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NON-HOMOGENEOUS AND TIME-CHANGED VERSIONS OF GENERALIZED COUNTING PROCESSES

KULDEEP KUMAR KATARIA,* *Indian Institute of Technology Bhilai*
MOSTAFIZAR KHANDAKAR,** *Indian Institute of Information Technology Guwahati*
PALANIAPPAN VELLAISAMY,*** *University of California, Santa Barbara*

Abstract

In this paper, we introduce a non-homogeneous version of the generalized counting process (GCP). We time-change this process by an independent inverse stable subordinator and derive the system of governing differential–integral equations for the marginal distributions of its increments. We then consider the GCP time-changed by a multistable subordinator and obtain its Lévy measure and the distribution of its first passage times. We discuss an application of a time-changed GCP, namely the time-changed generalized counting process-I (TCGCP-I) in ruin theory. A fractional version of the TCGCP-I is studied, and its long-range dependence property is established.

Keywords: Bernštein function; multistable subordinator; non-homogenous Poisson process; long-range dependence property; risk process

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1. Introduction

Di Crescenzo *et al.* [14] introduced and studied a Lévy process $\{M(t)\}_{t\geq 0}$, namely the generalized counting process (GCP). Its transition probabilities are given by

$$\mathbb{P}\{M(t+h)=n|M(t)=m\}=\begin{cases}1-\Lambda h+o(h), & n=m,\\ \lambda_j h+o(h), & n=m+j,\ j=1,2,\ldots,k,\\ o(h), & n>m+k,\end{cases} \tag{1}$$

where the λ_j are positive rates, $\Lambda=\lambda_1+\lambda_2+\cdots+\lambda_k$, and $o(h)\to 0$ as $h\to 0$. From (1), it is evident that the GCP performs k kinds of jumps of amplitude, $1,2,\ldots,k$, with rates $\lambda_1,\lambda_2,\ldots,\lambda_k$, respectively. Its fractional version, the generalized fractional counting process (GFCP), is obtained by time-changing the GCP by an independent inverse stable subordinator $\{Y_\beta(t)\}_{t\geq 0}$ for $0<\beta<1$, that is, $\{M(Y_\beta(t))\}_{t\geq 0}$. We denote the GFCP by $\{M^\beta(t)\}_{t\geq 0}$. Its state probabilities $p^\beta(n,t)=\mathbb{P}\{M^\beta(t)=n\}$ satisfy the following system of fractional differential

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* Postal address: Department of Mathematics, Indian Institute of Technology Bhilai, Durg-491002, India. Email: kuldeepk@iitbhilai.ac.in
** Postal address: Department of Science and Mathematics, Indian Institute of Information Technology Guwahati, Bongora-781015, India. Email: mostafizar.khandakar@iiitg.ac.in
*** Postal address: Department of Statistics and Applied Probability, University of California, Santa Barbara, CA-93106, USA. Email: pvellais@ucsb.edu

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equations (see [14]):

$$\frac{\mathrm{d}^{\beta}}{\mathrm{d}t^{\beta}}p^{\beta}(n,t) = -\Lambda p^{\beta}(n,t) + \sum_{j=1}^{n\wedge k}\lambda_j p^{\beta}(n-j,t), \quad n \geq 0, \tag{2}$$

with initial conditions

$$p^{\beta}(n,0) = \begin{cases} 1, & n = 0, \\ 0, & n \geq 1. \end{cases}$$

Here, $\frac{\mathrm{d}^{\beta}}{\mathrm{d}t^{\beta}}$ is the Caputo fractional derivative defined in (16) and $n \wedge k = \min\{n, k\}$.

For $\beta = 1$, the GFCP reduces to the GCP. For $k = 1$, the GFCP and the GCP reduce to the time-fractional Poisson process (TFPP) (see [2]) and the Poisson process, respectively. Recently, several authors have introduced and studied counting processes such as the Poisson process of order k (PPoK), the Pólya–Aeppli process of order k (PAPoK), the Bell–Touchard process, and the Poisson–logarithmic process. These processes, along with their fractional variants, are particular cases of the GFCP (see [21, 23]). For other fractional generalizations of the Poisson process, such as the space-fractional Poisson process (SFPP) and space–time-fractional Poisson process (STFPP), we refer the reader to [28]. The fractional versions of the Poisson process have applications in fields such as queueing theory, finance, fractional quantum mechanics, and the modeling of catastrophic events such as tsunamis and earthquakes.

Let $\overset{\mathrm{d}}{=}$ denote equality in distribution. Also, let $\{X_i\}_{i\geq 1}$ be a sequence of independent and identically distributed (i.i.d.) random variables with probability mass function (PMF)

$$\mathbb{P}\{X_1 = j\} = \lambda_j/\Lambda, \quad j = 1, 2, \ldots, k. \tag{3}$$

Di Crescenzo *et al.* [14] showed that the GCP is equal in distribution to the following compound Poisson process:

$$M(t) \overset{\mathrm{d}}{=} \sum_{i=1}^{N(t)} X_i, \tag{4}$$

where the Poisson process $\{N(t)\}_{t\geq 0}$ with intensity Λ is independent of $\{X_i\}_{i\geq 1}$.

Leonenko *et al.* [26] introduced and studied a non-homogeneous version of the TFPP. Buchak and Sakhno [9] obtained the governing equations for marginal distributions of a non-homogeneous Poisson process time-changed by an inverse subordinator. Recently, Kadankova *et al.* [18] studied non-homogeneous versions of the PPoK and the PAPoK along with their fractional extensions.

In Section 2, we give some preliminary results that will be used later. In Section 3, an alternative representation of the one-dimensional distribution of the GFCP in terms of the derivatives of the Mittag-Leffler function is obtained. We introduce a non-homogeneous version of the GCP, namely the non-homogeneous generalized counting process (NGCP), and obtain its probability generating function (PGF) and the system of differential equations that governs its state probabilities. We time-change the NGCP by an independent inverse stable subordinator to obtain its fractional version, which we call the non-homogeneous generalized fractional counting process (NGFCP). Also, we discuss the arrival times of the NGFCP. A generalization of the NGFCP is introduced by time-changing the NGCP with an independent

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inverse subordinator, for which we obtain the governing system of equations of marginal distributions involving a convolution-type derivative studied by Toaldo [34]. The results obtained in this section generalize and complement the results of [9], [18], and 26].

A multistable subordinator $\{H(t)\}_{t\geq 0}$ with time-dependent stability index $\beta(t) \in (0, 1)$ is an inhomogeneous subordinator. The Laplace transform of increments of an inhomogeneous subordinator $\{R(t)\}_{t\geq 0}$ is given by

$$\mathbb{E}\left(\mathrm{e}^{-u(R(t)-R(s))}\right) = \mathrm{e}^{-\int_s^t f(u,\tau)\,\mathrm{d}\tau}, \quad 0 \leq s \leq t,$$

which holds under some suitable conditions (see [30]). Here, $u \to f(u, t)$ is a Bernštein function for each $t \geq 0$ such that

$$f(u, t) = \int_0^\infty (1 - \mathrm{e}^{-ux})\nu_t(\mathrm{d}x). \tag{5}$$

A multistable subordinator is characterized by the Lévy measure (see [7])

$$\nu_t(\mathrm{d}x) = \frac{\beta(t)x^{-\beta(t)-1}}{\Gamma(1-\beta(t))}\,\mathrm{d}x, \quad x > 0, \tag{6}$$

where $t \to \beta(t)$ is a regular function. Its associated Bernštein function is $f(u, t) = u^{\beta(t)}$. The multistable subordinators extend the stable subordinators by considering the time-dependent stability index. These have proven to be useful in modeling phenomena in finance and the natural sciences. For more details on multistable subordinators, we refer the reader to [30].

In Section 4, we consider the GCP time-changed by an independent multistable subordinator, that is, $\{M(H(t))\}_{t\geq 0}$. We obtain the Lévy measure, the Bernštein function, and the distribution of its first passage times. For a particular case of the process $\{M(H(t))\}_{t\geq 0}$, we obtain its one-dimensional distribution and PGF.

Kataria and Khandakar [20] studied a time-changed version of the GFCP, namely the time-changed generalized fractional counting process-I (TCGFCP-I) $\{M_f^\beta(t)\}_{t\geq 0}$. It is defined as follows:

$$M_f^\beta(t) := M^\beta(D_f(t)),$$

where the GFCP is independent of the Lévy subordinator $\{D_f(t)\}_{t\geq 0}$. For $\beta = 1$, it reduces to the time-changed generalized counting process-I (TCGCP-I) $\{M_f(t)\}_{t\geq 0}$. From the transition probabilities of the TCGCP-I (see (48)), it can be observed that the jumps can take any positive value in a small time interval, so the TCGCP-I could be useful in modeling real-life situations where multiple occurrences are allowed in an infinitesimal interval of time.

In Section 5, we obtain some additional results for the TCGCP-I. The system of differential equations that governs its state probabilities, the distribution of its first passage times, and related governing equations are obtained. For a particular case of the TCGCP-I, we obtain the system of differential equations that governs its state probabilities. As an application, we consider a risk process in which the TCGCP-I is used to model the number of claims received by an insurance company. Some results for the joint distribution of the time to ruin and the deficit at the time of ruin are derived. We show that the one-dimensional distributions of the TCGFCP-I are not infinitely divisible. We discuss the conditional distribution of the kth order statistic from a sample whose size is modeled by a particular case of the TCGFCP-I, namely the time-fractional negative binomial process (see [36]).

In Section 6, we consider the following time-changed process:

$$\mathscr{M}_f^\alpha(t) := M_f(Y_\alpha(t)) = M(D_f(Y_\alpha(t))), \quad t \geq 0,$$

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where the Lévy subordinator $\{D_f(t)\}_{t\geq 0}$, the inverse stable subordinator $\{Y_\alpha(t)\}_{t\geq 0}$ for $0<\alpha<1$, and the GCP $\{M(t)\}_{t\geq 0}$ are independent of each other. We obtain its PGF, mean, variance, and covariance, as well as the system of differential equations that governs its state probabilities. Its long-range dependence (LRD) property is established. Also, it is shown that the process $\{\mathscr{M}_f^\alpha(t)\}_{t\geq 0}$ has the overdispersion property, so it can potentially be used when the empirical count data exhibit overdispersion. We prove some results for two particular cases of this time-changed process.

2. Preliminaries

In this section, we collect some preliminary results for some known processes and the definition of the LRD property.

2.1. Generalized counting process

Let $\mathbb{N}_0$ be the set of non-negative integers and let $\Lambda=\lambda_1+\lambda_2+\cdots+\lambda_k$. The state probabilities $p(n,t)=\mathbb{P}\{M(t)=n\}$ of a GCP are given by (see [14])

$$p(n,t)=\sum_{\Omega(k,n)}\left(\prod_{j=1}^{k}\frac{(\lambda_j t)^{x_j}}{x_j!}\right)\mathrm{e}^{-\Lambda t},\quad n\geq 0, \tag{7}$$

where $\Omega(k,n):=\{(x_1,x_2,\ldots,x_k):\ \sum_{j=1}^{k} jx_j=n,\ x_j\in\mathbb{N}_0\}$.

Its PGF $G(u,t)=\mathbb{E}(u^{M(t)})$ is given by (see [21])

$$G(u,t)=\exp\left(-\sum_{j=1}^{k}\lambda_j(1-u^j)t\right),\quad |u|\leq 1. \tag{8}$$

The following limiting result holds for the GCP (see [21, Equation (17)]):

$$\lim_{t\to\infty}\frac{M(t)}{t}=\sum_{j=1}^{k} j\lambda_j \text{ in probability.} \tag{9}$$

2.2. The subordinator and its inverse

Let

$$f(s)=c_1+c_2 s+\int_0^\infty\left(1-\mathrm{e}^{-sx}\right)\overline{\nu}(\mathrm{d}x),\quad s>0,\ c_1\geq 0,\ c_2\geq 0 \tag{10}$$

be a Bernštein function and let $\overline{\nu}(\cdot)$ be a non-negative Lévy measure on $(0,\infty)$ such that $\int_0^\infty (x\wedge 1)\,\overline{\nu}(\mathrm{d}x)<\infty$. A subordinator $\{D_f(t)\}_{t\geq 0}$ is a one-dimensional Lévy process characterized by the following Laplace transform (see [1, Section 1.3.2]):

$$\mathbb{E}\left(\mathrm{e}^{-sD_f(t)}\right)=\mathrm{e}^{-tf(s)}.$$

Its first passage time is called the inverse subordinator and is defined as

$$Y_f(t):=\inf\{x\geq 0: D_f(x)>t\},\quad t\geq 0. \tag{11}$$

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Let $h_f(t, x)$ be the density of $\{Y_f(t)\}_{t\geq 0}$. Its Laplace transform is given by (see [34])

$$\tilde{h}_f(s, x) = \int_0^\infty \mathrm{e}^{-st} h_f(t, x)\,\mathrm{d}t = \frac{f(s)}{s}\,\mathrm{e}^{-xf(s)}, \tag{12}$$

provided that the following condition holds.

Condition **I.** We have that $\overline{\nu}(0, \infty) = \infty$ and that the tail $\nu(x) = c_1 + \overline{\nu}(x, \infty)$ of the Lévy measure $\overline{\nu}(\cdot)$ is absolutely continuous.

Toaldo [34] discussed a new type of differential operator that generalizes the Caputo fractional derivative and Riemann–Liouville fractional derivative. Let $u(\cdot)$ be an absolutely continuous function. Its generalized Caputo derivative associated with the Bernštein function f is defined as (see [34, Definition 2.4])

$${}^f\mathcal{D}_t u(t) := c_2 \frac{\mathrm{d}}{\mathrm{d}t} u(t) + \int_0^t \frac{\partial}{\partial t} u(t-x)\nu(x)\,\mathrm{d}x. \tag{13}$$

It is related to the generalized Riemann–Liouville derivative ${}^f\mathscr{D}_t$ through the following relation (see [34, Proposition 2.7]):

$${}^f\mathscr{D}_t u(t) = {}^f\mathcal{D}_t u(t) + \nu(t)u(0). \tag{14}$$

If $|u(t)| \leq C\mathrm{e}^{s_0 t}$, where C and s_0 are some constants, then the Laplace transform of generalized Caputo derivative is given by (see [34, Lemma 2.5])

$$\mathcal{L}\left({}^f\mathcal{D}_t u(t); s\right) = f(s)\tilde{u}(s) - \frac{f(s)}{s}u(0), \quad s > s_0. \tag{15}$$

For $f(s) = s^\beta$ with $\beta \in (0, 1)$, the derivative (13) reduces to (see [34, Remark 2.6])

$${}^f\mathcal{D}_t u(t) = \frac{\mathrm{d}^\beta}{\mathrm{d}t^\beta} u(t),$$

where $\dfrac{\mathrm{d}^\beta}{\mathrm{d}t^\beta}$ is the Caputo fractional derivative defined as (see [24])

$$\frac{\mathrm{d}^\beta}{\mathrm{d}t^\beta} u(t) := \begin{cases} \dfrac{1}{\Gamma(1-\beta)} \displaystyle\int_0^t (t-s)^{-\beta} u'(s)\,\mathrm{d}s, & 0 < \beta < 1, \\ u'(t), & \beta = 1. \end{cases} \tag{16}$$

Its Laplace transform is given by (see [24, Equation (5.3.3)])

$$\mathcal{L}\left(\frac{\mathrm{d}^\beta}{\mathrm{d}t^\beta} u(t); s\right) = s^\beta \tilde{u}(s) - s^{\beta-1}u(0), \quad s > 0. \tag{17}$$

2.3. The stable subordinator and its inverse

A stable subordinator $\{D_\beta(t)\}_{t\geq 0}$ for $0 < \beta < 1$ is a non-decreasing Lévy process whose Laplace transform is given by $\mathbb{E}(\mathrm{e}^{-sD_\beta(t)}) = \mathrm{e}^{-ts^\beta}$ for $s > 0$. Its associated Bernštein function is $f_2(s) = s^\beta$. Its first passage time $\{Y_\beta(t)\}_{t\geq 0}$, which is defined as $Y_\beta(t) := \inf\{x \geq 0 : D_\beta(x) > t\}$, is called the inverse stable subordinator.

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Let $h_\beta(t, x)$ be the density of $\{Y_\beta(t)\}_{t\geq 0}$. Its Laplace transform is given by

$$\tilde{h}_\beta(s, x) = s^{\beta-1}\mathrm{e}^{-xs^\beta}. \tag{18}$$

The mean and variance of $\{Y_\beta(t)\}_{t\geq 0}$ are (see [25])

$$\mathbb{E}\big(Y_\beta(t)\big) = \frac{t^\beta}{\Gamma(\beta+1)}, \tag{19}$$

$$\mathrm{Var}\big(Y_\beta(t)\big) = \left(\frac{2}{\Gamma(2\beta+1)} - \frac{1}{\Gamma^2(\beta+1)}\right) t^{2\beta}. \tag{20}$$

For fixed t_1 and large t_2, the following asymptotic result holds for the covariance:

$$\mathrm{Cov}\big(Y_\beta(t_1), Y_\beta(t_2)\big) \sim \frac{t_1^{2\beta}}{\Gamma(2\beta+1)}. \tag{21}$$

2.4. Fractional negative binomial processes

The density function $g(x, t)$ of a gamma subordinator $\{Z(t)\}_{t\geq 0}$ is given by

$$g(x, t) = \frac{a^{bt}}{\Gamma(bt)} x^{bt-1}\mathrm{e}^{-ax}, \quad x > 0,$$

where $a > 0$ and $b > 0$. Its associated Bernštein function is (see [1, p. 55])

$$f_1(s) = b\log(1 + s/a), \quad s > 0. \tag{22}$$

The space–time-fractional negative binomial process is defined as (see [22])

$$\mathcal{N}_\beta^\alpha(t) := N^{\alpha,\beta}(Z(t)), \quad t \geq 0,$$

where the STFPP $\{N^{\alpha,\beta}(t)\}_{t\geq 0}$ is independent of the gamma subordinator. For $\alpha = 1$ it reduces to the time-fractional negative binomial process (see [36]), and for $\beta = 1$ it reduces to the space-fractional negative binomial process. The PGF of the time-fractional negative binomial process is (see [22])

$$\mathbb{E}(u^{\mathcal{N}_\beta^1(t)}) = \sum_{k=0}^{\infty} \frac{(-\lambda)^k(1-u)^k}{\Gamma(k\beta+1)} \frac{\Gamma(bt+k\beta)}{a^{k\beta}\Gamma(bt)}, \quad |u| \leq 1. \tag{23}$$

Its state probability is given by (see [36, Section 4.1])

$$\mathbb{P}\{\mathcal{N}_\beta^1(t) = n\} = \sum_{l=n}^{\infty} (-1)^{l+n} \binom{l}{n} \frac{\Gamma(l\beta+bt)}{\Gamma(bt)\Gamma(\beta l+1)} \left(\frac{\lambda}{a^\beta}\right)^l, \quad n \geq 0. \tag{24}$$

2.5. Long-range dependence property

Let $s > 0$ be fixed. The process $\{X(t)\}_{t\geq 0}$ is said to exhibit the LRD property if its correlation function has the following asymptotic behaviour (see [15, 27]):

$$\mathrm{Corr}\,(X(s), X(t)) \sim c(s)t^{-\theta} \quad \text{as } t \to \infty$$

for some $c(s) > 0$ and $\theta \in (0, 1)$. If $\theta \in (1, 2)$, then it is said to possesses the short-range dependence property.

