{2s,r}(\Omega)$ is compactly embedded in $L^\gamma(\Omega)$.

Suppose now $1 < r < \frac{2N}{N+2s}$, we first claim that the embedding $\mathcal{W}^{2s,r}(\Omega) \subset L^1(\Omega)$ is compact. Indeed, let $\{u_j\}$ be a bounded sequence in $\mathcal{W}^{2s,r}(\Omega)$, then $\{u_j\}$ is bounded in $L^{\frac{Nr}{N-2sr}}(\Omega)$ and the sequence $f_j = (-\Delta)^s u_j$ is bounded in $L^r(\Omega)$. Let $A > 0$ be chosen later, we denote $g_j = f_j \chi_{|f_j(x)| \le A}$ and $h_j = f_j \chi_{|f_j(x)| > A}$. By Hölder and Chebychev inequalities,

$$\|h_j\|_{L^1(\Omega;\delta^s dx)} \le C|h_j|_1 \le C|f_j|_r \Big(\int_{|f_j(x)|>A} dx \Big)^{1-\frac{1}{r}} \le C' A^{\frac{1}{r}-1}.$$

Let $\{v_j\}$ and $\{w_j\}$ be respectively L^1-weak solutions of

$$\begin{cases} (-\Delta)^s v_j = g_j & \text{in } \Omega, \\ v_j = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \qquad \begin{cases} (-\Delta)^s w_j = h_j & \text{in } \Omega, \\ w_j = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{2.8}$$

Clearly $u_j = v_j + w_j$. Given any $\varepsilon > 0$, we fix A large enough such that

$$|w_j|_1 \le C\|h_j\|_{L^1(\Omega;\delta^s dx)} \le C' A^{\frac{1}{r}-1} \le \varepsilon.$$

Moreover, $\{v_j\}$ is bounded in $X_0^s(\Omega)$ hence relatively compact in $L^1(\Omega)$. Therefore $\{u_j\}$ is precompact, or equivalently relatively compact in $L^1(\Omega)$, which means $\mathcal{W}^{2s,r}(\Omega)$ is compactly embedded in $L^1(\Omega)$. Applying again the interpolation inequality (2.7), $\{u_j\}$ is compact in $L^\gamma(\Omega)$ for $1 \le \gamma < \frac{Nr}{N-2sr}$. So we are done. □

2.3. A linking theorem

The Linking theorem 2.8 below was given by Felmer [22, Theorem 3.1], it is useful for the proof of theorem 1.4. For the sake of completeness, we recall also the definition of Palais–Smale condition.

https://doi.org/10.1017/prm.2025.10038 Published online by Cambridge University Press

DEFINITION 2.7. *For a Banach space X, $\mathcal{I} \in C^1(X,\mathbb{R})$ is said satisfying the Palais–Smale condition at level $c \in \mathbb{R}$ (for short $(PS)_c$) if any sequence $\{u_j\} \subset X$ satisfying*

$$\mathcal{I}(u_j) \to c \quad \text{and} \quad \mathcal{I}'(u_j) \to 0 \text{ in } X^*$$

*admits a convergent subsequence in X. We say that $\mathcal{I}$ satisfies the Palais–Smale condition ((*PS*) for short) if $(PS)_c$ is satisfied for all $c \in \mathbb{R}$.*

THEOREM 2.8. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space such that $H = H_1 \oplus H_2$. Suppose that $\mathcal{I} \in C^1(H)$ satisfies (*PS*) condition and $\mathcal{I}(z) = \frac{1}{2}\langle Lz, z\rangle - \mathcal{J}(z)$, where*

(i) $L : H \to H$ is a bounded, self-adjoint linear operator and $L(H_1) \subset H_1$; $\mathcal{J}' : H \to H^*$ is compact;

(ii) There exist two linear, bounded, invertible operators B_1, $B_2 : H \to H$ such that $\widehat{B}_\tau = P_2 B_1^{-1} e^{\tau L} B_2 : H_2 \to H_2$ is invertible for any $\tau \geq 0$. Here $P_2 : H \to H_2$ is the projection along H_1 and $e^{\tau L} = \sum_{n\in\mathbb{N}} \frac{(\tau L)^n}{n!}$;

(iii) Let $e_1 \in H_1$ with $\|e_1\| = 1$. Let $\rho > 0$, $R_1 > \frac{\rho}{\|B_1^{-1} B_2 e_1\|}$, $R_2 > \rho$ and define

$$S = \{B_1 z_1 : z_1 \in H_1, \|z_1\| = \rho\},$$
$$Q = \{B_2(te_1 + z_2) : 0 \leq t \leq R_1, z_2 \in H_2, \|z_2\| \leq R_2\}.$$

Suppose that $\mathcal{I}(z) \geq \sigma > 0$ on S and $\mathcal{I}(z) \leq 0$ on ∂Q.

Then $\mathcal{I}$ has a critical point $z_0 \in H$ such that $\mathcal{I}(z_0) \geq \sigma$.

2.4. Legendre–Fenchel transform

Let V be a Banach space and V^* be its dual space. For a function $G : V \to \mathbb{R} \cup \{+\infty\}$, $G \not\equiv +\infty$, the function $G^* : V^* \to \mathbb{R} \cup \{+\infty\}$ given by

$$G^*(u^*) = \sup\{\langle u^*, v\rangle - G(v) : v \in V\}, \quad \forall\, u^* \in V^*, \tag{2.9}$$

is named the Legendre–Fenchel transform of G. The following are some basic properties of G^*.

LEMMA 2.9. *Let G^* be the Legendre–Fenchel transform of G.*

(a) The Legendre–Fenchel transform reverses the order, that is, $\widetilde{G} \geq G$ implies $\widetilde{G}^* \leq G^*$.

(b) If $G \in C^1(V,\mathbb{R})$ is convex and $G^* \in C^1(V^*,\mathbb{R})$, then for $v \in V$, $u^* \in V^*$, there holds

$$G^*(u^*) + G(v) = \langle u^*, v\rangle \iff v = (G^*)'(u^*) \iff u^* = G'(v).$$

(c) If $V = \mathbb{R}^N$, G is lower semi-continuous, strictly convex, and $\lim_{|x|\to\infty} \frac{G(x)}{|x|} = +\infty$, then $G^* \in C^1(\mathbb{R}^N,\mathbb{R})$.

(d) If $V = \mathbb{R}^N$, $G(x) = \frac{1}{p+1}|x|^{p+1}$ with $p > 0$, then $G^*(x) = \frac{p}{p+1}|x|^{1+\frac{1}{p}}$. If $V = L^{p+1}(\Omega)$ with $p > 0$, $G(v) = \frac{1}{p+1}|v|_{p+1}^{p+1}$, then $G^*(u^*) = \frac{p}{p+1}|u^*|_{1+\frac{1}{p}}^{1+\frac{1}{p}}$.

https://doi.org/10.1017/prm.2025.10038 Published online by Cambridge University Press

(e) Let $a > 0$ and $G_a(u) := aG(u)$, then $G_a^*(u^*) = aG^*\big(\frac{u^*}{a}\big)$.

Proof. For (a)–(d), see respectively [37, Chapter I, Section 6.2], [37, Chapter I, Lemma 6.3], [28, Proposition 2.4] and [21, Proposition 4.2, Remark 4.1]. For (e), one can check directly by means of (2.9). □

3. Proof of theorem 1.4

In this section, we will use the interpolation space method to handle theorem 1.4. Throughout this section, we assume that H satisfies $(H1)$–$(H4)$.

Let us begin with the understanding of interpolation spaces E^α $(\alpha \geq 0)$ and interpolation operators A^α given in (1.14) and (1.15). First, $E^0 = L^2(\Omega)$ and $A^0 = \mathrm{id}_{L^2(\Omega)}$. Clearly for $\alpha \geq 0$, A^α is an isomorphism and its inverse $A^{-\alpha} : L^2(\Omega) \to E^\alpha$ can be denoted by

$$A^{-\alpha}u = \sum_{j\geq 1} \lambda_j^{\frac{-\alpha}{2s}} a_j\varphi_j \quad \forall\, u = \sum_{j\geq 1} a_j\varphi_j \in L^2(\Omega).$$

More generally, for convenience, we define formally, when it makes sense,

$$A^\beta\Big(\sum_{j\geq 1} a_j\varphi_j\Big) = \sum_{j\geq 1} \lambda_j^{\frac{\beta}{2s}} a_j\varphi_j, \quad \forall\, \beta \in \mathbb{R}. \tag{3.1}$$

Proposition 3.1. *For any $s \in (0,1)$, there hold $E^{2s} = \mathcal{H}^{2s}(\Omega)$, and $A^{2s}u = (-\Delta)^s u$ for all $u \in E^{2s}$.*

Proof. For any $u \in \mathcal{H}^{2s}(\Omega)$, let

$$u = \sum_{j\geq 1} a_j\varphi_j \in L^2(\Omega), \quad (-\Delta)^s u = \sum_{j\geq 1} b_j\varphi_j \in L^2(\Omega).$$

Observe that

$$b_j = \int_\Omega (-\Delta)^s u\varphi_j dx = \int_\Omega (-\Delta)^s \varphi_j u dx = \int_\Omega \lambda_j\varphi_j u dx = a_j\lambda_j.$$

So $(-\Delta)^s u = \sum_{j\geq 1} a_j\lambda_j\varphi_j$ and $\sum_{j\geq 1} a_j^2\lambda_j^2 < \infty$ as $u \in \mathcal{H}^{2s}(\Omega)$. It follows that $\mathcal{H}^{2s}(\Omega) \subset E^{2s}$ and $(-\Delta)^s u = A^{2s}u$ for all $u \in \mathcal{H}^{2s}(\Omega)$.

