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Convergence of an Anisotropic Perfectly Matched Layer Method for Helmholtz Scattering Problems

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Abstract. The anisotropic perfectly matched layer (APML) defines a continuous vector field outside a rectangle domain and performs the complex coordinate stretching along the vector field. Inspired by [Z. Chen *et al.*, Inverse Probl. Imag., 7, (2013):663–678] and based on the idea of the shortest distance, we propose a new approach to construct the vector field which still allows us to prove the exponential decay of the stretched Green function without the constraint on the thickness of the PML layer. Moreover, by using the reflection argument, we prove the stability of the PML problem in the PML layer and the convergence of the PML method. Numerical experiments are also included.

AMS subject classifications: 15A12; 65F10; 65F15

Key words: Anisotropic PML, Helmholtz equation, Reflection argument, Exponential convergence.

1. Introduction

We propose and study an anisotropic perfectly matched layer (APML) method for solving Helmholtz scattering problem with Dirichlet boundary condition

$$\Delta u + k^2 u = f \qquad \text{in } \Omega = \mathbb{R}^2 \setminus \bar{D}, \tag{1.1}$$

$$u = g \qquad \text{on } \Gamma_D, \tag{1.2}$$

$$\sqrt{r}\left(\frac{\partial u}{\partial r} - \mathbf{i}ku\right) \to 0 \quad \text{as } r = |x| \to \infty\,. \tag{1.3}$$

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Here $D \subset \mathbb{R}^2$ is a bounded domain with Lipschitz boundary Γ_D, $f \in (H^1(\Omega))'$ has the support inside $B(R_0) = \{x \in \mathbb{R}^2 : |x| \leq R_0\}$, where $(H^1(\Omega))'$ being the dual space of $H^1(\Omega)$, and $g \in H^{1/2}(\Gamma_D)$. Here we assume the wave number k is a constant. We remark that the results in this paper can be extended to the case when $k^2(x)$ is a variable inside some bounded domains or to solve the scattering problems with other boundary conditions such as Neumann or impedance boundary conditions on Γ_D.

Since the pioneering work of the B*érén*ger [1] which proposed a PML technique for solving the time dependent Maxwell equations, various constructions of PML absorbing layers have been proposed and studied in the literatures (see, e.g, Turkel and Yefet [24], Teixeira and Chew [22] for their reviews). Of particular importance to the development and the analysis PML method is the technique of complex coordinate stretching by Chew and Weedon [12]. Under the assumption that the solution is composed of outgoing waves only, the basic idea of the PML technique is to surround the computational domain by a fictitious layer of the finite thickness with specially designed model medium that absorbs all the waves that propagate from inside the computational domain.

The convergence of the PML method has drawn considerable attention in the literature. Lassa and Somersalo [17, 18] and Hohage *et al.* [15] studied the acoustic scattering problems for circular and smooth PML layers. The anisotropic PML method in which the PML layer is placed outside a rectangle or cuboid domain is of considerable interest as opposed to the circular PML method because it provides greater flexibility and efficiency to solve problems involving anisotropic scatters. The convergence of the uniaxial PML method has been considered recently in Chen and Wu [8], Kim and Pascial [16], Bramble and Pasciak [2] and Chen and Zheng [10] for the acoustic scattering problems. It is also proved in [2, 6, 8, 15–18] that the PML solution converges exponentially to the solution of the original scattering problem as the thickness of the PML layer tends to infinity. In the practical application of the PML method, the adaptive PML method has been proposed and studied in Chen and Wu [7] for grating, Chen and Liu [6], and Chen and Wu [8] for acoustic scattering problem, Chen and Chen [3], and Chen *et al.* [4] for Maxwell problem. The main idea of the adaptive PML method is to use a posteriori error estimate to determine the PML parameters and to use the adaptive finite element method to solve the PML equations. The adaptive PML method provides a complete numerical strategy to solve the scattering problems in the framework of finite element which produces automatically a coarse mesh size away from the fixed domain and thus makes the total computational costs insensitive to the thickness the absorbing PML layers.

The purpose of this paper is to propose a new construction of the anisotropic PML method and prove the convergence of this APML method. We will extend the idea in [4] for electromagnetic scattering problems and in [5] for acoustic scattering problem. The main idea in [4,5] is to define a continuous vector field outside the computational domain and perform the complex coordinate stretching along the direction of the vector field. This construction of the PML method by performing the complex coordinate stretching along a continuous vector field is different from the uniaxial PML method

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and seems to be more suitable for scattering problem with discontinuous wave numbers, see. Oskooi *et al.* [21]. The convergence of the anisotropic PML method in [4] is proved for a special construction of the vector field outside the rectangle domain which requires the thickness of the PML layer satisfying a uniform lower bound. This constraint is removed in the new construction of the vector field. The stability of the PML problem in the PML layer is proved by extending the reflection argument in [2] for the uniaxial PML method, then we prove the convergence of this anisotropic PML method based on the integral representation of the exterior Dirichlet problem for the Helmholtz equation and the idea of the complex coordinate stretching. We also refer to Trenve [23] and Zschiedrich *et al.* [25] for similar idea of defining anisotropic PML layer.

The paper is organized as follows. In Section 2, we introduce our new anisotropic PML formulation by following the method of the complex coordinates stretching in Chew and Weedon [12] for (1.1)-(1.3). In Section 3, we prove the exponential decay of the PML extension. In Section 4 we study the stability of the APML problem in the layer. In Section 5 we show the convergence of the APML method. In Section 6 we introduce the finite element method for solving the PML problem. Numerical experiments are presented in the final section, showing the competitive behavior of our new anisotropic PML method.

2. The PML equation

Let D be contained in the interior of the rectangle $B_1 = \{x \in \mathbb{R}^2 : |x_1| < L_1/2, |x_2| < L_2/2\}$, $\Gamma_1 = \partial B_1$ and $\mathbf{n}_1$ be the unit outer normal to Γ_1. We start by introducing the Dirichlet-to-Neumann operator $T : H^{1/2}(\Gamma_1) \to H^{-1/2}(\Gamma_1)$. Given $\chi \in H^{1/2}(\Gamma_1)$, we define $T\chi = \frac{\partial \xi}{\partial \mathbf{n}_1}$ on Γ_1, where ξ is a solution of the following exterior Dirichlet problem

$$\Delta\xi + k^2\xi = 0 \qquad \text{in } \mathbb{R}^2 \setminus \bar{B}_1, \tag{2.1}$$

$$\xi = \chi \qquad \text{on } \Gamma_1, \tag{2.2}$$

$$\sqrt{r}\left(\frac{\partial \xi}{\partial r} - \mathbf{i}k\xi\right) \to 0 \quad \text{as } \ r = |x| \to \infty\,. \tag{2.3}$$

It is well known that (2.1)-(2.3) has a unique solution $\xi \in H^1_{\text{loc}}(\mathbb{R}^2 \setminus \bar{B}_1)$ (e.g. Colton-Kress [14] and McLean [20]). Thus T is well defined and a continuous linear operator.

Let $a : H^1(\Omega_1) \times H^1(\Omega_1) \to \mathbb{C}$ be the sesquilinear form

$$a(\varphi, \psi) = \int_{\Omega_1} \left(\nabla\varphi \cdot \nabla\bar{\psi} - k^2\varphi\bar{\psi}\right) \mathrm{d}x - \langle T\varphi, \psi\rangle_{\Gamma_1}\,, \tag{2.4}$$

where $\Omega_1 = B_1 \setminus \bar{D}$ and $\langle\cdot,\cdot\rangle_{\Gamma_1}$ stands for the inner product on $L^2(\Gamma_1)$ or the duality pairing between $H^{-1/2}(\Gamma_1)$ and $H^{1/2}(\Gamma_1)$. The scattering problem (1.1)-(1.3) is equivalent to the following weak formulation [14]: Given $f \in \left(H^1(\Omega_1)\right)'$ with the support included in Ω_1 and $g \in H^{1/2}(\Gamma_D)$, find $u \in H^1(\Omega_1)$ such that $u = g$ on Γ_D and

$$a(u, \psi) = (f, \psi)_{\Omega_1}, \qquad \forall\, \psi \in H^1_{\Gamma_D}(\Omega_1)\,, \tag{2.5}$$

