Proceedings of the Royal Society of Edinburgh: Section A Mathematics, 1–31, *2025*
DOI:10.1017/prm.2025.10098

Normalized solutions for p-Laplacian equation with general nonlinearities on bounded domain

Fengshuang Gao
School of Mathematics, Nanjing University of Aeronautics and Astronautics, Jiangsu, P.R. China (gfs@nuaa.edu.cn) (corresponding author)

Yuxia Guo
Department of Mathematical Science, Tsinghua University, Beijing, P.R. China (yguo@mail.tsinghua.edu.cn)

(Received 20 March 2025; revised 19 October 2025; accepted 19 October 2025)

We consider the existence, nonexistence and multiplicity of normalized solutions to the p-Laplacian equation on a bounded domain

$$\begin{cases} -\Delta_p u + \lambda |u|^{p-2}u = g(u), & \text{in } \Omega, \\ \int_\Omega |u|^p = \rho, u = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.1}$$

where Ω is a bounded domain, $p \geq 2$. Firstly, under suitable assumptions on ρ, if g is at most mass-critical at infinity, we prove the existence of infinitely many solutions. Secondly, for ρ large, if g is mass-supercritical, we perform a blow-up analysis to show the nonexistence of finite Morse index solutions. At last, for ρ suitably small, combining with the monotonicity argument, we obtain a multiplicity result. In particular, when $p = 2$, we obtain the existence of infinitely many normalized solutions.

Keywords: p-Laplacian equation; normalized solution; existence and multiplicity; bounded domain; finite Morse index solution

2020 *Mathematics Subject Classification:* 35A01, 35J10, 35J62

1. Introduction

This paper concerns the following p-Laplacian problem:

$$\begin{cases} -\Delta_p u + \lambda |u|^{p-2}u = g(u), & \text{in } \Omega, \\ \int_\Omega |u|^p = \rho, u = 0, & \text{on } \partial\Omega, \end{cases} , \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with $0 \in \Omega$, $N \geq 3$, $2 \leq p < N$ and g is a general nonlinear term. Under some suitable assumptions on g, we investigate how

© The Author(s), 2025. Published by Cambridge University Press on behalf of The Royal Society of Edinburgh.

https://doi.org/10.1017/prm.2025.10098 Published online by Cambridge University Press

the conditions on ρ affect the existence and multiplicity of $(\lambda, u) \in \mathbb{R} \times W_0^{1,p}(\Omega)$ such that (1.1) holds.

When $p = 2$ and $\Omega = \mathbb{R}^N$, the problem (1.1) is reduced to the following problem:

$$\begin{cases} -\Delta u + \lambda u = g(u), & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 = \rho. \end{cases} \tag{1.2}$$

Since the pioneering work of Jeanjean [13], there has been extensive attention paid to the existence of normalized solutions for (1.2), and a lot of contributions have been made. While it is impossible to list all these outstanding works, we refer, for example, the interested reader to [3, 15, 16, 27, 28, 30] for the problem with combined nonlinearities, and [1, 8, 19] for the problems with potentials.

In order to solve (1.2), a natural idea is to minimize the functional

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 - \int_{\mathbb{R}^N} G(u),$$

on the submanifold

$$S(\rho) = \{u \in H^1(\mathbb{R}^N), \|u\|_2^2 = \rho\},$$

where $G(u)$ is the primitive function of g. This is indeed the case when g is mass-subcritical (see [27, 28]). However, if g is mass-supercritical, $I(u)$ is unbounded from below on $S(\rho)$, and one may look for minimax critical point instead. During this process, one of the major difficulties is to find a bounded $(P.S.)$ sequence. In case Ω is the entire space $\mathbb{R}^N$, Pohozaev identity is a powerful tool in getting over this difficulty. However, when Ω is a general bounded domain, Pohozaev identity is not so effective due to the appearance of boundary integral terms. For this reason, there are not so many results on bounded domain compared to the case of entire space $\mathbb{R}^N$.

To the best of the author's knowledge, the first contribution in this direction is credited to Noris, Tavares and Vezini [20]. Utilizing bifurcation discussions, they established two positive solutions when Ω is a ball and g is the pure power nonlinearity $|u|^{q-2}u$ with $2 + \frac{4}{N} < q < 2^*$. Further studies on general bounded domain were carried out in [22], where the authors established the first multiple result and claimed the existence of mountain pass solutions as well. Nevertheless, as pointed by themselves in [23], the proof of Proposition 4.4 contained there misses the boundedness of the $(P.S.)$ sequence. Fortunately, in [23], the authors successfully fixed this problem by exploiting the monotonicity argument and blow-up analysis. We note in all these papers, a change of variable is performed to move the parameter from the constraint to the equation, which seems to break down for equations with general nonlinearities. To this end, there has been some interesting works on special domains where the boundary integral terms in Pohozaev identity could be well estimated, among which, the works of Bartsch, Qi, Zou, Song [2, 24, 25] are worth mentioning.

As far as the authors are concerned, there are not so many results known for p-Laplacian problems. In case of $\Omega = \mathbb{R}^N$, existence of normalized solutions are obtained in [11, 17, 31]. However, little is known for the existence of normalized solutions for (1.1) on general bounded domain. The main purpose of the present

https://doi.org/10.1017/prm.2025.10098 Published online by Cambridge University Press

paper is to study the normalized solutions for (1.1) on general bounded domains, including the existence, non-existence and multiplicities. For this, we introduce the following assumptions on g :

1. (G_1) $g \in C^1(\mathbb{R})$, $g'(s) > 0$ for $s > 0$ and $g(-s) = -g(s)$.
2. (G_2) there exists some $(\alpha, \beta) \in \mathbb{R}^2_+$ satisfying $p < \alpha, \beta < p^*$ such that

$$\lim_{s\to 0^+} \frac{g'(s)}{s^{\alpha-2}} = \mu_1(\alpha - 2) > 0, \text{and } \lim_{s\to+\infty} \frac{g'(s)}{s^{\beta-2}} = \mu_2(\beta - 2) > 0.$$

Set

$$S_\rho = \{u \in W_0^{1,p}(\Omega) : \|u\|_p^p = \rho\}$$

and let

$$E(u) = \frac{1}{p}\int_\Omega |\nabla u|^p - \int_\Omega G(u),$$

where $G(u)$ denotes the primitive function of g. When g is at most L^p critical at infinity, under suitable assumptions on ρ, $E(u)$ is coercive on S_ρ. In this case, classical variational methods can be applied to deduce the existence and multiplicity result, yielding the following existence and multiplicity result. For the case of the semilinear systems, one may refer to Reference [21].

THEOREM 1.1. *Suppose g satisfies assumptions (G_1)-(G_2) and one of the following conditions holds*

(1) $p < \beta < p + \frac{p^2}{N}$ *and* $\rho > 0$.
(2) $\beta = p + \frac{p^2}{N}$ *and ρ satisfies* (3.3).

Then (1.1) *admits a solution which is a global minimizer on S_ρ for $E(u)$. Moreover,* (1.1) *admits infinitely many solutions in these cases.*

As it can be seen from Theorem 1.1, when g is L^p-subcritical at infinity, (1.1) admits solutions for every $\rho > 0$, each being a global minimizer of the energy functional. However, for the L^p-critical case, it is only proved that there exist solutions for ρ suitablely small. Hence, for an arbitrary $\rho > 0$, it is natural to ask whether there exist solutions when g is at least L^p-critical at infinity.

In the following, we consider this problem. In particular, we restrict our attention to the existence of solutions with finite Morse index, and give a negative answer to the above question. More precisely, we have the following:

THEOREM 1.2. *For any fixed $k \geq 0$,*

$$\mathcal{U}_k := \{\rho > 0,\ (1.1) \text{ admits solution having Morse index } m(u) \leq k\}$$

then $\sup\ \mathcal{U}_k = +\infty$ *if and only if* $p < \beta < p + \frac{p^2}{N}$.

REMARK 1.3. We remark that under the assumption $p > 2$, E is of C^2, thus Morse index is well-defined.

https://doi.org/10.1017/prm.2025.10098 Published online by Cambridge University Press

Following the ideas in [23], we will perform a blow-up analysis to deduce the conclusion. However, note that, in order to deal with general nonlinearities, the change of variable in [23] is not applicable for generating a blow-up parameter. Instead, as in [6, 16], we take the Lagrange multiplier λ as a natural parameter, and use λ to perform a blow-up analysis. In this process, some additional difficulties need to be overcome due to the nonlinearity and degeneracy of the p-Laplace operator. This part of analysis is of independent interest. Therefore, We present the proof in the Appendix.

Recall that, under suitable assumptions on ρ, we have obtained the existence of global minimizer for E when g is at most L^p-critical at infinity. A natural question is what happens when g becomes L^p-supercritical. As shown in Section 2, when g is mass-supercritical at infinity, the energy functional E is unbounded from below on S_ρ. Consequently, it seems not feasible to look for global minimizer in this case. When considering local minimizers, it is important to find an $r > 0$ such that

$$\inf_{B_r} E(u) < \inf_{\partial B_r} E(u),$$

where

$$B_r := \left\{ u \in S_\rho, \int_\Omega |\nabla u|^p \le r \right\},$$

and this is indeed the case for ρ suitably small.

THEOREM 1.4. *Suppose g satisfies the assumptions (G_1)–(G_2) and one of the following conditions holds:*

(1) $\beta > p + \frac{p^2}{N}$, $\alpha \le \beta$ *and* ρ *satisfies* (4.4),
(2) $\beta > p + \frac{p^2}{N}$, $\alpha > \beta$ *and* ρ *satisfies* (4.5),

then (1.1) *admits a solution u_ρ which is a local minimizer for $E(u)$.*

Once the local minimizer is obtained, noting that $\inf_{u \in S_\rho} E(u) = -\infty$, it is natural to look for a second solution which is of mountain-pass type. We define

$$m := \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} E(\gamma(t)) > \max\left\{E(u_\rho), E(v)\right\},$$

where

$$\Gamma := \left\{\gamma \in C([0,1], S_\rho) : \gamma(0) = u_\rho, \gamma(1) = v\right\},$$

and v satisfies $E(v) < 0$, whose existence is guaranteed by Lemma 4.2 below. We aim to show that m can be attained. To do this, the main difficulty lies in finding a bounded *P.S.* sequence. In order to overcome this difficulty, we adopt the so-called Struwe monotonicity arguments as in [4, 14]. Roughly speaking, this argument is to introduce a family of functional E_θ depending on a parameter θ, for *almost every* θ, E_θ admits a solution u_θ. Then by the blow-up analysis performed in Theorem 1.2, u_θ converges to a solution of the original problem. This strategy has been successfully applied on graphs by Chang, Jeanjean and Soave [6], and to

https://doi.org/10.1017/prm.2025.10098 Published online by Cambridge University Press

semilinear elliptic equation with $g(u) = |u|^{q-2}u$, $2+\frac{4}{N} < q \le 2^*$ in [23]. For general $p \ge 2$, we have the following multiplicity result.

THEOREM 1.5. *Let g, ρ satisfy the conditions in Theorem 1.4, then equation (1.1) has a second solution $\tilde{u}_\rho \not\equiv u_\rho$ and $\tilde{u}_\rho$ is of mountain pass type.*

As a byproduct, we have the following result for $p = 2$.

THEOREM 1.6. *Let $p = 2$ and assume that $(G_1)-(G_2)$, then for any n, there exists $\rho_n > 0$ such that for $0 < \rho < \rho_n$, (1.1) has n solutions at the level c_k, $k = 1, \dots, n$ and $c_n \to +\infty$ as $n \to \infty$.*

Our paper is organized as follows. We recall some definitions and lemmas in Section 2. In 3, we prove Theorem 1.1 and 1.2. Section 4 is devoted to the mass-supercritical case, where Theorem 1.4, 1.5 and 1.6 are proved. We put the blow-up arguments used in the proof of Section 3 in the Appendix A. This part is of independent interest, and we believe it can be used to deal with other problems related to p-Laplacian equations.

2. Preliminary

In this section, let us recall some definitions and results that will be used throughout this paper, we refer interested readers to [4, 14, 16] and references therein.

DEFINITION 2.1. *Let $\varphi : W_0^{1,p}(\Omega) \to \mathbb{R}$ be a C^2-functional, we say that φ' and φ'' are $\alpha-$Hölder continuous on bounded set if for any $R > 0$, one can find $M = M(R) > 0$ such that, for any u_1, $u_2 \in B_R(0)$:*

$$\|\varphi'(u_1) - \varphi'(u_2)\|_* \le M\|u_1 - u_2\|^\alpha, \|\varphi''(u_1) - \varphi''(u_2)\|_{**} \le M\|u_1 - u_2\|^\alpha,$$

where $\|\cdot\|_*$ and $\|\cdot\|_{**}$ are the operator norm of $\mathcal{L}(W_0^{1,p}(\Omega), \mathbb{R})$ and of $\mathcal{L}(W_0^{1,p}(\Omega), \mathcal{L}(W_0^{1,p}(\Omega), \mathbb{R}))$ respectively.