Conversely, let $u \in E^{2s}$, then $u = \sum_{j\geq 1} a_j\varphi_j$ with $\sum_{j\geq 1} \lambda_j^2 a_j^2 < \infty$. Let $\varphi \in \mathcal{H}^{2s}(\Omega)$, $\varphi = \sum_{j\geq 1} b_j\varphi_j$, we consider

$$f(\varphi) = \int_\Omega u(-\Delta)^s \varphi dx = \sum_{j\geq 1} a_j b_j \lambda_j. \tag{3.2}$$

Therefore

$$|f(\varphi)| \leq \Big|\sum_{j\geq 1} a_j^2\lambda_j^2\Big|^{\frac{1}{2}} \Big|\sum_{j\geq 1} b_j^2\Big|^{\frac{1}{2}} = \|u\|_{E^{2s}}|\varphi|_2.$$

Using Riesz's representation theorem and the density of $\mathcal{H}^{2s}(\Omega)$ in $L^2(\Omega)$ (all eigenfunctions belong to $\mathcal{H}^{2s}(\Omega)$), we deduce that there exists $v \in L^2(\Omega)$ such

https://doi.org/10.1017/prm.2025.10038 Published online by Cambridge University Press

that

$$f(\varphi) = \int_\Omega v\varphi dx. \tag{3.3}$$

Combining (3.2) and (3.3), we get $(-\Delta)^s u = v$, so $(-\Delta)^s u \in L^2(\Omega)$ and $u \in \mathcal{H}^{2s}(\Omega)$. □

REMARK 3.2. By the definition of E^α, if $0 < \alpha < 2s$, E^α is the real interpolation space $[L^2(\Omega), E^{2s}(\Omega)]_{\alpha/2s}$. Therefore, applying proposition 2.6 and proposition 3.1, $E^\alpha \subset L^{p+1}(\Omega)$ is a continuous embedding when $\frac{1}{p+1} \geq \frac{1}{2} - \frac{\alpha}{N}$ and this embedding is compact provided that the strict inequality is valid, see [30] and [1, Sections 7.22, 7.23].

By (1.17), (1.18), and remark 3.2, fix $\alpha \in (0, 2s)$ such that

$$N\left(\frac{1}{2} - \frac{1}{\max\{p,q\}+1}\right) < \alpha < N\left(\frac{1}{\min\{p,q\}+1} - \frac{N-4s}{2N}\right), \tag{3.4}$$

so that

$$\mathbf{E}_\alpha := E^\alpha \times E^{2s-\alpha} \subset X^*$$

and $\mathcal{E} : \mathbf{E}_\alpha \to \mathbb{R}$ given in (1.16) is well defined. In this section, we always consider $\mathbf{E}_\alpha$ with $\alpha \in (0, 2s)$ satisfying (3.4). $\mathbf{E}_\alpha$ is a Hilbert space with scalar product

$$((u_1, v_1), (u_2, v_2))_{\mathbf{E}_\alpha} = (u_1, u_2)_{E^\alpha} + (v_1, v_2)_{E^{2s-\alpha}}$$

and $\mathbf{E}_\alpha = \mathbf{E}_\alpha^+ \oplus \mathbf{E}_\alpha^-$, where

$$\mathbf{E}_\alpha^+ = \left\{(u, A^{-2s+\alpha} \circ A^\alpha u) : u \in E^\alpha\right\}, \quad \mathbf{E}_\alpha^- = \left\{(u, -A^{-2s+\alpha} \circ A^\alpha u) : u \in E^\alpha\right\}.$$

In the spirit of (3.1), we will write $A^{-2s+\alpha} \circ A^\alpha$ as $A^{-2s+2\alpha}$ for briefness, similarly we write $A^{2s-2\alpha}$ for $A^{-\alpha} \circ A^{2s-\alpha}$. If $z = (u, v) \in \mathbf{E}_\alpha$, denote

$$z^+ = (u^+, v^+) = \left(\frac{u + A^{2s-2\alpha}v}{2}, \frac{v + A^{-2s+2\alpha}u}{2}\right) \in \mathbf{E}_\alpha^+,$$
$$z^- = (u^-, v^-) = \left(\frac{u - A^{2s-2\alpha}v}{2}, \frac{v - A^{-2s+2\alpha}u}{2}\right) \in \mathbf{E}_\alpha^-,$$

then $z = z^+ + z^-$. Notice that although the quadratic part of $\mathcal{E}$ turns out to be strongly indefinite, it is positive definite in $\mathbf{E}_\alpha^+$ and negative definite in $\mathbf{E}_\alpha^-$. Consider the bilinear form $B : \mathbf{E}_\alpha \times \mathbf{E}_\alpha \to \mathbb{R}$ defined by

$$B((u, v), (\varphi, \psi)) = \int_\Omega A^\alpha u A^{2s-\alpha}\psi + A^{2s-\alpha}vA^\alpha\varphi\, dx. \tag{3.5}$$

It is easy to see that B is continuous and symmetric, which induces a self-adjoint bounded linear operator $L : \mathbf{E}_\alpha \to \mathbf{E}_\alpha$ satisfying

$$B((u, v), (\varphi, \psi)) = \left\langle L(u, v), (\varphi, \psi)\right\rangle_{\mathbf{E}_\alpha}, \quad \forall\, (u, v), (\varphi, \psi) \in \mathbf{E}_\alpha. \tag{3.6}$$

https://doi.org/10.1017/prm.2025.10038 Published online by Cambridge University Press

REMARK 3.3. By definition, we see easily that

$$L(u,v) = (A^{2s-2\alpha}v, A^{2\alpha-2s}u), \quad \forall\ (u,v) \in \mathbf{E}_\alpha,$$

and $L|_{\mathbf{E}_\alpha^+} = \mathrm{id}_{\mathbf{E}_\alpha^+}$, $L|_{\mathbf{E}_\alpha^-} = -\mathrm{id}_{\mathbf{E}_\alpha^-}$.

PROPOSITION 3.4. *Let p, q satisfy (1.17), (1.18), and $(u,v) \in \mathbf{E}_\alpha$ be a critical point of $\mathcal{E}$. Then (u,v) is a L^1-weak solution to (1.1).*

Proof. Since $(u,v) \in \mathbf{E}_\alpha$ is a critical point of $\mathcal{E}$, $\langle \mathcal{E}'(u,v), (\varphi,\psi)\rangle = 0$ for all $(\varphi,\psi) \in \mathbf{E}_\alpha$, i.e.

$$\int_\Omega A^\alpha u A^{2s-\alpha}\psi\,dx + \int_\Omega A^{2s-\alpha}vA^\alpha\varphi\,dx - \int_\Omega H_v(u,v)\psi\,dx - \int_\Omega H_u(u,v)\varphi\,dx = 0.$$

Taking $\varphi = 0$, we get

$$\int_\Omega A^\alpha u A^{2s-\alpha}\psi\,dx = \int_\Omega H_v(u,v)\psi\,dx. \tag{3.7}$$

It follows from proposition 3.1 that $\mathcal{T}_s(\Omega) \subset E^{2s} \subset E^{2s-\alpha}$. Let $\psi \in \mathcal{T}_s(\Omega)$ with $\psi = \sum_{j\geq 1} b_j\varphi_j$, and decompose $u = \sum_{j\geq 1} a_j\varphi_j$. By proposition 3.1,

$$\int_\Omega A^\alpha u A^{2s-\alpha}\psi\,dx = \sum_{j\geq 1} a_j b_j \lambda_j^s = \int_\Omega uA^{2s}\psi\,dx = \int_\Omega u(-\Delta)^s\psi\,dx. \tag{3.8}$$

Combining (3.7) with (3.8), we obtain

$$\int_\Omega u(-\Delta)^s\psi\,dx = \int_\Omega H_v(u,v)\psi\,dx.$$

Similarly, for any $\varphi \in \mathcal{T}_s(\Omega)$, there holds

$$\int_\Omega v(-\Delta)^s\varphi\,dx = \int_\Omega H_u(u,v)\varphi\,dx.$$

So (u,v) is a L^1-weak solution to (1.1). □

To find a nontrivial critical point of $\mathcal{E}$ by theorem 2.8, our next step is to check the (PS) condition for $\mathcal{E}$. We start from the following compactness property.

LEMMA 3.5. *Let* p, q *satisfy (1.17), (1.18), H satisfy $(H4)$ and α satisfy (3.4), then $H_z : \mathbf{E}_\alpha \to X$ is compact.*

Proof. By $(H4)$ and remark 3.2, H_z is well defined from $\mathbf{E}_\alpha$ to X^*. Let $\{(u_j,v_j)\}$ be bounded in $\mathbf{E}_\alpha$, we aim to prove that $\{(H_u(u_j,v_j), H_v(u_j,v_j))\}$ has a convergent subsequence in X.

Since $\mathbf{E}_\alpha$ is a Hilbert space, up to a subsequence, there exists $(u,v) \in \mathbf{E}_\alpha$ such that $(u_j,v_j) \rightharpoonup (u,v)$ weakly in $\mathbf{E}_\alpha$. By virtue of the compact embeddings

https://doi.org/10.1017/prm.2025.10038 Published online by Cambridge University Press

$E^\alpha \subset L^{p+1}(\Omega)$, $E^{2s-\alpha} \subset L^{q+1}(\Omega)$ (see remark 3.2 and (3.4)), there holds, up to a subsequence,

$$\begin{aligned} &u_j \to u \text{ in } L^{p+1}(\Omega), \quad v_j \to v \text{ in } L^{q+1}(\Omega) \quad \text{and} \\ &u_j(x) \to u(x), \quad v_j(x) \to v(x) \quad \text{a.e. in } \Omega.. \end{aligned}$$

Using [9, Theorem 4.9], there exists $(\Phi_1, \Phi_2) \in X^*$ such that $|u_j| \le \Phi_1, |v_j| \le \Phi_2$ a.e. for all j. Applying $(H4)$ again, we obtain

$$\begin{aligned} |H_u(u_j, v_j)| &\le C\big(|\Phi_1|^p + |\Phi_2|^{\frac{p(q+1)}{(p+1)}} + 1\big) \in L^{1+\frac{1}{p}}(\Omega), \\ |H_v(u_j, v_j)| &\le C\big(|\Phi_2|^q + |\Phi_1|^{\frac{q(p+1)}{(q+1)}} + 1\big) \in L^{1+\frac{1}{q}}(\Omega). \end{aligned}$$

Let

$$w_j = H_u(u_j, v_j), \quad w = H_u(u, v), \quad y_j = H_v(u_j, v_j), \quad y = H_v(u, v).$$

Using the Lebesgue dominated convergence theorem, we get $(w_j, y_j) \to (w, y)$ in X. The proof is done. □

Remark 3.6. Clearly, under the assumptions of lemma 3.5, $\mathcal{H}' : \mathbf{E}_\alpha \to \mathbf{E}_\alpha^*$ is compact, which can be seen as follows

$$\mathbf{E}_\alpha \underset{H_z}{\to} X \underset{\text{id}}{\to} \mathbf{E}_\alpha^*.$$

The first mapping above is compact by lemma 3.5, and the last one is continuous by the embedding $\mathbf{E}_\alpha \subset X^*$.