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3. Non-homogeneous GCP and its fractional variants

In this section, we introduce and study a non-homogeneous version of the GFCP, namely the NGFCP. We start by giving some additional results related to the GCP and its fractional variant, the GFCP.

Let Z_1 denote the time of first occurrence of a GCP event in $[0, t]$. Then

$$\begin{aligned}\mathbb{P}\{Z_1 \le x | M(t) = 1\} &= \frac{\mathbb{P}\{\text{exactly one occurrence in } [0, x],\ \text{no occurrence in } (x, t]\}}{\mathbb{P}\{M(t) = 1\}} \\ &= \frac{\mathbb{P}\{M(x) = 1, M(t-x) = 0\}}{\mathbb{P}\{M(t) = 1\}} \\ &= \frac{\mathbb{P}\{M(x) = 1\}\mathbb{P}\{M(t-x) = 0\}}{\mathbb{P}\{M(t) = 1\}} = \frac{x}{t}, \quad 0 \le x \le t,\end{aligned}$$

where in the penultimate step we have used the independent increment of GCP and the last step follows from using (7). So the distribution of Z_1 given that exactly one GCP event has occurred in $[0, t]$ is uniform in $[0, t]$.

Di Crescenzo *et al.* [14] showed that

$$M^{\beta}(t) \stackrel{\mathrm{d}}{=} M(T_{2\beta}(t)), \quad t > 0, \tag{25}$$

where the GCP is independent of the random process $\{T_{2\beta}(t)\}_{t>0}$ whose distribution is given by the folded solution of the Cauchy problem

$$\frac{\mathrm{d}^{2\beta}}{\mathrm{d}t^{2\beta}} u(x, t) = \frac{\partial^2}{\partial x^2} u(x, t), \quad t > 0,\ x \in \mathbb{R}, \tag{26}$$

with $u(x, 0) = \delta(x)$ for $0 < \beta \le 1$ and $\frac{\partial}{\partial t} u(x, 0) = 0$ for $1/2 < \beta \le 1$. The solution $u_{2\beta}(x, t)$ of (26) is given by (see [2])

$$u_{2\beta}(x, t) = \frac{1}{2t^{\beta}} W_{-\beta, 1-\beta}\left(-\frac{|x|}{t^{\beta}}\right), \quad t > 0,\ x \in \mathbb{R}, \tag{27}$$

where $W_{\nu,\gamma}(\cdot)$ is the Wright function defined as

$$W_{\nu,\gamma}(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!\Gamma(k\nu + \gamma)}, \quad \nu > -1,\ \gamma > 0,\ x \in \mathbb{R}.$$

Let

$$\bar{u}_{2\beta}(x, t) = \begin{cases} 2u_{2\beta}(x, t), & x > 0, \\ 0, & x < 0 \end{cases} \tag{28}$$

be the folded solution to (26). The following result gives an alternate version of the PMF of the GFCP in terms of derivatives of the Mittag-Leffler function. The two-parameter Mittag-Leffler function is defined as (see [24, p. 42])

$$E_{\nu,\gamma}(x) := \sum_{j=0}^{\infty} \frac{x^j}{\Gamma(j\nu + \gamma)}, \quad x \in \mathbb{R},$$

where $\nu > 0$ and $\gamma > 0$. It reduces to the Mittag-Leffler function for $\gamma = 1$.

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Proposition 1. *The PMF of the GFCP is given by*

$$p^{\beta}(n,t)=\sum_{\Omega(k,n)}\left(\prod_{j=1}^{k}\frac{\lambda_j^{x_j}}{x_j!}\right)\frac{1}{\Lambda^{z_k}}\frac{\mathrm{d}^{z_k}}{\mathrm{d}s^{z_k}}\left[s^{z_k-1}E_{\beta,1}\left(-\frac{\Lambda t^{\beta}}{s}\right)\right]_{s=1},$$

where $z_k=x_1+x_2+\cdots+x_k$.

Proof. From (25) and (28), we have

$$p^{\beta}(n,t)=\int_0^{\infty}p(n,x)\bar{u}_{2\beta}(x,t)\,\mathrm{d}x. \tag{29}$$

On using (7) and (27) in (29), we get

$$\begin{aligned}
p^{\beta}(n,t)&=\int_0^{\infty}\sum_{\Omega(k,n)}\left(\prod_{j=1}^{k}\frac{\lambda_j^{x_j}}{x_j!}\right)\mathrm{e}^{-\Lambda x}x^{z_k}t^{-\beta}W_{-\beta,1-\beta}(-xt^{-\beta})\,\mathrm{d}x\\
&=\sum_{\Omega(k,n)}\left(\prod_{j=1}^{k}\frac{\lambda_j^{x_j}}{x_j!}\right)\sum_{m=0}^{\infty}\frac{(-1)^m t^{-\beta-\beta m}}{m!\Gamma(-\beta m+1-\beta)}\int_0^{\infty}\mathrm{e}^{-\Lambda x}x^{m+z_k}\,\mathrm{d}x\\
&=\sum_{\Omega(k,n)}\left(\prod_{j=1}^{k}\frac{\lambda_j^{x_j}}{x_j!}\right)\frac{t^{-\beta}}{\Lambda^{z_k+1}}\sum_{m=0}^{\infty}\frac{(-t^{\beta}\Lambda)^{-m}}{\Gamma(-\beta m+1-\beta)}(m+z_k)(m+z_k-1)\cdots(m+1)\\
&=\sum_{\Omega(k,n)}\left(\prod_{j=1}^{k}\frac{\lambda_j^{x_j}}{x_j!}\right)\frac{t^{-\beta}}{\Lambda^{z_k+1}}\frac{\mathrm{d}^{z_k}}{\mathrm{d}s^{z_k}}\left[\sum_{m=0}^{\infty}s^{m+z_k}\frac{(-t^{\beta}\Lambda)^{-m}}{\Gamma(-\beta m+1-\beta)}\right]_{s=1}\\
&=\sum_{\Omega(k,n)}\left(\prod_{j=1}^{k}\frac{\lambda_j^{x_j}}{x_j!}\right)\frac{t^{-\beta}}{\Lambda^{z_k+1}}\frac{\mathrm{d}^{z_k}}{\mathrm{d}s^{z_k}}\left[s^{z_k}E_{-\beta,1-\beta}\left(-\frac{s}{\Lambda t^{\beta}}\right)\right]_{s=1}.
\end{aligned}$$

The result follows upon using Equation (5.1) of [2]. □

Remark 1. On substituting $k=1$ in Proposition 1, we get an equivalent expression for the PMF of the TFPP (see [2, p. 1801]). For $\lambda_j=\lambda$ with $1\le j\le k$, we get the following alternate version of the PMF of the time-fractional PPoK:

$$p^{\beta}(n,t)\big|_{\lambda_j=\lambda}=\sum_{\Omega(k,n)}\left(\prod_{j=1}^{k}\frac{(1/k)^{x_j}}{x_j!}\right)\frac{\mathrm{d}^{z_k}}{\mathrm{d}s^{z_k}}\left[s^{z_k-1}E_{\beta,1}\left(-\frac{k\lambda t^{\beta}}{s}\right)\right]_{s=1}.$$

Similarly, we can obtain alternate expressions for the PMFs of other particular and limiting cases of the GFCP, such as the fractional PAPoK, the fractional Bell–Touchard process, and the fractional Poisson–logarthmic process.

Remark 2. Let $G(u,t)=\mathbb{E}(u^{M(t)})=\sum_{n=0}^{\infty}u^n p(n,t)$ be the PGF of the GCP as given in (8), and let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with distribution $F(x)=\mathbb{P}\{X<x\}$. Then

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$$\begin{aligned}
&\mathbb{P}\Big\{\max_{1\le j\le M^{\beta}(t)} X_j < x\Big\}\\
&= \sum_{n=0}^{\infty} (\mathbb{P}\{X < x\})^n \mathbb{P}\{M^{\beta}(t) = n\}\\
&= \int_0^{\infty} G(F(x), s)\mathbb{P}\{T_{2\beta}(t) \in \mathrm{d}s\} \quad \text{(using (8) and (25))}\\
&= \int_0^{\infty} \exp\Big(-s\sum_{j=1}^{k}\lambda_j(1-F^j(x))\Big) t^{-\beta} W_{-\beta,1-\beta}(-st^{-\beta})\,\mathrm{d}s \quad \text{(using (28))}\\
&= E_{\beta,1}\Big(-\sum_{j=1}^{k}\lambda_j(1-F^j(x))t^{\beta}\Big), \qquad (30)
\end{aligned}$$

where in the last step we have used Equation (2.13) of [3]. Similarly, it can be shown that

$$\mathbb{P}\Big\{\min_{1\le j\le M^{\beta}(t)} X_j > x\Big\} = E_{\beta,1}\Big(-\sum_{j=1}^{k}\lambda_j\left(1-(1-F(x))^j\right)t^{\beta}\Big).$$

On taking $k = 1$ and $\lambda_1 = 1$ in (30), we get a known result (see [33, Equation (34)]):

$$\mathbb{P}\Big\{\max_{1\le j\le M^{\beta}(t)} X_j < x\Big\}\Big|_{k=1=\lambda_1} = E_{\beta,1}\big(-(1-F(x))t^{\beta}\big).$$

Further, for $\beta = 1$ and exponentially distributed X_i, this reduces to the Gumbel distribution

$$\mathbb{P}\Big\{\max_{1\le j\le M^{\beta}(t)} X_j < x\Big\}\Big|_{k=1=\lambda_1=\beta} = \exp\big(-\exp(-x)t\big).$$

Thus, the distribution obtained in (30) is a fractional generalization of the Gumbel distribution.

Next, we introduce a non-homogeneous version of the GCP. Let $\{\mathcal{M}(t)\}_{t\ge 0}$ be a counting process with deterministic and time-dependent intensity functions $\lambda_j(t) : [0, \infty) \to [0, \infty)$, for $1 \le j \le k$, such that $\mathcal{M}(0) = 0$. We call it the non-homogeneous generalized counting process (NGCP) if it has independent increments and its transition probabilities are given by

$$\mathbb{P}\{\mathcal{M}(t+h) = n | \mathcal{M}(t) = m\} = \begin{cases} 1 - \sum_{j=1}^{k}\lambda_j(t)h + o(h), & n = m,\\ \lambda_j(t)h + o(h), \ n = m+j, & j = 1, 2, \ldots, k,\\ o(h), & n > m+k, \end{cases}$$

where $o(h) \to 0$ as $h \to 0$. Thus, the system of differential equations that governs its state probabilities $q_n(t) = \mathbb{P}\{\mathcal{M}(t) = n\}$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}q_n(t) = -\sum_{j=1}^{k}\lambda_j(t)q_n(t) + \sum_{j=1}^{n\wedge k}\lambda_j(t)q_{n-j}(t), \quad n \ge 0, \qquad (31)$$

with initial conditions

$$q_n(0) = \begin{cases} 1, & n = 0,\\ 0, & n \ge 1. \end{cases}$$

Here, $q_j(t) = 0$ for all $j < 0$ and $t \ge 0$.

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For $0 \le s < t$ and $j \in \{1, 2, \ldots, k\}$, let $\Lambda_j(t) = \int_0^t \lambda_j(u)\,\mathrm{d}u < \infty$ be the cumulative rate function and let $\Lambda_j(s, t) = \int_s^t \lambda_j(u)\,\mathrm{d}u = \Lambda_j(t) - \Lambda_j(s)$. Using (31), the PGF of the NGCP can be obtained as

$$\mathbb{E}\big(u^{\mathcal{M}(t)}\big) = \exp\bigg(\sum_{j=1}^{k} \Lambda_j(t)(u^j - 1)\bigg).$$

Remark 3. The NGCP reduces to the GCP for constant rates, that is, $\lambda_j(t) = \lambda_j$ for $t \ge 0$ and $1 \le j \le k$. For $k = 1$, it reduces to the non-homogeneous Poisson process. For $\lambda_j(t) = \lambda(t)$ and $\lambda_j(t) = (1-\rho)\rho^{j-1}\lambda(t)/(1-\rho^k)$ $(1 \le j \le k)$, it reduces to the non-homogeneous PPoK and the non-homogeneous PAPoK, respectively (see [18]). By letting $k \to \infty$ in (31) and taking $\lambda_j(t) = \lambda(t)(1-\rho)\rho^{j-1}$ $(j \ge 1)$, the system (31) reduces to the governing system of differential equations for the state probabilities of the non-homogeneous Pólya–Aeppli process (PAP) (see [11]). Chukova and Minkova [11] showed an application of the non-homogeneous PAP in risk theory; along similar lines it can be shown that the NGCP has applications in risk theory.

The increment process of the NGCP for $v \ge 0$ is defined as

$$I(t, v) := \mathcal{M}\,(t+v) - \mathcal{M}(v), \quad t \ge 0.$$

Its marginal distribution $q_n(t, v) = \mathbb{P}\{I(t, v) = n\}$ is given by

$$q_n(t, v) = \sum_{\Omega(k,n)} \bigg(\prod_{j=1}^{k} \frac{(\Lambda_j(v, t+v))^{x_j}}{x_j!}\bigg) \mathrm{e}^{-\sum_{j=1}^{k} \Lambda_j(v,t+v)}, \quad n \ge 0.$$

Thus, its characteristic function can be obtained as

$$\begin{aligned}
\hat{q}_\xi(t, v) &= \sum_{n=0}^{\infty} \mathrm{e}^{\omega\xi n} q_n(t, v) = \sum_{n=0}^{\infty} \mathrm{e}^{\omega\xi n} \sum_{\Omega(k,n)} \bigg(\prod_{j=1}^{k} \frac{(\Lambda_j(v, t+v))^{x_j}}{x_j!}\bigg) \mathrm{e}^{-\sum_{j=1}^{k} \Lambda_j(v,t+v)} \\
&= \mathrm{e}^{-\sum_{j=1}^{k} \Lambda_j(v,t+v)} \sum_{n=0}^{\infty} \sum_{r=0}^{n} \sum_{\substack{x_1+x_2+\cdots+x_k=r \\ x_1+2x_2+\cdots+kx_k=n}} \prod_{j=1}^{k} \frac{(\Lambda_j(v, t+v))^{x_j}}{x_j!} \mathrm{e}^{\omega\xi j x_j} \\
&= \mathrm{e}^{-\sum_{j=1}^{k} \Lambda_j(v,t+v)} \sum_{r=0}^{\infty} \sum_{x_1+x_2+\cdots+x_k=r} \prod_{j=1}^{k} \frac{(\Lambda_j(v, t+v))^{x_j}}{x_j!} \mathrm{e}^{\omega\xi j x_j} \\
&= \mathrm{e}^{-\sum_{j=1}^{k} \Lambda_j(v,t+v)} \sum_{r=0}^{\infty} \frac{1}{r!} \bigg(\sum_{j=1}^{k} \Lambda_j(v, t+v) \mathrm{e}^{\omega\xi j}\bigg)^r \\
&= \exp\bigg(\sum_{j=1}^{k} \Lambda_j(v, t+v)(\mathrm{e}^{\omega\xi j} - 1)\bigg), \quad \omega = \sqrt{-1},\ \xi \in \mathbb{R}.
\end{aligned} \tag{32}$$

3.1. The NGCP time-changed by the inverse stable subordinator

For $0 < \beta < 1$, let $\{\mathcal{M}_\beta(t)\}_{t \ge 0}$ be a time-changed NGCP defined as follows:

$$\mathcal{M}_\beta(t) := \mathcal{M}(Y_\beta(t)), \quad t \ge 0,$$

where $\{\mathcal{M}(t)\}_{t \ge 0}$ is independent of the inverse stable subordinator $\{Y_\beta(t)\}_{t \ge 0}$. We call it the NGFCP. Its increment process is defined as

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$$I_\beta(t, v) := \mathcal{M}(Y_\beta(t) + v) - \mathcal{M}(v), \quad t \geq 0,$$

whose marginal distributions, $q_n^\beta(t, v) = \mathbb{P}\{I_\beta(t, v) = n\}$ for $n \geq 0$, are given by

$$\begin{aligned} q_n^\beta(t, v) &= \int_0^\infty q_n(u, v) h_\beta(t, u)\, \mathrm{d}u \\ &= \int_0^\infty \sum_{\Omega(k,n)} \left(\prod_{j=1}^k \frac{(\Lambda_j(v, u+v))^{x_j}}{x_j!} \right) \mathrm{e}^{-\sum_{j=1}^k \Lambda_j(v,u+v)} h_\beta(t, u)\, \mathrm{d}u. \end{aligned} \tag{33}$$

Here, $h_\beta(t, u)$ is the density of $\{Y_\beta(t)\}_{t\geq 0}$. So the marginal distributions of the NGFCP are given by

$$\begin{aligned} q_n^\beta(t) = q_n^\beta(t, 0) &= \int_0^\infty q_n(u, 0) h_\beta(t, u) \mathrm{d}u \\ &= \int_0^\infty \sum_{\Omega(k,n)} \left(\prod_{j=1}^k \frac{(\Lambda_j(u))^{x_j}}{x_j!} \right) \mathrm{e}^{-\sum_{j=1}^k \Lambda_j(u)} h_\beta(t, u)\, \mathrm{d}u, \end{aligned} \tag{34}$$

where we have used $\Lambda_j(0, u) = \Lambda_j(u)$.