DEFINITION 2.2. *For any $u \in S_\rho$ and $\xi \ge 0$, we use $T_u S_\rho$ to denote the tangent space at point u. We define the* approximate Morse index *and* Morse index *by*

$$\tilde{m}_\xi(u) = \max\left\{k : \exists V \subset T_u S_\rho,\ dim(V) = k : D^2\varphi(u)(v,v) < -\xi\|v\|^p \text{ for all } v \in V, v \not\equiv 0\right\},$$

and

$$m(u) = \max\left\{k : \exists V \subset C_c^1(\Omega),\ dim(V) = k : D^2\varphi(u)(v,v) < 0 \text{ for all } v \in V, v \not\equiv 0\right\},$$

where

$$D^2\varphi(u)(v,v) := \varphi''(u)(v,v) - \frac{\langle \varphi'(u), u\rangle}{\|u\|_p^p}(p-1)\int_\Omega |u|^{p-2}|v|^2.$$

We will use the following more general results on functional and Morse index due to [14].

https://doi.org/10.1017/prm.2025.10098 Published online by Cambridge University Press

THEOREM 2.3. *Let W be a Banach space equipped with the norm $\|\cdot\|$ and let $J \subset \mathbb{R}^+$ be an interval. We consider a family E_θ of C^1-functionals on W of the form*

$$E_\theta(u) = A(u) - \theta B(u), \theta \in J,$$

where $B(u) \geq 0$, $\forall u \in W$ and such that either $A(u) \to \infty$ or $B(u) \to \infty$ as $\|u\| \to \infty$. Let us assume that there are two points w_0, w_1 in S_ρ such that, setting

$$\Gamma = \{\gamma \in C([0,1], S_\rho), \gamma(0) = w_1, \gamma(1) = w_2\},$$

there holds, $\forall \theta \in J$,

$$m_\theta := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} E_\theta(\gamma(t)) > \max\{E_\theta(w_1), E_\theta(w_2)\}.$$

Then for almost every $\theta \in J$, there is a sequence $\{u_n\} \in S_\rho$ such that

(1) *$\{u_n\}$ is bounded in W.*
(2) *$E_\theta(u_n) \to m_\theta$ and $E'_\theta|_{S_\rho}(u_n) \to 0$ in W^{-1}.*

THEOREM 2.4. *Let W be a infinite-dimensional Hilbert space and $I \subset (0, \infty)$ be an interval, consider a family of C^2 functionals E_θ on W of the form*

$$E_\theta(u) = A(u) - \theta B(u), \theta \in I$$

where $B(u) \geq 0$ for all $u \in W$ and such that either $A(U) \to \infty$ or $B(u) \to \infty$ as $\|u\| \to +\infty$.

Suppose that E_θ is even for every $\theta \in I$, and moreover E'_θ and E''_θ are $\alpha-$Hölder continuous on bounded sets in the sense of Definition 2.1 for some $\alpha \in (0, 1]$. Finally, suppose that there exists a $k \in \mathbb{N}$, $k \geq 2$, given odd functions $\gamma_i : S^{k-2} \to S_\rho$ where $i = 1, 2$, such that the set

$$\Gamma_k = \{\gamma \in C([0,1] \times S^{k-2}, S_\rho) : \gamma(t, \cdot) \textit{is odd}\ , \gamma(0, \cdot) = \gamma_0, \gamma(1, \cdot) = \gamma_1\}, \quad (2.1)$$

is non-empty and that

$$c_{\theta,k} = \inf_{\gamma \in \Gamma_k} \max_{[0,1] \times S^{k-2}} E_\theta(\gamma(t, x)) > \max_{x \in S^{k-2}} \{E_\theta(\gamma_1(x)), E_\theta(\gamma_2(x))\}, \forall \theta \in I. \quad (2.2)$$

Then, for almost every $\theta \in I$, there exists sequence $\{u_n\} \subset S_\rho$ and $\xi_n \to 0^+$ such that as $n \to \infty$

(1) *$\{u_n\}$ is bounded in W.*
(2) *$E_\theta(u_n) \to c_{\theta,k}$ and $E'_\theta|_{S_\rho}(u_n) \to 0$ in W^{-1}.*
(3) *$\tilde{m}_{\xi_n}(u_n) \leq k$.*

Moreover, under the assumptions of $(G_1) - (G_2)$, by direct calculation, we can prove the following lemma as in [16].

LEMMA 2.5. *Let $N \geq 2$ and assume that $(G_1) - (G_2)$ hold, then*

https://doi.org/10.1017/prm.2025.10098 Published online by Cambridge University Press

(i) $\lim\limits_{|s|\to 0}\frac{g(s)}{|s|^{\alpha-2}s}=\mu_1$ *and* $\lim\limits_{|s|\to+\infty}\frac{g(s)}{|s|^{\beta-2}s}=\mu_2$.

(ii) *There exists some* $C>0$ *such that*

$$|g(s)|\le C(|s|^{\alpha-1}+|s|^{\beta-1}),$$

and for any $M>0$ *there exists some* $C_{M,1}\ge C_{M,2}>0$ *such that*

$$C_{M,2}|s|^{\alpha-2}s\le g(s)\le C_{M,1}|s|^{\alpha-2}s,\forall s\in[0,M].$$

(iii) *There exists* $C>0$ *such that*

$$G(s)\le C(|s|^{\alpha}+|s|^{\beta}),s\in\mathbb{R},\text{ where }G(s):=\int_0^s g(t)dt,$$

and for any $M>0$, *there exists some* $C_{M,1}\ge C_{M,2}>0$ *such that*

$$C_{M,2}|s|^{\alpha}\le G(s)\le C_{M,1}|s|^{\alpha},\forall s\in[0,M].$$

(iv) *If* $\alpha>\beta$, *we have, for some* $D>0$, $G(s)\le D|s|^{\beta}$, $s\in\mathbb{R}$.

LEMMA 2.6. *Let* $\beta>p+\frac{p^2}{N}$, *then there exists* $U_k\in S_\rho$ *such that as* $k\to\infty$,

$$E(U_k)\to-\infty.$$

Proof. Let $\phi\in C_c^\infty(B_1)$ with $\phi>0$ and $\int_{B_1}\phi^p=1$. For $x_0\in\Omega$, denote

$$U_k=\rho^{\frac{1}{p}}k^{\frac{N}{p}}\phi(k(x-x_0)),$$

then for k large, we have *supp* $U_k\subset B_{\frac{1}{k}}(x_0)\subset\Omega$.

Moreover for a.e. $x\in$ *supp* U_k, $\lim\limits_{k\to\infty}G(U_k(x))/U_k(x)^\beta=\frac{\mu_2}{\beta}$, thus

$$\int_\Omega G(U_k)\ge\frac{\mu_2}{2\beta}\int_\Omega U_k^\beta=\frac{\mu_2}{2\beta}\rho^{\frac{\beta}{p}}k^{N(\frac{\beta}{p}-1)}\int_\Omega|\phi|^\beta.$$

On the other hand, direct computation shows that

$$\int_\Omega|U_k|^p=\rho\text{and }\int_\Omega|\nabla U_k|^p=\rho k^p\int_{B_1}|\nabla\phi|^p,$$

then as $k\to\infty$, there holds

$$E(U_k)=\frac{1}{p}\int_\Omega|\nabla U_k|^p-\int_\Omega G(U_k)\le\frac{\rho k^p}{p}\int_{B_1}|\nabla\phi|^p-\frac{\mu_2}{2\beta}\rho^{\frac{\beta}{p}}k^{N(\frac{\beta}{p}-1)}\int_\Omega|\phi|^\beta\to-\infty,$$

we conclude the proof. □

https://doi.org/10.1017/prm.2025.10098 Published online by Cambridge University Press

3. Subcritical case for $p \geq 2$ and nonexistence result

In this section, we give the proof of Theorem 1.1 and 1.2. Note that under the assumptions of Theorem 1.1, E is coercive on S_ρ, and classical variation methods may be applied, so we are able to prove Theorem 1.1 as follows.

Proof of Theorem 1.1. By Lemma 2.5 and Gagliardo-Nirenberg inequality, we have

$$\begin{aligned} E(u) \geq & \frac{1}{p}\int_\Omega |\nabla u|^p - C\int_\Omega |u|^\alpha + |u|^\beta \\ \geq & \frac{1}{p}\|\nabla u\|_p^p - (C_{N,\alpha}C\|u\|_p^{1-\gamma_\alpha}\|\nabla u\|_p^{\gamma_\alpha})^\alpha - (C_{N,\beta}C\|u\|_p^{1-\gamma_\beta}\|\nabla u\|_p^{\gamma_\beta})^\beta, \end{aligned} \tag{3.1}$$

where $\gamma_s = N(\frac{1}{p} - \frac{1}{s})$.

Thus if $p < \alpha, \beta < p + \frac{p^2}{N}$, we have $\alpha\gamma_\alpha, \beta\gamma_\beta < p$ and $E(u)$ is coercive.

In the meanwhile, by (iv) in Lemma 2.5, this is also the case when $\alpha \geq p + \frac{p^2}{N} > \beta$.

If $\alpha < \beta = p + \frac{p^2}{N}$, then $\beta\gamma_\beta = p$. Let ρ satisfy

$$\left(C_{N,\beta}C\rho^{\frac{1-\gamma_\beta}{p}}\right)^\beta < \frac{1}{p},$$

then the coercivity of E holds.

On the other hand, if $\beta = p + \frac{p^2}{N} \leq \alpha$, by (iv) in Lemma 2.5 again, we obtain

$$E(u) \geq \frac{1}{p}\int_\Omega |\nabla u|^p - D\int_\Omega |u|^\beta \geq \frac{1}{p}\|\nabla u\|_p^p - (C_{N,\beta}D\|u\|_p^{1-\gamma_\beta}\|\nabla u\|_p^{\gamma_\beta})^\beta. \tag{3.2}$$

Denote

$$m(\rho) := \inf_{S_\rho} E(u),$$

then E is coercive and weakly lower semi-continuous on S_ρ if

$$\max\left\{\left(C_{N,\beta}C\rho^{\frac{1-\gamma_\beta}{p}}\right)^\beta, \left(C_{N,\beta}D\rho^{\frac{1-\gamma_\beta}{p}}\right)^\beta\right\} < \frac{1}{p}, \tag{3.3}$$

thus $m(\rho)$ can be achieved.

For multiplicity, we notice if $p < \alpha, \beta < p + \frac{p^2}{N}$, it follows from (3.1) that

$$E(u) \geq \frac{1}{p}\|\nabla u\|_p^p - \left(CC_{N,\alpha}\rho^{\frac{(1-\gamma_\alpha)}{p}}\right)^\alpha \|\nabla u\|_p^{\gamma_\alpha\alpha} - \left(CC_{N,\beta}\rho^{\frac{(1-\gamma_\beta)}{p}}\right)^\beta \|\nabla u\|_p^{\gamma_\beta\beta}.$$

Denote

$$\varphi(t) = \frac{1}{p}t^p - \left(CC_{N,\alpha}\rho^{\frac{(1-\gamma_\alpha)}{p}}\right)^\alpha t^{\gamma_\alpha\alpha} - \left(CC_{N,\beta}\rho^{\frac{(1-\gamma_\beta)}{p}}\right)^\beta t^{\gamma_\beta\beta},$$

then $\varphi \in C(\mathbb{R}^+)$ satisfies $\varphi \to 0^-$ as $t \to 0^+$ and $\varphi \to +\infty$ as $t \to +\infty$, from which, we deduce E is bounded from below on S_ρ. Moreover, it is easy to check that E satisfies $(P.S.)$ condition for general level c.

https://doi.org/10.1017/prm.2025.10098 Published online by Cambridge University Press

Therefore if we define

$$\mathcal{A} := \{A \subset W_0^{1,p}(\Omega) : A \text{ is closed}, A = -A\} \text{ and } \Gamma_k := \{A \in \mathcal{A} : A \subset S_\rho, \gamma(A) \geq k\},$$

where $\gamma(A)$ denotes the Krasnoselskii genus. Then by Theorem 5.7 in [26], E admits infinitely many solutions u_k at level $c_k := \inf\limits_{A\in\Gamma_k} \sup\limits_{u\in A} E(u)$.