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where $(\cdot,\cdot)_{\Omega_1}$ is the inner product on $L^2(\Omega_1)$ or the duality pairing between $\left(H^1(\Omega_1)\right)'$ and $H^1(\Omega_1)$, and

$$H^1_{\Gamma_D}(\Omega_1) = \left\{ v \in H^1(\Omega_1) : v = 0 \text{ on } \Gamma_D \right\}.$$

The sesquilinear form (2.4) associates with a bounded linear operator $\hat{A} : H^1_{\Gamma_D}(\Omega_1) \to \left(H^1_{\Gamma_D}(\Omega_1)\right)'$ such that

$$(\hat{A}\varphi, \psi)_{\Omega_1} = a(\varphi, \psi), \qquad \forall \varphi, \psi \in H^1_{\Gamma_D}(\Omega_1).$$

It follows from the uniqueness solution of the problem (2.5) that $\hat{A}$ is surjective and one-to-one. Thus, the open mapping theorem shows that there exist a constant $C > 0$ such that the following inf-sup condition is satisfied

$$\sup_{0 \neq \psi \in H^1_{\Gamma_D}(\Omega_1)} \frac{|a(\varphi, \psi)|}{\|\psi\|_{H^1(\Omega_1)}} \geq C \|\varphi\|_{H^1(\Omega_1)}, \qquad \forall \varphi \in H^1_{\Gamma_D}(\Omega_1). \tag{2.6}$$

2.1. An anisotropic PML coordinate stretching

Now we turn to introduce a new construction of the anisotropic PML complex coordinate stretching transform $\tilde{x} = F(x)$. This construction can be easily extended to the general case of piecewise constant wave numbers. We set in this section $F(x) = x$ in B_1. Let $B_2 = \{x \in \mathbb{R}^2 : |x_1| < L_1/2 + d_1, |x_2| < L_2/2 + d_2\}$ be the rectangle which contains B_1, $\Gamma_2 = \partial B_2$. The domain $\mathbb{R}^2 \setminus \bar{B}_1$ is divided into four trapezoids $U_i^{\pm}$, $i = 1, 2$, where

$$U_i^{\pm} = \left\{ x \in \mathbb{R}^2 : \pm x_i \geq \frac{L_i}{2},\ |x_j| \leq \frac{L_j}{2} + \left(|x_i| - \frac{L_i}{2} \right) \frac{d_j}{d_i}, \quad j \neq i, j = 1, 2 \right\}.$$

For any $x \in \mathbb{R}^2 \setminus \bar{B}_1$, define $\mathbf{P}(x)$ be the projection of x on Γ_1 which has the shortest distance to x on Γ_1. It is easy to see that

$$(\mathbf{P}(x))_k \triangleq P_k(x) = \operatorname{sgn}(x_k) \left[|x_k| - \left(|x_k| - \frac{L_k}{2} \right)_+ \right], \quad k = 1, 2, \tag{2.7}$$

where $\operatorname{sgn}(\cdot)$ is the sign function and $(t)_+ := \max(0, t)$ for any $t \in \mathbb{R}$. In fact, for $x = (x_1, x_2)^T \in U_1^{\pm}$, if $|x_2| < L_2/2$, $\mathbf{P}(x) = (\operatorname{sgn}(x_1) L_1/2, x_2)^T$; if $|x_2| > L_2/2$, $\mathbf{P}(x) = (\operatorname{sgn}(x_1) L_1/2, \operatorname{sgn}(x_2) L_2/2)^T$ and the shortest distance between x and B_1 is defined as

$$r(x) = |x - \mathbf{P}(x)| = \left(\sum_{k=1}^{2} \left(|x_k| - \frac{L_k}{2} \right)_+^2 \right)^{1/2}.$$

We define the complex coordinate stretching, for $t > 0$, let $\alpha(t) = \eta(t) + \mathbf{i}\sigma(t)$ be the model medium property, where $\eta(t) = 1 + \zeta\sigma(t)$ with a constant $\zeta \geq 0$, $\sigma(t) > 0$ for

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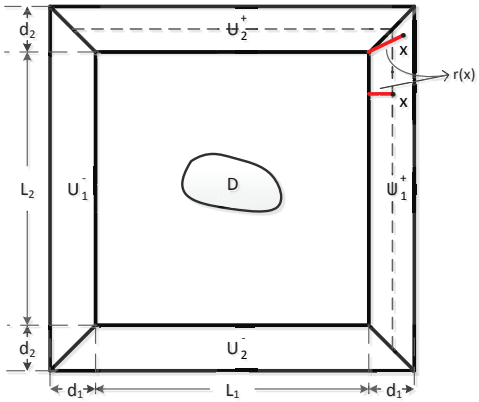


Figure 1: Setting of the scattering problem with the PML layer.

$t > 0$ and $\sigma(0) = 0$. We make the following assumption on the medium property which is rather mild in practical applications.

(H1) $\zeta \geq 1$, $\hat{\sigma}'(t) \geq 0$ for $t \geq 0$, $\sigma(t) = \hat{\sigma}(t) = \sigma_0$ for $t \geq r_0 > 0$, where $\sigma_0 \geq 0$ is a constant and $r_0 < \min(d_1, d_2)$.

The requirement that the medium property $\sigma = \hat{\sigma}$ is constant for $t \geq r_0$ has also been used in [4, 17] which is essential for using a reflection argument to prove the inf-sup condition for the PML problem in the truncated domain. One choose of the medium property is to take

$$\hat{\sigma}(t) = \sigma_0 \bigg(\int_0^t s^2 (r_0 - s)^2 \, \mathrm{d}s \bigg) \bigg(\int_0^{r_0} s^2 (r_0 - s)^2 \, \mathrm{d}s \bigg)^{-1}, \quad \forall t \in (0, r_0) \, .$$

Here $\sigma(t) = \hat{\sigma}(t) + t\hat{\sigma}'(t)$. Let $\mathbf{n}(x) = \big(x - \mathbf{P}(x)\big)/r(x)$, then the following identity is obvious

$$x = \mathbf{P}(x) + r(x)\mathbf{n}(x). \tag{2.8}$$

For $x \in \mathbb{R}^2 \setminus \bar{B}_1$, denote $\tilde{r}(x)$ as the complex stretching of $r(x)$

$$\tilde{r}(x) := \int_0^{r(x)} \alpha(t) \, \mathrm{d}t = r(x) + (\mathbf{i} + \zeta) \int_0^{r(x)} \sigma(t) \, \mathrm{d}t.$$

Now we define the complex coordinate $\tilde{x}(x) = (\tilde{x}_1(x), \tilde{x}_2(x))^T$ such that

$$\tilde{x} = \mathbf{P}(x) + \tilde{r}(x)\mathbf{n}(x) = \mathbf{P}(x) + \frac{\tilde{r}(x)}{r(x)}(x - \mathbf{P}(x)) \, .$$

Finally, we obtain

$$\tilde{x} = \mathbf{P}(x) + \beta(r(x))(x - \mathbf{P}(x)). \tag{2.9}$$

It is clear that for $t \geq 0$,

$$\beta(t) = \hat{\eta}(t) + \mathbf{i}\hat{\sigma}(t), \quad \hat{\eta}(t) = 1 + \zeta\hat{\sigma}(t), \quad \hat{\sigma}(t) = \frac{1}{t} \int_0^t \sigma(s) \, \mathrm{d}s.$$

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Here $r(x)$ is Lipschitz continuous in $\mathbb{R}^2 \setminus \bar{B}_1$ and thus the complex coordinate stretching function $\tilde{x}(x)$ is also Lipschitz continuous function in $\mathbb{R}^2 \setminus \bar{B}_1$.