Let $\delta_{n,0}$ denote the Kronecker delta function.

Theorem 1. *The PMF of the increment process of the NGFCP satisfies the following system of fractional differential–integral equations:*

$$\frac{\mathrm{d}^\beta}{\mathrm{d}t^\beta} q_n^\beta(t, v) = \int_0^\infty \left(-\sum_{j=1}^k \lambda_j(u+v) q_n(u, v) + \sum_{j=1}^{n\wedge k} \lambda_j(u+v) q_{n-j}(u, v) \right) h_\beta(t, u)\, \mathrm{d}u, \quad n \geq 0, \tag{35}$$

with initial condition $q_n^\beta(0, v) = \delta_{n,0}$.

Proof. The characteristic function of $I_\beta(t, v)$ can be written as

$$\hat{q}_\xi^\beta(t, v) = \int_0^\infty \hat{q}_\xi(u, v) h_\beta(t, u)\, \mathrm{d}u, \tag{36}$$

where $\hat{q}_\xi(u, v)$ is the characteristic function of $I(t, v)$ given in (32).

Let $\tilde{h}_\beta(s, u)$ be the Laplace transform of the density of $\{Y_\beta(t)\}_{t\geq 0}$ as given in (18). On taking the Laplace transform of (36), we get

$$\begin{aligned} \tilde{\hat{q}}_\xi^\beta(s, v) &= \int_0^\infty \hat{q}_\xi(u, v) \tilde{h}_\beta(s, u)\, \mathrm{d}u \\ &= s^{\beta-1} \int_0^\infty \exp\left(\sum_{j=1}^k \Lambda_j(v, u+v)(\mathrm{e}^{\omega\xi j} - 1) \right) \mathrm{e}^{-us^\beta}\, \mathrm{d}u \quad \text{(using (18) and (32))} \\ &= s^{\beta-1} \left\{ \left[-\frac{\mathrm{e}^{-us^\beta}}{s^\beta} \exp\left(\sum_{j=1}^k \Lambda_j(v, u+v)(\mathrm{e}^{\omega\xi j} - 1) \right) \right]_0^\infty \right. \\ &\qquad \left. + \frac{1}{s^\beta} \int_0^\infty \sum_{j=1}^k \lambda_j(u+v)(\mathrm{e}^{\omega\xi j} - 1) \exp\left(\sum_{j=1}^k \Lambda_j(v, u+v)(\mathrm{e}^{\omega\xi j} - 1) \right) \mathrm{e}^{-us^\beta}\, \mathrm{d}u \right\}. \end{aligned}$$

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Thus,

$$s^{\beta}\tilde{\hat{q}}_{\xi}^{\beta}(s,v)-s^{\beta-1}=\int_0^{\infty}\sum_{j=1}^{k}\lambda_j(u+v)(\mathrm{e}^{\omega\xi j}-1)\exp\Bigg(\sum_{j=1}^{k}\Lambda_j(v,u+v)(\mathrm{e}^{\omega\xi j}-1)\Bigg)s^{\beta-1}\mathrm{e}^{-us^{\beta}}\,\mathrm{d}u.$$

On taking the inverse Laplace transform and using (17) and (18), we get

$$\begin{aligned}\frac{\mathrm{d}^{\beta}}{\mathrm{d}t^{\beta}}\hat{q}_{\xi}^{\beta}(t,v)&=\int_0^{\infty}\sum_{j=1}^{k}\lambda_j(u+v)(\mathrm{e}^{\omega\xi j}-1)\exp\Bigg(\sum_{j=1}^{k}\Lambda_j(v,u+v)(\mathrm{e}^{\omega\xi j}-1)\Bigg)h_{\beta}(t,u)\,\mathrm{d}u\\&=\int_0^{\infty}\sum_{j=1}^{k}\lambda_j(u+v)(\mathrm{e}^{\omega\xi j}-1)\hat{p}_{\xi}(u,v)h_{\beta}(t,u)\,\mathrm{d}u,\end{aligned}$$

where we have used (32) in the last step. Finally, using the inversion formula for the characteristic function gives the required result. □

On substituting $v=0$ in Theorem 1, we get the system of differential equations that governs the PMF of the NGFCP.

Corollary 1. *The PMF $q_n^{\beta}(t)$ of the NGFCP satisfies*

$$\frac{\mathrm{d}^{\beta}}{\mathrm{d}t^{\beta}}q_n^{\beta}(t)=\int_0^{\infty}\Bigg(-\sum_{j=1}^{k}\lambda_j(u)q_n(u)+\sum_{j=1}^{n\wedge k}\lambda_j(u)q_{n-j}(u)\Bigg)h_{\beta}(t,u)\,\mathrm{d}u,\qquad n\geq 0,\tag{37}$$

with initial condition $q_n^{\beta}(0)=\delta_{n,0}$.

Remark 4. On choosing constant rates, that is, $\lambda_j(t)=\lambda_j$ for $1\leq j\leq k$, in (37) we get

$$\frac{\mathrm{d}^{\beta}}{\mathrm{d}t^{\beta}}p^{\beta}(n,t)=\int_0^{\infty}\Bigg(-\sum_{j=1}^{k}\lambda_j p(n,u)+\sum_{j=1}^{n\wedge k}\lambda_j p(n-j,u)\Bigg)h_{\beta}(t,u)\,\mathrm{d}u,$$

which reduces to (2). Similarly, by taking constant rates in (34), we get

$$p^{\beta}(n,t)=\int_0^{\infty}\sum_{\Omega(k,n)}\Bigg(\prod_{j=1}^{k}\frac{(\lambda_j u)^{x_j}}{x_j!}\Bigg)\mathrm{e}^{-\Lambda u}h_{\beta}(t,u)\,\mathrm{d}u,$$

which coincides with the PMF of the GFCP.

On taking $\lambda_j(t)=\lambda(t)$ and $\lambda_j(t)=(1-\rho)\rho^{j-1}\lambda(t)/(1-\rho^k)$ for $1\leq j\leq k$ in Theorem 1, we get the systems of differential–integral equations that govern the PMF of the corresponding increment process of the non-homogeneous fractional PPoK and the non-homogeneous fractional PAPoK, respectively (see [18, Theorems 3.3 and 4.4]).

Remark 5. For $\beta=1$, the Laplace transform of the inverse stable subordinator reduces to e^{-us}, whose inverse Laplace transform is $\delta(t-u)$, the Dirac delta distribution. Hence,

$$\int_0^{\infty}q_n(u,v)\delta(t-u)\,\mathrm{d}u=q_n(t,v).$$

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From (33) it follows that $q_n^1(t, v) = q_n(t, v)$, and the system of differential–integral equations in (35) reduces to

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}q_n(t, v) &= \int_0^\infty \Bigg(-\sum_{j=1}^{k}\lambda_j(u+v)q_n(u, v) + \sum_{j=1}^{n\wedge k}\lambda_j(u+v)q_{n-j}(u, v)\Bigg)\delta(t-u)\,\mathrm{d}u \\ &= -\sum_{j=1}^{k}\lambda_j(t+v)q_n(t, v) + \sum_{j=1}^{n\wedge k}\lambda_j(t+v)q_{n-j}(t, v).\end{aligned}$$

The mean, variance, and covariance of the NGCP are given by $\mathbb{E}(\mathcal{M}(t)) = \sum_{j=1}^{k} j\Lambda_j(t)$, $\mathrm{Var}\,(\mathcal{M}(t)) = \sum_{j=1}^{k} j^2\Lambda_j(t)$, and $\mathrm{Cov}\,(\mathcal{M}(s), \mathcal{M}(t)) = \sum_{j=1}^{k} j^2\Lambda_j(s\wedge t)$, respectively. As the NGCP has independent increments, it follows that the process $\{\mathcal{M}(t) - \sum_{j=1}^{k} j\Lambda_j(t)\}_{t\geq 0}$ is a martingale with respect to a natural filtration $\mathscr{F}_t = \sigma(\mathcal{M}(s), 0 < s \leq t)$.

Next, we discuss the arrival times of the NGCP. Let $\mathscr{T}_n = \min\{t \geq 0 : \mathcal{M}(t) = n\}$ be the arrival time of the nth NGCP event. Then

$$F_{\mathscr{T}_n}(t) = \mathbb{P}\{\mathscr{T}_n \leq t\} = \mathbb{P}\{\mathcal{M}(t) \geq n\} = 1 - \sum_{x=0}^{n-1}\sum_{\Omega(k,x)}\Bigg(\prod_{j=1}^{k}\frac{(\Lambda_j(t))^{x_j}}{x_j!}\Bigg)\mathrm{e}^{-\sum_{j=1}^{k}\Lambda_j(t)}.$$

We note that $F_{\mathscr{T}_n}(t)$ gives a distribution function if and only if the $\Lambda_j(t)$ satisfy the conditions given in Remark 5 of [26].

Similarly, let $\mathscr{T}_n^\beta = \min\{t \geq 0 : \mathcal{M}_\beta(t) = n\}$ be the arrival time of the nth NGFCP event. On using (34), its distribution can be written as

$$\begin{aligned}F_{\mathscr{T}_n^\beta}(t) &= \sum_{x=n}^{\infty}\int_0^\infty \sum_{\Omega(k,x)}\Bigg(\prod_{j=1}^{k}\frac{(\Lambda_j(u))^{x_j}}{x_j!}\Bigg)\mathrm{e}^{-\sum_{j=1}^{k}\Lambda_j(u)}h_\beta(t, u)\,\mathrm{d}u \\ &= \int_0^\infty\Bigg(1 - \sum_{x=0}^{n-1}\sum_{\Omega(k,x)}\Bigg(\prod_{j=1}^{k}\frac{(\Lambda_j(u))^{x_j}}{x_j!}\Bigg)\mathrm{e}^{-\sum_{j=1}^{k}\Lambda_j(u)}\Bigg)h_\beta(t, u)\,\mathrm{d}u \\ &= \int_0^\infty F_{\mathscr{T}_n}(u)h_\beta(t, u)\,\mathrm{d}u.\end{aligned}$$

3.2. The NGCP time-changed by the inverse subordinator

Let $\{Y_f(t)\}_{t\geq 0}$ be the inverse subordinator defined in (11) such that $c_1 = c_2 = 0$ in its associated Bernštein function given in (10). We consider a time-changed process defined as follows:

$$\mathcal{M}_f(t) := \mathcal{M}(Y_f(t)), \quad t \geq 0,$$

where the NGCP $\{\mathcal{M}(t)\}_{t\geq 0}$ is independent of $\{Y_f(t)\}_{t\geq 0}$.

For $f(s) = s^\beta$ with $0 < \beta < 1$, the process $\{\mathcal{M}_f(t)\}_{t\geq 0}$ reduces to the NGFCP. For $k = 1$, it reduces to the non-homogeneous Poisson process time-changed by the inverse subordinator that was introduced and studied by Buchak and Sakhno [9].

Let $v \geq 0$. The increment process of $\{\mathcal{M}_f(t)\}_{t\geq 0}$ is defined as

$$I_f(t, v) = I(Y_f(t), v) = \mathcal{M}\left(Y_f(t) + v\right) - \mathcal{M}(v), \quad t \geq 0.$$

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Its marginal distribution $q_n^f(t, v) = \mathbb{P}\{I_f(t, v) = n\}$ for $n \geq 0$ is given by

$$q_n^f(t, v) = \int_0^\infty q_n(u, v) h_f(t, u)\, \mathrm{d}u$$
$$= \int_0^\infty \sum_{\Omega(k,n)} \left(\prod_{j=1}^k \frac{(\Lambda_j(v, u+v))^{x_j}}{x_j!} \right) \mathrm{e}^{-\sum_{j=1}^k \Lambda_j(v,u+v)} h_f(t, u)\, \mathrm{d}u,$$

where $h_f(t, u)$ is the density of $\{Y_f(t)\}_{t\geq 0}$, which exists under Condition **I**. So the marginal distribution of $\{\mathcal{M}_f(t)\}_{t\geq 0}$, i.e. $q_n^f(t) = q_n^f(t, 0) = \mathbb{P}\{\mathcal{M}_f(t) = n\}$, is given by

$$q_n^f(t) = \int_0^\infty q_n(u) h_f(t, u)\, \mathrm{d}u$$
$$= \int_0^\infty \sum_{\Omega(k,n)} \left(\prod_{j=1}^k \frac{(\Lambda_j(u))^{x_j}}{x_j!} \right) \mathrm{e}^{-\sum_{j=1}^k \Lambda_j(u)} h_f(t, u)\, \mathrm{d}u.$$

Theorem 2. *Let ${}^f\mathcal{D}_t$ be the generalized Caputo derivative with respect to the Bernštein function f defined in (13), and suppose that Condition **I** holds. Then the PMF of $\{I_f(t, v)\}_{t\geq 0}$ satisfies the following system of differential–integral equations:*

$${}^f\mathcal{D}_t q_n^f(t, v) = \int_0^\infty \left(-\sum_{j=1}^k \lambda_j(u+v) q_n(u, v) + \sum_{j=1}^{n\wedge k} \lambda_j(u+v) q_{n-j}(u, v) \right) h_f(t, u)\, \mathrm{d}u,$$

with initial condition $q_n^f(0, v) = \delta_{n,0}$.

Proof. The proof follows similar lines to that of Theorem 1. We present it here for the sake of completeness.

The characteristic function of $I_f(t, v)$ can be written as

$$\hat{q}_\xi^f(t, v) = \int_0^\infty \hat{q}_\xi(u, v) h_f(t, u)\, \mathrm{d}u. \tag{38}$$

On taking the Laplace transform in (38) and using (12) and (32), we get

$$\tilde{\hat{q}}_\xi^f(s, v) = \frac{f(s)}{s} \int_0^\infty \exp\left(\sum_{j=1}^k \Lambda_j(v, u+v)(\mathrm{e}^{\omega\xi j} - 1) \right) \mathrm{e}^{-uf(s)}\, \mathrm{d}u$$
$$= \frac{f(s)}{s} \left\{ \left[-\frac{\mathrm{e}^{-uf(s)}}{f(s)} \exp\left(\sum_{j=1}^k \Lambda_j(v, u+v)(\mathrm{e}^{\omega\xi j} - 1) \right) \right]_0^\infty \right.$$
$$\left. + \frac{1}{f(s)} \int_0^\infty \sum_{j=1}^k \lambda_j(u+v)(\mathrm{e}^{\omega\xi j} - 1) \exp\left(\sum_{j=1}^k \Lambda_j(v, u+v)(\mathrm{e}^{\omega\xi j} - 1) \right) \mathrm{e}^{-uf(s)} \mathrm{d}u \right\}.$$

Thus,

$$f(s)\tilde{\hat{q}}_\xi^f(s, v) - \frac{f(s)}{s} = \int_0^\infty \sum_{j=1}^k \lambda_j(u+v)(\mathrm{e}^{\omega\xi j} - 1) \exp\left(\sum_{j=1}^k \Lambda_j(v, u+v)(\mathrm{e}^{\omega\xi j} - 1) \right) \frac{f(s)}{s} \mathrm{e}^{-uf(s)} \mathrm{d}u.$$

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On taking the inverse Laplace transform and using (12) and (15), we get

$$
\begin{aligned}
{}^{f}\mathcal{D}_t \hat{q}^f_\xi(t,v) &= \int_0^\infty \sum_{j=1}^k \lambda_j(u+v)(\mathrm{e}^{\omega\xi j}-1)\exp\Bigg(\sum_{j=1}^k \Lambda_j(v,u+v)(\mathrm{e}^{\omega\xi j}-1)\Bigg) h_f(t,u)\,\mathrm{d}u \\
&= \int_0^\infty \sum_{j=1}^k \lambda_j(u+v)(\mathrm{e}^{\omega\xi j}-1)\hat{q}_\xi(u,v) h_f(t,u)\,\mathrm{d}u,
\end{aligned}
$$

where we have used (32) in the last step. Now, the desired result follows on using the inversion formula for the characteristic function. □

On substituting $v=0$ in Theorem 2, we get the system of differential–integral equations that governs the PMF of $\{\mathcal{M}_f(t)\}_{t\geq 0}$.

Corollary 2. *Assume that Condition* **I** *holds. Then the PMF of* $\{\mathcal{M}_f(t)\}_{t\geq 0}$ *satisfies the following system of differential–integral equations:*

$$
{}^{f}\mathcal{D}_t q^f_n(t) = \int_0^\infty \Bigg(-\sum_{j=1}^k \lambda_j(u) q_n(u) + \sum_{j=1}^{n\wedge k} \lambda_j(u) q_{n-j}(u)\Bigg) h_f(t,u)\,\mathrm{d}u, \quad n\geq 0, \tag{39}
$$

with initial condition $q^f_n(0)=\delta_{n,0}$.