Similarly, we can prove (1.1) admits infinitely many solutions when $\beta < p + \frac{p^2}{N} \leq \alpha$ or under assumption (ii) in Theorem 1.1. □

Now we turn to the proof of Theorem 1.2, for this purpose, let us first study the asymptotic behavior of $(u_n, \lambda_n) \subset W_0^{1,p}(\Omega) \times \mathbb{R}$ satisfying:

$$\begin{cases} -\Delta_p u_n + \lambda_n |u_n|^{p-2} u_n = g(u_n), & \text{in } \Omega, \\ u_n = 0, & \text{on } \partial\Omega, \end{cases} \tag{3.4}$$

with $m(u_n)$ finite.

To proceed, we first consider the case when λ_n is upper bounded.

LEMMA 3.1. *Assume (u_n, λ_n) is a pair of solution to (3.4) and $\lambda_n \to -\infty$, then $\lim\limits_{n\to\infty} \|u_n\|_\infty = 0$.*

Proof. Let (u_n, λ_n) be a pair of solution to (3.4) with $\lambda_n \to -\infty$, suppose on the contrary that

$$\limsup_{n\to\infty} \|u_n\|_\infty = c > 0, \quad |u_n(x_n)| = \|u_n\|_\infty.$$

Denote v_k the eigenfunction corresponding to μ_k, where μ_k is the k^{th} eigenvalue of Dirichlet boundary problem for the Laplace equation. Let

$$\tilde{v}_k = \epsilon^{-\frac{N-2}{2}} v_k(\epsilon^{-1}(x - x_n)),$$

then $\tilde{v}_k = 0$ outside $\Omega_\epsilon := \epsilon\Omega + x_n$. Taking ϵ small enough, such that

$$|\nabla u_n| < \frac{1}{2} \text{ and } |u_n| > \frac{c}{2} \text{ in } \Omega_\epsilon,$$

https://doi.org/10.1017/prm.2025.10098 Published online by Cambridge University Press

we have

$$\begin{aligned}
D^2E(u_n)(\tilde{v}_k,\tilde{v}_k) =& \int_{\Omega_\epsilon} |\nabla u_n|^{p-2}|\nabla \tilde{v}_k|^2 + (p-2)\int_{\Omega_\epsilon} |\nabla u_n|^{p-4}(\nabla u_n, \nabla \tilde{v}_k)^2 \\
&+ \lambda_n(p-1)\int_{\Omega_\epsilon} |u_n|^{p-2}\tilde{v}_k^2 - \int_{\Omega_\epsilon} g'(u_n)\tilde{v}_k^2 \\
\leq& (p-1)\int_{\Omega} |\nabla u_n(\epsilon x + x_n)|^{p-2}|\nabla v_k|^2 \\
&+ \epsilon^2\lambda_n(p-1)\int_{\Omega} |u_n(\epsilon x + x_n)|^{p-2}v_k^2 - \int_{\Omega} g'(u_n(\epsilon x + x_n))v_k^2 \\
\leq& \frac{p-1}{2^{p-2}}\int_{\Omega} |\nabla v_k|^2 + \frac{c^{p-2}}{2^{p-2}}\epsilon^2\lambda_n(p-1)\int_{\Omega} v_k^2 \\
&- \int_{\Omega} g'(u_n(\epsilon x + x_n))v_k^2 \\
=& \frac{(p-1)}{2^{p-2}}\left(\mu_k + c^{p-2}\epsilon^2\lambda_n\right)\int_{\Omega} v_k^2 - \int_{\Omega} g'(u_n(\epsilon x + x_n))v_k^2,
\end{aligned}$$

taking λ_n negatively enough, we obtain $D^2E(u_n)(\tilde{v}_k,\tilde{v}_k) < 0$, for $k = 1, 2, \cdots$, this is a contradiction with the fact that $m(u_n)$ is finite. □

LEMMA 3.2. *Let $M > 0$, then for $|\lambda_n| \leq M$, there exists $C_M > 0$ such that $\|u_n\|_\infty \leq C_M$.*

Proof. Suppose on the contrary, there exists (λ_n, u_n) such that

$$-\Delta_p u_n + \lambda_n |u_n|^{p-2}u_n = g(u_n), \text{in } \Omega,$$

and $M_n := |u_n(x_n)| = \|u_n\|_\infty \to \infty$.

Let $v_n(x) = \frac{1}{M_n}u_n(M_n^{-\frac{\beta-p}{p}}x + x_n)$, then we have $\|v_n\|_\infty = 1$ and

$$-\Delta_p v_n + \frac{\lambda_n|v_n|^{p-2}v_n}{M_n^{\beta-p}} = \frac{g(M_n v_n)}{M_n^{\beta-1}}, \text{in } \Omega_n := M_n^{\frac{\beta-p}{p}}(\Omega - x_n).$$

By standard regularity result for p-Laplacian equation, we obtain, up to a subsequence,

$$v_n \to v, \text{in } C^1_{loc}(H)$$

and v satisfies

$$\begin{cases} -\Delta_p v = \mu_2|v|^{\beta-2}v, & \text{in } H, \\ v = 0, & \text{on } \partial H, \end{cases} \tag{3.5}$$

where $H = \mathbb{R}^N$ or $\mathbb{R}^N_+$. Moreover, as in [9], we can prove $m(v) \leq \sup_n m(v_n) \leq k$.

https://doi.org/10.1017/prm.2025.10098 Published online by Cambridge University Press

Now we **claim** that $v \in W_0^{1,p}(H)$, then as Lemma 2.1 in [10], we have

$$\left(1-\frac{1}{p}\right)\int_{B_R\cap\partial H}|\partial_N v|^p = \int_{H\cap\partial B_R}\left[\partial_N v|\nabla v|^{p-2}\nabla v\cdot\vec{n} - \frac{|\nabla v|^p}{p}\vec{n}_N\right] + \int_{H\cap\partial B_R}\frac{\mu_2|v|^\beta}{\beta},$$

since $v \in W_0^{1,p}(H)$, letting $R\to\infty$, we obtain $\partial_N v = 0$ on ∂H. Thus combining with the following Pohozaev identity [12]

$$\mu_2\left(1-\frac{N}{p}+\frac{N}{\beta}\right)\int_{\mathbb{R}^N_+}|v|^\beta = \left(1-\frac{1}{p}\right)\int_{\partial\mathbb{R}^N_+} x\cdot\nu\left|\frac{\partial v}{\partial n}\right|^p = 0.$$

we deduce $H=\mathbb{R}^N$, a contradiction with the results in [7].

In order to prove the claim, for $\varphi\in C_c^1(H)$, taking $\psi = v\varphi^p$ as a test function in (3.5), we obtain

$$\int_H|\nabla v|^p\varphi^p + p\int_H|\nabla v|^{p-2}\nabla v\cdot\nabla\varphi\varphi^{p-1}v = \mu_2\int_H|v|^\beta\varphi^p,$$

by Hölder's inequality, we have, for any $\epsilon>0$, there exists $C_\epsilon>0$ such that

$$\int_H|\nabla v|^{p-2}\nabla v\nabla\varphi\varphi^{p-1}v \le \epsilon\int_H|\nabla v|^p\varphi^p + C_\epsilon\int_H|\nabla\varphi|^p|v|^p,$$

thus

$$(1-\epsilon)\int_H|\nabla v|^p\varphi^p \le \mu_2\int_H|v|^\beta\varphi^p + C_\epsilon\int_H|\nabla\varphi|^p|v|^p. \tag{3.6}$$

To go further, we recall that if v is a finite Morse index solution, then v is stable outside a compact set $\mathcal{K}\subset H$, i.e. $\forall\phi\in C_c^1(H\setminus\mathcal{K})$,

$$L_v(\phi,\phi) := \int_H|\nabla v|^{p-2}|\nabla\phi|^2 + (p-2)\int_H|\nabla v|^{p-4}(\nabla v,\nabla\phi)^2 - \mu_2(\beta-1)\times\int_H|v|^{\beta-2}\phi^2 \ge 0.$$

For $\varphi\in C_c^1(H\setminus\mathcal{K})$, taking $\phi = v\varphi^{\frac{p}{2}}$, then

$$\int_H|\nabla v|^{p-4}(\nabla v,\nabla\phi)^2 \le \int_H|\nabla v|^{p-2}|\nabla\phi|^2 \le \int_H(1+\epsilon)|\nabla v|^p\varphi^p + C_\epsilon|\nabla\varphi|^p|v|^p.$$

From this, we deduce by (3.6) that

$$\begin{aligned}&(\beta-1)\mu_2\int_H|v|^\beta\varphi^p\\ &\le (p-1)\int_H(1+\epsilon)|\nabla v|^p\varphi^p + C_\epsilon|\nabla\varphi|^p|v|^p\\ &\le (p-1)(1+\epsilon)\mu_2\int_H|v|^\beta\varphi^p + C_\epsilon\int_H|\nabla\varphi|^p|v|^p.\end{aligned} \tag{3.7}$$

https://doi.org/10.1017/prm.2025.10098 Published online by Cambridge University Press

Since $\beta > p$, taking ϵ small enough, we obtain

$$\int_H |v|^\beta \varphi^p \le C \int_H |\nabla \varphi|^p |v|^p.$$

Then it yields from (3.6) that for $\forall \varphi \in C_c^1(H \setminus \mathcal{K})$

$$\int_H |\nabla v|^p \varphi^p + \int_H |v|^\beta \varphi^p \le C \int_H |\nabla \varphi|^p |v|^p. \tag{3.8}$$

Suppose $\mathcal{K} \subset B_{R_0}$, we take

$$\psi(x) = \begin{cases} 0, & \text{in } H \cap B_{R_0+1}, \\ 1, & \text{in } H \cap (B_R \setminus B_{R_0+2}), \\ 0, & \text{in } H \cap B_{2R}^c. \end{cases}$$

and $\varphi = \psi^k$ as a test function in (3.8), where $k = \frac{\beta}{\beta - p}$, then we have

$$\begin{aligned} &\int_H |\nabla v|^p \psi^{kp} + \int_H |v|^\beta \psi^{kp} \\ \le & C \int_H \psi^{(k-1)p} |\nabla \psi|^p |v|^p \\ \le & \epsilon \int_H \psi^{kp} |v|^\beta + C_\epsilon \int_H |\nabla \psi|^{pk}, \end{aligned}$$

thus

$$\int_H |\nabla v|^p \psi^{kp} + \int_H |v|^\beta \psi^{kp} \le C \int_H |\nabla \psi|^{pk} \le C + C R^{N - \frac{p\beta}{\beta - p}}, \tag{3.9}$$

letting $R \to \infty$ and noticing $\|v\|_\infty = 1$, we obtain $v \in W_0^{1,p}(H)$. □

Now we focus on the asymptotic behaviour as $\lambda_n \to +\infty$. The proof is of independent interest, thus we put it in Appendix A.

In the following, denote v a solution of

$$-\Delta_p v + |v|^{p-2} v = \mu_2 |v|^{\beta - 2} v, \text{in } H, \tag{3.10}$$

where $H = \mathbb{R}^N$ or $\mathbb{R}_+^N$, then we have

PROPOSITION 3.3. *Let u_n be a sequence of solutions of (3.4) corresponding to $\lambda_n \to +\infty$ and $m(u_n) \le \bar{k}$, then up to a subsequence, there exist $P_n^1, \dots, P_n^k$, $k \le \bar{k}$, so that for all $i, j = 1, \dots, k$, $i \ne j$, there holds:*

(1) $|u_n(P_n^i)| = \max\limits_{\Omega \cap B_{R_n \lambda_n^{-\frac{1}{p}}}(P_n^i)} |u_n|$ *for some $R_n \to +\infty$ and $\lambda_n^{\frac{1}{p}} |P_n^i - P_n^j| \to +\infty$ as $n \to +\infty$.*

(2) *For $x \in \Omega_{n,i} := \lambda_n^{\frac{1}{p}}(\Omega - P_n^i)$, defining $v_{i,n}(x) = \lambda_n^{-\frac{1}{\beta - p}} u_n(\lambda_n^{-\frac{1}{p}} x + P_n^i)$, then $v_{i,n} \to v_i$ in $C_{loc}^1(H)$, where v_i is a bounded solution of (3.10) with $m(v_i) \le \bar{k}$.*

https://doi.org/10.1017/prm.2025.10098 Published online by Cambridge University Press

Moreover, for any $q \geq p$,

$$\lambda_n^{\frac{N}{p}-\frac{q}{\beta-p}} \int_\Omega |u_n|^q \to \sum_{i=1}^{k} \int_H |v_i|^q. \tag{3.11}$$

By (3.11), we are able to derive the asymptotic behavior of $\{\|u_n\|_p\}$ as $\lambda_n \to +\infty$.