Lemma 3.7. *Let* p, q *satisfy* (1.17), (1.18), H *satisfy* $(H2)$ *and* $(H4)$. *Then the functional* $\mathcal{E}$ *satisfies* $(PS)_c$ *condition for any* $c \in \mathbb{R}$.

Proof. Let $\{(u_j, v_j)\} \subset \mathbf{E}_\alpha$ be any $(PS)_c$ sequence of $\mathcal{E}$ with $c \in \mathbb{R}$. By definition, when j goes to infinity,

$$\mathcal{E}(u_j, v_j) = \int_\Omega A^\alpha u_j A^{2s-\alpha} v_j \, dx - \int_\Omega H(u_j, v_j) \, dx = c + o(1), \tag{3.9}$$

and for all $(\varphi, \psi) \in \mathbf{E}_\alpha$,

$$\begin{aligned} \langle \mathcal{E}'(u_j, v_j), (\varphi, \psi) \rangle &= \int_\Omega A^\alpha u_j A^{2s-\alpha} \psi \, dx + \int_\Omega A^{2s-\alpha} v_j A^\alpha \varphi \, dx - \int_\Omega H_u(u_j, v_j) \varphi \, dx \\ &\quad - \int_\Omega H_v(u_j, v_j) \psi dx \\ &= o(1) \|(\varphi, \psi)\|_{\mathbf{E}_\alpha}. \end{aligned} \tag{3.10}$$

We first claim that $\{(u_j, v_j)\}$ is bounded in $\mathbf{E}_\alpha$. Take first

$$(\varphi, \psi) = \left(\frac{q+1}{p+q+2} u_j, \frac{p+1}{p+q+2} v_j \right).$$

https://doi.org/10.1017/prm.2025.10038 Published online by Cambridge University Press

From (3.10), (3.9), and $(H2)$, as j goes to infinity,

$$\begin{aligned}
&c + o(1)\|(u_j, v_j)\|_{\mathbf{E}_\alpha} + o(1)\\
&= \mathcal{E}(u_j, v_j) - \left\langle \mathcal{E}'(u_j, v_j), \left(\frac{q+1}{p+q+2}u_j, \frac{p+1}{p+q+2}v_j\right)\right\rangle\\
&= \frac{(p+1)(q+1)}{p+q+2}\\
&\quad \left(\frac{1}{p+1}\int_\Omega H_u(u_j, v_j)u_j\,dx + \frac{1}{q+1}\int_\Omega H_v(u_j, v_j)v_j\,dx - \int_\Omega H(u_j, v_j)\,dx\right)\\
&\quad + \frac{pq-1}{p+q+2}\int_\Omega H(u_j, v_j)dx\\
&\quad \geq \frac{pq-1}{p+q+2}\int_\Omega H(u_j, v_j)dx - C.
\end{aligned}$$

From $(H2)$ (see [22, Lemma 1.1]), there exist $C_1, C_2 > 0$ such that

$$H(u, v) \geq C_1(|u|^{p+1} + |v|^{q+1}) - C_2, \quad \forall\ (u, v) \in \mathbf{E}_\alpha.$$

Thus,

$$|u_j|_{p+1}^{p+1} + |v_j|_{q+1}^{q+1} \leq C + o(1)\|(u_j, v_j)\|_{\mathbf{E}_\alpha}. \tag{3.11}$$

On the other hand, it follows from (3.10) with $\psi = 0$ that

$$\int_\Omega A^{2s-\alpha}v_j A^\alpha\varphi dx = \int_\Omega H_u(u_j, v_j)\varphi dx + o(1)\|\varphi\|_{E^\alpha}, \tag{3.12}$$

and if $\varphi = 0$,

$$\int_\Omega A^\alpha u_j A^{2s-\alpha}\psi dx = \int_\Omega H_v(u_j, v_j)\psi dx + o(1)\|\psi\|_{E^{2s-\alpha}}. \tag{3.13}$$

Applying $(H4)$, Hölder inequality and $E^\alpha \subset L^{p+1}(\Omega)$, there holds

$$\int_\Omega |H_u(u_j, v_j)\varphi|\,dx \leq C\left(|u_j|_{p+1}^p + |v_j|_{q+1}^{p(q+1)/(p+1)} + 1\right)\|\varphi\|_{E^\alpha}. \tag{3.14}$$

Since $A^{2s-\alpha}$ is an isomorphism from $E^{2s-\alpha}$ to $L^2(\Omega)$ and due to (3.12), (3.14), we have

$$\begin{aligned}
\|v_j\|_{E^{2s-\alpha}} = |A^{2s-\alpha}v_j|_2 &= \sup_{\|\varphi\|_{E^\alpha}=1}\left|\int_\Omega A^{2s-\alpha}v_j A^\alpha\varphi dx\right|\\
&\leq C\left(|u_j|_{p+1}^p + |v_j|_{q+1}^{p(q+1)/(p+1)} + 1\right).
\end{aligned} \tag{3.15}$$

Similarly, we obtain

$$\|u_j\|_{E^\alpha} \leq C\left(|v_j|_{q+1}^q + |u_j|_{p+1}^{q(p+1)/(q+1)} + 1\right). \tag{3.16}$$

Combining (3.11) with (3.15) and (3.16), it is clear that $\{(u_j, v_j)\}$ is bounded in $\mathbf{E}_\alpha$. Therefore, up to a subsequence, there is $(u, v) \in \mathbf{E}_\alpha$ such that $(u_j, v_j) \rightharpoonup (u, v)$

https://doi.org/10.1017/prm.2025.10038 Published online by Cambridge University Press

weakly in $\mathbf{E}_\alpha$. Let

$$w_j = H_u(u_j, v_j), \quad w = H_u(u, v), \quad y_j = H_v(u_j, v_j), \quad y = H_v(u, v).$$

By lemma 3.5, up to a subsequence, $(w_j, y_j) \to (w, y)$ in X. In view of (3.12) and (3.13), let j tend to ∞, there holds

$$\int_\Omega A^{2s-\alpha} v A^\alpha \varphi dx = \int_\Omega w\varphi dx, \quad \int_\Omega A^\alpha v A^{2s-\alpha}\psi dx = \int_\Omega y\psi dx. \tag{3.17}$$

Thanks to (3.12), (3.17), Hölder inequality and $E^\alpha \subset L^{p+1}(\Omega)$, we arrive at

$$\begin{aligned} \|v_j - v\|_{E^{2s-\alpha}} &= \sup_{\|\varphi\|_{E^\alpha}=1} \int_\Omega A^{2s-\alpha}(v_j - v) A^\alpha \varphi dx \\ &= \sup_{\|\varphi\|_{E^\alpha}=1} \left(\int_\Omega (w_j - w)\varphi dx + o(1)\|\varphi\|_{E^\alpha} \right) \\ &\le C\left(|w_j - w|_{1+\frac{1}{p}} + o(1) \right) \end{aligned}$$

which implies $v_j \to v$ in $E^{2s-\alpha}$. In the same way, we obtain $u_j \to u$ in E^α. □

In the sequel, we shall verify that $\mathcal{E}$ possesses the linking structure stated in theorem 2.8.

LEMMA 3.8. *Let p, q satisfy (1.17), (1.18), H satisfy (H1)–(H4). Then there exist two linear, bounded and invertible operators B_1, $B_2 : \mathbf{E}_\alpha \to \mathbf{E}_\alpha$ such that for all $\tau \ge 0$, $\widehat{B}_\tau = P_2 B_1^{-1} e^{\tau L} B_2 : \mathbf{E}_\alpha^- \to \mathbf{E}_\alpha^-$ is invertible, where P_2 is the projection of $\mathbf{E}_\alpha$ onto $\mathbf{E}_\alpha^-$. Moreover, let*

$$e_1 = \left(\lambda_1^{-\frac{\alpha}{2s}} \varphi_1, \lambda_1^{\frac{-2s+\alpha}{2s}} \varphi_1 \right) \in \mathbf{E}_\alpha^+,$$

then there exist constants $R_2 > \rho > 0$, $R_1 > \rho / \|B_1^{-1} B_2 e_1\|_{\mathbf{E}_\alpha}$ and $\sigma > 0$ such that

$$(G1)\ \ \mathcal{E}(z) \ge \sigma > 0 \text{ on } S_\rho^+; \qquad (G2)\ \ \mathcal{E}(z) \le 0 \text{ on } \partial Q$$

where $S_\rho^+ := \{B_1 z^+ : z^+ \in \mathbf{E}_\alpha^+, \|z^+\| = \rho\}$ and

$$Q := \left\{ B_2(te_1 + z^-) : 0 \le t \le R_1,\ z^- \in \mathbf{E}_\alpha^-,\ \|z^-\|_{\mathbf{E}_\alpha} \le R_2 \right\}.$$

Proof. Since $\frac{1}{p+1} + \frac{1}{q+1} < 1$, we can select $\mu, \nu \ge 1$ satisfying

$$\frac{1}{p+1} < \frac{\mu}{\mu+\nu}, \quad \frac{1}{q+1} < \frac{\nu}{\mu+\nu}. \tag{3.18}$$

Let $R_1 > 1$, $0 < \rho < 1$ and define

$$B_1(u, v) = (\rho^{\mu-1} u, \rho^{\nu-1} v), \quad B_2(u, v) = (R_1^{\mu-1} u, R_1^{\nu-1} v), \quad \forall\ (u, v) \in \mathbf{E}_\alpha,$$

https://doi.org/10.1017/prm.2025.10038 Published online by Cambridge University Press

Since $L|_{\mathbf{E}_\alpha^-} = -id_{\mathbf{E}_\alpha^-}$, $\widehat{B}_\tau$ is invertible in $\mathbf{E}_\alpha^-$. According to the definitions of B_1 and B_2, we set

$$S_\rho^+ = \big\{(\rho^{\mu-1}u^+, \rho^{\nu-1}v^+) : \|(u^+, v^+)\|_{\mathbf{E}_\alpha} = \rho,\, z^+ = (u^+, v^+) \in \mathbf{E}_\alpha^+\big\},$$

and

$$\begin{aligned} Q = \Big\{ & t(R_1^{\mu-1}\widetilde{\varphi}_1, R_1^{\nu-1}\lambda_1^{\frac{-s+\alpha}{s}}\widetilde{\varphi}_1) + (R_1^{\mu-1}u^-, R_1^{\nu-1}v^-) : 0 \le t \le R_1, \\ & \|(u^-, v^-)\|_{\mathbf{E}_\alpha} \le R_2,\, z^- = (u^-, v^-) \in \mathbf{E}_\alpha^-\Big\}, \end{aligned}$$

where $\widetilde{\varphi}_1 = \lambda_1^{-\frac{\alpha}{2s}}\varphi_1$.