On choosing constant rates, that is, $\lambda_j(t)=\lambda_j$ for $1\leq j\leq k$, in (39) we get

$$
{}^{f}\mathcal{D}_t p^f_n(t) = -\Lambda p^f_n(t) + \sum_{j=1}^{n\wedge k} \lambda_j p^f_n(t), \tag{40}
$$

where $p^f_n(t)=\mathbb{P}\{M(Y_f(t))=n\}$ are the state probabilities of $\{M(Y_f(t))\}_{t\geq 0}$, a process studied in [20].

Remark 6. Alternatively, the system of differential equations (40) can be obtained as follows. Note that

$$
p^f_n(t) = \int_0^\infty p(n,u) h_f(t,u)\,\mathrm{d}u, \quad n\geq 0. \tag{41}
$$

On taking the generalized Riemann–Liouville derivative ${}^{f}\mathscr{D}_t$ and using the results

$$
{}^{f}\mathscr{D}_t h_f(t,u) = -\frac{\partial}{\partial u} h_f(t,u), \quad h_f(t,0)=v(t), \quad h_f(0,u)=\delta(u)
$$

(see [34, Theorem 4.1]) in (41), we get

$$
\begin{aligned}
{}^{f}\mathscr{D}_t p^f_n(t) &= -\int_0^\infty p(n,u)\frac{\partial}{\partial u} h_f(t,u)\,\mathrm{d}u \\
&= -\left[p(n,u)h_f(t,u)\right]_{u=0}^\infty + \int_0^\infty h_f(t,u)\frac{\mathrm{d}}{\mathrm{d}u} p(n,u)\,\mathrm{d}u \\
&= p(n,0)h_f(t,0) + \int_0^\infty h_f(t,u)\Bigg(-\Lambda p(n,u) + \sum_{j=1}^{n\wedge k}\lambda_j p(n-j,u)\Bigg)\,\mathrm{d}u \\
&= p(n,0)v(t) - \Lambda p^f_n(t) + \sum_{j=1}^{n\wedge k}\lambda_j p^f_{n-j}(t),
\end{aligned} \tag{42}
$$

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where in the penultimate step we have used (2) with $\beta = 1$. Also,

$$p_n^f(0) = \int_0^\infty p(n, u) h_f(0, u)\, \mathrm{d}u = \int_0^\infty p(n, u)\delta(u)\, \mathrm{d}u = p(n, 0).$$

From (14) we get

$${}^f\mathcal{D}_t p_n^f(t) = {}^f\mathscr{D}_t p_n^f(t) - \nu(t) p_n^f(0),$$

which on using in (42) gives the required result.

4. The GCP time-changed by the multistable subordinator

Let $\{H(t)\}_{t\geq 0}$ be a multistable subordinator with stability index $\beta(t) \in (0, 1)$. In this section we consider the time-changed process

$$X(t) := M(H(t)), \quad t \geq 0, \tag{43}$$

where the GCP $\{M(t)\}_{t\geq 0}$ is independent of $\{H(t)\}_{t\geq 0}$. We call it the generalized space-multifractional counting process (GSMCP).

For $k = 1$, the GCP reduces to the Poisson process. So the GSMCP reduces to a special process introduced and studied by Beghin and Ricciuti [7], namely the space-multifractional Poisson process (SMPP). Using Proposition 2.1 of [7], it follows that the GSMCP is a non-homogeneous subordinator.

Next, we obtain the Lévy measure of the GSMCP. Let $\nu_t(\cdot)$ and $\nu_t^*(\cdot)$ denote the Lévy measures of $\{H(t)\}_{t\geq 0}$ and $\{X(t)\}_{t\geq 0}$, respectively. Again, it follows from Equation (2.1) of [7] that

$$\begin{aligned}
\nu_t^*(\mathrm{d}x) &= \int_0^\infty \mathbb{P}\{M(s) \in \mathrm{d}x\} \nu_t(\mathrm{d}s) \\
&= \sum_{n=1}^\infty \sum_{\Omega(k,n)} \left(\prod_{j=1}^k \frac{\lambda_j^{x_j}}{x_j!} \right) \delta_n(\mathrm{d}x) \frac{\beta(t)}{\Gamma(1-\beta(t))} \int_0^\infty \mathrm{e}^{-\Lambda s} s^{z_k - \beta(t) - 1}\, \mathrm{d}s \quad \text{(using (6) and (7))} \\
&= \sum_{n=1}^\infty \sum_{\Omega(k,n)} \left(\prod_{j=1}^k \frac{\lambda_j^{x_j}}{x_j!} \right) \frac{z_k!(-1)^{z_k+1}}{\Lambda^{z_k - \beta(t)}} \binom{\beta(t)}{z_k} \delta_n(\mathrm{d}x),
\end{aligned} \tag{44}$$

where

$$\binom{\beta(t)}{z_k} = \frac{\Gamma(\beta(t)+1)}{z_k! \Gamma(\beta(t) - z_k + 1)}$$

is the generalized binomial coefficient and $\delta_n(\cdot)$ is the Dirac delta function centered at n.

Remark 7. For $k = 1$ and $\Lambda = \lambda_1 = \lambda$ (say), the Lévy measure (44) reduces to

$$\nu_t^*(\mathrm{d}x)\big|_{k=1} = \lambda^{\beta(t)} \sum_{n=1}^\infty (-1)^{n+1} \binom{\beta(t)}{n} \delta_n(\mathrm{d}x),$$

which agrees with the Lévy measure of the SMPP (see [7, Equation (3.1)]).

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On using (5) and (44), the Bernštein function $f^*(u,t)$ of GSMCP can be obtained as follows:

$$\begin{aligned} f^*(u,t) &= \int_0^\infty (1-\mathrm{e}^{-ux})\nu_t^*(\mathrm{d}x) \\ &= \sum_{n=1}^\infty \sum_{\Omega(k,n)} \left(\prod_{j=1}^k \frac{\lambda_j^{x_j}}{x_j!}\right) \frac{z_k!(-1)^{z_k+1}}{\Lambda^{z_k-\beta(t)}} \binom{\beta(t)}{z_k} \int_0^\infty (1-\mathrm{e}^{-ux})\delta_n(\mathrm{d}x) \\ &= \sum_{n=1}^\infty \sum_{\Omega(k,n)} \left(\prod_{j=1}^k \frac{\lambda_j^{x_j}}{x_j!}\right) \frac{z_k!(-1)^{z_k+1}}{\Lambda^{z_k-\beta(t)}} \binom{\beta(t)}{z_k} (1-\mathrm{e}^{-un}). \end{aligned}$$

The next result gives the distribution of first passage times of the GSMCP.

Proposition 2. *Let T_m be the time of the first upcrossing of the level m, i.e.,*

$$T_m = \inf\{s \geq 0 : X(s) \geq m\}.$$

Then

$$\mathbb{P}\{T_m > t\} = \sum_{n=0}^{m-1} \sum_{\Omega(k,n)} \left(\prod_{j=1}^k \frac{(-\lambda_j)^{x_j}}{x_j!}\right) \frac{\mathrm{d}^{z_k}}{\mathrm{d}\Lambda^{z_k}} \mathrm{e}^{-\int_0^t \Lambda^{\alpha(\tau)}\,\mathrm{d}\tau}. \tag{45}$$

Proof. Using $\mathbb{P}\{T_m > t\} = \mathbb{P}\{X(t) < m\}$ and (43), we get

$$\begin{aligned} \mathbb{P}\{T_m > t\} &= \sum_{n=0}^{m-1} \int_0^\infty \mathbb{P}\{M(s)=n\}\mathbb{P}\{H(t) \in \mathrm{d}s\} \\ &= \sum_{n=0}^{m-1} \sum_{\Omega(k,n)} \left(\prod_{j=1}^k \frac{\lambda_j^{x_j}}{x_j!}\right) \int_0^\infty \mathrm{e}^{-\Lambda s} s^{z_k} \mathbb{P}\{H(t) \in \mathrm{d}s\} \quad \text{(using (7))} \\ &= \sum_{n=0}^{m-1} \sum_{\Omega(k,n)} \left(\prod_{j=1}^k \frac{\lambda_j^{x_j}}{x_j!}\right) (-1)^{z_k} \frac{\mathrm{d}^{z_k}}{\mathrm{d}\Lambda^{z_k}} \int_0^\infty \mathrm{e}^{-\Lambda s} \mathbb{P}\{H(t) \in \mathrm{d}s\}. \end{aligned}$$

This completes the proof. □

Remark 8. On substituting $k=1$ in (45), we get the corresponding result for the SMPP (see [7, Section 3.3]).

4.1. Generalized space-fractional counting process

The multistable subordinator $\{H(t)\}_{t\geq 0}$ reduces to the stable subordinator $\{D_\beta(t)\}_{t\geq 0}$ when the stability index is a constant, that is, $\beta(t) = \beta \in (0,1)$ for all $t \geq 0$. In this case, the GSMCP reduces to the following time-changed process:

$$Y(t) = M(D_\beta(t)), \quad t \geq 0, \tag{46}$$

where the GCP is independent of the stable subordinator. We call the process $\{Y(t)\}_{t\geq 0}$ the generalized space-fractional counting process (GSFCP).

When $k=1$, the GSFCP reduces to the SFPP (see [28]). For $\lambda_j = \lambda$ $(j=1,2,\ldots,k)$, it reduces to the space-fractional PPoK (see [17]). For $0 \leq \rho < 1$ and $\lambda_j = \lambda(1-\rho)\rho^{j-1}/(1-\rho^k)$ $(j=1,2,\ldots,k)$, it reduces to the space-fractional version of the PAPoK.

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Let $l_\beta(t,\cdot)$ be the density of the stable subordinator and $G(u,t)$ the PGF of the GCP given in (8). From (46), the PGF of the GSFCP can be obtained as

$$\begin{aligned} G_\beta(u,t) &= \int_0^\infty G(u,x) l_\beta(t,x)\,\mathrm{d}x \\ &= \int_0^\infty \exp\Big(-\sum_{j=1}^k \lambda_j(1-u^j)x\Big) l_\beta(t,x)\,\mathrm{d}x \\ &= \exp\Big(-t\Big(\sum_{j=1}^k \lambda_j(1-u^j)\Big)^\beta\Big). \end{aligned} \tag{47}$$

Remark 9. Alternatively, the PGF (47) can be obtained by choosing the Bernštein function $f(s)=f_2(s)=s^\beta$ in Proposition 6 of [20].

On substituting $\lambda_j=\lambda$ for all $j=1,2,\ldots,k$ in (47), we get the PGF of the space-fractional PPoK (see [17, Equation (34)]).

The PGF of the GSFCP satisfies the differential equation

$$\frac{\partial}{\partial t}G_\beta(u,t) = -\Big(\sum_{j=1}^k \lambda_j(1-u^j)\Big)^\beta G_\beta(u,t), \quad G_\beta(u,0)=1.$$

Proposition 3. *Let X_i, $i\geq 1$ be i.i.d. uniform random variables in* [0, 1]. *Then*

$$G_\beta(u,t) = \mathbb{P}\Big\{\min_{0\leq i\leq N(t)} X_i^{1/\beta} \geq 1-\frac{1}{\Lambda}\sum_{j=1}^k \lambda_j u^j\Big\}, \quad 0<u<1,$$

where $\{N(t)\}_{t\geq 0}$ is a homogeneous Poisson process with rate Λ^β such that $\min_{0\leq i\leq N(t)} X_i^{1/\beta}=1$ *when $N(t)=0$.*

Proof. Observe that

$$\begin{aligned} \mathbb{P}\Big\{\min_{0\leq i\leq N(t)} X_i^{1/\beta} \geq 1-\frac{1}{\Lambda}\sum_{j=1}^k \lambda_j u^j\Big\} &= \sum_{n=0}^\infty \mathbb{P}\Big\{\min_{0\leq i\leq n} X_i^{1/\beta} \geq 1-\frac{1}{\Lambda}\sum_{j=1}^k \lambda_j u^j\Big\}\frac{(\Lambda^\beta t)^n}{n!}\mathrm{e}^{-t\Lambda^\beta} \\ &= \mathrm{e}^{-t\Lambda^\beta}\sum_{n=0}^\infty \Big(\mathbb{P}\Big\{X_1^{1/\beta} \geq 1-\frac{1}{\Lambda}\sum_{j=1}^k \lambda_j u^j\Big\}\Big)^n \frac{(\Lambda^\beta t)^n}{n!} \\ &= \mathrm{e}^{-t\Lambda^\beta}\sum_{n=0}^\infty \Big(1-\Big(1-\frac{1}{\Lambda}\sum_{j=1}^k \lambda_j u^j\Big)^\beta\Big)^n \frac{(\Lambda^\beta t)^n}{n!} \\ &= \exp\Big(-t\Big(\sum_{j=1}^k \lambda_j(1-u^j)\Big)^\beta\Big), \end{aligned}$$

which coincides with (47). This completes the proof. □

Note that the distribution of jumps of the TCGCP-I $\{M_f(t)\}_{t\geq 0}$ is given by (see [20])

$$\mathbb{P}\{M_f(h)=n\} = \begin{cases} 1-hf(\Lambda)+o(h), & n=0, \\ -h\displaystyle\sum_{\Omega(k,n)} f^{(z_k)}(\Lambda)\prod_{j=1}^k \frac{(-\lambda_j)^{x_j}}{x_j!}+o(h), & n\geq 1. \end{cases} \tag{48}$$

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Thus, from (48) and Remark 9, the jump distribution of the GSFCP is given by

$$\mathbb{P}\{Y(h)=n\}=\begin{cases}1-h\Lambda^{\beta}+o(h), & n=0,\\ -h\displaystyle\sum_{\Omega(k,n)}(\beta)_{z_k}\Lambda^{\beta-z_k}\prod_{j=1}^{k}\frac{(-\lambda_j)^{x_j}}{x_j!}+o(h), & n\geq 1,\end{cases}$$

where $(\beta)_{z_k}=\beta(\beta-1)\cdots(\beta-z_k+1)$.

The Lévy measure and the distribution of first passage times of the GSFCP follow from (44) and (45), respectively.

Remark 10. In view of (4), it can be observed that the GSFCP is equal in distribution to a space-fractional compound Poisson process, that is,

$$Y(t)\overset{\mathrm{d}}{=}\sum_{j=1}^{N(D_\beta(t))}X_j,$$

where $\{N(D_\beta(t))\}_{t\geq 0}$ is the SFPP with intensity Λ and is independent of the sequence of i.i.d. random variables $\{X_j\}_{j\geq 1}$ with distribution (3).

The state probabilities of the GSFCP can be obtained by putting $f(s)=s^{\beta}$ in Remark 9 of [20] as follows:

$$\begin{aligned}\mathbb{P}\{Y(t)=n\}&=\sum_{\Omega(k,n)}\left(\prod_{j=1}^{k}\frac{\lambda_j^{x_j}}{x_j!}\right)\frac{(-1)^{z_k}}{\Lambda^{z_k}}\frac{\mathrm{d}^{z_k}}{\mathrm{d}v^{z_k}}\mathrm{e}^{-t\Lambda^{\beta}v^{\beta}}\Big|_{v=1}\\&=\sum_{\Omega(k,n)}\left(\prod_{j=1}^{k}\frac{\lambda_j^{x_j}}{x_j!}\right)\frac{(-1)^{z_k}}{\Lambda^{z_k}}\sum_{r=0}^{\infty}\frac{(-\Lambda^{\beta}t)^r}{r!}\frac{\Gamma(\beta r+1)}{\Gamma(\beta r+1-z_k)}\\&=\sum_{\Omega(k,n)}\left(\prod_{j=1}^{k}\frac{\lambda_j^{x_j}}{x_j!}\right)\frac{(-1)^{z_k}}{\Lambda^{z_k}}\,{}_1\psi_1\left[\begin{matrix}(1,\beta)\\(1-z_k,\beta)\end{matrix}\Big|-\Lambda^{\beta}t\right],\quad n\geq 0,\end{aligned}\tag{49}$$

where ${}_1\psi_1(\cdot)$ is the generalized Wright function (see [24, Equation (1.11.14)]).

Further, for $k=1$ and $\Lambda=\lambda$ in (49), we get the PMF of the SFPP (see [28, Equation (2.15)]).

5. The GCP time-changed by the Lévy subordinator

In this section, we obtain some additional results for a time-changed version of the GCP, namely the TCGCP-I $\{M_f(t)\}_{t\geq 0}$. It is defined as (see [20])

$$M_f(t):=M(D_f(t)),\quad t\geq 0,$$

where the GCP $\{M(t)\}_{t\geq 0}$ is independent of the Lévy subordinator $\{D_f(t)\}_{t\geq 0}$. Here, the condition that the Lévy subordinator have finite moments is not essential.

The next result follows from (48).

Proposition 4. *The state probabilities $p_f(n,t)=\mathbb{P}\{M_f(t)=n\}$ for $n\geq 0$ of the TCGCP-I solve the following system of differential equations:*

$$\frac{\mathrm{d}}{\mathrm{d}t}p_f(n,t)=-f(\Lambda)p_f(n,t)-\sum_{m=1}^{n}\sum_{\Omega(k,m)}f^{(z_k)}(\Lambda)\left(\prod_{j=1}^{k}\frac{(-\lambda_j)^{x_j}}{x_j!}\right)p_f(n-m,t),\tag{50}$$

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with initial conditions

$$p_f(n,0)=\begin{cases}1, & n=0,\\ 0, & n\geq 1.\end{cases}$$

Remark 11. On substituting $k=1$ in (50), we get the system of differential equations that governs the state probabilities of a time-changed Poisson process $\{N(D_f(t))\}_{t\geq 0}$ (see [29, Equation (2.1)]).