COROLLARY 3.4. *Let (u_n, λ_n) be a sequence of solutions to (3.4) with $\lambda_n \to +\infty$, we have that*

(1) *if $p < \beta < p + \frac{p^2}{N}$, then $\|u_n\|_p^p \to \infty$.*
(2) *if $\beta = p + \frac{p^2}{N}$, then $\|u_n\|_p^p \to \sum_{i=1}^{k} \int_H |v_i|^p$.*
(3) *if $p + \frac{p^2}{N} < \beta < p^*$, then $\|u_n\|_p^p \to 0$.*

Proof of Theorem 1.2. On one side, for any fixed $k \geq 0$, if (1.1) admits solution u_n with $m(u_n) \leq k$ and $\|u_n\|_p \to \infty$, then (u_n, λ_n) is a sequence of solution for (3.4), by Lemma 3.1, 3.2 and Corollary 3.4, we obtain $p < \beta < p + \frac{p^2}{N}$.

On the other side, if $p < \beta < p + \frac{p^2}{N}$, by Theorem 1.1, $E(u)$ admits a global minimizer on S_ρ for any $\rho > 0$.

Since $W_0^{1,p} = u \oplus T_u S_\rho$, then $m(u) \leq 1$. This completes the proof of the Theorem. □

4. Supercritical case for $p \geq 2$

As discussed in Lemma 2.6, when g satisfies L^p-supercritical growth condition at infinity, the global minimizer of E does not exist, thus in this section, we may look for local minimizer instead.

For this purpose, let $\lambda_1(\Omega)$ be the first Dirichlet eigenvalue of $-\Delta_p$ in Ω and for $r \geq \lambda_1(\Omega)$, denote

$$B_r := \left\{ u \in S_\rho, \int_\Omega |\nabla u|^p \leq r\rho \right\},$$

and

$$\partial B_r := \left\{ u \in S_\rho, \int_\Omega |\nabla u|^p = r\rho \right\}.$$

Since B_r is weakly closed in S_ρ and g is Sobolev subcritical, it is easy to see $\inf\limits_{B_r} E(u)$ can be achieved for any $r \geq \lambda_1(\Omega)$, define

$$c_r = \inf_{B_r} E(u) \text{and } \hat{c}_r = \inf_{\partial B_r} E(u).$$

In order to find a solution of (1.1), it is sufficient to find r such that $c_r < \hat{c}_r$.

https://doi.org/10.1017/prm.2025.10098 Published online by Cambridge University Press

Proof of Theorem 1.4. We prove that for ρ sufficiently small, there exists $r > \lambda_1$ satisfying

$$c_r < \hat{c}_r. \tag{4.1}$$

Firstly, noting that

$$\int_\Omega G(u) \le C \int_\Omega |u|^\alpha + |u|^\beta,$$

then for $s > p$, letting $\theta_s = \frac{p^*-s}{p^*-p}$, we obtain

$$\int_\Omega |u|^s \le \left(\int_\Omega |u|^p\right)^{\theta_s} \left(\int_\Omega |u|^{p^*}\right)^{1-\theta_s} \le C_{N,s}\rho^{\theta_s} \left(\int_\Omega |\nabla u|^p\right)^{\frac{p^*}{p}(1-\theta_s)},$$

thus we can estimate $E(u)$ on ∂B_r as follows

$$\begin{aligned} &E(u)\\ &\ge \frac{1}{p}\int_\Omega |\nabla u|^p - CC_{N,\alpha}\rho^{\theta_\alpha}\left(\int_\Omega |\nabla u|^p\right)^{\frac{p^*}{p}(1-\theta_\alpha)} - CC_{N,\beta}\rho^{\theta_\beta}\left(\int_\Omega |\nabla u|^p\right)^{\frac{p^*}{p}(1-\theta_\beta)}\\ &\ge \frac{r\rho}{p} - CC_{N,\alpha}\rho^{\theta_\alpha}(r\rho)^{\frac{p^*}{p}(1-\theta_\alpha)} - CC_{N,\beta}\rho^{\theta_\beta}(r\rho)^{\frac{p^*}{p}(1-\theta_\beta)}. \end{aligned} \tag{4.2}$$

On the other hand, by definition

$$c_r = \min_{s\in[\lambda_1,r]} \hat{c}_s \le \hat{c}_{\lambda_1},$$

therefore it remains to prove that $\hat{c}_{\lambda_1} < \hat{c}_r$ for ρ sufficiently small.

To see it, denote φ_1 the eigenfunction corresponding to λ_1, that is

$$-\Delta_p\varphi_1 = \lambda_1\varphi_1^{p-1} \text{ and } \|\varphi_1\|_p^p = 1,$$

then a direct calculation shows

$$\|\rho^{\frac{1}{p}}\varphi_1\|_p^p = \rho \text{ and } \|\nabla\rho^{\frac{1}{p}}\varphi_1\|_p^p = \rho\lambda_1.$$

Hence, we deduce

$$\hat{c}_{\lambda_1} \le E(\rho^{\frac{1}{p}}\varphi_1) \le \frac{\rho\lambda_1}{p}.$$

Next, it is sufficient to find $r > \lambda_1$ such that

$$\frac{\rho\lambda_1}{p} < \frac{r\rho}{p} - CC_{N,\alpha}\rho^{\theta_\alpha}(r\rho)^{\frac{p^*}{p}(1-\theta_\alpha)} - CC_{N,\beta}\rho^{\theta_\beta}(r\rho)^{\frac{p^*}{p}(1-\theta_\beta)}.$$

In case of $\alpha \le \beta$, it is equivalent to,

$$CC_{N,\alpha}r^{\frac{p^*}{p}(\theta_\beta-\theta_\alpha)}\rho^{(\frac{p^*}{p}-1)(1-\theta_\alpha)} + CC_{N,\beta}\rho^{(\frac{p^*}{p}-1)(1-\theta_\beta)} < f(r) := \frac{r-\lambda_1}{r^{\frac{p^*}{p}(1-\theta_\beta)}}. \tag{4.3}$$

https://doi.org/10.1017/prm.2025.10098 Published online by Cambridge University Press

For the right hand, we have

$$\max_{r\geq\lambda_1} f(r) = f\left(\frac{a}{a-1}\lambda_1\right) = \frac{(a-1)^{a-1}}{a^a}\lambda_1^{-(a-1)},$$

where $a = \frac{p^*}{p}(1-\theta_\beta)$.

For the left hand side of (4.3), noticing $r \geq \lambda_1$, we have

$$\begin{aligned}&CC_{N,\alpha} r^{\frac{p^*}{p}(\theta_\beta-\theta_\alpha)}\rho^{(\frac{p^*}{p}-1)(1-\theta_\alpha)} + CC_{N,\beta}\rho^{(\frac{p^*}{p}-1)(1-\theta_\beta)}\\ \leq& CC_{N,\alpha}\lambda_1^{\frac{p^*}{p}(\theta_\beta-\theta_\alpha)}\rho^{(\frac{p^*}{p}-1)(1-\theta_\alpha)} + CC_{N,\beta}\rho^{(\frac{p^*}{p}-1)(1-\theta_\beta)},\end{aligned}$$

thus (4.1) holds if

$$CC_{N,\alpha}\lambda_1^{\frac{p^*}{p}(\theta_\beta-\theta_\alpha)}\rho^{(\frac{p^*}{p}-1)(1-\theta_\alpha)} + CC_{N,\beta}\rho^{(\frac{p^*}{p}-1)(1-\theta_\beta)} < \frac{(a-1)^{a-1}}{a^a}\lambda_1^{-(a-1)}. \quad (4.4)$$

In case of $\alpha > \beta$, it can be deduced similarly, and we obtain (4.1) holds if

$$CC_{N,\alpha}\rho^{(\frac{p^*}{p}-1)(1-\theta_\alpha)} + CC_{N,\beta}\lambda_1^{\frac{p^*}{p}(\theta_\alpha-\theta_\beta)}\rho^{(\frac{p^*}{p}-1)(1-\theta_\beta)} < \frac{(a-1)^{a-1}}{a^a}\lambda_1^{-(a-1)}, \quad (4.5)$$

where $a = \frac{p^*}{p}(1-\theta_\alpha)$. □

In the following, we will find a second solution which is of mountain pass type. For this purpose, we introduce a family of functionals

$$E_\theta(u) = \frac{1}{p}\int_\Omega |\nabla u|^p - \theta\int_\Omega G(u),$$

where $\theta \in [\frac{1}{2}, 1]$. Denote u_ρ the solution of (1.1) obtained in Theorem 1.4, then

$$E(u_\rho) = \inf_{B_{r_0}} E(u) < \inf_{\partial B_{r_0}} E(u),$$

for some $r_0 > 0$. In the following, we first study asymptotic properties of $E_\theta(u)$ as $\theta \to 1^-$.

LEMMA 4.1. *For any fixed $r > 0$, let $\partial B_r := \{u \in S_\rho : \|\nabla u\|_p^p = \rho r\}$ and assumptions $(G_1)-(G_2)$ hold, then*

$$\lim_{\theta\to 1^-} E_\theta(u_\rho) = E(u_\rho), \quad (4.6)$$

and

$$\lim_{\theta\to 1^-}\inf_{\partial B_r} E_\theta(u) = \inf_{\partial B_r} E(u). \quad (4.7)$$

Proof. We first prove (4.7). Note that $E_\theta(u) \geq E(u)$ for all $\theta \in [\frac{1}{2}, 1]$. Hence, we have

$$\inf_{\partial B_r} E_\theta(u) \geq \inf_{\partial B_r} E(u), \forall\theta \in [\frac{1}{2}, 1].$$

In the following, it is sufficient to prove $\limsup_{\theta\to 1^-}\inf_{\partial B_r} E_\theta \leq \inf_{\partial B_r} E$.

https://doi.org/10.1017/prm.2025.10098 Published online by Cambridge University Press

Taking a minimizing sequence $\{u_n\} \subset \partial B_r$ such that $\lim_{n\to\infty} E(u_n) = \inf_{\partial B_r} E$. Since $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$, one gets

$$\begin{aligned}\inf_{\partial B_r} E_\theta \leq & \liminf_{n\to\infty} E_\theta(u_n) = \lim_{n\to\infty} E(u_n) + o_\theta(1)\\ = & \inf_{\partial B_r} E + o_\theta(1),\end{aligned}$$

where $o_\theta(1) \to 0$ as $\theta \to 1^-$. Sending $\theta \to 1^-$, we have

$$\limsup_{\theta\to 1^-} \inf_{\partial B_r} E_\theta \leq \inf_{\partial B_r} E,$$

and this completes the proof to (4.7).

The proof of (4.6) is similar and much easier, thus we omit it here. □

LEMMA 4.2. *Under the assumptions of Theorem 1.4, there exist $\epsilon > 0$ and $v \in S_\rho \setminus B_{r_0}$ such that for $\theta \in [1-\epsilon, 1]$,*

$$m_\theta := \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} E_\theta(\gamma(t)) > \max\left\{E_\theta(u_\rho), E_\theta(v)\right\},$$

where

$$\Gamma := \left\{\gamma \in C([0,1], S_\rho) : \gamma(0) = u_\rho, \gamma(1) = v\right\},$$

is independent of θ.

Proof. By the proof of Theorem 1.4, we have that there exists $r_0 > 0$ satisfying

$$E(u_\rho) = \inf_{B_{r_0}} E(u) < \inf_{\partial B_{r_0}} E(u),$$

thus by Lemma 4.1, there exists $\epsilon > 0$ such that for $\theta \in [1-\epsilon, 1]$

$$E_\theta(u_\rho) < \inf_{\partial B_{r_0}} E_\theta(u). \tag{4.8}$$

Moreover, for any $u \in S_\rho$, define

$$u^t = t^{\frac{N}{p}} u(tx) \in S_\rho, t \geq 1, \tag{4.9}$$

we have as $t \to +\infty$

$$\begin{aligned}E_\theta(u^t) = & \frac{1}{p}\int_\Omega |\nabla u^t|^p - \theta\int_\Omega G(u^t)\\ \leq & \frac{1}{p}\int_\Omega |\nabla u^t|^p - \frac{\mu_2}{2\beta}\int_\Omega |u^t|^\beta\\ = & \frac{t^p}{p}\int_\Omega |\nabla u|^p - \frac{\mu_2}{2\beta} t^{\frac{\beta N}{p} - N}\int_\Omega |u|^\beta\\ \to & -\infty \text{ uniformly w.r.t } \theta \in [\frac{1}{2}, 1],\end{aligned}$$

https://doi.org/10.1017/prm.2025.10098 Published online by Cambridge University Press

thus there exists $T_0 > 0$ such that $E_\theta(u^t) < E_\theta(u_\rho)$ for $t > T_0$. Taking

$$T := \max\left\{1, T_0, \frac{(\rho r_0)^{\frac{1}{p}}}{\|\nabla u\|_p}\right\},$$

and $v = u^t$, by direct calculation, we have

$$\|\nabla v\|_p^p = t^p \|\nabla u\|_p^p > \rho r_0 \text{ for } t > T.$$

Then for any $\gamma \in \Gamma$, there exists $t^* \in (0,1)$ such that $\gamma(t^*) \in \partial B_{r_0}$. Hence

$$\inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} E_\theta(\gamma(t)) \geq \inf_{\partial B_{r_0}} E_\theta(u),$$

by (4.8), we complete the proof. □

Proposition 4.3. *Under the assumptions of Theorem 1.4, there exists a critical point u_θ of E_θ constrained on S_ρ at the level c_θ for almost every $\theta \in [1-\epsilon, 1]$, where ϵ is given by Lemma 4.2. Moreover, u_θ solves*

$$\begin{cases} -\Delta_p u_\theta + \lambda_\theta u_\theta = \theta g(u_\theta), & \text{in } \Omega, \\ u_\theta = 0, \text{ on } \partial\Omega, \end{cases} \tag{4.10}$$

for some λ_θ.