For any $z \in \mathbb{C}_{++} = \{z \in \mathbb{C} : \mathrm{Re}\,(z) \geq 0, \mathrm{Im}\,(z) \geq 0\}$, denote $\tilde{x}_z = x + z\hat{\sigma}(r(x))(x - \mathbf{P}(x))$. Obviously, $z = \zeta + \mathbf{i} \in \mathbb{C}_{++}$ and denote $\tilde{x}_{\zeta+\mathbf{i}} \triangleq \tilde{x}$, we write $\tilde{x}_z = (\tilde{x}_1^z, \tilde{x}_2^z)^T$, $\tilde{y}_z = (\tilde{y}_1^z, \tilde{y}_2^z)^T$ and define the complex distance

$$d(\tilde{x}_z, \tilde{y}_z) = \Big[(\tilde{x}_1^z - \tilde{y}_1^z)^2 + (\tilde{x}_2^z - \tilde{y}_2^z)^2\Big]^{1/2}.$$

Here and in the following, for any $z \in \mathbb{C}$, $z^{1/2}$ is the analytic branch of $\sqrt{z}$ such that $\mathrm{Re}\,(z^{1/2}) \geq 0$ for any $z \in \mathbb{C} \setminus (-\infty, 0]$. The following lemma is critical to the development of this paper.

Lemma 2.1. *For any $z \in U := \{z \in \mathbb{C}_{++} : \mathrm{Re}\,(z) \geq \mathrm{Im}\,(z)\}$, we have*

$$|x - y| \leq |d(\tilde{x}_z, \tilde{y}_z)| \leq C(1 + |z|\sigma_0)|x - y|,$$

where the constant C is independent of σ_0 and ζ.

Proof. We only proof the case $x, y \in \mathbb{R}^2 \setminus \bar{B}_1$, the other cases are easier to prove. We assume $x, y \in \mathbb{R}^2 \setminus \bar{B}_1$. By the definition of (2.9) we have

$$\tilde{x}_k^z(x) = x_k + z a_k(x), \quad a_k(x) = \hat{\sigma}(r(x))(x_k - P_k(x)), \quad k = 1, 2,$$

where P_k is defined in (2.7). Straightforward calculation show that

$$\begin{aligned}
\mathrm{Re}\,\left[d(\tilde{x}, \tilde{y})^2\right] &= \mathrm{Re} \sum_{k=1}^{2} \Big(\tilde{x}_k^z(x) - \tilde{y}_k^z(y)\Big)^2 \\
&= |x - y|^2 + \big(\mathrm{Re}\,(z)^2 - \mathrm{Im}\,(z)^2\big) \sum_{k=1}^{2} \Big(a_k(x) - a_k(y)\Big)^2 \\
&\quad + 2\,\mathrm{Re}\,(z) \sum_{k=1}^{2} (x_k - y_k)\Big(a_k(x) - a_k(y)\Big).
\end{aligned}$$

Recall new representation in (2.8) and consequently

$$\begin{aligned}
&\sum_{k=1}^{2} (x_k - y_k)\Big(a_k(x) - a_k(y)\Big) \\
&= (x - y) \cdot \Big(\tau(r(x))\mathbf{n}(x) - \tau(r(y))\mathbf{n}(y)\Big) \\
&= (\mathbf{P}(x) - \mathbf{P}(y)) \cdot \Big(\tau(r(x))\mathbf{n}(x) - \tau(r(y))\mathbf{n}(y)\Big) \\
&\quad + \Big(r(x)\mathbf{n}(x) - r(y)\mathbf{n}(y)\Big) \cdot \Big(\tau(r(x))\mathbf{n}(x) - \tau(r(y))\mathbf{n}(y)\Big).
\end{aligned}$$

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By the convexity of the domain B_1, we have $(\mathbf{P}(x)-\mathbf{P}(y))\cdot\mathbf{n}(x) \geq 0$ for any $x \in \mathbb{R}^2\setminus\bar{B}_1$. Thus $(\mathbf{P}(x)-\mathbf{P}(y))\cdot(\tau(r(x))\mathbf{n}(x)-\tau(r(y))\mathbf{n}(y)) \geq 0$ for any $x, y \in \mathbb{R}^2\setminus\bar{B}_1$. Moreover, since $|\mathbf{n}(x)| = |\mathbf{n}(y)| = 1$ and $\tau(t) = r(t)\hat{\sigma}(t)$ is non-decreasing, we obtain

$$\begin{aligned}&\big(r(x)\mathbf{n}(x)-r(y)\mathbf{n}(y)\big)\cdot\Big(\tau(r(x))\mathbf{n}(x)-\tau(r(y))\mathbf{n}(y)\Big)\\ &\geq (r(x)-r(y))\Big(\tau(r(x))-\tau(r(y))\Big)\geq 0.\end{aligned}$$

This shows the left inequality in this lemma. To show the right one, denote $F_k(x) = \tilde{x}_k^z(x) = x_k + z\hat{\sigma}(r(x))(|x_k| - L_k/2)_+\mathrm{sgn}(x_k)$ and notice that

$$|F_k(x)-F_k(y)| \leq \max_{x\in\mathbb{R}^2}|\nabla F_k(x)||x-y|, \quad k=1,2.$$

By using the fact $r(x)\hat{\sigma}'(r(x)) \leq C\sigma_0$, one can easily show that $|\nabla F_k(x)| \leq C(1+|z|\sigma_0)$. This completes the proof since $d(\tilde{x}_z,\tilde{y}_z) = |\sum_{k=1}^2(F_k(x)-F_k(y))^2|^{1/2}$. □

Remark 2.1. Here $z = \zeta + \mathbf{i} \in U$ under the assumption (H1) .

2.2. Derivation of the PML equation

The derivation of the PML equation depends crucially on the idea of the complex stretching transform $\tilde{x} = F(x)$ outside the B_1 (see e.g Chew and Weedon [12]), where $F : \mathbb{R}^2\setminus\bar{B}_1 \to \mathbb{C}^2$ is a complex valued function as before.

For the sake of completeness, we recall the derivation of the PML equation in the paper [8]. By the third Green formula, the solution ξ of the exterior Dirichlet problem (2.1)-(2.3) satisfies

$$\xi = -\Psi_{\mathrm{SL}}(\lambda)(x) + \Psi_{\mathrm{DL}}(\chi)(x), \qquad \text{in } \mathbb{R}^2\setminus\bar{B}_1, \tag{2.10}$$

where $\lambda = T\chi \in H^{-1/2}(\Gamma_1)$ is the Neumann trace of ξ on Γ_1, and $\Psi_{\mathrm{SL}}, \Psi_{\mathrm{DL}}$ are respectively the single and double layer potential

$$\begin{aligned}\Psi_{\mathrm{SL}}(\lambda)(x) &= \int_{\Gamma_1} G(x,y)\lambda(y)\,\mathrm{d}s(y), &&\forall\lambda\in H^{-1/2}(\Gamma_1),\\ \Psi_{\mathrm{DL}}(\chi)(x) &= \int_{\Gamma_1}\frac{\partial G(x,y)}{\partial\mathbf{n}_1(y)}\chi(y)\,\mathrm{d}s(y), &&\forall\chi\in H^{1/2}(\Gamma_1).\end{aligned}$$

Here $G(x,y) = \frac{\mathbf{i}}{4}H_0^{(1)}(k|x-y|)$ is the fundamental solution of the constant wave number Helmholtz equation satisfying the Sommerefeld radiation condition.

We follow the method of the complex coordinates stretching [12] to introduce the PML equation. We define the modified single and double layer potential

$$\tilde{\Psi}_{\mathrm{SL}}(\lambda)(x) = \int_{\Gamma_1}\tilde{G}(x,y)\lambda(y)\,\mathrm{d}s(y), \qquad \forall\lambda\in H^{-1/2}(\Gamma_1), \tag{2.11}$$

$$\tilde{\Psi}_{\mathrm{DL}}(\chi)(x) = \int_{\Gamma_1}\frac{\partial\tilde{G}(x,y)}{\partial\mathbf{n}_1(y)}\chi(y)\,\mathrm{d}s(y), \quad \forall\chi\in H^{1/2}(\Gamma_1)\,. \tag{2.12}$$

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Let $J(y) = \det(DF(y))$ and $\tilde{G}(x, y)$ be the modified Green function defined as

$$\tilde{G}(x, y) = \frac{\mathbf{i}}{4} J(y) H_0^{(1)}\big(kd(\tilde{x}, \tilde{y})\big)\,.$$

Since $\tilde{G}(x, y)$ is smooth for $x \in \mathbb{R}^2 \setminus \bar{B}_1$ and $y \in \bar{B}_1$, it is clear $\tilde{\Psi}_{\mathrm{SL}}(\lambda), \tilde{\Psi}_{\mathrm{DL}}(\chi)$ are smooth in $\mathbb{R}^2 \setminus \bar{B}_1$. For any $\chi \in H^{1/2}(\Gamma_1)$, the PML extension $\mathbb{E}(\chi)(x)$ is defined as

$$\mathbb{E}(\chi)(x) = -\tilde{\Psi}_{\mathrm{SL}}(\lambda)(x) + \tilde{\Psi}_{\mathrm{DL}}(\chi)(x), \qquad \text{in } \mathbb{R}^2 \setminus \bar{B}_1. \tag{2.13}$$

By (2.10) we know that $\mathbb{E}(\chi) = \chi$ on Γ_1 for any $\chi \in H^{1/2}(\Gamma_1)$. By Lemma 2.1, $|d(\tilde{x}, \tilde{y})| \geq |x - y|$ for $x \in \mathbb{R}^2, y \in \mathbb{R}^2$, $\mathbb{E}(\chi)$ is the Lipshcitz continuous in $\mathbb{R}^2 \setminus \bar{B}_1$.