Verification of $(G1)$. For any $(\rho^{\mu-1}u^+, \rho^{\nu-1}v^+) \in S_\rho^+$, by $(H3)$ and $\mathbf{E}_\alpha \subset L^{p+1}(\Omega) \times L^{q+1}(\Omega)$, when ρ is small, by (3.18), there exist $C_1, C_2 > 0$ such that

$$\begin{aligned} \mathcal{E}(\rho^{\mu-1}u^+, \rho^{\nu-1}v^+) &\ge \frac{1}{2}\rho^{\mu+\nu-2}\|z^+\|_{\mathbf{E}_\alpha}^2 - C_1\rho^{(\mu-1)(p+1)} \\ &\quad \int_\Omega |u^+|^{p+1}dx - C_1\rho^{(\nu-1)(q+1)}\int_\Omega |v^+|^{q+1}dx \\ &\ge \frac{1}{2}\rho^{\mu+\nu-2}\|z^+\|_{\mathbf{E}_\alpha}^2 - C_2\rho^{(\mu-1)(p+1)}\|z^+\|_{\mathbf{E}_\alpha}^{p+1} \\ &\quad - C_2\rho^{(\nu-1)(q+1)}\|z^+\|_{\mathbf{E}_\alpha}^{q+1} \\ &\ge \frac{1}{2}\rho^{\mu+\nu} - C_2\rho^{\mu(p+1)} - C_2\rho^{\nu(q+1)}, \end{aligned}$$

which yields that there exist $\sigma,\, \rho > 0$ satisfying $\mathcal{E}(\rho^{\mu-1}u^+, \rho^{\nu-1}v^+) \ge \sigma > 0$.

Verification of $(G2)$. We proceed by the following three steps.

Step 1. For any $(R_1^{\mu-1}u^-, R_1^{\nu-1}v^-) \in Q \cap \{t = 0\}$, it follows from $(H1)$ that

$$\mathcal{E}(R_1^{\mu-1}u^-, R_1^{\nu-1}v^-) \le R_1^{\mu+\nu-2}\int_\Omega A^\alpha u^- A^{2s-\alpha}v^-\,dx \le -R_1^{\mu+\nu-2}\int_\Omega |A^\alpha u^-|^2 dx \le 0.$$

Step 2. For any $\widetilde{z}_{R_1} = (R_1^\mu\widetilde{\varphi}_1 + R_1^{\mu-1}u^-,\, R_1^\nu\lambda_1^{\frac{-s+\alpha}{s}}\widetilde{\varphi}_1 + R_1^{\nu-1}v^-) \in Q \cap \{t = R_1\}$, we write $u^- = r\varphi_1 + w$ where $w \in E^\alpha$ is orthogonal to φ_1 in $L^2(\Omega)$.

Let $r \ge 0$. By direct computations,

$$r + \lambda_1^{-\frac{\alpha}{2s}}t = \int_\Omega (t\widetilde{\varphi}_1 + u^-)\varphi_1 dx \le |t\widetilde{\varphi}_1 + u^-|_{p+1}|\varphi_1|_{\frac{p+1}{p}} \le C|t\widetilde{\varphi}_1 + u^-|_{p+1}. \quad (3.19)$$

https://doi.org/10.1017/prm.2025.10038 Published online by Cambridge University Press

Set $\widetilde{z}_t = t(R_1^{\mu-1}\widetilde{\varphi}_1, R_1^{\nu-1}\lambda_1^{\frac{-s+\alpha}{s}}\widetilde{\varphi}_1) + (R_1^{\mu-1}u^-, R_1^{\nu-1}v^-)$ with $t \geq 0$. Using $(H2)$ (see [22, Lemma 1.1]) and (3.19), we have

$$\begin{aligned}\mathcal{H}(\widetilde{z}_t) &\geq C_1 R_1^{(p+1)(\mu-1)} \int_\Omega |t\widetilde{\varphi}_1 + u^-|^{p+1} dx + C_1 R_1^{(q+1)(\nu-1)} \\ &\int_\Omega |t\lambda_1^{\frac{-s+\alpha}{s}}\widetilde{\varphi}_1 + v^-|^{q+1} dx - C_2 \\ &\geq C_3 R_1^{(p+1)(\mu-1)}(r + \lambda_1^{-\frac{\alpha}{2s}} t)^{p+1} - C_2 \\ &\geq C_4 R_1^{(p+1)(\mu-1)} t^{p+1} - C_2.\end{aligned} \tag{3.20}$$

If $r < 0$, let $v^- = -A^{-2s+\alpha} \circ A^\alpha u^- = -A^{-2s+2\alpha}u^-$,

$$\begin{aligned}\int_\Omega v^-\varphi_1 dx = \int_\Omega (-A^{-2s+2\alpha}u^-)\varphi_1 dx &= \int_\Omega -A^{-2s+2\alpha}(r\varphi_1 + w)\varphi_1 dx \\ &= -\lambda_1^{\frac{-s+\alpha}{s}} r - \int_\Omega \varphi_1 A^{-2s+2\alpha} w dx \\ &= -\lambda_1^{\frac{-s+\alpha}{s}} r.\end{aligned}$$

Thus,

$$\lambda_1^{\frac{-s+\alpha}{s}}(\lambda_1^{-\frac{\alpha}{2s}} t - r) = \int_\Omega (t\lambda_1^{\frac{-s+\alpha}{s}}\widetilde{\varphi}_1 + v^-)\varphi_1 \, dx \leq |t\lambda_1^{\frac{-s+\alpha}{s}}\widetilde{\varphi}_1 + v^-|_{q+1}|\varphi_1|_{\frac{q+1}{q}}.$$

Hence, similar to (3.20), one has

$$\mathcal{H}(\widetilde{z}_t) \geq C_5 R_1^{(q+1)(\nu-1)} t^{q+1} - C_6. \tag{3.21}$$

Therefore, either (3.20) or (3.21) holds, which yields

$$\mathcal{E}(\widetilde{z}_{R_1}) \leq R_1^{\mu+\nu} - \frac{R_1^{\mu+\nu-2}}{2}\|z^-\|^2_{\mathbf{E}_\alpha} - C_4 R_1^{(p+1)\mu} + C_2,$$

or

$$\mathcal{E}(\widetilde{z}_{R_1}) \leq R_1^{\mu+\nu} - \frac{R_1^{\mu+\nu-2}}{2}\|z^-\|^2_{\mathbf{E}_\alpha} - C_5 R_1^{(q+1)\nu} + C_6.$$

In both cases, we can choose $R_2 = 2R_1$ large so that $\mathcal{E}(\widetilde{z}_{R_1}) \leq 0$.

Step 3. For any $\|(u^-, v^-)\|_{\mathbf{E}_\alpha} = R_2$, there holds

$$\widetilde{z}_t = t(R_1^{\mu-1}\widetilde{\varphi}_1, R_1^{\nu-1}\lambda_1^{\frac{-s+\alpha}{s}}\widetilde{\varphi}_1) + (R_1^{\mu-1}u^-, R_1^{\nu-1}v^-) \in Q \cap \{\|z^-\|_{\mathbf{E}_\alpha} = R_2\}.$$

Since either (3.20) or (3.21) holds, we get either

$$\mathcal{E}(\widetilde{z}_t) \leq R_1^{\mu+\nu-2}t^2 - \frac{R_1^{\mu+\nu-2}}{2}R_2^2 - C_4 R_1^{(p+1)(\mu-1)} t^{p+1} + C_2,$$

or

$$\mathcal{E}(\widetilde{z}_t) \leq R_1^{\mu+\nu-2}t^2 - \frac{R_1^{\mu+\nu-2}}{2}R_2^2 - C_5 R_1^{(q+1)(\nu-1)} t^{q+1} + C_6.$$

Choosing $R_2 = 2R_1$ large, we get $\mathcal{E}(\widetilde{z}_t) \leq 0$ for $0 \leq t \leq R_1$. □

https://doi.org/10.1017/prm.2025.10038 Published online by Cambridge University Press

LEMMA 3.9. *Under the assumption (H4) with subcritical (p,q), any weak solution $(u,v)\in X^*$ of (1.1) is a classical solution.*

Proof. Since $(u,v)\in X^*$ and $(H4)$, one has $H_z(u,v)\in X$, so

$$(u,v)\in \mathcal{W}^{2s,1+\frac{1}{q}}(\Omega)\times \mathcal{W}^{2s,1+\frac{1}{p}}(\Omega). \tag{3.22}$$

Consider first $p,q>\frac{2s}{N-2s}$, proposition 2.6 yields then that $u\in L^{\frac{N(q+1)}{Nq-2s(q+1)}}(\Omega)$ and $v\in L^{\frac{N(p+1)}{Np-2s(p+1)}}(\Omega)$. The assumption $(H4)$ deduces that $(u,v)\in \mathcal{W}^{2s,r_1}(\Omega)\times \mathcal{W}^{2s,t_1}(\Omega)$ where

$$r_1=\min\left\{\frac{N(p+1)}{[Nq-2s(p+1)]q},\frac{N(q+1)^2}{[Nq-2s(q+1)](p+1)q}\right\},$$

and

$$t_1=\min\left\{\frac{N(q+1)}{[Nq-2s(q+1)]p},\frac{N(p+1)^2}{[Np-2s(p+1)](q+1)p}\right\}.$$

Since (p,q) is subcritical, we have

$$r_1>r_0:=1+\frac{1}{q},\quad t_1>t_0:=1+\frac{1}{p}. \tag{3.23}$$

In the same way, whenever $r_n,t_n<\frac{N}{2s}$, let

$$r_{n+1}=\min\left\{\frac{Nt_n}{(N-2st_n)q},\frac{Nr_n(q+1)}{(N-2sr_n)(p+1)q}\right\}$$

and

$$t_{n+1}=\min\left\{\frac{Nr_n}{(N-2sr_n)p},\frac{Nt_n(p+1)}{(N-2st_n)(q+1)p}\right\},$$

then $(u,v)\in \mathcal{W}^{2s,r_{n+1}}(\Omega)\times \mathcal{W}^{2s,t_{n+1}}(\Omega)$, and $r_n<r_{n+1},t_n<t_{n+1}$ due to (3.23). We claim that

$$\text{there exists } n \text{ such that } (u,v)\in \mathcal{W}^{2s,r_n}(\Omega)\times \mathcal{W}^{2s,t_n}(\Omega) \text{ with } \min\{r_n,t_n\}>\frac{N}{2s}. \tag{3.24}$$

This claim can be proved by contradiction. Assume for example $r_n\to r\le\frac{N}{2s}$, $t_n\to t\le\frac{N}{2s}$, other cases can be ruled out similarly. We can check that

$$r=\frac{N}{2s}\frac{pq-1}{pq+q},\quad t=\frac{N}{2s}\frac{pq-1}{pq+p},$$

which contradicts $r>r_0$ since (p,q) is subcritical.