Remark 12. The system of differential equations in (50) can equivalently be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}p_f(n,t)=-f\left(\Lambda\left(I-\frac{1}{\Lambda}\sum_{j=1}^{n\wedge k}\lambda_j B^j\right)\right)p_f(n,t), \tag{51}$$

where B is backward shift operator such that $Bp_f(n,t)=p_f(n-1,t)$. This can be shown as follows:

$$\begin{aligned}
&-f\left(\Lambda\left(I-\frac{1}{\Lambda}\sum_{j=1}^{n\wedge k}\lambda_j B^j\right)\right)p_f(n,t)\\
&=-\int_0^\infty\left(I-\mathrm{e}^{-\Lambda s\left(I-\frac{1}{\Lambda}\sum_{j=1}^{n\wedge k}\lambda_j B^j\right)}\right)\overline{\nu}(\mathrm{d}s)p_f(n,t)\\
&=-\int_0^\infty\left(p_f(n,t)-\mathrm{e}^{-\Lambda s}\sum_{r=0}^{\infty}\frac{s^r}{r!}\left(\sum_{j=1}^{n\wedge k}\lambda_j B^j\right)^r p_f(n,t)\right)\overline{\nu}(\mathrm{d}s)\\
&=-\int_0^\infty\left(p_f(n,t)-\mathrm{e}^{-\Lambda s}\sum_{r=0}^{\infty}s^r\sum_{x_1+x_2+\cdots+x_{n\wedge k}=r}\left(\prod_{j=1}^{n\wedge k}\frac{\lambda_j^{x_j}}{x_j!}B^{jx_j}\right)p_f(n,t)\right)\overline{\nu}(\mathrm{d}s)\\
&=-\int_0^\infty\left(p_f(n,t)-\mathrm{e}^{-\Lambda s}\sum_{r=0}^{\infty}s^r\sum_{x_1+x_2+\cdots+x_k=r}\left(\prod_{j=1}^{k}\frac{\lambda_j^{x_j}}{x_j!}B^{jx_j}\right)p_f(n,t)\right)\overline{\nu}(\mathrm{d}s)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad(p_f(-j,t)=0,\ j\geq 1)\\
&=-\int_0^\infty\left(p_f(n,t)-\mathrm{e}^{-\Lambda s}\sum_{r=0}^{\infty}s^r\sum_{m=r}^{\infty}\sum_{\substack{x_1+x_2+\cdots+x_k=r\\ x_1+2x_2+\cdots+kx_k=m}}\left(\prod_{j=1}^{k}\frac{\lambda_j^{x_j}}{x_j!}B^{jx_j}\right)p_f(n,t)\right)\overline{\nu}(\mathrm{d}s)\\
&=-\int_0^\infty\left(p_f(n,t)-\mathrm{e}^{-\Lambda s}\sum_{m=0}^{\infty}\sum_{r=0}^{m}s^r\sum_{\substack{x_1+x_2+\cdots+x_k=r\\ x_1+2x_2+\cdots+kx_k=m}}\left(\prod_{j=1}^{k}\frac{\lambda_j^{x_j}}{x_j!}B^{jx_j}\right)p_f(n,t)\right)\overline{\nu}(\mathrm{d}s)\\
&=-\int_0^\infty\left(p_f(n,t)-\mathrm{e}^{-\Lambda s}\sum_{m=0}^{\infty}\sum_{\Omega(k,m)}s^{z_k}\left(\prod_{j=1}^{k}\frac{\lambda_j^{x_j}}{x_j!}\right)B^m p_f(n,t)\right)\overline{\nu}(\mathrm{d}s)\\
&=-\int_0^\infty\left(1-\mathrm{e}^{-\Lambda s}\right)p_f(n,t)\overline{\nu}(\mathrm{d}s)+\sum_{m=1}^{n}\sum_{\Omega(k,m)}\left(\prod_{j=1}^{k}\frac{\lambda_j^{x_j}}{x_j!}\right)p_f(n-m,t)\int_0^\infty s^{z_k}\mathrm{e}^{-\Lambda s}\overline{\nu}(\mathrm{d}s)\\
&=-f(\Lambda)p_f(n,t)-\sum_{m=1}^{n}\sum_{\Omega(k,m)}f^{(z_k)}(\Lambda)\left(\prod_{j=1}^{k}\frac{(-\lambda_j)^{x_j}}{x_j!}\right)p_f(n-m,t).
\end{aligned}\tag{52}$$

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Remark 13. If we choose $f(s) = f_2(s) = s^\beta$ in (51), we get the system of differential equations that governs the state probabilities of the GSFCP in the following form:

$$\frac{\mathrm{d}}{\mathrm{d}t} p_{f_2}(n, t) = -\Lambda^\beta \left(I - \frac{1}{\Lambda} \sum_{j=1}^{n \wedge k} \lambda_j B^j \right)^\beta p_{f_2}(n, t), \tag{53}$$

with $p_{f_2}(n, 0) = \delta_{n,0}$. Also, taking $\lambda_j = \lambda$ for all $1 \le j \le k$ in (53), we get the system of differential equations that governs the state probabilities of the space-fractional PPoK.

For the Bernštein function f_1 given in (22), the TCGCP-1 reduces to the GCP time-changed by an independent gamma subordinator, that is,

$$M_{f_1}(t) := M(Z(t)), \quad t \ge 0. \tag{54}$$

For $x \ge 0$ and $t \ge 0$, the density function of the gamma subordinator satisfies (see [6, Equation (9)])

$$\begin{cases} \dfrac{\partial}{\partial x} g(x, t) = -b(1 - \mathrm{e}^{-\partial_t/a}) g(x, t), \\ g(x, 0) = \delta(x), \\ \lim_{|x| \to +\infty} g(x, t) = 0, \end{cases} \tag{55}$$

where $\partial_t = \frac{\partial}{\partial t}$, $\delta(x)$ is the Dirac delta function, and $\mathrm{e}^{-\partial_t/a}$ is a shift operator, which for any analytic function $f : \mathbb{R} \to \mathbb{R}$ and $c \in \mathbb{R}$ is defined as

$$\mathrm{e}^{c\partial_t} f(t) := \sum_{n=0}^{\infty} \frac{(c\partial_t)^n}{n!} f(t) = f(t + c).$$

Proposition 5. *The PMF* $p_{f_1}(n, t) = \mathbb{P}\{M_{f_1}(t) = n\}$ $(n \ge 0)$ *of* $\{M_{f_1}(t)\}_{t \ge 0}$ *satisfies the system*

$$\mathrm{e}^{-\partial_t/a} p_{f_1}(n, t) = (1 + \Lambda/b) p_{f_1}(n, t) - \frac{1}{b} \sum_{j=1}^{n \wedge k} \lambda_j p_{f_1}(n - j, t),$$

with initial conditions

$$p_{f_1}(n, 0) = \begin{cases} 1, & n = 0, \\ 0, & n > 1. \end{cases}$$

Proof. From (54), we can write

$$p_{f_1}(n, t) = \int_0^\infty p(n, x) g(x, t) \, \mathrm{d}z, \tag{56}$$

where $p(n, x)$ is the PMF of the GCP and $g(x, t)$ is the density function of the gamma subordinator. By applying the shift operator $\mathrm{e}^{-\partial_t/a}$ to both sides of (56) and using (55), we get

$$\begin{aligned} \mathrm{e}^{-\partial_t/a} p_{f_1}(n, t) &= \int_0^\infty p(n, x) \left(g(x, t) + \frac{1}{b} \frac{\partial}{\partial x} g(x, t) \right) \mathrm{d}x \\ &= p_{f_1}(n, t) + \frac{1}{b} \left[p(n, x) g(x, t) \right]_{x=0}^\infty - \frac{1}{b} \int_0^\infty g(x, t) \frac{\mathrm{d}}{\mathrm{d}x} p(n, x) \, \mathrm{d}x \\ &= p_{f_1}(n, t) - \frac{1}{b} \int_0^\infty g(x, t) \left(-\Lambda p(n, x) + \sum_{j=1}^{n \wedge k} \lambda_j p(n - j, x) \right) \mathrm{d}x, \end{aligned}$$

where in the last step we have used (2) with $\beta = 1$. The proof concludes in view of (56). □

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5.1. First passage times of the TCGCP-I

Here, we study the first passage times of the TCGCP-I, defined as

$$T_f^n := \inf\{s \geq 0 : M_f(s) \geq n\}.$$

The distribution of T_f^n can be written as

$$\mathbb{P}\{T_f^n < s\} = \mathbb{P}\{M_f(s) \geq n\} = \sum_{m=n}^{\infty} \int_0^{\infty} \sum_{\Omega(k,m)} \left(\prod_{j=1}^{k} \frac{(\lambda_j x)^{x_j}}{x_j!} \right) \mathrm{e}^{-\Lambda x} \mathbb{P}\{D_f(s) \in \mathrm{d}x\},$$

so its density function $\mathscr{P}_f(n, s) = \mathbb{P}\{T_f^n \in \mathrm{d}s\}/\mathrm{d}s$ can be obtained as

$$\begin{aligned} \mathscr{P}_f(n, s) &= \frac{\mathrm{d}}{\mathrm{d}s} \sum_{m=n}^{\infty} \int_0^{\infty} \sum_{\Omega(k,m)} \left(\prod_{j=1}^{k} \frac{(\lambda_j x)^{x_j}}{x_j!} \right) \mathrm{e}^{-\Lambda x} \mathbb{P}\{D_f(s) \in \mathrm{d}x\} \qquad (57) \\ &= \frac{\mathrm{d}}{\mathrm{d}s} \int_0^{\infty} \left(1 - \sum_{l=0}^{n-1} \sum_{\Omega(k,l)} \left(\prod_{j=1}^{k} \frac{(\lambda_j x)^{x_j}}{x_j!} \right) \mathrm{e}^{-\Lambda x} \right) \mathbb{P}\{D_f(s) \in \mathrm{d}x\} \\ &= -\frac{\mathrm{d}}{\mathrm{d}s} \sum_{l=0}^{n-1} \sum_{\Omega(k,l)} \left(\prod_{j=1}^{k} \frac{(-\lambda_j)^{x_j}}{x_j!} \right) \int_0^{\infty} \frac{\mathrm{d}^{z_k}}{\mathrm{d}\Lambda^{z_k}} \mathrm{e}^{-\Lambda x} \mathbb{P}\{D_f(s) \in \mathrm{d}x\} \\ &= -\frac{\mathrm{d}}{\mathrm{d}s} \sum_{l=0}^{n-1} \sum_{\Omega(k,l)} \left(\prod_{j=1}^{k} \frac{(-\lambda_j)^{x_j}}{x_j!} \right) \frac{\mathrm{d}^{z_k}}{\mathrm{d}\Lambda^{z_k}} \mathrm{e}^{-sf(\Lambda)}. \qquad (58) \end{aligned}$$

Equivalently,

$$\mathbb{P}\{T_f^n \in \mathrm{d}s\} = \mathbb{P}\{T_f^{n-1} \in \mathrm{d}s\} - \sum_{\Omega(k,n-1)} \left(\prod_{j=1}^{k} \frac{(-\lambda_j)^{x_j}}{x_j!} \right) \frac{\mathrm{d}}{\mathrm{d}s} \frac{\mathrm{d}^{z_k}}{\mathrm{d}\Lambda^{z_k}} \mathrm{e}^{-sf(\Lambda)} \mathrm{d}s,$$

which gives a recursive relation between the distributions of T_f^n.

Remark 14. From (58), we have

$$\mathbb{P}\{T_f^1 \in \mathrm{d}s\} = f(\Lambda) \mathrm{e}^{-sf(\Lambda)} \, \mathrm{d}s,$$

that is, the waiting time of the first event for the TCGCP-I is exponential.

Next, we derive the system of differential equations that governs the distribution of T_f^n. The following result will be used.

Proposition 6. *Let* $\mathcal{O}_f(u, t) = \sum_{n=1}^{\infty} u^n \mathscr{P}_f(n, t)$ *for* $|u| < 1$. *Then*

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{O}_f(u, t) = -f\left(\sum_{j=1}^{k} (1 - u^j) \lambda_j \right) \mathcal{O}_f(u, t).$$

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Proof. On using (57), we get

$$\begin{aligned}
\mathcal{O}_f(u,t) &= \frac{\mathrm{d}}{\mathrm{d}t}\sum_{m=1}^{\infty}\sum_{n=1}^{m} u^n \int_0^{\infty} \sum_{\Omega(k,m)} \left(\prod_{j=1}^{k} \frac{(\lambda_j x)^{x_j}}{x_j!}\right) \mathrm{e}^{-\Lambda x} \mathbb{P}\{D_f(t) \in \mathrm{d}x\} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\frac{u}{u-1}\sum_{m=1}^{\infty}(u^m-1)\int_0^{\infty} \sum_{\Omega(k,m)} \left(\prod_{j=1}^{k} \frac{(\lambda_j x)^{x_j}}{x_j!}\right) \mathrm{e}^{-\Lambda x} \mathbb{P}\{D_f(t) \in \mathrm{d}x\} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\frac{u}{u-1}\int_0^{\infty} \left(\mathrm{e}^{\sum_{j=1}^{k}(u^j-1)\lambda_j x} - 1\right) \mathbb{P}\{D_f(t) \in \mathrm{d}x\} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\frac{u}{u-1}\mathrm{e}^{-tf\left(\sum_{j=1}^{k}(1-u^j)\lambda_j\right)} \\
&= \frac{u}{1-u} f\left(\sum_{j=1}^{k}(1-u^j)\lambda_j\right) \mathrm{e}^{-tf\left(\sum_{j=1}^{k}(1-u^j)\lambda_j\right)}.
\end{aligned}$$

The result follows on taking the derivative with respect to t. □

Theorem 3. *The density function of the first passage times of the TCGCP-I solves the following system of differential equations:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{P}_f(n,t) = -f\left(\Lambda\left(I - \frac{1}{\Lambda}\sum_{j=1}^{n\wedge k}\lambda_j B^j\right)\right)\mathscr{P}_f(n,t), \quad n \geq 1.$$

Proof. On using (52) for $\mathscr{P}_f(n,t)$, we get

$$\begin{aligned}
&\sum_{n=1}^{\infty} u^n f\left(\Lambda\left(I - \frac{1}{\Lambda}\sum_{j=1}^{n\wedge k}\lambda_j B^j\right)\right)\mathscr{P}_f(n,t) \\
&\quad = f(\Lambda)\mathcal{O}_f(u,t) - \sum_{n=1}^{\infty} u^n \sum_{m=1}^{n}\sum_{\Omega(k,m)}\left(\prod_{j=1}^{k}\frac{\lambda_j^{x_j}}{x_j!}\right)\mathscr{P}_f(n-m,t)\int_0^{\infty} s^{zk}\mathrm{e}^{-\Lambda s}\overline{\nu}(\mathrm{d}s) \\
&\quad = f(\Lambda)\mathcal{O}_f(u,t) - \sum_{m=1}^{\infty}\sum_{n=m}^{\infty} u^n \sum_{\Omega(k,m)}\left(\prod_{j=1}^{k}\frac{\lambda_j^{x_j}}{x_j!}\right)\mathscr{P}_f(n-m,t)\int_0^{\infty} s^{zk}\mathrm{e}^{-\Lambda s}\overline{\nu}(\mathrm{d}s) \\
&\quad = f(\Lambda)\mathcal{O}_f(u,t) - \mathcal{O}_f(u,t)\sum_{m=1}^{\infty} u^m \sum_{\Omega(k,m)}\left(\prod_{j=1}^{k}\frac{\lambda_j^{x_j}}{x_j!}\right)\int_0^{\infty} s^{zk}\mathrm{e}^{-\Lambda s}\overline{\nu}(\mathrm{d}s) \\
&\quad = f(\Lambda)\mathcal{O}_f(u,t) - \mathcal{O}_f(u,t)\int_0^{\infty} \mathrm{e}^{-\Lambda s}\left(\mathrm{e}^{\sum_{j=1}^{k}u^j\lambda_j s} - 1\right)\overline{\nu}(\mathrm{d}s) \\
&\quad = f\left(\sum_{j=1}^{k}(1-u^j)\lambda_j\right)\mathcal{O}_f(u,t).
\end{aligned}$$

The result follows on applying Proposition 6. □

Let $\mathcal{T}_f^n$ be the first-hitting times of the TCGCP-I, that is,

$$\mathcal{T}_f^n := \inf\{s \geq 0 : M_f(s) = n\}, \quad n \geq 1.$$

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As the TCGCP-I is a Lévy process, it has independent increments, so the hitting-time distribution for state n of the TCGCP-I has the form

$$\begin{aligned}\mathbb{P}\{\mathcal{T}_f^n \in \mathrm{d}s\} &= \mathbb{P}\left\{\bigcup_{l=0}^{n-1}\{M_f(s)=l, M_f[s, s+\mathrm{d}s)=n-l)\}\right\} \\ &= \sum_{l=0}^{n-1}\sum_{\Omega(k,l)}\left(\prod_{j=1}^{k}\frac{\lambda_j^{x_j}}{x_j!}\right)\frac{(-1)^{z_k}}{\Lambda^{z_k}}\frac{\mathrm{d}^{z_k}}{\mathrm{d}v^{z_k}}\mathrm{e}^{-sf(\Lambda v)}\Big|_{v=1}\cdot(-\mathrm{d}s)\sum_{\Omega(k,n-l)}f^{(z_k)}(\Lambda)\prod_{j=1}^{k}\frac{(-\lambda_j)^{x_j}}{x_j!},\end{aligned}$$

where we have used Remark 9 of [20] and (48) in the last step. Therefore,

$$\mathbb{P}\{\mathcal{T}_f^n < \infty\} = \sum_{l=0}^{n-1}\sum_{\Omega(k,l)}\left(\prod_{j=1}^{k}\frac{\lambda_j^{x_j}}{x_j!}\right)\frac{(-1)^{z_k+1}}{\Lambda^{z_k}}\frac{\mathrm{d}^{z_k}}{\mathrm{d}v^{z_k}}\frac{1}{f(\Lambda v)}\Big|_{v=1}\sum_{\Omega(k,n-l)}f^{(z_k)}(\Lambda)\prod_{j=1}^{k}\frac{(-\lambda_j)^{x_j}}{x_j!}. \tag{59}$$

On taking $k=1$ and $\Lambda=\lambda$ in (59), we get

$$\mathbb{P}\{\mathcal{T}_f^n < \infty\}\big|_{k=1} = \sum_{l=0}^{n-1}\frac{(-1)^{n+1}}{l!(n-l)!}\frac{\mathrm{d}^l}{\mathrm{d}v^l}\frac{1}{f(\lambda v)}\Big|_{v=1}\lambda^{n-l}f^{(n-l)}(\lambda),$$

which agrees with Equation (2.7) of [16].