Proof. We apply Theorem 2.3 with

$$W = W_0^{1,p}(\Omega), A(u) = \frac{1}{p}\int_\Omega |\nabla u|^p, B(u) = \int_\Omega G(u).$$

Together with Lemma 4.2, we have for almost every $\theta \in [1-\epsilon, 1]$, there exists a bounded $(P.S.)$ sequence $\{u_n\} \subset S_\rho$ satisfying

$$E_\theta(u_n) \to c_\theta, (E_\theta|_{S_\rho})'(u_n) \to 0, \text{as } n \to \infty.$$

Up to a subsequence, we may assume that

$$\begin{aligned} u_n &\rightharpoonup u_\theta \quad \text{weakly in } W_0^{1,p}(\Omega), \\ u_n &\to u_\theta \quad \text{strongly in } L^q(\Omega) \text{ for } p \leq q < p^*, \\ u_n &\to u_\theta \quad \text{almost everwhere in } \Omega. \end{aligned}$$

It can be verified that $u_\theta \in S_\rho$ is a critical point of E_θ constrained on S_ρ.

Let λ_θ be the Lagrange multiplier corresponding to u_θ, then $(u_\theta, \lambda_\theta)$ satisfies (4.10).

The proof is completed. □

https://doi.org/10.1017/prm.2025.10098 Published online by Cambridge University Press

Denote

$$\Gamma = \{\gamma \in C([0,1], S_\rho), \gamma(0) = u_\rho, \gamma(1) = v\},$$

and

$$m(\rho) = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} E(\gamma(t)).$$

Next we prove $m(\rho)$ can be obtained.

Proof of Theorem 1.5. For $\theta \in [1-\epsilon, 1]$, let $(u_\theta, \lambda_\theta)$ be solutions obtained in Proposition 4.3, then (4.10) is satisfied. Denote (u_n, λ_n) the sequence of solutions corresponding to θ_n, where $\theta_n \to 1^-$ as $n \to \infty$, then (u_n, λ_n) satisfies (3.4), that is

$$\begin{cases} -\Delta_p u_n + \lambda_n |u_n|^{p-2} u_n = \theta_n g(u_n), & \text{in } \Omega, \\ u_n = 0, & \text{on } \partial\Omega, \end{cases} \tag{4.11}$$

with $\|u_n\|_p^p = \rho$, moreover, we see that the sequence $\{u_n\}$ has Morse index uniformly bounded since it is made of mountain pass solutions.

Note that $\theta_n g(u)$ satisfies assumptions $(G_1) - (G_2)$ with μ_1, μ_2 substituded by $\theta_n\mu_1$, $\theta_n\mu_2$ respectively, applying Lemma 3.1 and Corollary 3.4, we have $\{\lambda_n\}$ is uniformly bounded, furthermore by Lemma 3.2, there exists $C > 0$ independent of n such that $\|u_n\|_\infty \le C$.

On the other hand, multiplying u_n on both sides of (4.11), we have

$$\int_\Omega |\nabla u_n|^p = \theta_n \int_\Omega g(u_n)u_n - \lambda_n \int_\Omega |u_n|^p. \tag{4.12}$$

Since $\|u_n\|_\infty \le C$, it can be seen from (4.12) that the sequence $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$.

Finally, as in Proposition 4.3, we conclude that the existence of $\tilde{u}_\rho$ such that

$$u_n \to \tilde{u}_\rho \text{in } W_0^{1,p}(\Omega),$$

and $\tilde{u}_\rho$ solves (1.1) for some $\lambda \in \mathbb{R}$. □

Now, as in Section 3, we intend to study multiplicity of solutions in the mass-supercritical case. However, in the absence of an abstract result similar to Theorem 2.4, we only focus our attentions to the case when $p = 2$ and leave the general case of $p > 2$ to be open.

For this purpose, it remains to show the existence of γ_i, $i = 1, 2$ such that (2.2) holds.

LEMMA 4.4. *Assume the hypotheses $(G_1) - (G_2)$, then for any $k \ge 2$ and $\rho > 0$, there exists $\gamma_1 : S^{k-2} \to S_\rho$ such that for any $u \in \gamma_1(S^{k-2})$, there holds*

$$\|\nabla u\|_2^2 \le \rho\Lambda_{k-1} \text{ and } E_\theta(u) < \frac{1}{2}\rho\Lambda_{k-1},$$

where Λ_i, $i = 1, \dots, k$ denotes the fist k eigenvalues of the Dirichlet Laplacian in Ω.

https://doi.org/10.1017/prm.2025.10098 Published online by Cambridge University Press

Proof. For $i = 1, \dots, k-1$, let φ_i be eigenfunctions corresponding to Λ_i, moreover for $(a_1, \dots, a_{k-1}) \in S^{k-2}$ define

$$\gamma_1 : \quad S^{k-2} \to S_\rho, \{a_i\} \mapsto \sum_{i=1}^{k-1} a_i \varphi_i$$

then for any $u \in \gamma_1(S^{k-2})$

$$E_\theta(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \theta \int_\Omega G(u) \le \frac{1}{2}\rho\Lambda_{k-1} - C\int_\Omega |u|^\alpha.$$

On the other hand, since for $2 < \alpha < 2^*$,

$$\|u\|_\alpha^\alpha \ge \|u\|_2^\alpha |\Omega|^{-\frac{\alpha-2}{2}},$$

we obtain

$$E_\theta(u) \le \frac{1}{2}\rho\Lambda_{k-1} - C\rho^{\frac{\alpha}{2}}|\Omega|^{-\frac{\alpha-2}{2}}.$$

□

LEMMA 4.5. *Assume that $(G_1) - (G_2)$, then for any $k \ge 2$, there exists $\rho_k > 0$ such that for any $0 < \rho < \rho_k$, there is $t_0 > \rho\Lambda_{k-1}$ satisfying $E_\theta(u) \ge \frac{1}{2}\rho\Lambda_{k-1}$ for $u \in S_\rho$ and $\|\nabla u\|_2^2 = t_0$.*

Proof. For any $q \in (2, 2^*)$, by the Gagliardo-Nirenberg inequality, we have

$$\|v\|_q^q \le C_{N,q}\|\nabla v\|_2^{\frac{N(q-2)}{2}}\|v\|_2^{q-\frac{N(q-2)}{2}},$$

so

$$E_\theta(u) \ge \frac{1}{2}\|\nabla u\|_2^2 - CC_{N,\alpha}\|\nabla u\|_2^{\frac{N(\alpha-2)}{2}}\rho^{\frac{\alpha}{2}-\frac{N(\alpha-2)}{4}} - CC_{N,\beta}\|\nabla u\|_2^{\frac{N(\beta-2)}{2}}\rho^{\frac{\beta}{2}-\frac{N(\beta-2)}{4}}.$$

Denote

$$\psi(t) := \frac{1}{2}t - CC_{N,\alpha}t^{\frac{N(\alpha-2)}{4}}\rho^{\frac{\alpha}{2}-\frac{N(\alpha-2)}{4}} - CC_{N,\beta}t^{\frac{N(\beta-2)}{4}}\rho^{\frac{\beta}{2}-\frac{N(\beta-2)}{4}},$$

then ψ achieves its maximum at t_0 where t_0 solves

$$\frac{1}{2} - CC_{N,\alpha}\frac{N(\alpha-2)}{4}t_0^{\frac{N(\alpha-2)}{4}-1}\rho^{\frac{\alpha}{2}-\frac{N(\alpha-2)}{4}} - CC_{N,\beta}\frac{N(\beta-2)}{4}t_0^{\frac{N(\beta-2)}{4}-1}\rho^{\frac{\beta}{2}-\frac{N(\beta-2)}{4}} = 0, \tag{4.13}$$

and

$$\psi(t_0) = \left(\frac{N(\alpha-2)}{4} - 1\right)CC_{N,\alpha}\rho^{\frac{\alpha}{2}-\frac{N(\alpha-2)}{4}}t_0^{\frac{N(\alpha-2)}{4}} + \left(\frac{N(\beta-2)}{4} - 1\right) \times CC_{N,\beta}\rho^{\frac{\beta}{2}-\frac{N(\beta-2)}{4}}t_0^{\frac{N(\beta-2)}{4}}.$$

https://doi.org/10.1017/prm.2025.10098 Published online by Cambridge University Press

By (4.13), we have

$$\max\left\{CC_{N,\alpha}\frac{N(\alpha-2)}{4}t_0^{\frac{N(\alpha-2)}{4}-1}\rho^{\frac{\alpha}{2}-\frac{N(\alpha-2)}{4}}, CC_{N,\beta}\frac{N(\beta-2)}{4}t_0^{\frac{N(\beta-2)}{4}-1}\rho^{\frac{\beta}{2}-\frac{N(\beta-2)}{4}}\right\} \geq \frac{1}{4},$$

Hence

$$t_0 \geq \min\left\{\left[\frac{1}{CC_{N,\alpha}N(\alpha-2)}\rho^{\frac{N(\alpha-2)}{4}-\frac{\alpha}{2}}\right]^{\frac{4}{N(\alpha-2)-4}}, \left[\frac{1}{CC_{N,\beta}N(\beta-2)}\rho^{\frac{N(\beta-2)}{4}-\frac{\beta}{2}}\right]^{\frac{4}{N(\beta-2)-4}}\right\},$$

therefore, we obtain

$$\begin{aligned}\psi(t_0) > \min\Bigg\{&\left(\frac{N(\alpha-2)-4}{4\left[N(\alpha-2)\right]^{\frac{N(\alpha-2)}{N(\alpha-2)-4}}}\right)\left(CC_{N,\alpha}\rho^{\frac{\alpha}{2}-\frac{N(\alpha-2)}{4}}\right)^{-\frac{4}{N(\alpha-2)-4}},\\ &\left(\frac{N(\beta-2)-4}{4\left[N(\beta-2)\right]^{\frac{N(\beta-2)}{N(\beta-2)-4}}}\right)\left(CC_{N,\beta}\rho^{\frac{\beta}{2}-\frac{N(\beta-2)}{4}}\right)^{-\frac{4}{N(\beta-2)-4}}\Bigg\}.\end{aligned} \tag{4.14}$$

Finally, note that for $\alpha, \beta < 2^*$,

$$\max\left\{\frac{\alpha}{2}-\frac{N(\alpha-2)}{4}, \frac{\beta}{2}-\frac{N(\beta-2)}{4}\right\} > 0$$

we may take ρ satisfying

$$\begin{aligned}\rho\Lambda_{k-1} < \min\Bigg\{&\left(\frac{N(\alpha-2)-4}{2\left[N(\alpha-2)\right]^{\frac{N(\alpha-2)}{N(\alpha-2)-4}}}\right)\left(CC_{N,\alpha}\rho^{\frac{\alpha}{2}-\frac{N(\alpha-2)}{4}}\right)^{-\frac{4}{N(\alpha-2)-4}},\\ &\left(\frac{N(\beta-2)-4}{2\left[N(\beta-2)\right]^{\frac{N(\beta-2)}{N(\beta-2)-4}}}\right)\left(\frac{C_{N,\beta}}{\beta}\rho^{\frac{\beta}{2}-\frac{N(\beta-2)}{4}}\right)^{-\frac{4}{N(\beta-2)-4}}\Bigg\},\end{aligned} \tag{4.15}$$

then we can check that $E_\theta(u) > \frac{1}{2}\rho\Lambda_{k-1}$ for $\|\nabla u\|_2^2 = t_0$ and $t_0 > \rho\Lambda_{k-1}$. □