Therefore, after finite number of iterations, (3.24) holds true. Then $u,v\in C^s(\overline{\Omega})$, thanks again to proposition 2.6 and lemma 1.1.

Other situations with p or $q\le\frac{2s}{N-2s}$ can be handled easily using proposition 2.6 and lemma 1.1, so we omit the details. □

https://doi.org/10.1017/prm.2025.10038 Published online by Cambridge University Press

Proof of theorem 1.4 completed. By lemmas 3.7–3.8, remarks 3.3, 3.6, applying theorem 2.8 and proposition 3.4, there exists a L^1-weak solution $(u, v) \in X^*$. By lemma 3.9, (u, v) is a classical solution, so the proof is completed. □

4. Dual method and proof of theorem 1.5

This section is devoted to the proof of theorem 1.5. We state first some basic properties for H, H^*, $\mathcal{H}$, and $\mathcal{H}^*$. As

$$H(u,v) = \int_0^1 H_u(tu,tv)u + H_v(tu,tv)vdt, \tag{4.1}$$

there holds readily

LEMMA 4.1. *For H satisfying $(H5)$, there are positive constants A_1, A_2 such that*

$$A_1\left(|u|^{p+1}+|v|^{q+1}\right) \le H(u,v) \le A_2\left(|u|^{p+1}+|v|^{q+1}\right), \quad \forall\,(u,v)\in\mathbb{R}^2.$$

Let H^* be the Legendre–Fenchel transform of convex functional H, and $\mathcal{H}^*$ be the Legendre–Fenchel transform of $\mathcal{H}: X^* \to \mathbb{R}$, i.e.

$$H^*(u,v) = \sup_{(t,s)\in\mathbb{R}^2}\left\{tu+sv-H(t,s)\right\}, \quad \forall\,(u,v)\in\mathbb{R}^2,$$

and

$$\mathcal{H}^*(f,g) = \sup_{(u,v)\in X^*}\left\{\int_\Omega (fu+gv)dx - \mathcal{H}(u,v)\right\}, \quad \forall\,(f,g)\in X. \tag{4.2}$$

Using lemma 4.1 and lemma 2.9 (d)(e), recalling that the Legendre–Fenchel transform reverses the order by lemma 2.9 (a), we have

LEMMA 4.2. *Assume that H is convex satisfying $(H5)$, then there are positive constants A_3, A_4 such that*

$$A_3\left(|u|^{1+\frac{1}{p}}+|v|^{1+\frac{1}{q}}\right) \le H^*(u,v) \le A_4\left(|u|^{1+\frac{1}{p}}+|v|^{1+\frac{1}{q}}\right), \quad \forall\,(u,v)\in\mathbb{R}^2$$

and

$$A_3\left(|f|_{1+\frac{1}{p}}^{1+\frac{1}{p}}+|g|_{1+\frac{1}{q}}^{1+\frac{1}{q}}\right) \le \mathcal{H}^*(f,g) \le A_4\left(|f|_{1+\frac{1}{p}}^{1+\frac{1}{p}}+|g|_{1+\frac{1}{q}}^{1+\frac{1}{q}}\right), \quad \forall\,(f,g)\in X.$$

Consider ∇H and ∇H^* as mappings of $\mathbb{R}^2$. If ∇H is invertible, we denote by $(\nabla H)^{-1}$ its inverse.

LEMMA 4.3. *Assume that H satisfies $(H5)$–$(H6)$, then $H^* \in C^1(\mathbb{R}^2,\mathbb{R})$. Moreover $\nabla H: \mathbb{R}^2 \to \mathbb{R}^2$ is a homeomorphism, and $(\nabla H)^{-1} = \nabla H^*$.*

Proof. By lemma 4.1, there holds $\lim_{|z|\to\infty}\frac{H(z)}{|z|} = \infty$. Applying lemma 2.9 (c), $H^* \in C^1(\mathbb{R}^2)$.

https://doi.org/10.1017/prm.2025.10038 Published online by Cambridge University Press

As $\nabla H : \mathbb{R}^2 \to \mathbb{R}^2$ is strictly monotone, ∇H is injective. We claim that ∇H is surjective. This fact should be known, we give a proof here for the sake of completeness. Indeed, for any $R_1 > 0$, by $(H5)$, there exists $R_2 > 0$ such that if $|z| \geq R_2$, then $|\nabla H(z)| > R_1$. Obviously, for any $w \in B_{R_1}$, $\deg(\nabla H, B_{R_2}, w)$ is well defined and

$$\deg(\nabla H, B_{R_2}, w) = \deg(\nabla H, B_{R_2}, 0), \tag{4.3}$$

where 'deg' denotes topological degree, see [38, Appendix]. We define a homotopy $F : B_{R_2} \times [0,1] \to \mathbb{R}^2$,

$$F(z,t) = \nabla H\left(\frac{z}{1+t}\right) - \nabla H\left(\frac{-tz}{1+t}\right). \tag{4.4}$$

For any $z \in \partial B_{R_2}$ and $t \in [0,1]$, we have $\frac{z}{1+t} \neq \frac{-tz}{1+t}$, hence $F(z,t) \neq 0$ as ∇H is injective. Therefore,

$$\deg(F(\cdot,1), B_{R_2}, 0) = \deg(F(\cdot,0), B_{R_2}, 0) = \deg(\nabla H, B_{R_2}, 0).$$

According to Borsuk theorem [38, Theorem D.17], $\deg(F(\cdot,1), B_{R_2}, 0)$ is odd, which implies

$$\deg(\nabla H, B_{R_2}, 0) \neq 0.$$

So by (4.3) we get

$$\deg(\nabla H, B_{R_2}, w) \neq 0, \quad \forall\, w \in B_{R_1},$$

which deduces that there exists some $z \in B_{R_2}$ such that $\nabla H(z) = w$. Due to the arbitrariness of R_1, we complete the proof of this claim.

By Brouwer's invariance of domain theorem, it's known that any continuous bijection of Euclidean space is indeed a homeomorphism; we can get this fact also by noting that ∇H is proper, i.e. the preimage of any bounded set is bounded seeing $(H5)$, so ∇H is a closed mapping, hence a homeomorphism. Finally, it follows from lemma 2.9 (b) that $(\nabla H)^{-1} = \nabla H^*$. □

Now we consider the functional version of the previous lemma. Recall that $H_z(u,v) = \nabla H(u,v)$ and $H_z^*(u,v) = \nabla H^*(u,v)$.

LEMMA 4.4. *H_z is a homeomorphism from X^* onto X, and $H_z^{-1} = H_z^*$.*

Proof. We claim that H_z is continuous. Consider $(u_n, v_n) \to (u,v)$ in X^*, as before, up to a subsequence, there exists $(\Phi_1, \Phi_2) \in X^*$ such that $|u_n| \leq \Phi_1$, $|v_n| \leq \Phi_2$ a.e. for all n; $u_n \to u$ and $v_n \to v$ a.e. in Ω. Using $(H5)$ once more, we obtain

$$|H_u(u_n, v_n)| \leq C\Big(|\Phi_1|^p + |\Phi_1|^{\alpha-1}|\Phi_2|^\beta\Big) \in L^{1+\frac{1}{p}}(\Omega)$$

and

$$|H_v(u_n, v_n)| \leq C\Big(|\Phi_2|^q + |\Phi_1|^\alpha|\Phi_2|^{\beta-1}\Big) \in L^{1+\frac{1}{q}}(\Omega).$$

By Lebesgue's theorem, we deduce that $H_z(u_n, v_n) \to H_z(u,v)$ in X, which means H_z is continuous.

https://doi.org/10.1017/prm.2025.10038 Published online by Cambridge University Press

Given any $(f,g)\in X$, $(u,v)=(\nabla H)^{-1}(f,g)=\nabla H^*(f,g)\in X^*$ thanks to $(H5)$, since $|u(x)|^p\le C|f(x)|$ and $|v(x)|^q\le C|g(x)|$. The bijectivity of H_z is an easy consequence by definition. The continuity of $H_z^{-1}=H_z^*$ can be proved as for H_z, we omit the details. □

Finally, we state properties of the Legendre–Fenchel transform $\mathcal{H}^*$ for the Hamiltonian functional $\mathcal{H}$.