For the solution u of the scattering problem (2.5), let $\tilde{u} = \mathbb{E}(u\big|_{\Gamma_1})$ be the PML extension of $u\big|_{\Gamma_1}$, which satisfies $\tilde{u} = u$ on Γ_1. It is obvious that $\tilde{u}$ satisfies

$$\frac{\partial^2 \tilde{u}}{\partial \tilde{x}_1^2} + \frac{\partial^2 \tilde{u}}{\partial \tilde{x}_2^2} + k^2 \tilde{u} = 0 \quad \text{ in } \mathbb{R}^2 \setminus \bar{B}_1.$$

This implies the corresponding PML equation can be obtained by the chain rule

$$\nabla \cdot (A \nabla \tilde{u}) + J k^2 \tilde{u} = 0 \quad \text{ in } \mathbb{R}^2 \setminus \bar{B}_1, \tag{2.14}$$

where $A = J(x) DF^{-1}(x) DF^{-T}(x), J = \det(DF(x))$ and DF is the Jacboic matrix. The PML equation (2.14) should be understood in the weak sense and the PML solution $\hat{u}$ in $\Omega_2 = B_2 \setminus \bar{D}$ is defined as the solution of the following systems

$$\nabla \cdot (A \nabla \hat{u}) + J k^2 \hat{u} = f \quad \text{ in } \Omega_2, \tag{2.15}$$

$$\hat{u} = g \ \text{ on } \Gamma_D, \quad \hat{u} = 0 \quad \text{ on } \Gamma_2. \tag{2.16}$$

The stability of the PML problem (2.15)-(2.16) and the convergence of the PML solution $\hat{u}$ in Ω_1 to the solution u of the original scattering problem (1.1)-(1.3) in Ω_1 will be studied in the next sections.

To conclude this section, for the sake of later reference, we write down the explicit formula for the matrix A in the domain $\mathbb{R}^2 \setminus \bar{B}_1$. For $x \in \mathbb{R}^2 \setminus \bar{B}_1$, it is easy to check that

$$DF = \Lambda + (\alpha - \beta) q q^T, \quad q_k = \frac{\mathrm{sgn}(x_k)\big(|x_k| - \frac{L_k}{2}\big)_+}{r(x)}. \tag{2.17}$$

$\Lambda = \mathrm{diag}(a_1, a_2)$, $q = (q_1, q_2)$, where $\alpha = \alpha(r(x)),\ \beta = \beta(r(x))$ and a_k is defined as

$$a_k = 1 + (\beta - 1) H\left(|x_k| - \frac{L_k}{2}\right) = \begin{cases} \beta, & \text{if } |x_k| \geq \frac{L_k}{2}, \\ 1, & \text{if } |x_k| < \frac{L_k}{2}, \end{cases}$$

where $H(x)$ is the Heaviside function. By Sherman-Morrison theorem, we know that

$$B = DF^{-1} = \Lambda^{-1} - \frac{\alpha - \beta}{\alpha \beta} q q^T\,, \tag{2.18}$$

where $B_{kl} = a_k^{-1} \delta_{kl} - \frac{\alpha - \beta}{\alpha \beta} q_k q_l$ and δ_{kl} is the Kronecker delta.

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Lemma 2.2. *The eigenvalue of DF is $\{\alpha, a_1 + a_2 - \beta\}$ for $x \in \mathbb{R}^2 \setminus \bar{B}_1$.*

Proof. We notice that $\Lambda q = \beta q$, then $DFq = \big(\Lambda + (\alpha - \beta) q q^T\big) q = \alpha q$. One eigenvalue is $\lambda_1 = \alpha$, the other one λ_2 is determined by the relation $\lambda_1 + \lambda_2 = a_1 + a_2 + (\alpha - \beta)$, this implies $\lambda_2 = a_1 + a_2 - \beta$. This completes the proof. □

Now it is easy to see that

$$\begin{aligned} J(x) &= \det DF(x) = \alpha(a_1 + a_2 - \beta) \\ &= \begin{cases} \alpha\beta, & \text{if } |x_1| \geq \frac{L_1}{2},\ |x_2| \geq \frac{L_2}{2}, \\ \alpha, & \text{others.} \end{cases} \end{aligned} \tag{2.19}$$

3. Estimates for the PML extension

Throughout the whole paper, for any bounded domain $D \subset \mathbb{R}^2$ with boundary Γ_D, we use the weighted H^1-norm

$$\|\psi\|_{H^1(D)} = \Big(\|\nabla \psi\|^2_{L^2(D)} + \|k\psi\|^2_{L^2(D)} \Big)^{1/2}$$

and the weighted $H^{1/2}$-norm

$$\|v\|_{H^{1/2}(\Gamma_D)} = \Big(d_D^{-1} \|v\|^2_{L^2(\Gamma_D)} + |v|^2_{\frac{1}{2},\Gamma_D} \Big)^{1/2},$$

where d_D is the diameter of D and

$$|v|^2_{\frac{1}{2},\Gamma_D} = \int_{\Gamma_D} \int_{\Gamma_D} \frac{|v(x) - v(x')|^2}{|x - x'|^2} \, \mathrm{d}s(x) \, \mathrm{d}s(x').$$

It is obvious that

$$\|v\|_{H^{1/2}(\Gamma_D)} \leq (|\Gamma_D| d_D^{-1})^{1/2} \|v\|_{L^\infty(\Gamma_D)} + |\Gamma|_D \|\nabla v\|_{L^\infty(\Gamma_D)}, \quad \forall\, v \in W^{1,\infty}(\Gamma_D). \tag{3.1}$$

The following lemma is proved in [6, Lemma 2.2] which indicates the first Hankel function decays exponentially away from the upper half complex plane.

Lemma 3.1. *For any $\nu \in \mathbb{R}, z \in \mathbb{C}_{++} = \{z \in \mathbb{C} : \mathrm{Re}\,(z) \geq 0, \mathrm{Im}\,(z) \geq 0\}$ and $\Theta \in \mathbb{R}$ such that $0 < \Theta \leq |z|$, we have*

$$|H^{(1)}_\nu(z)| \leq e^{-\mathrm{Im}\,(z)\left(1 - \frac{\Theta^2}{|z|^2}\right)^{1/2}} |H^{(1)}_\nu(\Theta)|.$$

From Lemma 3.1 and (2.13), to ensure the exponential decay of the stretched Green solution $\tilde{G}(x, y)$, we need an uniform lower bound of $\mathrm{Im}\, d(\tilde{x}, y)$ which tends to infinity when the thickness of the PML layer or the PML medium property tends to infinity. We first recall the following lemma from [8].