5.2. An application to risk theory

Recently, several authors have introduced and studied risk processes in which the TFPP is used to model the number of claims received by a insurance company (see [5, 10]). If the claims arrive in groups of size less than or equal to a fixed number k, then the GCP can be used to model the arrivals of these claims (see [21]). In real life there are situations in which a large number of claims arrive in an infinitesimal time interval. It is important to note that the TCGCP-I allows arbitrary arrivals in a short span of time.

Consider the following risk model with the TCGCP-I as the counting process:

$$X_f(t) = ct - \sum_{j=1}^{M_f(t)} Z_j, \quad t \ge 0, \tag{60}$$

where $c>0$ denotes the constant premium rate and $\{Z_j\}_{j\ge1}$ is a sequence of positive i.i.d. random variables with distribution F. Here, the Z_j represent the claim sizes, which are independent of $\{M_f(t)\}_{t\ge0}$. The expected value of $\{X_f(t)\}_{t\ge0}$ is given by

$$\mathbb{E}(X_f(t)) = ct - \mu\mathbb{E}(M_f(t)),$$

where $\mu=\mathbb{E}(Z_j)$. The relative safety loading factor ρ for (60) is given by

$$\rho = \frac{\mathbb{E}(X_f(t))}{\mathbb{E}\left(\sum_{j=1}^{M_f(t)} Z_j\right)} = \frac{ct}{\mu\mathbb{E}(M_f(t))} - 1.$$

Let $U(t)=u+X_f(t)$ for $t\ge0$ be the surplus process, where $u\ge0$ is the initial capital, let τ be the time to ruin of an insurance company, i.e. $\tau=\inf\{t>0 : U(t)<0\}$ with $\inf\phi=\infty$, and let $\psi(u)=\mathbb{P}_u\{\tau<\infty\}$ be the probability of ruin. Let

$$K(u,y) = \mathbb{P}_u\{\tau<\infty, D\le y\}, \quad y\ge0 \tag{61}$$

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be the joint probability that the ruin occurs in finite time and the deficit, i.e. $D = |U(\tau)|$, at the time of ruin is not more than y. Also, let $u' = u + ch$ and let F^{*j} be the distribution of $Z_1 + Z_2 + \cdots + Z_j$ for all $j \geq 1$. Now, using (48) and (61), we get

$$K(u, y) = (1 - hf(\Lambda))K(u', y) + o(h) - h\sum_{n=1}^{\infty}\sum_{\Omega(k,n)} f^{(z_k)}(\Lambda)\left(\prod_{j=1}^{k}\frac{(-\lambda_j)^{x_j}}{x_j!}\right)$$
$$\times\left(\int_0^{u'} K(u' - x, y)\,\mathrm{d}F^{*n}(x) + F^{*n}(u' + y) - F^{*n}(u')\right).$$

After rearranging the terms, we have

$$\frac{K(u', y) - K(u, y)}{ch} = \frac{f(\Lambda)}{c}K(u', y) + \frac{o(h)}{h} - \frac{1}{c}\sum_{n=1}^{\infty}\sum_{\Omega(k,n)} f^{(z_k)}(\Lambda)\left(\prod_{j=1}^{k}\frac{(-\lambda_j)^{x_j}}{x_j!}\right)$$
$$\times\left(\int_0^{u'} K(u' - x, y)\,\mathrm{d}F^{*n}(x) + F^{*n}(u' + y) - F^{*n}(u')\right).$$

On taking $h \to 0$, we get

$$\frac{\partial K(u, y)}{\partial u} = \frac{f(\Lambda)}{c}K(u, y) - \frac{1}{c}\sum_{n=1}^{\infty}\sum_{\Omega(k,n)} f^{(z_k)}(\Lambda)\left(\prod_{j=1}^{k}\frac{(-\lambda_j)^{x_j}}{x_j!}\right)$$
$$\times\left(\int_0^{u} K(u - x, y)\,\mathrm{d}F^{*n}(x) + F^{*n}(u + y) - F^{*n}(u)\right). \tag{62}$$

Note that

$$\sum_{n=1}^{\infty}\sum_{\Omega(k,n)} f^{(z_k)}(\Lambda)\prod_{j=1}^{k}\frac{(-\lambda_j)^{x_j}}{x_j!} = \sum_{r=0}^{\infty}\frac{(-\Lambda)^r}{r!}f^{(r)}(\Lambda) - f(\Lambda) = f(0) - f(\Lambda) = -f(\Lambda), \tag{63}$$

as $f(0) = 0$. Now, let

$$W(x) = -\frac{1}{f(\Lambda)}\sum_{n=1}^{\infty}\sum_{\Omega(k,n)} f^{(z_k)}(\Lambda)\left(\prod_{j=1}^{k}\frac{(-\lambda_j)^{x_j}}{x_j!}\right)F^{*n}(x) \tag{64}$$

be the mixer distribution of the aggregated claims. Thus, (62) reduces to

$$\frac{\partial K(u, y)}{\partial u} = \frac{f(\Lambda)}{c}\left(K(u, y) + W(u) - W(u + y) - \int_0^{u} K(u - x, y)\,\mathrm{d}W(x)\right). \tag{65}$$

As $\lim_{y\to\infty} K(u, y) = \psi(u)$, by letting $y \to \infty$ in (65) we get the following governing equation for the probability of ruin:

$$\frac{\mathrm{d}}{\mathrm{d}u}\psi(u) = \frac{f(\Lambda)}{c}\left(\psi(u) + W(u) - 1 - \int_0^{u}\psi(u - x)\,\mathrm{d}W(x)\right).$$

Theorem 4. *The joint probability of the time to ruin and the deficit at the time of ruin with zero initial capital is given by*

$$K(0, y) = \frac{f(\Lambda)}{c}\int_0^{y}(1 - W(u))\,\mathrm{d}u. \tag{66}$$

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Proof. On integrating (65) with respect to u on $(0, \infty)$, we get

$$K(\infty, y) - K(0, y) = \frac{f(\Lambda)}{c}\left(\int_0^\infty K(u, y)\,\mathrm{d}u + \int_0^\infty (W(u) - W(u+y))\,\mathrm{d}u\right) - \frac{f(\Lambda)}{c}\int_0^\infty \int_0^u K(u-x, y)\,\mathrm{d}W(x)\,\mathrm{d}u.$$

Hence, using $K(\infty, y) = 0$ and the change of variable yields

$$K(0, y) = \frac{f(\Lambda)}{c}\int_0^\infty (W(u+y) - W(u))\,\mathrm{d}u = \frac{f(\Lambda)}{c}\int_0^y (1 - W(u))\,\mathrm{d}u.$$

This completes the proof. □

Remark 15. On taking $y \to \infty$ in (66), we get $\psi(0) = \frac{f(\Lambda)}{c}\int_0^\infty (1 - W(u))\,\mathrm{d}u$.

Remark 16. Suppose that the claim sizes Z_j follow the exponential distribution with mean $\mu > 0$, that is, $F(x) = 1 - \mathrm{e}^{-x/\mu}$ for $x \geq 0$. Then

$$F^{*n}(x) = 1 - \sum_{m=0}^{n-1} \frac{(x/\mu)^m}{m!}\mathrm{e}^{-x/\mu}$$

is an Erlang distribution function. Hence, from (64) we get

$$\begin{aligned} W(x) &= -\frac{1}{f(\Lambda)}\sum_{n=1}^\infty \sum_{\Omega(k,n)} f^{(z_k)}(\Lambda)\left(\prod_{j=1}^k \frac{(-\lambda_j)^{x_j}}{x_j!}\right)\left(1 - \sum_{m=0}^{n-1}\frac{(x/\mu)^m}{m!}\mathrm{e}^{-x/\mu}\right) \\ &= 1 + \frac{1}{f(\Lambda)}\sum_{n=1}^\infty \sum_{\Omega(k,n)} f^{(z_k)}(\Lambda)\left(\prod_{j=1}^k \frac{(-\lambda_j)^{x_j}}{x_j!}\right)\sum_{m=0}^{n-1}\frac{(x/\mu)^m}{m!}\mathrm{e}^{-x/\mu}, \end{aligned} \tag{67}$$

where in the last step we have used (63). Thus, on substituting (67) into (66), we obtain the value of $K(0, \mathrm{y})$ in the following form:

$$K(0, y) = \frac{1}{c}\sum_{n=1}^\infty \sum_{\Omega(k,n)} f^{(z_k)}(\Lambda)\left(\prod_{j=1}^k \frac{\lambda_j^{x_j}}{x_j!}\right)(-1)^{z_k+1}\sum_{m=0}^{n-1}\frac{\mu}{m!}\gamma(m+1, y/\mu),$$

where $\gamma(s, x) = \int_0^x t^{s-1}\mathrm{e}^{-t}\,\mathrm{d}t$ for $s > 0$ is the lower incomplete gamma function.

5.3. The GFCP time-changed by the Lévy subordinator

Kataria and Khandakar [20] introduced the following time-changed process, the TCGFCP-I:

$$M_f^\beta(t) := M^\beta(D_f(t)), \quad t \geq 0, \tag{68}$$

where the GFCP $\{M^\beta(t)\}_{t\geq 0}$ is independent of the Lévy subordinator $\{D_f(t)\}_{t\geq 0}$, whose moments are finite.

First, we give an additional result for $\{M_f^\beta(t)\}_{t\geq 0}$.

Proposition 7. *The one-dimensional distributions of the TCGFCP-I are not infinitely divisible.*

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Proof. Using the self-similarity property of the inverse stable subordinator, we get

$$M_f^\beta(t) = M(Y_\beta(D_f(t))) \stackrel{\mathrm{d}}{=} M\big((D_f(t))^\beta Y_\beta(1)\big).$$

Because $D_f(t) \to \infty$ as $t \to \infty$ almost surely, we have

$$\begin{aligned}\lim_{t\to\infty} \frac{M_f^\beta(t)}{t^\beta} &\stackrel{\mathrm{d}}{=} Y_\beta(1) \lim_{t\to\infty} \frac{M\left((D_f(t))^\beta Y_\beta(1)\right)}{(D_f(t))^\beta Y_\beta(1)} \left(\frac{D_f(t)}{t}\right)^\beta \\ &\stackrel{\mathrm{d}}{=} \sum_{j=1}^k j\lambda_j Y_\beta(1) \lim_{t\to\infty} \left(\frac{D_f(t)}{t}\right)^\beta \quad \text{(using (9))} \\ &\stackrel{\mathrm{d}}{=} \sum_{j=1}^k j\lambda_j Y_\beta(1) \left(\mathbb{E}(D_f(1))\right)^\beta,\end{aligned}$$

where in the last step we have used the strong law of large numbers for a Lévy subordinator (see [8, p. 92]). Thus, the result follows as $Y_\beta(1)$ is not infinitely divisible (see [35]). □

In (68), we take the Bernštein function f_1 to obtain the GFCP time-changed by an independent gamma subordinator, that is,

$$M_{f_1}^\beta(t) := M^\beta(Z(t)), \quad t \geq 0.$$

The mean, variance, covariance, and LRD property of $\{M_{f_1}^\beta(t)\}_{t\geq 0}$ follow from Section 4.1 of [20]. It can be shown that the increment process $\{M_{f_1}^\beta(t+h) - M_{f_1}^\beta(t)\}_{t\geq 0}$, for $h > 0$ fixed, exhibits the short-range dependence property. The proof follows similar lines to that of Theorem 4 of [27].

For $k = 1$, the process $\{M_{f_1}^\beta(t)\}_{t\geq 0}$ reduces to the time-fractional negative binomial process $\{\mathcal{N}_\beta^1(t)\}_{t\geq 0}$ with intensity λ (see [36]). For notational convenience, we write $Q^\beta(t,\lambda) = \mathcal{N}_\beta^1(t)$ for $t \geq 0$. Using (23), its mth factorial moment is given by

$$\mathbb{E}\big(Q^\beta(t,\lambda)(Q^\beta(t,\lambda)-1)\cdots(Q^\beta(t,\lambda)-m+1)\big) = \frac{\lambda^m m!\Gamma(bt+m\beta)}{\Gamma(m\beta+1)a^{m\beta}\Gamma(bt)}, \quad m \geq 1. \tag{69}$$

On taking $\beta = 1$ in (69), we get the mth factorial moment of the negative binomial process (see [29, Remark 4.7]).

The next result gives the conditional distribution of $X_{(k)}^{Q^\beta(t,\lambda)}$, a kth order statistic from the sample of size $Q^\beta(t,\lambda)$, where $k \in \{1, 2, \ldots, Q^\beta(t,\lambda)\}$.

Theorem 5. *Let $\{X_i\}_{i\geq 1}$ be a sequence of i.i.d. random variables with distribution function F. Then*

$$\mathbb{P}\big\{X_{(k)}^{Q^\beta(t,\lambda)} < z \,\big|\, Q^\beta(t,\lambda) \geq k\big\} = \frac{\mathbb{P}\{Q^\beta(t,\lambda F(z)) \geq k\}}{\mathbb{P}\{Q^\beta(t,\lambda) \geq k\}}.$$

Proof. For $1 \leq k \leq n$, we will use the following standard result for the distribution of kth order statistics:

$$\mathbb{P}\{X_{(k)}^n < z\} = \sum_{j=k}^n \binom{n}{j} F^j(z)\,(1-F(z))^{n-j}. \tag{70}$$

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Now,

$$\mathbb{P}\big\{X_{(k)}^{Q^\beta(t,\lambda)} < z \,\big|\, Q^\beta(t,\lambda) \geq k\big\} = \frac{\sum_{n=k}^{\infty} \mathbb{P}\{X_{(k)}^{Q^\beta(t,\lambda)} < z,\, Q^\beta(t,\lambda) = n\}}{\mathbb{P}\{Q^\beta(t,\lambda) \geq k\}}$$
$$= \frac{\sum_{n=k}^{\infty} \mathbb{P}\{X_{(k)}^{Q^\beta(t,\lambda)} < z \mid Q^\beta(t,\lambda) = n\}\, \mathbb{P}\{Q^\beta(t,\lambda) = n\}}{\mathbb{P}\{Q^\beta(t,\lambda) \geq k\}}. \quad (71)$$

Using (24) and (70), the quantity in the numerator on the right-hand side of (71) can be evaluated as follows:

$$\sum_{n=k}^{\infty} \mathbb{P}\big\{X_{(k)}^{Q^\beta(t,\lambda)} < z \,\big|\, Q^\beta(t,\lambda) = n\big\} \mathbb{P}\{Q^\beta(t,\lambda) = n\}$$
$$= \sum_{n=k}^{\infty} \left(\sum_{j=k}^{n} \binom{n}{j} F^j(z)\,(1-F(z))^{n-j} \right) \sum_{l=n}^{\infty} (-1)^{l+n} \binom{l}{n} \frac{\Gamma(l\beta+bt)}{\Gamma(bt)\Gamma(\beta l+1)} \left(\frac{\lambda}{a^\beta}\right)^l$$
$$= \sum_{l=k}^{\infty} \frac{(-1)^l \Gamma(l\beta+bt)}{\Gamma(bt)\Gamma(\beta l+1)} \left(\frac{\lambda}{a^\beta}\right)^l \sum_{n=k}^{l} \binom{l}{n} (-1)^n \sum_{j=k}^{n} \binom{n}{j} F^j(z)\,(1-F(z))^{n-j}$$
$$= \sum_{l=k}^{\infty} \frac{(-1)^l \Gamma(l\beta+bt)}{\Gamma(bt)\Gamma(\beta l+1)} \left(\frac{\lambda}{a^\beta}\right)^l \sum_{j=k}^{l} F^j(z) \sum_{n=j}^{l} \binom{l}{n} \binom{n}{j} (-1)^n\,(1-F(z))^{n-j}$$
$$= \sum_{l=k}^{\infty} \frac{(-1)^l \Gamma(l\beta+bt)}{\Gamma(bt)\Gamma(\beta l+1)} \left(\frac{\lambda}{a^\beta}\right)^l \sum_{j=k}^{l} \binom{l}{j} F^j(z) \sum_{n=0}^{l-j} \binom{l-j}{n} (-1)^{n+j}\,(1-F(z))^n$$
$$= \sum_{l=k}^{\infty} \frac{(-1)^l \Gamma(l\beta+bt)}{\Gamma(bt)\Gamma(\beta l+1)} \left(\frac{\lambda}{a^\beta}\right)^l F^l(z) \sum_{j=k}^{l} \binom{l}{j} (-1)^j$$
$$= \sum_{j=k}^{\infty} \sum_{l=j}^{\infty} \binom{l}{j} (-1)^{l+j} \left(\frac{F(z)\lambda}{a^\beta}\right)^l \frac{\Gamma(l\beta+bt)}{\Gamma(bt)\Gamma(\beta l+1)}$$
$$= \sum_{j=k}^{\infty} \mathbb{P}\{Q^\beta(t,\lambda F(z)) = j\} = \mathbb{P}\{Q^\beta(t,\lambda F(z)) \geq k\}.$$

The proof follows upon substituting the above expression into (71). □

Similarly, it can be shown that the space-fractional and space–time-fractional negative binomial processes exhibit the kth order statistic property as in Theorem 5. Also, this result holds for a more general process, namely the time-changed fractional Poisson process studied by Kataria and Khandakar [22].