LEMMA 4.5. $\mathcal{H}^*\in C^1(X,\mathbb{R})$. *More precisely,*

$$\mathcal{H}^*(f,g)=\int_\Omega H^*(f,g)dx,\quad \left\langle (\mathcal{H}^*)'(f,g),(\widetilde{f},\widetilde{g})\right\rangle=\int_\Omega H_u^*(f,g)\widetilde{f}+H_v^*(f,g)\widetilde{g}dx. \tag{4.5}$$

Proof. For any $(f,g)\in X$, by lemma 2.9 (b) and lemma 4.3, there exist functions u,v such that

$$H^*(f,g)=fu+gv-H(u,v),$$

and $(u,v)=H_z^*(f,g)=\nabla H^*(f,g)$. By lemma 4.4, we have $(u,v)\in X^*$. In view of (4.2), it follows that

$$\begin{aligned}\mathcal{H}^*(f,g)&=\sup_{(\widetilde{u},\widetilde{v})\in X^*}\int_\Omega f\widetilde{u}+g\widetilde{v}-H(\widetilde{u},\widetilde{v})dx\\&\le\int_\Omega\sup_{(t,s)\in\mathbb{R}^2}\{f(x)t+g(x)s-H(t,s)\}\,dx\\&=\int_\Omega H^*(f,g)dx\\&=\int_\Omega fu+gv-H(u,v)dx\\&\le\mathcal{H}^*(f,g),\end{aligned}$$

which implies the expression of $\mathcal{H}^*$. By $H^*\in C^1(\mathbb{R}^2,\mathbb{R})$ and lemma 4.4, we get the expression of $(\mathcal{H}^*)'$ and $\mathcal{H}^*\in C^1(X^*,\mathbb{R})$. □

Next, we explain how to find weak solutions to (1.1). Clearly, $(-\Delta)^s$ is an isomorphism of $\mathcal{W}^{2s,r}(\Omega)$ onto $L^r(\Omega)$. Therefore we can denote its inverse by $\mathcal{A}:L^r(\Omega)\to\mathcal{W}^{2s,r}(\Omega)$. Define $\mathcal{J}:X\to\mathbb{R}$ by

$$\mathcal{J}(f,g)=\mathcal{H}^*(f,g)-\int_\Omega g\mathcal{A}fdx.$$

Using proposition 2.6, $\mathcal{J}$ is well defined.

https://doi.org/10.1017/prm.2025.10038 Published online by Cambridge University Press

LEMMA 4.6. *$\mathcal{A}$ is self-adjoint in the following sense:*

$$\int_\Omega g\mathcal{A}f dx = \int_\Omega f\mathcal{A}g dx, \quad \forall\, (f,g) \in X. \tag{4.6}$$

Proof. In fact, this is a direct consequence of the self-adjointness of $(-\Delta)^s$ over $X_0^s(\Omega)$ and density argument. Let $\{f_n\}, \{g_n\} \subset C_c^\infty(\Omega)$ satisfying $(f_n, g_n) \to (f, g)$ in X. Set $(u_n, v_n) = (\mathcal{A}f_n, \mathcal{A}g_n) \in X_0^s(\Omega)^2$. Then

$$\int_\Omega g_n\mathcal{A}f_n dx = \int_\Omega u_n(-\Delta)^s v_n dx = \int_\Omega v_n(-\Delta)^s u_n dx = \int_\Omega f_n\mathcal{A}g_n dx.$$

Taking $n \to \infty$, we get (4.6). □

We are now in position to present the dual method setting.

PROPOSITION 4.7. *If $(f,g) \in X$ is a critical point of $\mathcal{J}$, then $H_z^*(f,g)$ is a L^1-weak solution to (1.1).*

Proof. Since $(f,g) \in X$ is a critical point of $\mathcal{J}$,

$$\begin{aligned} 0 =& \langle \mathcal{J}'(f,g), (f_1,g_1)\rangle = \\ & \langle (\mathcal{H}^*)'(f,g), (f_1,g_1)\rangle - \int_\Omega g\mathcal{A}f_1 dx - \int_\Omega g_1\mathcal{A}f dx, \quad \forall\, (f_1,g_1) \in X. \end{aligned} \tag{4.7}$$

Let $(u,v) = H_z^*(f,g)$, then by lemma 4.5, we have

$$\langle (\mathcal{H}^*)'(f,g), (f_1,g_1)\rangle = \int_\Omega uf_1 dx + \int_\Omega vg_1 dx. \tag{4.8}$$

As $H_z^* = H_z^{-1}$, we obtain $(f,g) = (H_u(u,v), H_v(u,v))$. If we choose $g_1 = 0$, then using (4.7), (4.8), and lemma 4.6, it holds that

$$\int_\Omega uf_1 dx = \langle (\mathcal{H}^*)'(f,g), (f_1,0)\rangle = \int_\Omega g\mathcal{A}f_1 dx = \int_\Omega f_1\mathcal{A}g dx, \quad \forall\, f_1 \in L^{1+\frac{1}{p}}(\Omega).$$

Hence $u = \mathcal{A}g$, i.e. $(-\Delta)^s u = g = H_v(u,v)$ in the weak sense. Similarly, $(-\Delta)^s v = H_u(u,v)$. □

4.1. Proof of theorem 1.5 completed

Now we establish a mountain pass structure to get existence of nontrivial critical points of $\mathcal{J}$. Set

$$S_\rho := \{(\rho^{k-1}f, \rho^{l-1}g) : (f,g) \in X,\ \|(f,g)\|_X = \rho\}$$

where $k, l > 1$ satisfy

$$\frac{p}{p+1} > \frac{k}{k+l}, \quad \frac{q}{q+1} > \frac{l}{k+l}. \tag{4.9}$$

https://doi.org/10.1017/prm.2025.10038 Published online by Cambridge University Press

Assume $0 < \rho < 1$. For any $(\rho^{k-1}f, \rho^{l-1}g) \in S_\rho$, by lemma 4.2 and proposition 2.6,

$$\begin{aligned}\mathcal{J}(\rho^{k-1}f, \rho^{l-1}g) &= \mathcal{H}^*(\rho^{k-1}f, \rho^{l-1}g) - \rho^{k+l-2}\int_\Omega g\mathcal{A}f dx \\ &\geq C_1\rho^{(k-1)(1+1/p)}|f|^{1+\frac{1}{p}}_{1+\frac{1}{p}} + C_1\rho^{(l-1)(1+1/q)}|g|^{1+\frac{1}{q}}_{1+\frac{1}{q}} - C_2\rho^{k+l} \\ &\geq C_3\rho^{\max\{k(1+1/p),\, l(1+1/q)\}} - C_2\rho^{k+l}.\end{aligned}$$

According to (4.9), there exist $\rho_0 > 0, \beta > 0$ such that $\mathcal{J}(\rho_0^{k-1}f, \rho_0^{l-1}g) > \beta$ if $(\rho_0^{k-1}f, \rho_0^{l-1}g) \in S_{\rho_0}$. On the other hand, we fix some $(f_0, g_0) \in X$ with

$$\int_\Omega f_0\mathcal{A}g_0 dx > 0.$$

By lemma 4.2,

$$\mathcal{J}(\rho^k f_0, \rho^l g_0) \leq C_4\rho^{k(1+1/p)}|f_0|^{1+\frac{1}{p}}_{1+\frac{1}{p}} + C_4\rho^{l(1+1/q)}|g_0|^{1+\frac{1}{q}}_{1+\frac{1}{q}} - \rho^{k+l}\int_\Omega f_0\mathcal{A}g_0 dx.$$

Consequently, due to (4.9), there exists $\rho_1 > \rho_0$ such that $\mathcal{J}(\rho_1^k f_0, \rho_1^l g_0) < 0$. Let

$$\Gamma := \{\gamma \in C([0,1], X) : \gamma(0) = 0,\ \gamma(1) = (\rho_1^k f_0, \rho_1^l g_0)\}.$$

It is clear that $\gamma([0,1]) \cap S_{\rho_0} \neq \varnothing$ for any $\gamma \in \Gamma$, from which we obtain a mountain pass structure of $\mathcal{J}$ around 0, and define the mountain pass level by

$$c := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\mathcal{J}(\gamma(t)) \geq \beta > 0.$$

LEMMA 4.8. *$\mathcal{J}$ satisfies the Palais–Smale condition.*

Proof. Let $\{(f_n, g_n)\} \subset X$ be a Palais–Smale consequence of $\mathcal{J}$ at level c, that is,

$$\mathcal{J}(f_n, g_n) \to c \ \text{ and } \ \mathcal{J}'(f_n, g_n) \to 0 \ \text{ in } X^*, \quad \text{as } n \to \infty. \tag{4.10}$$

From (4.10) and (4.6), it follows that

$$\mathcal{H}^*(f_n, g_n) - \langle(\mathcal{H}^*)'(f_n, g_n), ((1-\theta)f_n, \theta g_n)\rangle = c + o(1)\|(f_n, g_n)\|_X + o(1) \tag{4.11}$$

where θ is given in $(H7)$. Let $(u_n, v_n) = H_z^*(f_n, g_n)$. According to lemma 2.9 (b) and lemma 4.5,

$$\mathcal{H}^*(f_n, g_n) = \int_\Omega f_n u_n + g_n v_n dx - \mathcal{H}(u_n, v_n), \tag{4.12}$$

https://doi.org/10.1017/prm.2025.10038 Published online by Cambridge University Press

which together with (4.11) and $(H7)$ implies that

$$\begin{aligned} & c + o(1)\|(f_n, g_n)\|_X + o(1) \\ &= \int_\Omega f_n u_n + g_n v_n dx - \mathcal{H}(u_n, v_n) - \langle (\mathcal{H}^*)'(f_n, g_n), ((1-\theta)f_n, \theta g_n)\rangle \\ &= \theta \int_\Omega H_u(u_n, v_n)u_n dx + (1-\theta)\int_\Omega H_v(u_n, v_n)v_n dx - \mathcal{H}(u_n, v_n) \\ &\geq C_1 \int_\Omega |u_n|^{p+1} dx + C_1 \int_\Omega |v_n|^{q+1} dx - C_2|\Omega|. \end{aligned}$$

Combining the above inequality, (4.12), lemma 4.1, lemma 4.2, and Young's inequality, $\{(f_n, g_n)\}$ is bounded in X.

Furthermore, by proposition 2.6, $\mathcal{A} : X \to X^*$ is compact, hence $\{(\mathcal{A}f_n, \mathcal{A}g_n)\}$ is compact in X^*. By (4.10),

$$\mathcal{J}'(f_n, g_n) = H_z^*(f_n, g_n) - (\mathcal{A}g_n, \mathcal{A}f_n) = o(1) \quad \text{in } X^*. \tag{4.13}$$

Since H_z^* is a homeomorphism from X onto X^*, $\{(f_n, g_n)\}$ is compact in X. The proof is done. □

As $\mathcal{J}$ satisfies the (PS) condition, applying mountain pass theorem [2] and proposition 4.7, we get a nontrivial solution $(u, v) \in X^*$ of (1.1). Since $(H5)$ implies $(H4)$, by lemma 3.9, (u, v) is a classical solution.

5. Positive solutions for fractional Lane–Emden system

As a very special case of (1.1), we consider the system (1.24) under the subcritical assumption (1.25). Using lemma 3.9, we know that any energy solution of (1.26) belongs to $C^s(\overline{\Omega})$.