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Lemma 3.2. *For any* $z_1 = a_1 + \mathbf{i}b_1,\ z_2 = a_2 + \mathbf{i}b_2$ *with* $a_1, b_1, a_2, b_2 \in \mathbb{R}$ *such that* $a_1b_1 + a_2b_2 > 0$*, we have*

$$\mathrm{Im}(z_1^2 + z_2^2)^{1/2} \geq \frac{a_1b_1 + a_2b_2}{\sqrt{a_1^2 + a_2^2}}.$$

Lemma 3.3. *Let* $z_j = \tilde{x}_j - y_j,\ j = 1, 2$*. For any* $x \in \Gamma_2,\ y \in \bar{B}_1$*, we have*

$$\mathrm{Im}d(\tilde{x}, y) \geq \frac{\sigma_0 \min(d_1^2, d_2^2)}{\sqrt{(L_1 + d_1)^2 + (L_2 + d_2)^2}} = \gamma_0.$$

Proof. By the assumption (H1), $\hat{\sigma}(r(x)) = \sigma_0$. We have from (2.7),(2.9) that

$$\begin{aligned} z_j = \tilde{x}_j - y_j &= (x_j - y_j) + \zeta\sigma_0 \left(x_j - P_j(x)\right) + \mathbf{i}\sigma_0 \left(x_j - P_j(x)\right) \\ &= (x_j - y_j) + \zeta\sigma_0 \mathrm{sgn}(x_j) \left(|x_j| - \frac{L_j}{2}\right)_+ + \mathbf{i}\sigma_0 \mathrm{sgn}(x_j) \left(|x_j| - \frac{L_j}{2}\right)_+. \end{aligned}$$

It is easy to check that for $y \in \bar{B}_1$,

$$\mathrm{sgn}(x_j)(x_j - y_j) \left(|x_j| - \frac{L_j}{2}\right)_+ \geq 0\,.$$

Thus by Lemma 3.2, we have

$$\begin{aligned} \mathrm{Im}d(\tilde{x}, y) &\geq \frac{\sigma_0 \sum_{j=1}^2 \left(|x_j - y_j| \big(|x_j| - \frac{L_j}{2}\big)_+ + \zeta\sigma_0 \big(|x_j| - \frac{L_j}{2}\big)_+^2\right)}{(1 + \zeta\sigma_0)|x - y|} \\ &\geq \frac{\sigma_0 \sum_{j=1}^2 \big(|x_j| - \frac{L_j}{2}\big)_+^2}{|x - y|} \\ &\geq \frac{\sigma_0 \min(d_1^2, d_2^2)}{\sqrt{(L_1 + d_1)^2 + (L_2 + d_2)^2}}\,, \end{aligned}$$

where we have used $|x_j - y_j| \geq (|x_j| - L_j/2)_+$. This completes the proof. □

We remark that in this lemma we do not require the thickness of the PML layer d_1, d_2 should have an uniform lower bound as required in [4, Lemma 3]. This is the main advantage of our new complex stretching formulation in this paper.

The following lemma shows the estimate of the fundamental solution $\tilde{G}$ of the PML equation.

Lemma 3.4. *Let* (H1) *be satisfied and* γ_0 *be chosen that* $\gamma_0 k \geq 1$*. Then there exists a constant* $C > 0$ *depending only on* γ_0*, such that for any* $x \in \Gamma_2, y \in \bar{B}_1$*,*

(i) $|\tilde{G}(x, y)| \leq Ce^{-(\gamma_0 k - 1)}$;

(ii) $\left|\frac{\partial \tilde{G}(x, y)}{\partial x_j}\right| \leq Cke^{-(\gamma_0 k - 1)}, \quad j = 1, 2;$

(iii) $\left|\frac{\partial \tilde{G}(x, y)}{\partial y_j}\right| \leq Cke^{-(\gamma_0 k - 1)}, \quad j = 1, 2;$

(iv) $\left|\frac{\partial^2 \tilde{G}(x, y)}{\partial x_i \partial y_j}\right| \leq Ck^2 e^{-(\gamma_0 k - 1)}, \quad i, j = 1, 2.$

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Proof. Due to $\tilde{G}(x,y) = \frac{\mathbf{i}}{4}H_0^{(1)}(kd(\tilde{x},y))$ when $x \in \Gamma_2, y \in \bar{B}_1$, then the proof is similar to the paper [8, Lemma 3.4]. We omit the details here. □

Based on Lemma 3.1, Lemma 3.2, the estimate (3.1) and the result in [8, Lemma 3.5-3.6], we know the modified single and double layer potentials $\tilde{\Psi}_{\mathrm{SL}}, \tilde{\Psi}_{\mathrm{DL}}$ are exponential decay as long as γ_0 large enough. The following theorem is the main result of this section.

Theorem 3.1. *Let* (H1) *and* $\gamma_0 k > 1$ *be satisfied. For any* $f \in H^{1/2}(\Gamma_1)$*, let* $\mathbb{E}(f)$ *be the* PML *extension defined in* (2.13)*. Then there exists a constant C depending only on* σ_0*, such that*

$$\|\mathbb{E}(f)\|_{H^{1/2}(\Gamma_2)} \leq C(1+kL)^2 e^{-(\gamma_0 k-1)}\|f\|_{H^{1/2}(\Gamma_1)}.$$

The stability of the PML equation in the PML layer and the convergence of the PML solution $\hat{u}$ to the solution of (1.1)-(1.3) will be proved in the next section.

4. The PML equation in the PML layer

In this section we consider the Dirichlet problem of the PML equation in the layer. Let $q \in H^{1/2}(\Gamma_2)$ and w satisfies

$$\nabla \cdot (A\nabla w) + Jk^2 w = 0 \quad \text{in } \Omega_{\mathrm{PML}} = B_2 \setminus \bar{B}_1, \tag{4.1}$$

$$w = 0 \ \text{ on } \Gamma_1, \quad w = q \quad \text{on } \Gamma_2. \tag{4.2}$$

Introduce the sesquilinear form $c : H^1(\Omega_{\mathrm{PML}}) \times H^1(\Omega_{\mathrm{PML}}) \to \mathbb{C}$ as

$$c(\phi,\psi) := \big(A\nabla\phi, \nabla\psi\big)_{\Omega_{\mathrm{PML}}} - k^2\big(J\phi,\psi\big)_{\Omega_{\mathrm{PML}}}, \quad \forall\, \phi,\psi \in H_0^1(\Omega_{\mathrm{PML}}). \tag{4.3}$$

Then the corresponding weak formulation of (4.1)-(4.2) is: Find $w \in H^1(\Omega_{\mathrm{PML}})$ such that $w = 0$ on Γ_1, $w = q$ on Γ_2, and satisfy

$$c(w,\psi) = 0, \qquad \forall\, \psi \in H_0^1(\Omega_{\mathrm{PML}}). \tag{4.4}$$

We will extend the idea in [2], [9] to prove the inf-sup condition of the PML problem (4.4), that is, under the medium property assumption (H1), the sesquilinear form associated with (4.3) satisfies the inf-sup condition,

$$\sup_{\psi \in H_0^1(\Omega_{\mathrm{PML}})} \frac{|c(\phi,\psi)|}{\|\psi\|_{H^1(\Omega_{\mathrm{PML}})}} \geq \hat{C}\|\phi\|_{H^1(\Omega_{\mathrm{PML}})}, \qquad \forall \phi \in H_0^1(\Omega_{\mathrm{PML}}), \tag{4.5}$$

where constant $\hat{C} > 0$ depends on the domain Ω_{PML} and wave number k.

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4.1. Ellipticity of $A(x)$

The first important property is the ellipticity of the PML coefficient matrix. We start the representation of the matrix

$$A = JBB^T = J\left(\Lambda^{-2} + \frac{\beta^2 - \alpha^2}{\alpha^2 + \beta^2} qq^T\right), \tag{4.6}$$

where J, B and q are defined as (2.19), (2.18) and (2.17), respectively.