LEMMA 4.6. *Assume that $(G_1)-(G_2)$, then for any $k \geq 2$, $t_0 > 0$ and $\rho > 0$, there exists $\gamma_2 : S^{k-2} \to S_\rho$ such that for any $u \in \gamma_2(S^{k-2})$, there holds*

$$\|\nabla u\|_2^2 > t_0 \text{ and } E_\theta(u) < 0.$$

Proof. Since $0 \in \Omega$, there exists $\delta > 0$ such that $B_\delta(0) \subset \Omega$. Moreover, let $\{x_1, \ldots, x_{k-1}\}$ be any sequence of points in Ω satisfying $B_\delta(x_i) \subset \Omega$ and $B_\delta(x_i) \cap B_\delta(x_j) = \emptyset$ if $i \neq j$.

https://doi.org/10.1017/prm.2025.10098 Published online by Cambridge University Press

For any $\rho > 0$, taking $\varphi \in C_c^\infty(B_\delta(0))$ with $\|\varphi\|_2^2 = \rho$, then if $t \geq 1$, $\bar{\varphi}^t(x) := t^{\frac{N}{2}}\varphi(tx)$ satisfies $\|\bar{\varphi}\|_2^2 = \rho$ and $supp\ \ \bar{\varphi} \subset B_{\frac{\delta}{t}}(0)$. Denote

$$\bar{\varphi}_i = \bar{\varphi}(x - x_i), \qquad i = 1, \dots, k-1,$$

then $\gamma_2 : S^{k-2} \to S_\rho$ can be chosen as

$$\gamma_2((a_1, \dots, a_{k-1})) = \sum_{i=1}^{k-1} a_i \bar{\varphi}_i.$$

Now we can calculate for any $u \in \gamma_2(S^{k-2})$,

$$\begin{aligned} E_\theta\left(\sum_{i=1}^{k-1} a_i\bar{\varphi}_i\right) =& \frac{1}{2}\sum_i a_i^2 \int_{B_{\frac{\delta}{t}}(x_i)} |\nabla\bar{\varphi}_i|^2 - \theta\sum_i \int_{B_{\frac{\delta}{t}}(x_i)} G(a_i\bar{\varphi}_i) \\ \leq& \frac{t^2}{2}\sum_i a_i^2 \int_{B_\delta(0)} |\nabla\varphi|^2 - \frac{\mu_2}{4\beta} t^{\frac{N\beta}{2}-N} \sum_i |a_i|^\beta \int_{B_\delta(0)} |\varphi|^\beta \to -\infty, \end{aligned}$$

as $t \to +\infty$.

Thus there exists $T_0 > 0$ such that for all $t > T_0$, we have $E_\theta(u) < 0$ if $u \in \gamma_2(S^{k-2})$. As a result, for any $t_0 > 0$, we can choose

$$T := \max\left\{1, \frac{t_0^{\frac{1}{2}}}{\|\nabla\varphi\|_2}, T_0\right\},$$

then γ_2 satisfies all the desired properties if $t > T$. □

In the following, we denote

$$c_{\theta,k} = \inf_{\gamma\in\Gamma_k} \max_{[0,1]\times S^{k-2}} E_\theta(\gamma(t,x)),$$

where

$$\Gamma_k = \{\gamma \in C([0,1] \times S^{k-2}, S_\rho) : \gamma(t,\cdot)\text{is odd }, \gamma(0,\cdot) = \gamma_1, \gamma(1,\cdot) = \gamma_2\} \tag{4.16}$$

and γ_1, γ_2 are defined in Lemma 4.4, 4.6 respectively.

PROPOSITION 4.7. *Under the assumptions of* $(G_1) - (G_2)$, *for any* $k \geq 2$ *and* $0 < \rho < \rho_k$, *there exists a critical point* $u_{\theta,k}$ *of* E_θ *constrained on* S_ρ *at the level* $c_{\theta,k}$ *for almost every* $\theta \in \left(\frac{1}{2}, 1\right]$. *Moreover,* $u_{\theta,k}$ *solves*

$$\begin{cases} -\Delta u_{\theta,k} + \lambda_\theta u_{\theta,k} = \theta g(u_{\theta,k}), & in\ \Omega, \\ u_{\theta,k} = 0, on\ \partial\Omega, \end{cases} \tag{4.17}$$

for some $\lambda_\theta \in \mathbb{R}$ *and* $m(u_{\theta,k}) \leq k+1$. *Furthermore,* $E_\theta(u_{\theta,k}) \to +\infty$ *as* $k \to \infty$. *In particular there are infinitely many distinct values of* $c_{\theta,k}$.

https://doi.org/10.1017/prm.2025.10098 Published online by Cambridge University Press

Proof. Setting

$$A(u) = \int_\Omega |\nabla u|^2 \text{and } B(u) = \int_\Omega G(u),$$

we have $A(u) \to +\infty$ as $\|u\| \to +\infty$. Note that B'', whence E''_θ is $\min\{\alpha - 2, \beta - 2, 1\}$-Hölder continuous on bounded sets of $H_0^1(\Omega)$.

By Lemma 2.8 in [5] and Remark 4.5 in [22], we can see the set defined by (4.16) is non void. Moreover, by Lemmas 4.4–4.6, we have

$$\inf_{\gamma\in\Gamma_k}\max_{[0,1]\times S^{k-2}} E_\theta(\gamma(t,x)) \geq \inf_{\|\nabla u\|_2^2=t_0} E_\theta(u) > \max_{x\in S^{k-2}}\{E_\theta(\gamma_1(x)), E_\theta(\gamma_2(x))\},$$
$$\forall\theta \in \left[\frac{1}{2}, 1\right].$$

Thus for almost every $\theta \in \left[\frac{1}{2}, 1\right]$, there exists a bounded (*P.S.*) sequence $\{u_n\} \subset S_\rho$ and $\xi_n \to 0$ such that $\tilde{m}_{\xi_n}(u_n) \leq k$. As in the proof of Proposition 4.3, we obtain the existence of $u_{\theta,k}$ such that (4.17) holds, moreover,

$$c_{\theta,k} = E_\theta(u_{\theta,k}) > \frac{1}{2}\rho\Lambda_{k-1} \to +\infty \text{ as } k \to +\infty.$$

Since $W_0^{1,2}(\Omega) = u_{\theta,k} \oplus T_{u_{\theta,k}} S_\rho$, we deduce $m(u_{\theta,k}) \leq \tilde{m}_0(u_{\theta,k}) + 1$, now it remains to prove $\tilde{m}_0(u_{\theta,k}) \leq k$, the proof can be deduced as Remark 2.6 in [5], for reader's convenience, we include the details here.

Suppose by contradiction, there exists $W_0 \subset T_{u_{\theta,k}} S_\rho$ with $\dim W_0 = k + 1$ such that

$$D^2 E(u_{\theta,k})(w, w) < 0, \text{ for all } w \in W_0 \setminus \{0\}.$$

Since W_0 is of finite dimension, its unit sphere is compact and there exists $\xi > 0$ such that

$$D^2 E_\theta(u_{\theta,k})(w, w) < -\xi\|w\|^2, \text{ for all } w \in W_0 \setminus \{0\}.$$

Note that E''_θ is $\alpha-$Hölder continuous on bounded sets for some $\alpha \in (0, 1]$, it follows that there exists $\delta > 0$ such that for any $v \in S_\rho$ satisfying $\|v - u_{\theta,k}\| \leq \delta$, and

$$D^2 E_\theta(v)(w, w) < -\frac{\xi}{2}\|w\|^2 \text{ for all } w \in W_0 \setminus \{0\}.$$

In particular, for n large enough

$$\|u_n - u_{\theta,k}\| \leq \delta \text{ and } \xi_n < \frac{\xi}{2}.$$

So the previous inequality implies

$$D^2 E_\theta(u_n)(w, w) < -\xi_n\|w\|^2, \text{ for all } w \in W_0 \setminus \{0\}.$$

A contradiction with (3) in Proposition 2.4. □

Proof of Theorem 1.6. The proof is exactly the same as Theorem 1.5, thus we omit it here. □

https://doi.org/10.1017/prm.2025.10098 Published online by Cambridge University Press

Funding.
Gao is partially supported by NSFC (No.12201293) and Natural Science Foundation of Jiangsu Province (No.BK20220866), Guo is partially supported by NSFC (No.12271283).

Data availability statement.
The research has no associated data.

Conflict of interest.
The authors declare that they have no competing interest.

References

1 T. Bartsch, R. Molle, M. Rizzi and G. Verzini. Normalized solutions of mass supercritical Schrödinger equations with potential. *Comm. Partial Differential Equations*. **46** (2021), 1729–1756.
2 T. Bartsch, S. Qi and W. Zou. Normalized solutions to Schrödinger equations with potential and inhomogeneous nonlinearities on large smooth domains. *Math. Ann* **390** (2024), 1–47.
3 T. Bartsch, X. Zhong and W. Zou. Normalized solutions for a coupled Schrödinger system. *Math. Ann*. **380** (2021), 1713–1740.
4 J. Borthwick, X. Chang, L. Jeanjean and N. Soave. Bounded Palais-Smale sequences with Morse type information for some constrained functionals. *Trans. Am. Math. Soc*. **377** (2024), 4481–4517.
5 P. Carrillo, D. Galant, L. Jeanjean and C. Troestler. Infinitely many normalized solutions of L^2-supercritical NLS equations on noncompact metric graphs with localized nonlinearities. *Discrete Cont. Dyn*. (2025).
6 X. Chang, L. Jeanjean and N. Soave. Normalized solutions of L^2-supercritical NLS equations on compact metric graphs. *Ann. Inst. H. Poincaré Anal. Non Linéaire*. **41** (2024), 933–959.
7 L. Damascelli, A. Farina, B. Sciunzi and E. Valdinoci. Liouville results for m-Laplace equations of Lane-Emden-Fowler type. *Ann. Inst. H. Poincaré C Anal. Non Linéaire*. **26** (2009), 1099–1119.
8 Y. Ding and X. Zhong. Normalized solution to the Schrödinger equation with potential and general nonlinear term: mass super-critical case. *J. Differ. Equations*. **334** (2022), 194–215.
9 P. Esposito and M. Petralla. Pointwise blow-up phenomena for a Dirichlet problem. *Comm. Partial Differ.Equ*. **36** (2011), 1654–1682.
10 A. Farina, C. Mercuri and M. Willem. A Liouville theorem for the p-Laplacian and related questions. *Calc. Var. Partial Differential Equations*. **58** (2019), 1–13.
11 L. Gu, X. Zeng and H. Zhou. Eigenvalue problem for a p-Laplacian equation with trapping potentials. *Nonlinear Anal*. **148** (2017), 212–227.
12 M. Guedda and L. Veron. Quasilinear elliptic equations involving critical Sobolev exponents. *Nonlinear Anal. Theory Methods Applic*. **13** (1989), 879–902.
13 L. Jeanjean. Existence of solutions with prescribed norm for semilinear elliptic equations. *Nonlinear Anal. Theory Methods Applic*. **28** (1997), 1633–1659.
14 L. Jeanjean. On the existence of bounded Palais-Smale sequences and application to a Landesman-Lazer-type problem set on $\mathbb{R}^N$. *P. Roy. Soc. Edinb. A*. **129** (1999), 787–809.
15 L. Jeanjean and T. T. Le. Multiple normalized solutions for a Sobolev critical Schrödinger equation. *Math. Ann*. **384** (2022), 101–134.
16 L. Jeanjean, J. Zhang and X. Zhong. A global branch approach to normalized solutions for the Schrödinger equation. *J. Math. Pures Appl*. **183** (2024), 44–75.
17 G. Li and S. Yan. Eigenvalue problems for quasilinear elliptic-equations on $\mathbb{R}^N$. *Commun. Partial Differ. Equ*. **14** (1991), 1291–1414.
18 X. Liu and J. Zhao. p-Laplacian equations in $\mathbb{R}^N$ with finite potential via the truncation method. *Adv. Nonlinear Stud*. **17** (2017), 595–610.
19 R. Molle, G. Riey and G. Verzini. Normalized solutions to mass supercritical Schrödinger equations with negative potential. *J. Diff. Equations*. **333** (2022), 302–331.

https://doi.org/10.1017/prm.2025.10098 Published online by Cambridge University Press