Proposition 5.1. *Let p, q satisfy (1.25). Then u is an energy solution to (1.26) if and only if (u, v) is a classical solution to (1.24), where $v = |(-\Delta)^s u|^{\frac{1}{q}-1}(-\Delta)^s u$.*

Proof. First, assume that u is an energy solution to (1.26). It follows from (1.27) that

$$\int_\Omega v(-\Delta)^s \varphi dx = \int_\Omega |u|^{p-1} u\varphi dx \quad \forall\, \varphi \in \mathcal{T}_s(\Omega). \tag{5.1}$$

Next, since $v = |(-\Delta)^s u|^{\frac{1}{q}-1}(-\Delta)^s u$ and $u \in \mathcal{W}^{2s,1+\frac{1}{q}}(\Omega)$, we have

$$\int_\Omega u(-\Delta)^s \varphi dx = \int_\Omega |v|^{q-1} v\varphi dx \quad \forall\, \varphi \in \mathcal{T}_s(\Omega). \tag{5.2}$$

By (5.1) and (5.2), (u, v) is a weak solution to (1.24). Since u is an energy solution to (1.26), using lemma 3.9, u, $v \in C^s(\overline{\Omega})$.

https://doi.org/10.1017/prm.2025.10038 Published online by Cambridge University Press

Conversely, if (u, v) is a classical solution to (1.24), (5.1), and (5.2) will hold, which implies

$$\int_\Omega |(-\Delta)^s u|^{\frac{1}{q}-1}(-\Delta)^s u(-\Delta)^s \varphi dx = \int_\Omega |u|^{p-1} u\varphi dx \quad \forall\, \varphi \in \mathcal{W}^{2s,1+\frac{1}{q}}(\Omega). \tag{5.3}$$

Thus, u is an energy solution to (1.26). □

We denote

$$\mathcal{R}(u) := \langle \mathcal{I}'(u), u\rangle = |(-\Delta)^s u|^{1+\frac{1}{q}}_{1+\frac{1}{q}} - |u|^{p+1}_{p+1}, \quad \forall\, u \in \mathcal{W}^{2s,1+\frac{1}{q}}(\Omega). \tag{5.4}$$

For any $u \in \mathcal{N}_{\mathcal{I}}$ (see (1.28)), there holds

$$\langle \mathcal{R}'(u), u\rangle = \left(1+\frac{1}{q}\right)|(-\Delta)^s u|^{1+\frac{1}{q}}_{1+\frac{1}{q}} - (p+1)|u|^{p+1}_{p+1} = \left(\frac{1}{q}-p\right)|(-\Delta)^s u|^{1+\frac{1}{q}}_{1+\frac{1}{q}},$$

which implies

$$\begin{cases} \langle \mathcal{R}'(u), u\rangle < 0, & \text{if } \ pq > 1, \\ \langle \mathcal{R}'(u), u\rangle > 0, & \text{if } \ pq < 1. \end{cases} \tag{5.5}$$

Using the implicit function theorem, if $pq \neq 1$, $\mathcal{N}_{\mathcal{I}}$ is a C^1-submanifold of $\mathcal{W}^{2s,1+\frac{1}{q}}(\Omega)$ with codimension 1.

LEMMA 5.2. *If $pq \neq 1$, then $\mathcal{N}_{\mathcal{I}}$ is non-empty. Moreover, when $pq > 1$, $\mathcal{N}_{\mathcal{I}}$ is far away from zero and $\mathcal{I}$ constrained on $\mathcal{N}_{\mathcal{I}}$ has a positive lower bound; when $pq < 1$, $\mathcal{N}_{\mathcal{I}}$ is bounded in $\mathcal{W}^{2s,1+\frac{1}{q}}(\Omega)$.*

Proof. For any $u \in \mathcal{W}^{2s,1+\frac{1}{q}}(\Omega)\backslash\{0\}$, when $pq>1$ (resp. $pq<1$), let t_u be the maximum (resp. minimum) point of $\mathcal{I}(tu)$ for $t>0$. Then $t_u u \in \mathcal{N}_{\mathcal{I}}$, so $\mathcal{N}_{\mathcal{I}}$ is non empty. For any $u \in \mathcal{N}_{\mathcal{I}}$, using (5.4) and $\langle \mathcal{I}'(u), u\rangle = 0$, we have

$$|(-\Delta)^s u|^{1+\frac{1}{q}}_{1+\frac{1}{q}} = |u|^{p+1}_{p+1} \le C\,|(-\Delta)^s u|^{p+1}_{1+\frac{1}{q}}. \tag{5.6}$$

Therefore if $pq>1$, there exists some $C_0>0$ such that

$$|(-\Delta)^s u|_{1+\frac{1}{q}} > C_0, \quad \forall\, u \in \mathcal{N}_{\mathcal{I}}. \tag{5.7}$$

Next, if $u \in \mathcal{N}_{\mathcal{I}}$, by (5.4),

$$\mathcal{I}(u) = \left(\frac{q}{q+1} - \frac{1}{p+1}\right)|(-\Delta)^s u|^{1+\frac{1}{q}}_{1+\frac{1}{q}}.$$

From (5.7) and $pq>1$, it follows that $\mathcal{I}$ has a positive lower bound on $\mathcal{N}_{\mathcal{I}}$. When $pq<1$, we know from (5.6) that $\mathcal{N}_{\mathcal{I}}$ is bounded in $\mathcal{W}^{2s,1+\frac{1}{q}}(\Omega)$. □

https://doi.org/10.1017/prm.2025.10038 Published online by Cambridge University Press

In the sequel, we consider the functional $\mathcal{I}$ constrained on $\mathcal{N}_{\mathcal{I}}$. A constrained critical point u of $\mathcal{I}|_{\mathcal{N}_{\mathcal{I}}}$, means that there exists a Lagrange multiplier $\lambda \in \mathbb{R}$ such that

$$\mathcal{I}'(u) = \lambda \mathcal{R}'(u) \quad \text{in } \mathcal{W}^{2s,1+\frac{1}{q}}(\Omega)^*, \tag{5.8}$$

where $\mathcal{W}^{2s,1+\frac{1}{q}}(\Omega)^*$ is the dual space of $\mathcal{W}^{2s,1+\frac{1}{q}}(\Omega)$. In particular, u is a critical point of $\mathcal{I}$ whenever $\lambda = 0$.

LEMMA 5.3. *Assume that $pq \neq 1$. Then any constrained critical point of $\mathcal{I}|_{\mathcal{N}_{\mathcal{I}}}$ is a critical point of $\mathcal{I}$.*

Proof. Suppose that u is a constrained critical point, there is $\lambda \in \mathbb{R}$ such that (5.8) holds. Thus,

$$0 = \langle \mathcal{I}'(u), u\rangle = \lambda \langle \mathcal{R}'(u), u\rangle = \lambda \left(\frac{1}{q} - p\right) |(-\Delta)^s u|^{1+\frac{1}{q}}_{1+\frac{1}{q}}, \tag{5.9}$$

which deduces $\lambda = 0$. Consequently, u is a critical point of $\mathcal{I}$ □

LEMMA 5.4. *For $pq \neq 1$, $\mathcal{I}|_{\mathcal{N}_{\mathcal{I}}}$ satisfies the (*PS*) condition.*

Proof. Let $\{u_n\}$ be a (PS) sequence at level $c \in \mathbb{R}$ for $\mathcal{I}|_{\mathcal{N}_{\mathcal{I}}}$, that is $\{u_n\} \subset \mathcal{N}_{\mathcal{I}}$, $\lambda_n \in \mathbb{R}$ such that

$$\mathcal{I}(u_n) = c + o(1) \quad \text{and} \quad \mathcal{I}'(u_n) - \lambda_n \mathcal{R}'(u_n) = o(1) \ \text{ in } \mathcal{W}^{2s,1+\frac{1}{q}}(\Omega)^*. \tag{5.10}$$

Therefore

$$c + o(1) = \mathcal{I}(u_n) - \frac{1}{p+1}\mathcal{R}(u_n) = \left(\frac{q}{q+1} - \frac{1}{p+1}\right) |(-\Delta)^s u_n|^{1+\frac{1}{q}}_{1+\frac{1}{q}}.$$

By $pq \neq 1$, one can conclude the boundedness of $\{u_n\}$ in $\mathcal{W}^{2s,1+\frac{1}{q}}(\Omega)$, which yields also that $\{\langle \mathcal{R}'(u_n), u_n\rangle\}$ is bounded. Up to a subsequence, assume that

$$\langle \mathcal{R}'(u_n), u_n\rangle \to m \in \mathbb{R}. \tag{5.11}$$

Let $m \neq 0$, (5.10) yields $o(1) = \langle \mathcal{I}'(u_n) - \lambda_n \mathcal{R}'(u_n), u_n\rangle = \lambda_n m + o(1)$, which yields $\lambda_n \to 0$. Up to a new subsequence, there exists $u \in \mathcal{W}^{2s,1+\frac{1}{q}}(\Omega)$ such that $(-\Delta)^s u_n \rightharpoonup (-\Delta)^s u$ in $L^{\frac{q+1}{q}}(\Omega)$, and $u_n \to u$ in $L^{p+1}(\Omega)$, so $u \in \mathcal{N}_{\mathcal{I}}$. By direct computations,

$$o(1) = |u_n|^{p+1}_{p+1} - |u|^{p+1}_{p+1} = |(-\Delta)^s u_n|^{1+\frac{1}{q}}_{1+\frac{1}{q}} - |(-\Delta)^s u|^{1+\frac{1}{q}}_{1+\frac{1}{q}},$$

which together with the weak convergence of $(-\Delta)^s u_n$ in $L^{1+\frac{1}{q}}(\Omega)$ yields that $u_n \to u$ in $\mathcal{W}^{2s,1+\frac{1}{q}}(\Omega)$.