Lemma 4.1. *Let* (H1) *be satisfied. For any* $\xi = (\xi_1, \xi_2) \in \mathbb{C}^2$*, we have*

$$\mathrm{Re}\left(A\xi \cdot \bar{\xi}\right) \geq \min\left(\frac{\eta\hat{\eta} + \sigma\hat{\sigma}}{|\alpha|^2}, \frac{\eta}{|\alpha|^2}, \eta\right)|\xi|^2.$$

Proof. For any $\xi = (\xi_1, \xi_2) \in \mathbb{C}^2$, we have

$$A\xi \cdot \bar{\xi} = J\left(\Lambda^{-1} + \frac{\beta^2 - \alpha^2}{\alpha^2\beta^2}\right)\xi \cdot \bar{\xi} = J\sum_{j=1} a_j^{-2}|\xi_j|^2 + J\frac{\beta^2 - \alpha^2}{\alpha^2\beta^2}|q \cdot \xi|^2.$$

Case 1. When $|x_1| \geq \frac{L_1}{2}, |x_2| \geq \frac{L_2}{2}$, then $J = \alpha\beta$, $a_1 = a_2 = \beta$,

$$\begin{aligned}
\mathrm{Re}\left(A\xi \cdot \bar{\xi}\right) &= \mathrm{Re}\left(\frac{\alpha}{\beta}\right)|\xi|^2 + \mathrm{Re}\left(\frac{\beta}{\alpha} - \frac{\alpha}{\beta}\right)|q \cdot \xi|^2 \\
&\geq Re\left(\frac{\alpha}{\beta}\right)|\xi|^2 + \mathrm{Re}\left(\frac{\beta}{\alpha} - \frac{\alpha}{\beta}\right)|\xi|^2 \\
&= \frac{\eta\hat{\eta} + \sigma\hat{\sigma}}{|\alpha|^2}|\xi|^2,
\end{aligned}$$

where we have used $\mathrm{Re}\left(\frac{\beta}{\alpha} - \frac{\alpha}{\beta}\right) = (\eta\hat{\eta} + \sigma\hat{\eta})\left(|\alpha|^{-2} - |\beta|^{-2}\right) \leq 0$.

Case 2. When $|x_1| \geq \frac{L_1}{2}, |x_2| \leq \frac{L_2}{2}$, then $J = \alpha$, $a_1 = \beta, a_2 = 1, q_1 = 1, q_2 = 0$,

$$\begin{aligned}
\mathrm{Re}\left(A\xi \cdot \bar{\xi}\right) &= \mathrm{Re}\left(\frac{\alpha}{\beta^2}\right)|\xi_1|^2 + \mathrm{Re}\,(\alpha)|\xi_2|^2 + \mathrm{Re}\left(\frac{1}{\alpha} - \frac{\alpha}{\beta^2}\right)|\xi_1|^2 \\
&= \mathrm{Re}\left(\frac{1}{\alpha}\right)|\xi_1|^2 + \eta|\xi_2|^2 \\
&\geq \min\left(\frac{\eta}{|\alpha|^2}, \eta\right)|\xi|^2.
\end{aligned}$$

For the case $|x_1| \leq \frac{L_1}{2}, |x_2| \geq \frac{L_2}{2}$, the result is similar to the case 2. We omit the details here. This completes the proof. □

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4.2. Inf-Sup conditions

In this subsection, we will prove the $\inf-\sup$ condition (4.5). We start with the following elementary lemma.

Lemma 4.2. *Let assumption* (H1) *be satisfied. Then we have*

$$\max_{x\in\mathbb{R}^2}\int_{\mathbb{R}^2}|G(\tilde{x},\tilde{y})|\,\mathrm{d}y\le C_1(k),\quad \max_{x\in\mathbb{R}^2}\int_{\mathbb{R}^2}|\nabla_x G(\tilde{x},\tilde{y})|\,\mathrm{d}y\le C_2(k).$$

Since $|d(\tilde{x},\tilde{y})|\ge|x-y|$ by lemma 2.1. The proof is similar to the UPML in paper [9]. We omit the details here.

The first step is to prove the following inf-sup condition of the PML problem in the whole space $\mathbb{R}^2$.

Lemma 4.3. *Let assumption* (H1) *be satisfied. There exists a constant* $\hat{C}>0$ *depending on* k *that*

$$\sup_{\psi\in H^1(\mathbb{R}^2)}\frac{|(A\nabla\phi,\nabla\psi)_{\mathbb{R}^2}-k^2(J\phi,\psi)_{\mathbb{R}^2}|}{\|\psi\|_{H^1(\mathbb{R}^2)}}\ge\hat{C}\|\phi\|_{H^1(\mathbb{R}^2)},\quad\forall\,\phi\in H^1(\mathbb{R}^2).$$

Proof. In fact, this inf-sup condition is equivalent to show that for any $F\in H^1(\mathbb{R}^2)'$, there exists a unique solution $w\in H^1(\mathbb{R}^2)$ such that

$$(A\nabla w,\nabla\psi)_{\mathbb{R}^2}-k^2\,(Jw,\psi)_{\mathbb{R}^2}=\langle F,\psi\rangle_{\mathbb{R}^2},\quad\forall\psi\in H^1(\mathbb{R}^2),\tag{4.7}$$

and satisfies $\|w\|_{H^1(\mathbb{R}^2)}\le C\|F\|_{H^1(\mathbb{R}^2)'}$. First we need to prove the uniqueness of the problem (4.7). We introduce the Newton potential, for any $\Phi\in L^2(\mathbb{R}^2)$,

$$N(\Phi)(x)=\int_{\mathbb{R}^2}J(y)G(\tilde{x},\tilde{y})\Phi(y)\,\mathrm{d}y,\quad\forall x\in\mathbb{R}^2.$$

By Lemma 4.2, it is easy to see that $N(\Phi)\in H^1(\mathbb{R}^2)$ for any $\Phi\in L^2(\mathbb{R}^2)$. Since $\tilde{G}(x,y)=J(y)G(\tilde{x},\tilde{y})$ is the fundamental solution of the PML equation, which satisfies,

$$\big(AN(\Phi),\nabla\psi\big)_{\mathbb{R}^2}-k^2(JN(\Phi),\psi)_{\mathbb{R}^2}=\big(J\Phi,\psi\big)_{\mathbb{R}^2},\quad\forall\psi\in H^1(\mathbb{R}^2).\tag{4.8}$$

By the basic relationships $\big(A\nabla\varphi,\nabla\psi\big)_{\mathbb{R}^2}=\big(A\nabla\bar{\psi},\nabla\bar{\varphi}\big)_{\mathbb{R}^2}$ and $\big(J\varphi,\psi\big)_{\mathbb{R}^2}=\big(J\bar{\psi},\bar{\varphi}\big)_{\mathbb{R}^2}$, we take $\psi=\bar{u}$ in (4.8),

$$\begin{aligned}\big(J\Phi,\bar{u}\big)_{\mathbb{R}^2}&=\big(A\nabla N(\Phi),\nabla\bar{u}\big)_{\mathbb{R}^2}-\big(JN(\Phi),\bar{u}\big)_{\mathbb{R}^2}\\&=\big(A\nabla u,\nabla\overline{N(\Phi)}\big)_{\mathbb{R}^2}-\big(Ju,\overline{N(\Phi)}\big)_{\mathbb{R}^2}.\end{aligned}$$

Then

$$\big(J\Phi,\bar{u}\big)_{\mathbb{R}^2}=\big(A\nabla u,\nabla\overline{N(\Phi)}\big)_{\mathbb{R}^2}-k^2\big(Ju,\overline{N(\Phi)}\big)_{\mathbb{R}^2}=0,\quad\forall\,\Phi\in L^2(\mathbb{R}^2).$$

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This implies that $u \equiv 0$ and shows the uniqueness of the problem (4.7). Now we turn to prove the $\inf - \sup$ condition of the problem (4.7). Since $\mathrm{Re}\, A$ is elliptic, for any $F \in H^1(\mathbb{R}^2)'$, there exists a unique solution $v \in H^1(\mathbb{R}^2)$ of the following problem by Lax-Milgram lemma such that

$$\big(A\nabla v, \nabla\psi\big)_{\mathbb{R}^2} + k^2\big(Jv, \psi\big)_{\mathbb{R}^2} = \langle F, \psi\rangle_{\mathbb{R}^2}, \quad \forall\psi \in H^1(\mathbb{R}^2), \tag{4.9}$$

and satisfies the estimate $\|v\|_{H^1(\mathbb{R}^2)} \leq C\|F\|_{H^1(\mathbb{R}^2)'}$. We construct $F_1 = k^2 J^{-1}v + k^2 v \in L^2(\mathbb{R}^2)$, then the Newton potential $N(F_1) = \int_{\mathbb{R}^2} J(y)G(\tilde{x}, \tilde{y})F_1(y)\,\mathrm{d}y$ satisfies the following PML equation,

$$\big(A\nabla N(F_1), \nabla\psi\big)_{\mathbb{R}^2} - k^2\big(JN(F_1), \psi\big)_{\mathbb{R}^2} = \langle JF_1, \psi\rangle_{\mathbb{R}^2}, \quad \forall\psi \in H^1(\mathbb{R}^2). \tag{4.10}$$

