Math. Proc. Camb. Phil. Soc. (1976), **79**, 533
MPCPS 79–47
Printed in Great Britain

A geometrical proof that causality implies the Lorentz group

BY S. NANDA
Engineering College, Rourkela 769008, *India*

(*Received* 28 *April* 1975, *revised* 15 *July* 1975)

1. *Introduction.* That causality implies the Lorentz group was first proved by Zeeman (1). Later, Barruchi (2) and Teppati (3) proved a generalised version of the theorem by using completely different arguments. In this paper, we give a very simple and elementary geometric proof of the same theorem.

2. *Notation and terminology.* Let M denote Minkowski space of Special Relativity with characteristic quadratic form Q:

$$M = \{(x_0, x_1, x_2, x_3); x_i \text{ are reals}\}$$

$$Q(x) = x_0^2 - x_1^2 - x_2^2 - x_3^2.$$

Let $\quad K = \{x \in M : Q(x) > 0, x_0 > 0\}$ and $L = \{x \in M : Q(x) \geqslant 0, x_0 > 0\}$.

It is easy to verify that K satisfies the following three conditions.

(i) $K + K = \{x + y : x \in K, y \in K\} \subset K$,

(ii) $aK = \{ax : x \in K\} \subset K$ for every positive number a,

(iii) $K \cap (-K) = \varnothing$, where $-K = \{-x : x \in K\}$.

L also satisfies (i), (ii) and (iii). K and L are so-called positive cones and each generates a partial order on M as follows:

$$x < y \Leftrightarrow y - x \in K,$$

$$x \ll y \Leftrightarrow y - x \in L.$$

There is also another relation $<.$ on M defined by $x <. y \Leftrightarrow Q(y-x) = 0$ and $y_0 - x_0 > 0$, i.e. $y - x$ is a future-pointing lightlike (null) vector. It is easy to establish that

$$x \ll y \Leftrightarrow \begin{cases} x < y & \text{or} \\ x <. y. \end{cases}$$

For each $x \in M$, we also have the following cones:

$$K(x) = K + x, \quad L(x) = L + x, \quad K^*(x) = K(x) \cup \{x\}$$

and

$$L^*(x) = L(x) \cup \{x\}.$$

A map f of M into itself is said to be $<$-order-preserving if $x < y \Rightarrow f(x) < f(y)$. A one-one map f of M onto itself which is $<$-order-preserving and whose inverse is also $<$-order-preserving is called a $<$-automorphism. Similarly, we define a $\ll$-automorphism.

https://doi.org/10.1017/S0305004100052543 Published online by Cambridge University Press

Let G_0 be the group generated by (i) the orthochronous Lorentz group, i.e. all linear maps which leave the quadratic form Q invariant and preserve time-orientation, (ii) translations and (iii) positive dilatations (i.e. multiplication by positive scalars). It is well known that causality is represented by the partial order $<$. Zeeman's theorem states that the group of $<$-automorphisms of M is precisely G_0.

It is easy to see that f is a $<$-automorphism if and only if $f(K^*(x)) = K^*(fx)$ for every x and similarly, f is a $\ll$-automorphism if and only if $f(L^*(x)) = L^*(fx)$.

3. *Proof of Zeeman's theorem.* We start by showing that any $<$-automorphism of M is necessarily a homeomorphism of M with respect to the Euclidean topology. Denote by M^E the space M equipped with the Euclidean topology.

Definition. Let $O_{x,y} = \{z : x < z < y\}$. The sets $\{O_{x,y}\}_{x,y \in M}$ form a basis of neighbourhoods for a topology on M, called the 'interval topology'. Denote this topology by I and the Euclidean topology E.

LEMMA 1. *The interval topology on M coincides with the Euclidean topology.*

Proof. Since each $O_{x,y}$ is open in M^E, it follows that $I \leqslant E$ (where $\leqslant$ is the partial order on the collection of all topologies on M). On the other hand, for every open sphere O in M^E and every $z \in O$, it is possible to choose $x, y \in O$ such that $z \in O_{x,y} \subset O$. This implies that $E \leqslant I$. Hence the equality.

LEMMA 2. *Any $<$-automorphism of M is a homeomorphism of M^E.*

Proof. Observe that $O_{x,y} = K(x) \cap (-K(y))$. If f is a $<$-automorphism, then $f(O_{x,y}) = K(fx) \cap (-K(fy)) = O_{fx,fy}$. Thus f maps the basic open sets of M^E to its basic open sets; so does f^{-1}. This proves the desired result.