6. A time-changed version of $M_f(t)$

In this section, we consider the time-changed process

$$\mathscr{M}_f^\alpha(t) := M_f(Y_\alpha(t)) = M\big(D_f(Y_\alpha(t))\big), \quad t \geq 0, \quad (72)$$

where the Lévy subordinator $\{D_f(t)\}_{t\geq 0}$, the inverse stable subordinator $\{Y_\alpha(t)\}_{t\geq 0}$ for $0 < \alpha < 1$, and the GCP $\{M(t)\}_{t\geq 0}$ are independent of each other.

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For $k=1$, the process $\{\mathscr{M}_f^\alpha(t)\}_{t\geq 0}$ reduces to a time-changed Poisson process, that is, $\{N(D_f(Y_\alpha(t)))\}_{t\geq 0}$ (see [4, Section 3] and [29, Remark 2.3]).

Let $h_\alpha(t,x)$ be the density of $\{Y_\alpha(t)\}_{t\geq 0}$, and let $p_f^\alpha(n,t)=\mathbb{P}\{\mathscr{M}_f^\alpha(t)=n\}$, for $n\geq 0$, be the state probabilities of $\{\mathscr{M}_f^\alpha(t)\}_{t\geq 0}$. From (72) we have

$$p_f^\alpha(n,t)=\int_0^\infty p_f(n,x)h_\alpha(t,x)\,\mathrm{d}x. \tag{73}$$

On taking the Caputo fractional derivative in (73) and using (50), it can be shown that the state probabilities $p_f^\alpha(n,t)$ solve the following system of fractional differential equations:

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}p_f^\alpha(n,t)=-f(\Lambda)p_f^\alpha(n,t)-\sum_{m=1}^{n}\sum_{\Omega(k,m)}f^{(z_k)}(\Lambda)\left(\prod_{j=1}^{k}\frac{(-\lambda_j)^{x_j}}{x_j!}\right)p_f^\alpha(n-m,t),$$

with initial conditions

$$p_f^\alpha(n,0)=\begin{cases}1, & n=0,\\ 0, & n\geq 1.\end{cases}$$

In view of (51), the above system of differential equations can be written as

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}p_f^\alpha(n,t)=-f\left(\Lambda\left(I-\frac{1}{\Lambda}\sum_{j=1}^{n\wedge k}\lambda_j B^j\right)\right)p_f^\alpha(n,t).$$

On using Proposition 6 of [20], the PGF of $\{\mathscr{M}_f^\alpha(t)\}_{t\geq 0}$ is given by

$$\begin{aligned}\mathscr{G}_f^\alpha(u,t)&=\int_0^\infty \exp\left(-xf\Big(\sum_{j=1}^{k}\lambda_j(1-u^j)\Big)\right)h_\alpha(t,x)\,\mathrm{d}x\\&=E_{\alpha,1}\left(-f\Big(\sum_{j=1}^{k}\lambda_j(1-u^j)\Big)t^\alpha\right).\end{aligned} \tag{74}$$

It is known that the Mittag-Leffler function is an eigenfunction of the Caputo fractional derivative. So it follows that

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}\mathscr{G}_f^\alpha(u,t)=-f\left(\sum_{j=1}^{k}\lambda_j(1-u^j)\right)\mathscr{G}_f^\alpha(u,t),\ \ \mathscr{G}_f^\alpha(u,0)=1.$$

Remark 17. On taking $k=1$ and $\lambda_1=\lambda$, the PGF (74) reduces to $\mathscr{G}_f^\alpha(u,t)\big|_{k=1}=E_{\alpha,1}(-f(\lambda(1-u))t^\alpha)$, which agrees with the PGF of $\{N(D_f(Y_\alpha(t)))\}_{t\geq 0}$ (see [29, Equation (2.9)]).

Now we give the mean, variance, and covariance function of $\{\mathscr{M}_f^\alpha(t)\}_{t\geq 0}$. We assume that the second-order moments of $\{D_f(t)\}_{t\geq 0}$ are finite. The mean and variance of the TCGCP-I are given by $\mathbb{E}(M_f(t))=\sum_{j=1}^{k}j\lambda_j\mathbb{E}(D_f(t))$ and $\mathrm{Var}\,(M_f(t))=\left(\sum_{j=1}^{k}j\lambda_j\right)^2\mathrm{Var}\,(D_f(t))+\sum_{j=1}^{k}j^2\lambda_j\mathbb{E}(D_f(t))$ (see [20, Section 4.1]).

Let $q_1=\mathbb{E}(M_f(1))$ and $q_2=\mathrm{Var}\,(M_f(1))$. Then, by using Theorem 2.1 of [25], we get

$$\begin{aligned}\mathbb{E}\big(\mathscr{M}_f^\alpha(t)\big)&=q_1\mathbb{E}(Y_\alpha(t)),\\ \mathrm{Var}\big(\mathscr{M}_f^\alpha(t)\big)&=q_2\mathbb{E}(Y_\alpha(t))+q_1^2\,\mathrm{Var}\,(Y_\alpha(t)),\end{aligned} \tag{75}$$

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$$\operatorname{Cov}\big(\mathscr{M}_f^{\alpha}(s), \mathscr{M}_f^{\alpha}(t)\big) = q_2\mathbb{E}(Y_\alpha(s)) + q_1^2 \operatorname{Cov}(Y_\alpha(s), Y_\alpha(t)), \quad 0 < s \le t. \tag{76}$$

As $\operatorname{Var}(\mathscr{M}_f^{\alpha}(t)) - \mathbb{E}(\mathscr{M}_f^{\alpha}(t)) > 0$ for $t > 0$, the process $\{\mathscr{M}_f^{\alpha}(t)\}_{t\ge 0}$ exhibits the overdispersion property.

Theorem 6. *The process $\{\mathscr{M}_f^{\alpha}(t)\}_{t\ge 0}$ has the LRD property.*

Proof. From (75) and (76), we get

$$\operatorname{Corr}\big(\mathscr{M}_f^{\alpha}(s), \mathscr{M}_f^{\alpha}(t)\big) = \frac{q_2\mathbb{E}(Y_\alpha(s)) + q_1^2 \operatorname{Cov}(Y_\alpha(s), Y_\alpha(t))}{\sqrt{\operatorname{Var}\big(\mathscr{M}_f^{\alpha}(s)\big)}\sqrt{q_2\mathbb{E}(Y_\alpha(t)) + q_1^2 \operatorname{Var}(Y_\alpha(t))}}.$$

On using (19)–(21) for fixed s and large t, we get

$$\begin{aligned}\operatorname{Corr}\big(\mathscr{M}_f^{\alpha}(s), \mathscr{M}_f^{\alpha}(t)\big) &\sim \frac{q_2\mathbb{E}(Y_\alpha(s)) + \frac{q_1^2 s^{2\alpha}}{\Gamma(2\alpha+1)}}{\sqrt{\operatorname{Var}\big(\mathscr{M}_f^{\alpha}(s)\big)}\sqrt{\frac{q_2 t^{\alpha}}{\Gamma(\alpha+1)} + \frac{2q_1^2 t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{q_1^2 t^{2\alpha}}{\Gamma^2(\alpha+1)}}} \\ &\sim c(s)t^{-\alpha},\end{aligned}$$

where

$$c(s) = \frac{q_2\mathbb{E}(Y_\alpha(s)) + \frac{q_1^2 s^{2\alpha}}{\Gamma(2\alpha+1)}}{\sqrt{\operatorname{Var}\big(\mathscr{M}_f^{\alpha}(s)\big)}\sqrt{\frac{2q_1^2}{\Gamma(2\alpha+1)} - \frac{q_1^2}{\Gamma^2(\alpha+1)}}}.$$

This proves the result. □

Remark 18. It is known that processes that possess the LRD property have applications in various fields such as finance (see [12]), econometrics (see [31]), hydrology (see [13, pp. 461–472]), and internet data traffic modeling (see [19]). As the process $\{\mathscr{M}_f^{\alpha}(t)\}_{t\ge 0}$ exhibits the LRD property, it has potential applications in these areas.

6.1. The GCP time-changed by a fractional gamma process

Beghin [6] introduced a fractional gamma process $\{Z_\alpha(t)\}_{t\ge 0}$ by time-changing the gamma subordinator $\{Z(t)\}_{t\ge 0}$ with an independent inverse stable subordinator, that is,

$$Z_\alpha(t) := Z(Y_\alpha(t)).$$

For $x \ge 0$ and $t \ge 0$, the following holds for its density function $g_\alpha(x, t)$:

$$\begin{cases} \dfrac{\partial}{\partial x} g_\alpha(x, t) = -b(1 - \mathscr{O}^{\alpha}_{-1,t}) g_\alpha(x, t), \\ g_\alpha(x, 0) = 0, \\ \lim_{x\to+\infty} g_\alpha(x, t) = 0, \end{cases} \tag{77}$$

where $\mathscr{O}^{\alpha}_{-1,t}$ is the fractional shift operator (see [6, Theorem 5 and Equation (10)]).

For the Bernštein function f_1 given in (22), the process $\{\mathscr{M}_f^{\alpha}(t)\}_{t\ge 0}$ reduces to

$$\mathscr{M}_{f_1}^{\alpha}(t) := M(Z(Y_\alpha(t))) = M(Z_\alpha(t)), \quad t \ge 0. \tag{78}$$

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Proposition 8. *The PMF* $p^{\alpha}_{f_1}(n,t) = \mathbb{P}\{\mathscr{M}^{\alpha}_{f_1}(t) = n\}$ $(n \geq 0)$ *of* $\{\mathscr{M}^{\alpha}_{f_1}(t)\}_{t\geq 0}$ *satisfies the fractional differential equations*

$$\mathscr{O}^{\alpha}_{-1,t} p^{\alpha}_{f_1}(n,t) = (1 + \Lambda/b) p^{\alpha}_{f_1}(n,t) - \frac{1}{b} \sum_{j=1}^{n\wedge k} \lambda_j p^{\alpha}_{f_1}(n-j,t), \tag{79}$$

with initial conditions

$$p^{\alpha}_{f_1}(n,0) = \begin{cases} 1, & n = 0, \\ 0, & n \geq 1. \end{cases}$$

Proof. From (78), we can write

$$p^{\alpha}_{f_1}(n,t) = \int_0^{\infty} p(n,x) g_{\alpha}(x,t)\,\mathrm{d}x. \tag{80}$$

On both sides of (80) we apply the fractional shift operator $\mathscr{O}^{\alpha}_{-1,t}$ and use (77) to obtain

$$\begin{aligned} \mathscr{O}^{\alpha}_{-1,t} p^{\alpha}_{f_1}(n,t) &= \int_0^{\infty} p(n,x) \Big(g_{\alpha}(x,t) + \frac{1}{b} \frac{\partial}{\partial x} g_{\alpha}(x,t) \Big)\,\mathrm{d}x \\ &= p^{\alpha}_{f_1}(n,t) + \frac{1}{b} \left[p(n,x) g_{\alpha}(x,t) \right]_{x=0}^{\infty} - \frac{1}{b} \int_0^{\infty} g_{\alpha}(x,t) \frac{\mathrm{d}}{\mathrm{d}x} p(n,x)\,\mathrm{d}x \\ &= p^{\alpha}_{f_1}(n,t) - \frac{1}{b} \int_0^{\infty} g_{\alpha}(x,t) \Big(-\Lambda p(n,x) + \sum_{j=1}^{n\wedge k} \lambda_j p(n-j,x) \Big)\,\mathrm{d}x, \end{aligned}$$

where in the last step we have used (2) with $\beta = 1$. This coincides with the desired result by considering (80). □

6.2. Space–time-fractional version of the GCP

We take the Bernštein function f_2 in (72) to obtain the space–time-fractional version of the GCP, namely the generalized space–time-fractional counting process (GSTFCP) $\{\mathscr{M}^{\alpha}_{f_2}(t)\}_{t\geq 0}$, where

$$\mathscr{M}^{\alpha}_{f_2}(t) := M(D_{\beta}(Y_{\alpha}(t))), \quad t \geq 0.$$

For $1 \leq j \leq k$, if $\lambda_j = \lambda$ or $\lambda_j = \lambda(1-\rho)\rho^{j-1}/(1-\rho^k)$ for $0 \leq \rho < 1$, then the GSTFCP reduces to the space–time-fractional version of the PPoK or PAPoK, respectively.

From (74), the PGF of the GSTFCP is given by

$$\mathscr{G}^{\alpha}_{f_2}(u,t) = E_{\alpha,1}\left(-\Big(\sum_{j=1}^{k} (1-u^j)\lambda_j \Big)^{\beta} t^{\alpha} \right). \tag{81}$$

For $k = 1$, it reduces to the PGF of the STFPP (see [28, Equation (2.28)]). It solves the differential equations

$$\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}} \mathscr{G}^{\alpha}_{f_2}(u,t) = -\left(\sum_{j=1}^{k} (1-u^j)\lambda_j \right)^{\beta} \mathscr{G}^{\alpha}_{f_2}(u,t), \quad \mathscr{G}^{\alpha}_{f_2}(u,0) = 1.$$

Let X_i for $i \geq 1$ be i.i.d. uniform random variables in [0,1], and let $\{N_{\alpha}(t, \Lambda^{\beta})\}_{t\geq 0}$ be the TFPP with intensity Λ^{β} such that $\min_{0\leq i\leq N_{\alpha}(t,\Lambda^{\beta})} X_i^{1/\beta} = 1$ when $N_{\alpha}(t, \Lambda^{\beta}) = 0$. Then for $0 < u < 1$, the PGF of GSTFCP can be written as

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$$\begin{aligned}
\mathscr{G}^{\alpha}_{f_2}(u,t) &= \sum_{m=0}^{\infty} \frac{(-t^{\alpha}\Lambda^{\beta})^m}{\Gamma(m\alpha+1)} \left(1-\frac{1}{\Lambda}\sum_{j=1}^{k}\lambda_j u^j\right)^{\beta m} \\
&= \sum_{m=0}^{\infty} \frac{(-t^{\alpha}\Lambda^{\beta})^m}{\Gamma(m\alpha+1)} \sum_{r=0}^{m} (-1)^r \binom{m}{r} \left(1-\left(1-\frac{1}{\Lambda}\sum_{j=1}^{k}\lambda_j u^j\right)^{\beta}\right)^r \\
&= \sum_{r=0}^{\infty} \left(1-\left(1-\frac{1}{\Lambda}\sum_{j=1}^{k}\lambda_j u^j\right)^{\beta}\right)^r \sum_{m=r}^{\infty} (-1)^{m-r} \binom{m}{r} \frac{(t^{\alpha}\Lambda^{\beta})^m}{\Gamma(m\alpha+1)} \\
&= \sum_{r=0}^{\infty} \left(\mathbb{P}\left\{X_r^{1/\beta} \geq 1-\frac{1}{\Lambda}\sum_{j=1}^{k}\lambda_j u^j\right\}\right)^r \mathbb{P}\{N_{\alpha}(t,\Lambda^{\beta})=r\} \\
&= \mathbb{P}\left\{\min_{0\leq r\leq N_{\alpha}(t,\Lambda^{\beta})} X_r^{1/\beta} \geq 1-\frac{1}{\Lambda}\sum_{j=1}^{k}\lambda_j u^j\right\}. \qquad (82)
\end{aligned}$$

The result in (82) is of particular interest as it connects the GSTFCP with the TFPP via uniform random variables. On using the generalized binomial theorem to expand the right-hand side of (81), we obtain the PMF of the GSTFCP in the following form:

$$\mathbb{P}\{\mathscr{M}^{\alpha}_{f_2}(t)=n\} = \sum_{\Omega(k,n)} \left(\prod_{j=1}^{k} \frac{\lambda_j^{x_j}}{x_j!}\right) \frac{(-1)^{z_k}}{\Lambda^{z_k}} \sum_{m=0}^{\infty} \frac{(-\Lambda^{\beta}t^{\alpha})^m}{\Gamma(\alpha m+1)} \frac{\Gamma(\beta m+1)}{\Gamma(\beta m+1-z_k)}, \quad n\geq 0. \qquad (83)$$

The steps involved to obtain (83) are similar to those in the proof of (51).

Next, we obtain an alternate form of the PMF of the GSTFCP. The PMF (7) of the GCP can be rewritten as

$$p(n,t) = \sum_{\Omega(k,n)} \left(\prod_{j=1}^{k} \frac{\lambda_j^{x_j}}{x_j!}\right) (-\partial_{\Lambda})^{z_k} \mathrm{e}^{-\Lambda t}, \quad n\geq 0.$$

Then, from (72), we have

$$\begin{aligned}
\mathbb{P}\{\mathscr{M}^{\alpha}_{f_2}(t)=n\} &= \int_0^{\infty} \sum_{\Omega(k,n)} \left(\prod_{j=1}^{k} \frac{\lambda_j^{x_j}}{x_j!}\right) (-\partial_{\Lambda})^{z_k} \mathrm{e}^{-\Lambda s} \mathbb{P}\{D_{\beta}(Y_{\alpha}(t)) \in \mathrm{d}s\} \\
&= \sum_{\Omega(k,n)} \left(\prod_{j=1}^{k} \frac{\lambda_j^{x_j}}{x_j!}\right) (-\partial_{\Lambda})^{z_k} \mathbb{E}\left(\mathrm{e}^{-\Lambda D_{\beta}(Y_{\alpha}(t))}\right) \qquad (84) \\
&= \sum_{\Omega(k,n)} \left(\prod_{j=1}^{k} \frac{\lambda_j^{x_j}}{x_j!}\right) (-\partial_{\Lambda})^{z_k} E_{\alpha,1}\left(-\Lambda^{\beta}t^{\alpha}\right), \qquad (85)
\end{aligned}$$

where in the last step we have used Lemma 3.1 of [4]. Thus, the PGF of the GSTFCP can alternatively be obtained as follows:

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$$\begin{aligned}\mathscr{G}^{\alpha}_{f_2}(u,t) &= \sum_{n=0}^{\infty} u^n \sum_{\Omega(k,n)} \left(\prod_{j=1}^{k} \frac{\lambda_j^{x_j}}{x_j!}\right)(-\partial_\Lambda)^{z_k} \mathbb{E}\left(\mathrm{e}^{-\Lambda D_\beta(Y_\alpha(t))}\right) \quad \text{(using (84))} \\ &= \sum_{r=0}^{\infty} \left(\sum_{j=1}^{k} \lambda_j u^j\right)^r \frac{(-\partial_\Lambda)^r}{r!} \mathbb{E}\left(\mathrm{e}^{-\Lambda D_\beta(Y_\alpha(t))}\right) \\ &= \mathbb{E}\left(\mathrm{e}^{-\sum_{j=1}^{k}(1-u^j)\lambda_j D_\beta(Y_\alpha(t))}\right),\end{aligned}$$

which reduces to (81) on using Lemma 3.1 of [4].