Now (4.9) and (4.10) leads to

$$\big(A\nabla(v + N(F_1)), \nabla\psi\big)_{\mathbb{R}^2} - k^2(J(v + N(F_1)), \psi)_{\mathbb{R}^2} = \langle F, \psi\rangle_{\mathbb{R}^2}.$$

This implies $u = v + N(F_1)$ is a solution of (4.7). This completes the existence. It is left to show $\|N(F_1)\|_{H^1(\mathbb{R}^2)} \leq C(k)\,\|F_1\|_{L^2(\mathbb{R}^2)}$. For any $g \in L^2(\mathbb{R}^2)$, by Cauchy-Schwarz inequality

$$\begin{aligned}
&\big|(N(F_1), g)_{\mathbb{R}^2}\big| = \left|\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} J(y)G(\tilde{x}, \tilde{y})F_1(y)g(x)\,\mathrm{d}x\,\mathrm{d}y\right| \\
&\leq C\left(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |G(\tilde{x}, \tilde{y})||F_1(y)|^2\,\mathrm{d}x\,\mathrm{d}y\right)^{1/2}\left(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |G(\tilde{x}, \tilde{y})||g(x)|^2\,\mathrm{d}x\,\mathrm{d}y\right)^{1/2} \\
&\leq C(k)\|F_1\|_{L^2(\mathbb{R}^2)}\|g\|_{L^2(\mathbb{R}^2)}.
\end{aligned}$$

It follows from the lemma 4.2 that $\|N(F_1)\|_{L^2(\mathbb{R}^2)} \leq C(k)\|F_1\|_{L^2(\mathbb{R}^2)}$. Similarly, one can show $\|\nabla N(F_1)\|_{L^2(\mathbb{R}^2)} \leq C(k)\|F_1\|_{L^2(\mathbb{R}^2)}$. For $F_1 = k^2 J^{-1}v + k^2 v$ and $\|v\|_{L^2(\mathbb{R}^2)} \leq C\|F\|_{H^1(\mathbb{R}^2)'}$, we obtain,

$$\|N(F_1)\|_{H^1(\mathbb{R}^2)} \leq C(k)\|v\|_{L^2(\mathbb{R}^2)} \leq C(k)\|F\|_{H^1(\mathbb{R}^2)'}.$$

Finally we have

$$\|u\|_{H^1(\mathbb{R}^2)} \leq \|v\|_{H^1(\mathbb{R}^2)} + \|N(F_1)\|_{H^1(\mathbb{R}^2)} \leq C(k)\|F\|_{H^1(\mathbb{R}^2)'}.$$

This completes the proof. □

The next lemma plays a key role in proving the inf-sup condition (4.5). Before starting this lemma, we need some notations which will be used later. Let $\bar{L}_1 = \frac{L_1}{2} + d_1$, $\bar{L}_2 = \frac{L_2}{2} + d_2$, $\bar{l}_1 = \frac{L_1}{2} + r_0$, $\bar{l}_2 = \frac{L_2}{2} + r_0$, where r_0 is defined in assumption (H1), obviously, $\bar{l}_j < \bar{L}_j$, $j = 1, 2$. The next result develops the inf-sup condition for the square domain $B_2 = (-\bar{L}_1, \bar{L}_1) \times (-\bar{L}_2, \bar{L}_2)$.

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Lemma 4.4. *Let assumption* (H1) *be satisfied and $\gamma_0 k$ be sufficiently large. Then there exist a constant $C > 0$ such that*

$$\sup_{\psi\in H_0^1(B_2)} \frac{|(A\nabla\phi,\nabla\psi)_{B_2} - k^2(J\phi,\psi)_{B_2}|}{\|\psi\|_{H^1(B_2)}} \geq C\|\phi\|_{H^1(B_2)}, \quad \forall\, \phi \in H_0^1(B_2).$$

Proof. We will use the reflection argument [2], [9] to prove this lemma. For any $\phi \in H_0^1(B_2)$, define $F \in H^1(B_2)'$ as

$$\langle F,\psi\rangle_{B_2} = (A\nabla\phi,\nabla\psi)_{B_2} - k^2(J\phi,\psi)_{B_2}, \quad \forall\psi \in H_0^1(B_2).$$

Then the inf-sup condition is equivalent to show $\|\phi\|_{H^1(B_2)} \leq C\|F\|_{H^1(B_2)'}$.

We introduce an extension of ϕ to the domain $B_2^E := (-2\bar{L}_1+\bar{l}_1, 2\bar{L}_1-\bar{l}_1)\times(-2\bar{L}_2+\bar{l}_2, 2\bar{L}_2-\bar{l}_2)$ as follows. Let

$$\phi_1(x) = \begin{cases} -\phi(2\bar{L}_1 - x_1, x_2), & \text{if } x \in (\bar{L}_1, 2\bar{L}_1-\bar{l}_1)\times(-\bar{L}_2,\bar{L}_2),\\ \phi(x_1,x_2), & \text{if } x \in (-\bar{L}_1,\bar{L}_1)\times(-\bar{L}_2,\bar{L}_2),\\ -\phi(-2\bar{L}_1 - x_1, x_2), & \text{if } x \in (-2\bar{L}_1+\bar{l}_1, -\bar{L}_1)\times(-\bar{L}_2,\bar{L}_2),\end{cases}$$

where ϕ_1 is the extension of ϕ to $(-2\bar{L}_1+\bar{l}_1, 2\bar{L}_1-\bar{l}_1)\times(-\bar{L}_2,\bar{L}_2)$ by odd reflection with respect ot $x_1 = \pm\bar{L}_1$, then $\tilde{\phi}$ is defined as the extension of ϕ_1 to B_2^E by odd reflection with respect to $x_2 = \pm\bar{L}_2$

$$\tilde{\phi}(x) = \begin{cases} -\phi_1(x_1, 2\bar{L}_2 - x_2), & \text{if } x \in (-2\bar{L}_1+\bar{l}_1, 2\bar{L}_1-\bar{l}_1)\times(\bar{L}_2, 2\bar{L}_2-\bar{l}_2),\\ \phi_1(x_1,x_2), & \text{if } x \in (-2\bar{L}_1+\bar{l}_1, 2\bar{L}_1-\bar{l}_1)\times(-\bar{L}_2,\bar{L}_2),\\ -\phi_1(x_1, -2\bar{L}_2 - x_2), & \text{if } x \in (-2\bar{L}_1+\bar{l}_1, 2\bar{L}_1-\bar{l}_1)\times(-2\bar{L}_2+\bar{l}_2, -\bar{L}_2).\end{cases}$$

Define the functional $\tilde{F} \in H^1(B_2^E)$ as

$$\langle\tilde{F},\psi\rangle_{B_2^E} = (A\nabla\tilde{\phi},\nabla\psi)_{B_2^E} - k^2(J\tilde{\phi},\psi)_{B_2^E}, \quad \forall\psi \in H_0^1(B_2^E).$$

Actually, $\tilde{F}$ is an extension of $F \in H^1(B_2)$ and satisfies $\|\tilde{F}\|_{H^1(B_2^E)'} \leq C\|F\|_{H^1(B_2)'}$.