LEMMA 3. *Any $<$-automorphism of M is a $\ll$-automorphism.*

Proof. Let f be a $<$-automorphism and let an upper bar denote closure with respect to the Euclidean topology. Then we have

$$\begin{aligned} f(L^*(x)) &= f(\overline{K^*(x)}) = \overline{f(K^*(x))} \\ &= \overline{K^*(fx)} = L^*(fx). \end{aligned}$$

This proves that f is a $\ll$-automorphism.

LEMMA 4. *Null lines are mapped to null lines by any $<$-automorphism f.*

Proof. Let N be a null line and $x, y \in N$ with $x < . y$. Denote by $[x, y]$ the closed interval joining x and y. Clearly,

$$[x, y] = L^*(x) \cap (-L^*(y)).$$

Therefore,

$$\begin{aligned} f[x, y] &= f(L^*(x)) \cap f(-L^*(y)) \\ &= L^*(fx) \cap (-L^*(fy)) \neq \varnothing. \end{aligned}$$

This implies that $fx \ll fy$. Since any $z_1, z_2 \in [x, y]$ are not comparable (i.e. neither $z_1 < z_2$ nor $z_2 < z_1$) it follows that neither $fz_1 < fz_2$ nor $fz_2 < fz_1$. Therefore we must have $fx < . fy$. Since x and y are arbitrary in N the proof is complete.

https://doi.org/10.1017/S0305004100052543 Published online by Cambridge University Press

LEMMA 5. *A tangent hyperplane to the cone $L^*(x)$ touching a null line N is mapped by a $<$-automorphism f to a tangent hyperplane to the cone $L^*(fx)$ touching the null line $f(N)$.*

Proof. Consider the set $A = \bigcup_{z \in N} L^*(z)$ and its image by the map

$$f: f(A) = \bigcup_{z \in N} f(L^*(z))$$
$$= \bigcup_{fz \in fN} L^*(fz).$$

It is easy to see geometrically that the boundary of the set A is the tangent hyperplane to $L^*(x)$ touching along the generator N and similarly the boundary of the set fA is the corresponding tangent hyperplane to $L^*(fx)$ touching it along the null line fN. Since f is a homeomorphism, boundary points must be mapped to boundary points (4) and therefore, the lemma is proved.

COROLLARY 1. *Since every spacelike two-plane at a point x can be realized as the intersection of two null hyperplanes (i.e. tangent hyperplanes as described above), it follows that f maps every spacelike two-plane to a spacelike two-plane.*

COROLLARY 2. *Observe that every spacelike line through a point x can be realized as the intersection of two spacelike two-planes; therefore, every spacelike line is mapped by f to a spacelike line (by application of Corollary 1).*

COROLLARY 3. *Any two-plane generated by two spacelike lines is mapped to a two-plane.*

Proof. Let w_1 and w_2 be two spacelike lines and let W be the two-plane generated by them. Choose a third spacelike line w_3 meeting w_1 and w_2. By Corollary 2 above, all these lines are mapped to lines and therefore the image two-plane is completely determined.

COROLLARY 4. *Any timelike (and similarly lightlike) line is mapped by f to a straight line.*

Proof. Let T be a timelike (or lightlike) line through x. Clearly T can be realized as the intersection of two two-planes, each of which is generated by two spacelike lines. Applying Corollary 3, one gets the required result.

Thus every line through x (whether spacelike, timelike or lightlike) is mapped by f to a line through fx and therefore, every two-plane at x is mapped by f to a two-plane at fx. Hence by using a well-known result (5) (theorem 2·4, p. 282), we have the following:

THEOREM 1. *Every $<$-automorphism of M is affine.*

Thus f is a composition of a linear map with a translation. But it is known that any linear transformation which leaves the light cones invariant is the composition of a dilatation with an element of the Lorentz group (6). Since any $<$-automorphism f keeps the light cones invariant, it follows that $f \in G_0$. Therefore, we have Zeeman's theorem:

THEOREM 2. *The group of $<$-automorphisms of M is G_0.*

https://doi.org/10.1017/S0305004100052543 Published online by Cambridge University Press

REFERENCES

(1) ZEEMAN, E. C. *J. Math. Phys.* **5** (1964), 490.
(2) BARRUCHI, G. *Nuovo Cimento* A (10), **55** (1968), 385.
(3) TEPPATI, G. *Nuovo Cimento* A (10), **54** (1968), 800.
(4) ALEXANDROFF, A. D. *Combinatorial topology*, vol. 1 (Rochester, New York, Graylock Press 1956.)
(5) LEVY, H. *Projective and related geometries* (New York, Macmillan, 1964).
(6) PAULI, W. *Theory of relativity* (New York, Pergamon Press, 1958).

https://doi.org/10.1017/S0305004100052543 Published online by Cambridge University Press