Let now $m = 0$. It is clear from (5.11) and $u_n \in \mathcal{N}_{\mathcal{I}}$ that

$$o(1) = \langle \mathcal{R}'(u_n), u_n\rangle = \left(\frac{1}{q} - p\right) |(-\Delta)^s u_n|^{1+\frac{1}{q}}_{1+\frac{1}{q}},$$

which deduces that $|(-\Delta)^s u_n|_{1+\frac{1}{q}} \to 0$. So we are done. □

https://doi.org/10.1017/prm.2025.10038 Published online by Cambridge University Press

Proof of theorem 1.9 completed. (i) We apply a deformation lemma on C^1 manifold (see [7]) to $\mathcal{I}|_{\mathcal{N}_\mathcal{I}}$. By lemma 5.2, it is easy to see that $c_\mathcal{I} < 0$ if $pq < 1$ and $c_\mathcal{I} > 0$ if $pq < 1$. By lemma 5.4, $c_\mathcal{I}$ is attained by some $u \in \mathcal{N}_\mathcal{I}$. Hence, u is a constrained critical point of $\mathcal{I}|_{\mathcal{N}_\mathcal{I}}$. Using lemma 5.3, u is also a critical point of $\mathcal{I}$, hence an energy solution of (1.26). Let w be the weak solution of

$$(-\Delta)^s w = |(-\Delta)^s u| \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^N \setminus \Omega.$$

By lemma 2.1, $w \geq \pm u$, so $w \geq |u|$. Consequently, $\mathcal{I}(tw) \leq \mathcal{I}(tu)$ for all $t > 0$. Thus there exists a unique $t_w > 0$ such that $t_w w \in \mathcal{N}_\mathcal{I}$ and $\mathcal{I}(t_w w) \leq \mathcal{I}(u) = c_\mathcal{I}$. Then $t_w w$ is a minimizer for $c_\mathcal{I}$. By means of lemma 2.2, one has $t_w w > 0$. The proof can be concluded by proposition 5.1.

(ii) Assume that $u_1, u_2 \in \mathcal{W}^{2s,1+\frac{1}{q}}(\Omega)$ are two distinct positive energy solutions of (1.26). We know that $u_1, u_2 \in C^s(\overline{\Omega})$. Denote

$$\beta_0 := \sup\{l \in \mathbb{R} : u_2 \geq l u_1 \text{ a.e. in } \Omega\} \quad \text{and} \quad \beta_1 := \sup\{l \in \mathbb{R} : u_1 \geq l u_2 \text{ a.e. in } \Omega\}.$$

From lemma 2.2 and lemma 1.1, there exist $C_1, C_2 > 0$ such that $C_1 \delta^s \leq u_1, u_2 \leq C_2 \delta^s$. Therefore $0 < \beta_0, \beta_1 < \infty$. In order to prove $u_1 = u_2$, it suffices to prove that $\beta_0 \geq 1$ and $\beta_1 \geq 1$. Notice that

$$\begin{aligned}(-\Delta)^s \left(|(-\Delta)^s u_1|^{\frac{1}{q}-1}(-\Delta)^s u_1\right) &= u_1^p \geq \beta_1^p u_2^p \\ &= \beta_1^p (-\Delta)^s \left(|(-\Delta)^s u_2|^{\frac{1}{q}-1}(-\Delta)^s u_2\right) \\ &= (-\Delta)^s \left(|(-\Delta)^s(\beta_1^{pq} u_2)|^{\frac{1}{q}-1}(-\Delta)^s(\beta_1^{pq} u_2)\right).\end{aligned}$$

By comparison principle, we get $u_1 \geq \beta_1^{pq} u_2$. So $\beta_1 \geq \beta_1^{pq}$, hence $\beta_1 \geq 1$. A similar argument gives $\beta_0 \geq 1$. By proposition 5.1, the classical solution of (1.24) is unique.

(iii) Replacing $\mathcal{W}^{2s,1+\frac{1}{q}}(B_R)$ by $\mathcal{W}^{2s,1+\frac{1}{q}}_{\text{rad}}(B_R)$, we can check that all the above steps work, so we have a positive classical radially symmetric solution. □

Acknowledgements

The author completed this paper during his Ph.D. in East China Normal University. The author would like to thank Prof. Dong Ye for careful reading of the manuscript.

References

1 R. Adams and J. Fournier. *Sobolev spaces, Second edition, Pure and Applied Mathematics 140.* (Elsevier/Academic Press, Amsterdam, 2003).

2 A. Ambrosetti and P. H. Rabinowitz. Dual variational methods in critical point theory and applications. *J. Funct. Anal.* **14** (1973), 349–381.

3 V. Benci and P. H. Rabinowitz. Critical point theorems for indefinite functionals. *Invent. Math.* **52** (1979), 241–273.

4 U. Biccari, M. Warma and E. Zuazua. Local elliptic regularity for the Dirichlet fractional Laplacian. *Adv. Nonlinear Stud.* **17** (2017), 387–409.

5 D. Bonheure, E. Moreira dos Santos and M. Ramos. Ground state and non-ground state solutions of some strongly coupled elliptic systems. *Trans. Amer. Math. Soc.* **364** (2012), 447–491.

https://doi.org/10.1017/prm.2025.10038 Published online by Cambridge University Press

6 D. Bonheure, E. Moreira dos Santos and H. Tavares. Hamiltonian elliptic systems: a guide to variational frameworks. *Port. Math.* **71** (2014), 301–395.

7 A. Bonnet. A deformation lemma on a C^1 manifold. *Manuscripta Math.* **81** (1993), 339–359.

8 L. Brasco and G. Franzina. Convexity properties of Dirichlet integrals and Picone-type inequalities. *Kodai Math. J.* **37** (2014), 769–799.

9 H. Brezis. *Functional analysis, Sobolev Spaces and Partial Differential Equations.* (Springer, New York, 2010).

10 H. Brezis, T. Cazenave, Y. Martel and A. Ramiandrisoa. Blow up for $u_t - \Delta u = g(u)$ revisited. *Adv. Differential Equations.* **1** (1996), 73–90.

11 L. Caffarelli, J. M. Roquejoffre and Y. Sire. Variational problems with free boundaries for the fractional Laplacian. *J. Eur. Math. Soc.* **12** (2010), 1151–1179.

12 W. Choi. On strongly indefinite systems involving the fractional Laplacian. *Nonlinear Anal.* **120** (2015), 127–153.

13 W. Choi and S. Kim. Minimal energy solutions to the fractional Lane-Emden system: existence and singularity formation. *Rev. Mat. Iberoam.* **35** (2019), 731–766.

14 P. H. Clément, D. G. de Fegueiredo and E. Mitedieri. Positive solutions of semilinear elliptic systems. *Comm. Partial Differ. Equ.* **17** (1992), 923–940.

15 P. H. Clément and R. C. A. M. van der Vorst. On a semilinear elliptic system. *Differential Integral Equations.* **8** (1995), 13171–329.

16 E. Clarke and I Ekeland. Hamiltonian trajectories having prescribed minimal period. *Comm. Pure Appl. Math.* **33** (1980), 103–116.

17 R. Dalmasso. Existence and uniqueness of positive solutions of semilinear elliptic systems. *Nonlinear Anal.* **39** (2000), 559–568.

18 D. G. de Figueiredo, J. M. do Ó and B. Ruf. An Orlicz-space approach to superlinear elliptic systems. *J. Funct. Anal.* **224** (2005), 471–496.

19 D. G. de Figueiredo and P. L. Felmer. On superquadratic elliptic systems. *Trans. Amer. Math. Soc.* **343** (1994), 99–116.

20 E. Di Nezza, G. Palatucci and E. Valdinoci. Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. Math.* **136** (2012), 521–573.

21 I Ekeland and R. Temam. *Convex Analysis and Variational problems, Society for Industrial and Applied Mathematics (SIAM).* Philadelphia, (1999).

22 P. L. Felmer. Periodic solutions of "superquadratic" Hamiltonian systems. *J. Differential Equations.* **102** (1993), 188–207.

23 V. Ferone and B. Volzone. Symmetrization for fractional elliptic problems: a direct approach. *Arch. Ration. Mech. Anal.* **239** (2021), 1733–1770.

24 J. Hulshof and R. C. A. M. van der Vorst. Differential systems with strongly indefinite variational structure. *J. Funct. Anal.* **114** (1993), 32–58.

25 J. Hulshof, E. Mitidieri and R. C. A. M. van der Vorst. Strongly indefinite systems with critical Sobolev exponents. *Trans. Amer. Math. Soc.* **350** (1998), 2349–2365.

26 E. J. F. Leite and M. Montenegro. On positive viscosity solutions of fractional Lane-Emden systems. *Topol. Methods Nonlinear Anal.* **53** (2019), 407–425.

27 T. Leonori, I Peral, A. Primo and F. Soria. Basic estimates for solutions of a class of nonlocal elliptic and parabolic equations. *Discrete Contin. Dyn. Syst.* **35** (2015), 6031–6068.

28 J. Mawhin and M. Willem. *Critical Point Theory and Hamilton System.* (Springer, Berlin, 1989).

29 E. Mitidieri. A Rellich type identity and applications: Identity and applications. *Comm. Partial Differential Equations.* **18** (1993), 125–151.

30 A. Persson. Compact linear mappings between interpolation spaces. *Ark. Mat.* **5** (1964), 215–219.

31 L. D. Pezzo and A. Quaas. A Hopf's lemma and a strong minimum principle for the fractional p-Laplacian. *J. Differential Equations.* **263** (2017), 765–778.

32 M. Ramos and H. Tavares. Solutions with multiple spike patterns for an elliptic system. *Calc. Var. Partial Differential Equations.* **31** (2008), 1–25.

33 M. Ramos, H. Tavares and W. Zou. A Bahri-Lions theorem revisited. *Adv. Math.* **222** (2009), 2173–2195.

https://doi.org/10.1017/prm.2025.10038 Published online by Cambridge University Press

34 X. Ros-Oton and J. Serra. The extremal solution for the fractional Laplacian. *Calc. Var. Partial Differential Equations*. **50** (2014), 723–750.

35 X. Ros-Oton and J. Serra. The Dirichlet problem for the fractional Laplacian: regularity up to the boundary. *J. Math. Pures Appl.* **101** (2014), 275–302.

36 L. Silvestre. Regularity of the obstacle problem for a fractional power of the Laplace operator. *Commun. Pure Appl. Math.* **60** (2007), 67–112.

37 M. Struwe. *Variational methods, Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems*. 4th edn, (Springer-Verlag, Berlin, 2008).

38 M. Willem. *Minimax Theorems*. (Birkhäuser, Boston, 1996).

https://doi.org/10.1017/prm.2025.10038 Published online by Cambridge University Press