20 B. Noris, H. Tavares and G. Verzini. Existence and orbital stability of the ground states with prescribed mass for the L^2-critical and supercritical NLS on bounded domains. *Anal. PDE*. **7** (2014), 1807–1838.
21 B. Noris, H. Tavares and G. Verzini. Normalized solutions for nonlinear Schrödinger systems on bounded domains. *Nonlinearity*. **32** (2019), 1044–1072.
22 D. Pierotti and G. Verzini. Normalized bound states for the nonlinear Schrödinger equation in bounded domains. *Calc. Var. Partial Differential Equations*. **56** (2017), 27 133.
23 D. Pierotti, G. Verzini and J. Yu. Normalized solutions for Sobolev critical Schrödinger equations on bounded domains. *Siam J. Math. Ana.* **57** (2025), 262–285.
24 S. Qi and W. Zou. Normalized solutions of nonhomogeneous mass supercritical Schrödinger equations in bounded domains. *J. Geom. Anal*. **34** (2024), 59.
25 L. Song and W. Zou. Two positive normalized solutions on star-shaped bounded domains to the Brézis-Nirenberg problem. *I: Existence, arXiv preprint arXiv:2404.11204*, 2024.
26 M. Struwe. *Variational methods: Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems 4th ed*. (Springer-Verlag, 2008).
27 C. A. Stuart. Bifuration from the continuous spectrum in L^2-theory of elliptic equations on $\mathbb{R}^N$. In *Recent Methods in Nonlinear Analysis and Applications.*, Liguori, Napoli (1981)
28 C. A. Stuart. Bifurcation in $L^p(R_N)$ for a semilinear elliptic equation. *Proc. London Math. Soc*. **57** (1988), 511–541.
29 J. L. Vazquez. A strong maximum principle for some quasilinear elliptic equations. *Applied Mathematics and optimization*. **12** (1984), 191–202.
30 J. Wei and Y. Wu. Normalized solutions for Schrödinger equations with critical Sobolev exponent and mixed nonlinearities. *J. Funct. Anal*. **283** (2022), 46 109574.
31 Z. Zhang and Z. Zhang. Normalized solutions to p-Laplacian equations with combined nonlinearities. *Nonlinearity*. **35** (2022), 5621–5663.

Appendix A. Aymptotic behavior

In this section, we focus on the asymptotic behaviour of solutions for (3.4). First, we have the following estimate on $\|u_n\|_\infty$.

LEMMA A.1. *Let* (u_n, λ_n) *be a sequence of solution with* $\lambda_n \to +\infty$, *then* $\liminf_{n\to\infty} \frac{\|u_n\|_\infty^{\beta-p}}{\lambda_n} > 0$

Proof. We first prove $\liminf_{n\to\infty} \frac{\|u_n\|_\infty^{\beta-p}+\|u_n\|_\infty^{\alpha-p}}{\lambda_n} > 0$.

Denote

$$M_n = |u_n(x_n)| = \|u_n\|_\infty$$

and

$$v_n(x) = \frac{1}{M_n} u_n(\lambda_n^{-\frac{1}{p}} x + x_n), \text{ for } x \in \Omega_n := \lambda_n^{\frac{1}{p}}(\Omega - x_n),$$

then v_n satisfies $|v_n(0)| = 1$ and

$$\begin{cases} -\Delta_p v_n + |v_n|^{p-2} v_n = \frac{g(M_n v_n)}{M_n^{p-1}\lambda_n}, & \text{in } \Omega_n, \\ v_n = 0, & \text{on } \partial\Omega_n. \end{cases} \tag{A.1}$$

For $M >> 1$, denote $\varphi = e^{-\frac{|x|}{M}}$, taking $v_n\varphi^p$ as a test function in (A.1), then

$$\int_{\Omega_n} |\nabla v_n|^p \varphi^p + p\int_{\Omega_n} |\nabla v_n|^{p-2}\nabla v_n \nabla\varphi v_n \varphi^{p-1} + \int_{\Omega_n} |v_n|^p \varphi^p = \int_{\Omega_n} \frac{g(M_n v_n) v_n}{M_n^{p-1}\lambda_n}\varphi^p,$$

https://doi.org/10.1017/prm.2025.10098 Published online by Cambridge University Press

it yields from Hölder's inequality that

$$(1-\epsilon)\int_{\Omega_n}|\nabla v_n|^p\varphi^p+\int_{\Omega_n}|v_n|^p\varphi^p\le C_\epsilon\int_{\Omega_n}|v_n|^p|\nabla\varphi|^p+\int_{\Omega_n}\frac{g(M_nv_n)v_n}{M_n^{p-1}\lambda_n}\varphi^p.$$

Note that

$$\int_{\Omega_n}|v_n|^p|\nabla\varphi|^p\le\frac{1}{M^p}\int_{\Omega_n}|v_n|^p\varphi^p,$$

we may choose M sufficiently large such that $\frac{C_\epsilon}{M^p}<\frac{1}{2}$, thus

$$(1-\epsilon)\int_{\Omega_n}|\nabla v_n|^p\varphi^p+\frac{1}{2}\int_{\Omega_n}|v_n|^p\varphi^p\le\int_{\Omega_n}\frac{g(M_nv_n)v_n}{M_n^{p-1}\lambda_n}\varphi^p.$$

Suppose on the contrary $\liminf\limits_{n\to\infty}\frac{M_n^{\beta-p}+M_n^{\alpha-p}}{\lambda_n}=0$, then we deduce, up to a subsequence,

$$\left|\frac{g(M_nv_n)v_n}{M_n^{p-1}\lambda_n}\right|\le\frac{M_n^{\beta-p}+M_n^{\alpha-p}}{\lambda_n}\to 0,\text{as } n\to\infty.$$

By Lebesgue dominated convergence theorem, we have as $n\to\infty$

$$\int_{\Omega_n}|\nabla v_n|^p\varphi^p+\int_{\Omega_n}|v_n|^p\varphi^p\to 0,$$

a contradiction with $|v_n(0)|=1$.

Now if $\alpha\le\beta$, the conclusion trivially holds. If $\alpha>\beta$, $|g(u)|\le C|u|^{\beta-1}$, and we also obtain the desired result. □

LEMMA A.2. *Let (u_n,λ_n) be a sequence of solution with $\lambda_n\to+\infty$, then* $\limsup\limits_{n\to\infty}\frac{\|u_n\|_\infty^{\beta-p}}{\lambda_n}\le M<+\infty$

Proof. Let us set $M_n=|u_n(x_n)|=\|u_n\|_\infty$ and $v_n(x)=\frac{1}{M_n}u_n(M_n^{-\frac{\beta-p}{p}}x+x_n)$, then v_n satisfies

$$-\Delta_pv_n+\frac{\lambda_n|v_n|^{p-2}v_n}{M_n^{\beta-p}}=\frac{g(M_nv_n)}{M_n^{\beta-1}},\text{ in }\Omega_n.$$

Suppose on the contrary that $\limsup\limits_{n\to\infty}\frac{\|u_n\|_\infty^{\beta-p}}{\lambda_n}=+\infty$, then up to a subsequence,

$$v_n\to v\text{ in }C^1_{loc}(H)$$

and v satisfies

$$-\Delta_pv=\mu_2|v|^{\beta-2}v,\text{ in }H,$$

a contradiction follows as in Lemma 3.2. □

https://doi.org/10.1017/prm.2025.10098 Published online by Cambridge University Press

Following the same pattern as in [9] and [22], we are now in the position to give the local description of the asymptotic behaviour of u_n with bounded Morse index.

Let u_n be solution sequence of (3.4) with $m(u_n) < +\infty$. Defining

$$|u_n(P_n)| = \|u_n\|_\infty \text{ and } v_n(x) = \lambda_n^{-\frac{1}{\beta-p}} u_n(\lambda_n^{-\frac{1}{p}} x + P_n),$$

then v_n satisfies

$$\begin{cases} -\Delta_p v_n + |v_n|^{p-2} v_n = \lambda_n^{-\frac{\beta-1}{\beta-p}} g(\lambda_n^{\frac{1}{\beta-p}} v_n), & \text{in } \Omega_n \\ v_n = 0, & \text{on } \partial\Omega_n. \end{cases} \tag{A.2}$$

where by Lemma A.2, $\lambda_n^{-\frac{\beta-1}{\beta-p}} g(\lambda_n^{\frac{1}{\beta-p}} v_n) \in L^\infty(\Omega_n)$, thus as before, we have

$$v_n \to v \text{ in } C^1_{loc}(H)$$

with $H = \mathbb{R}^N$ or $\mathbb{R}^N_+$, $m(v) < +\infty$ and by Lemma A.1, $v \not\equiv 0$ solves

$$\begin{cases} -\Delta_p v + |v|^{p-2} v = \mu_2 |v|^{\beta-2} v, & \text{in } H \\ v = 0, & \text{on } \partial H. \end{cases} \tag{A.3}$$

For later use, we collect some properties of v as follows .

Proposition A.3. *Let v be a nontrivial solution to (A.3) with $m(v) < +\infty$, then*

(1) $v \in W^{1,p}_0(H) \cap C^1_{loc}(H)$.
(2) $m(v) \geq 1$, *moreover, if* $m(v) = 1$, *then* $H = \mathbb{R}^N$.
(3) *For all* $\varphi \in W^{1,p}_0(H)$, $\varphi \geq 0$, $V = |v|$ *satisfies*

$$\int_H |\nabla V|^{p-2} \nabla V \nabla \varphi + \int_H V^{p-1} \varphi \leq \mu_2 \int_H V^{\beta-1} \varphi,$$

and there exist C and $\gamma > 0$ such that

$$V(x) \leq C e^{-\gamma |x|}. \tag{A.4}$$

Proof. The proof of (1) is the same as Lemma 3.2, thus we omit it here. In the following, we give a brief proof of statements (2) and (3).

(2) It is easy to see $m(v) \geq 1$ once we show $\#\mathcal{N}(v) \leq m(v)$, where $\mathcal{N}(v)$ denotes the set of nodal domains of v and $\#\mathcal{N}(v)$ its cardinality.

Step 1. Let ω be a nodal domain of v, and $v|_\omega$ its restriction to v, by Lemma 3.1 in [10], we see there exists a sequence $\{v_n\} \subset C^1_c(\omega)$ such that

(i) $v_n \to v|_\omega$ in $L^q(\omega)$ for $p \leq q \leq p^*$ and everywhere in ω.
(ii) $\nabla v_n \to \nabla v|_\omega$ in $L^p(\omega)$ and almost everywhere in ω.

https://doi.org/10.1017/prm.2025.10098 Published online by Cambridge University Press

Extending v_n by zero outside ω and taking them as test functions in (A.3), one gets

$$\int_\omega |\nabla v|^{p-2}\nabla v\nabla v_n + \int_\omega |v|^{p-2}vv_n = \mu_2 \int_\omega |v|^{\beta-2}vv_n,$$

let $n \to \infty$, we have

$$\int_\omega |\nabla v|^p + |v|^p = \mu_2 \int_\omega |v|^\beta.$$

Step 2. Taking $\{v_n\}$ in the linearized functional

$$L_v(\phi,\phi) := \int_H |\nabla v|^{p-2}|\nabla\phi|^2 + (p-2)\int_H |\nabla v|^{p-4}(\nabla v,\nabla\phi)^2 + (p-1)\int_H |v|^{p-2}|\phi|^2 - \mu_2(\beta-1)\int_H |v|^{\beta-2}\phi^2,$$

and letting $n \to \infty$, we obtain

$$L_v[v_n,v_n] \to (p-1)\int_\omega (|\nabla v|^p + |v|^p) - \mu_2(\beta-1)\int_\omega |v|^\beta.$$

Combine this together with the results obtsined in Step 1 yield

$$L_v[v_n,v_n] \to \mu_2(p-\beta)\int_\omega |v|^\beta < 0,$$

that is, for every nodal domain ω, there is a v_n such that $L_v[v_n,v_n] < 0$. Thus $\#\mathcal{N}(v) \le m(v)$.

Step 3. Now we claim that if $v \in W_0^{1,p}(\mathbb{R}^N_+)$ is a solution to

$$-\Delta_p v + |v|^{p-2}v = \mu_2|v|^{\beta-2}v \text{in } \mathbb{R}^N_+, \tag{A.5}$$

and $m(v) = 1$, then $v \equiv 0$.

In fact, since $\#\mathcal{N}(v) \le m(v)$, we may assume $v \ge 0$ if $m(v) = 1$.

On the other hand, by (1), we have the following Pohozaev identity

$$\left(1-\frac{1}{p}\right)\int_{B_R\cap\partial\mathbb{R}^N_+} |\partial_N v|^p d\sigma = \int_{\mathbb{R}^N_+\cap\partial B_R}\left[\partial_N v|\nabla v|^{p-2}\nabla v\cdot\nu - \frac{|\nabla v|^p}{p}\nu_N\right]d\sigma + \int_{\mathbb{R}^N_+\cap\partial B_R}\left(\frac{\mu_2|v|^\beta}{\beta} - \frac{|v|^p}{p}\right)\nu_N d\sigma,$$

the right hand is bounded by a function $M(R)$ such that $M(R_k) \to 0$ for some sequence $R_k \to \infty$, thus $\partial_N v = 0$ everywhere on $\partial\mathbb{R}^N_+$, which is a contradiction with the Hopf's Lemma [29].