Let $\{N^{\alpha,\beta}(t)\}_{t\geq 0}$ denote the STFPP with intensity λ. For notational convenience, let $N^{\alpha,\beta}(t,\lambda) = N^{\alpha,\beta}(t)$. On taking $k=1$ and $\Lambda = \lambda_1 = \lambda$ (say) in (83), we get

$$\mathbb{P}\{N^{\alpha,\beta}(t,\lambda)=n\} = \frac{(-1)^n}{n!} \sum_{m=0}^{\infty} \frac{(-\lambda^\beta t^\alpha)^m \Gamma(\beta m+1)}{\Gamma(\beta m+1-n)\Gamma(\alpha m+1)}, \tag{86}$$

which agrees with the PMF of the STFPP (see [28, Equation (2.29)]). Again taking $k=1$ and $\Lambda=\lambda$ in (85), we get an alternate version of the PMF of the STFPP (see [4, Equation (3.19)]).

Theorem 7. *Let $X_{(k)}^{N^{\alpha,\beta}(t,\lambda)}$ be a kth order statistic from a sample of size $N^{\alpha,\beta}(t,\lambda)$, for $k \in \{1,2,\ldots,N^{\alpha,\beta}(t,\lambda)\}$, and let $\{X_i\}_{i\geq 1}$ be a sequence of i.i.d. random variables with distribution function F. Then*

$$\mathbb{P}\left\{X_{(k)}^{N^{\alpha,\beta}(t,\lambda)} < z \,\middle|\, N^{\alpha,\beta}(t,\lambda)\geq k\right\} = \frac{\mathbb{P}\{N^{\alpha,\beta}(t,\lambda F(z))\geq k\}}{\mathbb{P}\{N^{\alpha,\beta}(t,\lambda)\geq k\}}.$$

Proof. On using the law of conditional probability, we get

$$\begin{aligned}&\mathbb{P}\left\{X_{(k)}^{N^{\alpha,\beta}(t,\lambda)} < z \,\middle|\, N^{\alpha,\beta}(t,\lambda)\geq k\right\} \\ &\quad = \frac{\sum_{n=k}^{\infty} \mathbb{P}\{X_{(k)}^{N^{\alpha,\beta}(t,\lambda)} < z \mid N^{\alpha,\beta}(t,\lambda)=n\}\,\mathbb{P}\{N^{\alpha,\beta}(t,\lambda)=n\}}{\mathbb{P}\{N^{\alpha,\beta}(t,\lambda)\geq k\}}.\end{aligned} \tag{87}$$

Using (70) and (86), the quantity in the numerator on the right-hand side of (87) can be evaluated as follows:

$$\begin{aligned}&\sum_{n=k}^{\infty} \mathbb{P}\left\{X_{(k)}^{N^{\alpha,\beta}(t,\lambda)} < z \,\middle|\, N^{\alpha,\beta}(t,\lambda)=n\right\} \mathbb{P}\{N^{\alpha,\beta}(t,\lambda)=n\} \\ &\quad = \sum_{n=k}^{\infty}\sum_{j=k}^{n} \binom{n}{j} F^j(z)\,(1-F(z))^{n-j}\,\frac{(-1)^n}{n!} \sum_{m=0}^{\infty} \frac{(-\lambda^\beta t^\alpha)^m \Gamma(\beta m+1)}{\Gamma(\beta m+1-n)\Gamma(\alpha m+1)} \\ &\quad = \sum_{j=k}^{\infty}\sum_{m=0}^{\infty}\sum_{n=j}^{\infty} \frac{F^j(z)}{(n-j)!j!}\,(1-F(z))^{n-j}\,\frac{(-1)^n(-\lambda^\beta t^\alpha)^m \Gamma(\beta m+1)}{\Gamma(\beta m+1-n)\Gamma(\alpha m+1)} \\ &\quad = \sum_{j=k}^{\infty} \frac{(-1)^j F^j(z)}{j!} \sum_{m=0}^{\infty} \frac{(-\lambda^\beta t^\alpha)^m \Gamma(\beta m+1)}{\Gamma(\alpha m+1)} \sum_{n=0}^{\infty} \frac{(-1)^n\,(1-F(z))^n}{n!\Gamma(\beta m+1-n-j)}\end{aligned}$$

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$$= \sum_{j=k}^{\infty} \frac{(-1)^j}{j!} \sum_{m=0}^{\infty} \frac{(-\lambda^\beta t^\alpha)^m \Gamma(\beta m+1)}{\Gamma(\alpha m+1)} \frac{(F(z))^{\beta m}}{\Gamma(\beta m+1-j)}$$

$$= \sum_{j=k}^{\infty} \frac{(-1)^j}{j!} \sum_{m=0}^{\infty} \frac{(-\lambda^\beta F^\beta(z) t^\alpha)^m \Gamma(\beta m+1)}{\Gamma(\alpha m+1)\Gamma(\beta m+1-j)}$$

$$= \sum_{j=k}^{\infty} \mathbb{P}\{N^{\alpha,\beta}(t, \lambda F(z)) = j\} = \mathbb{P}\{N^{\alpha,\beta}(t, \lambda F(z)) \geq k\}.$$

The proof follows upon substituting the above expression into (87). □

Remark 19. Polito and Scalas [32] studied a time-changed Poisson process that generalizes the STFPP. Its state probabilities (see [32, Equation (3.19)]) satisfy a system of differential equations that involves the Prabhakar derivative. It can be shown that this process also has the kth order statistic property, as discussed in Theorem 7.

Abbreviations

GCP	generalized counting process
GFCP	generalized fractional counting process
GSMCP	generalized space-multifractional counting process
GSFCP	generalized space-fractional counting process
GSTFCP	generalized space–time-fractional counting process
i.i.d.	independent and identically distributed
LRD	long-range dependence
NGCP	non-homogeneous generalized counting process
NGFCP	non-homogeneous generalized fractional counting process
PAP	Pólya–Aeppli process
PAPoK	Pólya–Aeppli process of order k
PGF	probability generating function
PMF	probability mass function
PPoK	Poisson process of order k
SFPP	space-fractional Poisson process
SMPP	space-multifractional Poisson process
STFPP	space–time-fractional Poisson process
TCGCP-I	time-changed generalized counting process-I
TCGFCP-I	time-changed generalized fractional counting process-I
TFPP	time-fractional Poisson process

7. Concluding remarks

In this paper, we have obtained an alternate form of the PMF of the GFCP $\{M^\beta(t)\}_{t\geq 0}$ for $0 < \beta < 1$. We showed that the distribution of $\max_{1\leq j\leq M^\beta(t)} X_j$ is a fractional generalization of the Gumbel distribution. This has potential applications in geophysics (see [33]). We introduced and studied some distributional properties of the NGCP, the NGFCP, and a generalization of the NGFCP. The results obtained for these processes generalize and complement the results of [9, 18, 26]. As several time-changed variants of the NGCP, including the NGFCP, generalize the

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TABLE 1. Several special cases of the GCP and its time-changed and non-homogeneous versions

Process	Condition	Reduced process
GCP	$k=1$	Poisson process (PP)
GFCP	$k=1$	TFPP [2]
GFCP	$\lambda_j=\lambda,\ 1\le j\le k$	Time-fractional PPoK [18]
TCGCP-I	$k=1$	$N(D_f(t))$ [29]
NGCP	$k=1$	Non-homogeneous PP
NGCP	$\lambda_j(t)=\lambda(t),\ 1\le j\le k$	Non-homogeneous PPoK [18]
NGCP	$\lambda_j(t)=\frac{(1-\rho)\rho^{j-1}\lambda(t)}{(1-\rho^k)},\ 1\le j\le k$	Non-homogeneous PAPoK [18]
NGCP	$\lambda_j(t)=\lambda(t)(1-\rho)\rho^{j-1},\ k\to\infty, j\ge 1$	Non-homogeneous PAP [11]
$\mathcal{M}_f(t)$	$k=1$	Non-homogeneous PP time-changed by $Y_f(t)$ [9]
GSMCP	$k=1$	SMPP [7]
GSFCP	$k=1$	SFPP [28]
GSFCP	$\lambda_j=\lambda,\ 1\le j\le k$	Space-fractional PPoK [17]
$M^\beta(Z(t))$	$k=1$	Time-fractional negative binomial process [36]
$M\left(D_f(Y_\alpha(t))\right)$	$k=1$	$N(D_f(Y_\alpha(t)))$ [4, 29]
GSTFCP	$k=1$	STFPP [28]

non-homogeneous TFPP, we expect these processes to have potential applications in modeling systems in which anomalous waiting times have non-stationary distributions (see [26]).

We introduced the GSMCP, which is obtained by time-changing the GCP with an independent multistable subordinator. We obtained its Lévy measure, the Bernštein function, and the distribution of its first passage times. For a particular case of the GSMCP, namely the GSFCP, we obtained its one-dimensional distribution and PGF. Also, we expressed the PGF of the GSFCP in terms of the distribution of the minimum of i.i.d. uniform random variables. This result connects the GSFCP with the Poisson process via uniform random variables.

It is shown that the one-dimensional distribution of the TCGFCP-I $\{M_f^\beta(t)\}_{t\ge 0}$ is not infinitely divisible. If f_1 is the Bernštein function associated with the gamma subordinator, then $\{M_f^\beta(t)\}_{t\ge 0}$ reduces to a generalization of the time-fractional negative binomial process $\{Q^\beta(t,\lambda)\}_{t\ge 0}$. We obtained the conditional distribution of the kth order statistic from a sample of size $Q^\beta(t,\lambda)$ where $k\in\{1,2,\ldots,Q^\beta(t,\lambda)\}$. For $\beta=1$, the TCGFCP-I reduces to the TCGCP-I. For this process, we obtained the governing system of differential equations for its state probabilities and the distribution of its first passage times. A risk process is studied in which the number of claims received by an insurance company is modeled using the TCGCP-I. Some implicit expressions for the joint distribution of the time to ruin and the deficit at the time of ruin are obtained.

We then considered the fractional version of the TCGCP-I, that is, $\{\mathscr{M}_f^\alpha(t)\}_{t\ge 0}$. We obtained its PGF, mean, variance, and covariance, as well as the system of differential equations that governs its state probabilities. It is shown that the process $\{\mathscr{M}_f^\alpha(t)\}_{t\ge 0}$ has the LRD and overdispersion properties. So this process has applications in several areas, such as internet data traffic modeling, finance, econometrics, and hydrology, and it is useful for modeling overdispersed

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data. Finally, we discussed two particular cases of $\{\mathscr{M}_f^{\alpha}(t)\}_{t\geq 0}$, namely the GCP time-changed by a fractional gamma process and the GSTFCP.

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References

[1] APPLEBAUM, D. (2009). *Lévy Processes and Stochastic Calculus*. Cambridge University Press, Cambridge.
[2] BEGHIN, L. AND ORSINGHER, E. (2009). Fractional Poisson processes and related planar random motions. *Electron. J. Probab.* **14**, 1790–1827.
[3] BEGHIN, L. AND ORSINGHER, E. (2010). Poisson-type processes governed by fractional and higher-order recursive differential equations. *Electron. J. Probab.* **15**, 684–709.
[4] BEGHIN, L. AND D'OVIDIO, M. (2014). Fractional Poisson process with random drift. *Electron. J. Probab.* **19**, 26 pp.
[5] BEGHIN, L. AND MACCI, C. (2013). Large deviations for fractional Poisson processes. *Statist. Probab. Lett.* **83**, 1193–1202.
[6] BEGHIN, L. (2015). Fractional gamma and gamma-subordinated processes. *Stoch. Anal. Appl.* **33**, 903–926.
[7] BEGHIN, L. AND RICCIUTI, C. (2019). Time-inhomogeneous fractional Poisson processes defined by the multistable subordinator. *Stoch. Anal. Appl.* **37**, 171–188.
[8] BERTOIN, J. (1996). *Lévy Processes*. Cambridge University Press, Cambridge.
[9] BUCHAK, K. V. AND SAKHNO, L. M. (2019). On the governing equations for Poisson and Skellam processes time-changed by inverse subordinators. *Theory Probab. Math. Statist.* (98), 91–104.
[10] CONSTANTINESCU, C. D., RAMIREZ, J. M. AND ZHU, W. R. (2019). An application of fractional differential equations to risk theory. *Finance Stoch.* **23**, 1001–1024.
[11] CHUKOVA, S. AND MINKOVA, L. D. (2019). Non-homogeneous Pólya-Aeppli process. *Comm. Statist. Simulation Comput.* **48**, 2955–2967.
[12] DING, Z., GRANGER, C. W. J. AND ENGLE, R. F. (1993). A long memory property of stock market returns and a new model. *J. Empir. Finance* **1**, 83–106.
[13] DOUKHAN, P., OPPENHEIM, G. AND TAQQU, M. S. (Eds.). (2003). *Theory and Applications of Long-Range Dependence*. Birkhäuser, Boston.
[14] DI CRESCENZO, A., MARTINUCCI, B. AND MEOLI, A. (2016). A fractional counting process and its connection with the Poisson process. *ALEA Lat. Am. J. Probab. Math. Stat.* **13**, 291–307.
[15] D'OVIDIO, M. AND NANE, E. (2014). Time dependent random fields on spherical non-homogeneous surfaces. *Stochastic Process. Appl.* **124**, 2098–2131.
[16] GARRA, R., ORSINGHER, E. AND SCAVINO, M. (2017). Some probabilistic properties of fractional point processes. *Stoch. Anal. Appl.* **35**, 701–718.
[17] GUPTA, N. AND KUMAR, A. (2023). Fractional Poisson processes of order k and beyond. *J. Theoret. Probab.* **36**, 2165–2191.
[18] KADANKOVA, T., LEONENKO, N. AND SCALAS, E. (2023). Fractional non-homogeneous Poisson and Pólya-Aeppli processes of order k and beyond. *Comm. Statist. Theory Methods* **52**, 2682–2701.
[19] KARAGIANNIS, T., MOLLE, M. AND FALOUTSOS, M. (2004). Long-range dependence ten years of internet traffic modeling. *IEEE Internet Comput.* **8**, 57–64.
[20] KATARIA, K. K. AND KHANDAKAR, M. (2022). Skellam and time-changed variants of the generalized fractional counting process. *Fract. Calc. Appl. Anal.* **25**, 1873–1907.
[21] KATARIA, K. K. AND KHANDAKAR, M. (2022). Generalized fractional counting process. *J. Theoret. Probab.* **35**, 2784–2805.

https://doi.org/10.1017/apr.2025.21 Published online by Cambridge University Press

[22] KATARIA, K. K. AND KHANDAKAR, M. (2022). Time-changed space–time fractional Poisson process. *Stoch. Anal. Appl.* **40**, 246–267.
[23] KHANDAKAR, M. AND KATARIA, K. K. (2023). Some compound fractional Poisson processes. *Fractal Fract.* **7**, 15.
[24] KILBAS, A. A., SRIVASTAVA, H. M. AND TRUJILLO, J. J. (2006). *Theory and Applications of Fractional Differential Equations.* Elsevier Science B.V., Amsterdam.
[25] LEONENKO, N. N., MEERSCHAERT, M. M., SCHILLING, R. L. AND SIKORSKII, A. (2014). Correlation structure of time-changed Lévy processes. *Commun. Appl. Ind. Math.* **6**, e-483, 22 pp.
[26] LEONENKO, N., SCALAS, E. AND TRINH, M. (2017). The fractional non-homogeneous Poisson process. *Statist. Probab. Lett.* **120**, 147–156.
[27] MAHESHWARI, A. AND VELLAISAMY, P. (2016). On the long-range dependence of fractional Poisson and negative binomial processes. *J. Appl. Probab.* **53**, 989–1000.
[28] ORSINGHER, E. AND POLITO, F. (2012). The space fractional Poisson process. *Statist. Probab. Lett.* **82**, 852–858.
[29] ORSINGHER, E. AND TOALDO, B. (2015). Counting processes with Bernštein intertimes and random jumps. *J. Appl. Probab.* **52**, 1028–1044.
[30] ORSINGHER, E., RICCIUTI, C. AND TOALDO, B. (2016). Time-inhomogeneous jump processes and variable order operators. *Potential Anal.* **45**, 435–461.
[31] PAGAN, A. (1996). The econometrics of financial markets. *J. Empir. Finance* **3**, 15–102.
[32] POLITO, F. AND SCALAS, E. (2016). A generalization of the space fractional Poisson process and its connection to some Lévy processes. *Electron. Commun. Probab.* **21**, 14 pp.
[33] SCALAS, E. (2017). Continuous-time statistics and generalized relaxation equations. *Eur. Phys. J. B* **90**, 5 pp.
[34] TOALDO, B. (2015). Convolution-type derivatives, hitting-times of subordinators and time-changed C_0-semigroups. *Potential Anal.* **42**, 115–140.
[35] VELLAISAMY, P. AND KUMAR, A. (2018). First-exit times of an inverse Gaussian process. *Stochastics* **90**, 29–48.
[36] VELLAISAMY, P. AND MAHESHWARI, A. (2018). Fractional negative binomial and Pólya processes. *Probab. Math. Statist.* **38**, 77–101.

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