Thus if v is a solution of (A.5) with $m(v) = 1$, then $v \equiv 0$.

(3) The proof is similar to Lemma 2.2 and Lemma 2.4 in [18]. For later use, we briefly recall the proof of (A.4) there .

https://doi.org/10.1017/prm.2025.10098 Published online by Cambridge University Press

Step 1. By Moser's iteration, it is standard to prove that

$$\|V\|_{L^\infty(B_1(x))} \le C\|V\|_{L^\beta(B_2(x))}, \text{for all } x \in H,$$

where C is a constant independent of x, hence $V(x) \to 0$ as $|x| \to \infty$.

Step 2. Choosing R_0 large enough such that

$$\mu_2 V^{\beta-p}(x) < \frac{1}{2} \text{ for } |x| \ge R_0.$$

Then for $R > R_0$, we have

$$\int_{H\setminus B_R} |\nabla V|^{p-2}\nabla V\nabla\varphi + \frac{1}{2}\int_{H\setminus B_R} V^{p-1}\varphi \le 0, \text{for all } \varphi \in W_0^{1,p}(H\setminus B_R), \varphi \ge 0. \tag{A.6}$$

For any $R > 0$, define η_R as $\eta_R(x) = 0$ for $|x| \le R$, $\eta(x) = 1$ for $|x| \ge R+1$ and $|\nabla\eta_R| \le 2$ for $x \in H$. Taking $\varphi = \eta_R^2 V$, we have

$$\begin{aligned}\int_{H\setminus B_R} (|\nabla V|^p + V^p)\eta_R^2 &\le C\left|\int_{H\setminus B_R} \eta_R\nabla\eta_R V|\nabla V|^{p-2}\nabla V\right| \\ &\le C\int_{H\cap(B_{R+1}\setminus B_R)} (|\nabla V|^p + V^p).\end{aligned}$$

For $R_0 > 0$, let $R_n = R_0 + n$. Defining a sequence

$$a_n = \int_{H\setminus B_{R_0+n}} (|\nabla V|^p + V^p),$$

then $a_{n+1} \le C(a_n - a_{n+1})$, which implies $a_n \le C\theta^n$ for $\theta = \frac{C}{C+1} < 1$. Hence

$$\int_{H\setminus B_R} (|\nabla V|^p + V^p) \le Ce^{-\gamma R},$$

and

$$\int_{H\setminus B_R} V^\beta \le Ce^{-\gamma R}.$$

Step 3. By Step 1 again, we obtain the decay estimate. □

As a corollary of (2), we obtain that there exists $\phi_n \in C_c^1(\Omega)$ satisfying

$$supp\phi_n \subset B_{R\lambda_n^{-\frac{1}{p}}}(P_n) \cap \Omega, R > 0,$$

so that for n large

$$D^2E(u_n)(\phi_n, \phi_n) < 0. \tag{A.7}$$

https://doi.org/10.1017/prm.2025.10098 Published online by Cambridge University Press

To see this, we have by $m(v) \geq 1$ that there is $\varphi \in C_c^1(H)$ with $supp\varphi$ in $B_R(0) \cap H$ and

$$\int_H |\nabla v|^{p-2}|\nabla\varphi|^2 + (p-2)\int_H |\nabla v|^{p-4}(\nabla v, \nabla\varphi)^2 + (p-1)\int_H |v|^{p-2}\varphi^2 - \mu_2(\beta-1) \times \int_H |v|^{\beta-2}\varphi^2 < 0,$$

thus $\phi_n = \varphi(\lambda_n^{\frac{1}{p}}(x - P_n))$ is what we are looking for.

Having a description of the local behaviour around a blow up sequence P_n, we are able to analyze the global behaviour of $\{u_n\}$ and show Proposition 3.3.

Proof of Proposition 3.3. The proof is divided into three steps.

*Step*1. By the previous discussion, we obtain $|u_n(P_n)| = \|u_n\|_\infty$.

Denote $P_n^1 = P_n$ and consider the quantity

$$h_1(R) := \limsup_{n\to\infty} \lambda_n^{-\frac{1}{\beta-p}} \max_{|x-P_n^1| \geq R\lambda_n^{-\frac{1}{p}}} |u_n(x)|.$$

If $\lim_{R\to\infty} h_1(R) = 0$, then $v_n^1(x) := \lambda_n^{-\frac{1}{\beta-p}} u_n(\lambda_n^{-\frac{1}{p}}x + P_n^1) \to 0$ uniformly as $|x| \to \infty$.

Similar as in the proof of (3) in Proposition A.3 , we can prove

$$\int_{\Omega_n^1 \setminus B_R(0)} |v_n^1(x)|^q \leq Ce^{-\gamma R} \text{for some } \gamma > 0,$$

where $\Omega_n^1 := \lambda_n^{\frac{1}{p}}(\Omega - P_n^1)$. As a consequence, we have

$$\begin{aligned}
&\left|\lambda_n^{\frac{N}{p}-\frac{q}{\beta-p}}\int_\Omega |u_n|^q - \int_{B_R(0)\cap\Omega_n^1} |v_n^1|^q\right| \\
=&\lambda_n^{\frac{N}{p}-\frac{q}{\beta-p}}\left|\int_\Omega |u_n|^q - \int_{\Omega\cap B_{R\lambda_n^{-\frac{1}{p}}}(P_n^1)} |u_n|^q\right| \\
=&\lambda_n^{\frac{N}{p}-\frac{q}{\beta-p}}\int_{\{|x-P_n^1|\geq R\lambda_n^{-\frac{1}{p}}\}\cap\Omega} |u_n|^q \\
=&\int_{\Omega_n^1\setminus B_R(0)} |v_n^1(x)|^q \leq Ce^{-\gamma R}.
\end{aligned}$$

Letting $n \to \infty$, we obtain, up to a subsequence

$$\begin{aligned}
\left|\lambda_n^{\frac{N}{p}-\frac{q}{\beta-p}}\int_\Omega |u_n|^q - \int_{B_R\cap H} |v_1|^q\right| &= \lim_{n\to\infty}\left|\lambda_n^{\frac{N}{p}-\frac{q}{\beta-p}}\int_\Omega |u_n|^q - \int_{B_R(0)\cap\Omega_n^1} |v_n^1|^q\right| \\
&\leq Ce^{-\gamma R},
\end{aligned}$$

taking $R \to \infty$, we see (3.11) holds.

https://doi.org/10.1017/prm.2025.10098 Published online by Cambridge University Press

Step 2. Otherwise, if

$$\lim_{R\to\infty} h_1(R) = 4\delta > 0,$$

then for arbitrarily large R, up to a subsequence, we may assume there is $P_n^2 \in \Omega \setminus B_{R\lambda_n^{-\frac{1}{p}}}(P_n^1)$ satisfying

$$|u_n(P_n^2)| = \max_{\Omega\setminus B_{R\lambda_n^{-\frac{1}{p}}}(P_n^1)} |u_n(x)| \text{ and } |v_n^1(\lambda_n^{\frac{1}{p}}(P_n^2 - P_n^1))| = \lambda_n^{-\frac{1}{\beta-p}}|u_n(P_n^2)| > 2\delta.$$

On the other hand, noticing that

$$v_n^1 \to v \text{ in } C^1_{loc}(H) \text{ and } |v(x)| \le \delta \text{ for } |x| \ge R >> 1,$$

if $\{\lambda_n^{\frac{1}{p}}|P_n^1 - P_n^2|\}$ is bounded, then up to a subsequence, there is $R' \ge R$ such that

$$\lambda_n^{\frac{1}{p}}|P_n^1 - P_n^2| \to R',$$

we obtain the existence of x', $|x'| = R'$ and $|v_n^1(\lambda_n^{\frac{1}{p}}(P_n^2 - P_n^1))| \to |v(x')| \le \delta$, a contradiction. Thus $\lambda_n^{\frac{1}{p}}|P_n^1 - P_n^2| \to \infty$.

Now we claim that $|u_n(P_n^2)| = \max_{\Omega\cap B_{R_n\lambda_n^{-\frac{1}{p}}}(P_n^2)} |u_n(x)|$ for some $R_n \to +\infty$.

To see this, let $\epsilon_n = |u_n(P_n^2)|^{-(\beta-p)}$ and $R_n^{(2)} = \frac{1}{2}\frac{|P_n^1 - P_n^2|}{\epsilon_n^{\frac{1}{p}}}$, then $\epsilon_n \to 0$.

Moreover, $\epsilon_n \le (2\delta)^{p-\beta}\lambda_n^{-1}$. Thus

$$R_n^{(2)} \ge \frac{1}{2}(2\delta)^{\frac{1}{p}(\beta-p)}\lambda_n^{\frac{1}{p}}|P_n^1 - P_n^2| \to +\infty,$$

and $\forall x \in B_{R_n^{(2)}\epsilon_n^{\frac{1}{p}}}(P_n^2)$

$$|x - P_n^1| \ge |P_n^2 - P_n^1| - |x - P_n^2| \ge \frac{1}{2}|P_n^2 - P_n^1| \ge R\lambda_n^{-\frac{1}{p}},$$

that is

$$\Omega \cap B_{R_n^{(2)}\epsilon_n^{\frac{1}{p}}}(P_n^2) \subset \Omega \setminus B_{R\lambda_n^{-\frac{1}{p}}}(P_n^1)$$

and therefore

$$|u_n(P_n^2)| = \max_{\Omega\cap B_{R_n^{(2)}\epsilon_n^{\frac{1}{p}}}(P_n^2)} |u_n(x)|.$$

Denote $R_n := R_n^{(2)}\epsilon_n^{\frac{1}{p}}\lambda_n^{\frac{1}{p}}$, then $R_n = \frac{1}{2}\lambda_n^{\frac{1}{p}}|P_n^1 - P_n^2| \to \infty$ and

$$|u_n(P_n^2)| = \max_{\Omega\cap B_{R_n\lambda_n^{-\frac{1}{p}}}(P_n^2)} |u_n(x)|.$$

https://doi.org/10.1017/prm.2025.10098 Published online by Cambridge University Press

Step 3. Iterating the above steps, we obtain k local maxima $P_n^1, \dots, P_n^k$ such that (1) holds true and

$$h_k(R) := \limsup_{n\to\infty} \left(\lambda_n^{-\frac{1}{\beta-p}} \max_{d_{n,k} \geq R\lambda_n^{-\frac{1}{p}}} |u_n(x)| \right) \text{ satisfies } \lim_{R\to\infty} h_k(R) := 0, \quad \text{(A.8)}$$

where

$$d_{n,k}(x) := \min\left\{ |x - P_n^i|, i = 1, \dots, k. \right\}.$$

We now prove $k \leq \bar{k}$. In fact, Noticing by (A.7) that for every P_n^i, there exists ϕ_n^i satisfying

$$supp\phi_n^i \subset B_{R_i\lambda_n^{-\frac{1}{p}}}(P_n^i) \cap \Omega$$

and

$$D^2E(u_n)(\phi_n^i, \phi_n^i) < 0, \text{ for } n \text{ large},$$

since $m(u_n) \leq \bar{k}$, we obtain $k \leq \bar{k}$.

Finally, taking $U_n(x) := \lambda_n^{-\frac{1}{\beta-p}} |u_n(\lambda_n^{-\frac{1}{p}} x)|$, then

$$\int_{\Omega_n} |\nabla U_n|^{p-2} \nabla U_n \nabla \varphi + \int_{\Omega_n} U_n^{p-1} \varphi \leq C \int_{\Omega_n} U_n^{\alpha-1} \varphi + U_n^{\beta-1} \varphi, \text{ for all } \varphi \in W_0^{1,p}(\Omega_n), \varphi \geq 0,$$

where $\Omega_n := \lambda_n^{\frac{1}{p}} \Omega$.

Moreover by (A.8), $U_n(x) \to 0$ if $\min\{|x - \lambda_n^{\frac{1}{p}} P_n^i|\} \to \infty$.

Similar as in Step 1, we conclude

$$\int_{\Omega_n \setminus \cup B_R(\lambda_n^{\frac{1}{p}} P_n^i)} U_n^q \leq Ce^{-\gamma R},$$

and (3.11) holds. □

https://doi.org/10.1017/prm.2025.10098 Published online by Cambridge University Press