Now we extend $\tilde{F}$ to a functional $F_1 \in H^1(\mathbb{R}^2)'$ by the Hahn-Banach theorem such that

$$\|F_1\|_{H^1(\mathbb{R}^2)'} = \|\tilde{F}\|_{H^1(B_2^E)'} \leq C\|F\|_{H^1(B_2)'}.$$

For $F_1 \in H^1(\mathbb{R}^2)'$, by lemma 4.3, there exists a solution $v \in H^1(\mathbb{R}^2)$ such that

$$(A\nabla v,\nabla\psi)_{\mathbb{R}^2} - k^2(Jv,\psi)_{\mathbb{R}^2} = \langle F_1,\psi\rangle_{\mathbb{R}^2}, \qquad \forall\psi \in H^1(\mathbb{R}^2),$$

which satisfies $\|v\|_{H^1(\mathbb{R}^2)} \leq C\|F_1\|_{H^1(\mathbb{R}^2)} \leq C\|F\|_{H^1(B_2)'}$. Thus

$$\|\phi\|_{H^1(B_2)} \leq \|\tilde{\phi} - v\|_{H^1(B_2)} + \|v\|_{H^1(\mathbb{R}^2)}. \tag{4.11}$$

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Due to $\tilde{\phi}$ and v satisfy the following PML equation when restricted to B_2^E, for any $\psi \in H_0^1(B_2^E)$,

$$\big(A\nabla\tilde{\phi},\nabla\psi\big)_{B_2^E} - k^2\big(J\tilde{\phi},\psi\big)_{B_2^E} = \langle\tilde{F},\psi\rangle_{B_2^E},$$
$$\big(A\nabla v,\nabla\psi\big)_{B_2^E} - k^2\big(Jv,\psi\big)_{B_2^E} = \langle F_1,\psi\rangle_{B_2^E},$$

where $F_1(\psi) = \tilde{F}(\psi)$ when restricted to $H_0^1(B_2^E)$. Let $v_1 = v - \tilde{\phi}$, it is easy to see that $v_1 \in H^1(B_2^E)$ satisfies the following homogeneous PML equation

$$\big(A\nabla v_1,\nabla\psi\big)_{B_2^E} - k^2\big(Jv_1,\psi\big)_{B_2^E} = 0, \qquad \forall\,\psi \in H_0^1(B_2^E).$$

Then for any $x \in B_2^E$, v_1 has the integral representation as follows

$$v_1(x) = \int_{\partial B_2^E} A\nabla_y v_1\cdot\mathbf{n}_1(y)\tilde{G}(x,y) - A\nabla_y\big(\tilde{G}(x,y)\big)\cdot\mathbf{n}_1(y)v_1\,\mathrm{d}s(y).$$

It suffices to prove that $\|v_1\|_{H^1(B_2)} \le Ce^{-(\tilde{c}_0k\gamma_0-1)}\|v_1\|_{H^1(B_2^E)}$, where $\tilde{c}_0$ is a constant larger than zero, combine with (4.11), we have

$$\begin{aligned}\|\phi\|_{H^1(B_2)} &\le \|v\|_{H^1(\mathbb{R}^2)} + Ce^{-(\tilde{c}_0k\gamma_0-1)}\|v_1\|_{H^1(B_2^E)}\\ &\le \hat{C}\|F\|_{H^1(B_2)'} + Ce^{-(\tilde{c}_0k\gamma_0-1)}\Big(\|\phi\|_{H^1(B_2)} + \|\tilde{\phi}\|_{H^1(B_2^E\setminus\bar{B}_2)}\Big).\end{aligned}$$

It's easy to check $\|\tilde{\phi}\|_{H^1(B_2^E\setminus\bar{B}_2)} \le C\|\phi\|_{H^1(B_2)}$ and make $Ce^{-(\tilde{c}_0k\gamma_0-1)} \le 1/2$ as long as $\gamma_0 k$ large enough, then $\|\phi\|_{H^1(B_2)} \le \hat{C}\|F\|_{H^1(B_2)'}$. This completes the proof. □

It is in the position to prove the inf-sup condition (4.5) of the truncated PML problem in the PML layer $\Omega_{\mathrm{PML}} = B_2\setminus\bar{B}_1$.

Theorem 4.1. *Let assumption* (H1) *be satisfied and $\gamma_0 k$ be sufficiently large. Then there exist a constant $C > 0$ such that*

$$\sup_{\psi\in H_0^1(\Omega_{\mathrm{PML}})}\frac{|\big(A\nabla\phi,\nabla\psi\big)_{\Omega_{\mathrm{PML}}} - k^2(J\phi,\psi)_{\Omega_{\mathrm{PML}}}|}{\|\psi\|_{H^1(\Omega_{\mathrm{PML}})}} \ge C\|\phi\|_{H^1(\Omega_{\mathrm{PML}})}, \quad \forall\,\phi\in H_0^1(\Omega_{\mathrm{PML}}).$$

Furthermore, the PML *problem in the* PML *layer* (4.1)-(4.2) *has a unique weak solution $w \in H^1(\Omega_{\mathrm{PML}})$.*

Proof. For any $\phi \in H_0^1(\Omega_{\mathrm{PML}})$ which extends to be zero in B_1 and hence obtain a function (still denote as ϕ) in $H_0^1(B_2)$. By using lemma 4.4

$$\|\phi\|_{H^1(\Omega^{\mathrm{PML}})} = \|\phi\|_{H^1(B_2)} \le C\sup_{\psi\in H_0^1(B_2)}\frac{\big|(A\nabla\phi,\nabla\psi)_{\Omega_{\mathrm{PML}}} - k^2(J\phi,\psi)_{\Omega_{\mathrm{PML}}}\big|}{\|\psi\|_{H^1(B_2)}}.$$

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Here we have used the function $\phi = 0$ in B_1, so the integration in the sequilinear form $(A\nabla\phi, \nabla\psi)_{B_2} - k^2(J\phi, \psi)_{B_2}$ is restricted to Ω_{PML}. Now for any $\psi \in H_0^1(B_2)$, define $w = \overline{\mathbb{E}(\bar{\psi}|_{\Gamma_1})}$ be the PML extension (2.13) which satisfies

$$(A\nabla\phi, \nabla w)_{\Omega_{\mathrm{PML}}} - k^2(J\phi, w)_{\Omega_{\mathrm{PML}}} = 0\,.$$

Let $\chi_0 \in C^\infty(\mathbb{R}^2)$ be the cut-off function such that $\chi_0 = 1$ in $B_{r_0} := \{x \in \mathbb{R}^2 : |x_1| \le L_1/2 + r_0,\ |x_2| \le L_2/2 + r_0\}$, $\chi_0 = 0$ on Γ_2 and $|\nabla\chi_0| \le C(d - r_0)^{-1}$. Then define $\psi_1 = \psi - \chi_0 w \in H_0^1(\Omega_{\mathrm{PML}})$ and satisfies

$$\begin{aligned}
&\left|(A\nabla\phi, \nabla\psi)_{\Omega_{\mathrm{PML}}} - k^2(J\phi, \psi)_{\Omega_{\mathrm{PML}}}\right| \\
\le\ &\left|(A\nabla\phi, \nabla\psi_1)_{\Omega_{\mathrm{PML}}} - k^2(J\phi, \psi_1)_{\Omega_{\mathrm{PML}}}\right| \\
&\quad + \left|(A\nabla\phi, \nabla(1-\chi_0)w)_{\Omega_{\mathrm{PML}}} - k^2(J\phi, (1-\chi_0)w)_{\Omega_{\mathrm{PML}}}\right| \\
\le\ &\left|(A\nabla\phi, \nabla\psi_1)_{\Omega_{\mathrm{PML}}} - k^2(J\phi, \psi_1)_{\Omega_{\mathrm{PML}}}\right| + C\|\phi\|_{H^1(\Omega_{\mathrm{PML}})}\|w\|_{H^1(B_2\setminus\bar{B}_{r_0})}.
\end{aligned}$$

Since $\sigma(t) = \sigma_0$ when $t > r_0$, let $\bar{d}_i = d_i - r_0$, $i = 1, 2$, by lemma 3.3, for any $x \in B_2 \setminus \bar{B}_{r_0}$, $y \in \Gamma_1$,

$$\operatorname{Im} d(\tilde{x}, y) \ge \frac{\sigma_0 r_0^2}{\sqrt{(L_1 + d_1)^2 + (L_2 + d_2)^2}} := \tilde{\gamma}_0.$$

By the definition of the PML extension (2.13) and lemma 3.4, we have

$$\begin{aligned}
&\|w\|_{H^1(B_2\setminus\bar{B}_{r_0})} \le C(d)\left(\bar{d}^{-1}\|w\|_{L^\infty(B_2\setminus\bar{B}_{r_0})} + \|\nabla w\|_{L^\infty(B_2\setminus\bar{B}_{r_0})}\right) \\
\le\ &C(d)\max_{x\in B_2\setminus\bar{B}_{r_0}}\left(\|\tilde{G}(x,\cdot)\|_{L^\infty(\Gamma_1)} + \|\nabla_y\tilde{G}(x,\cdot)\|_{L^\infty(\Gamma_1)}\right)\|\psi\|_{H^{1/2}(\Gamma_1)} \\
\le\ &C(d)ke^{-(\tilde{\gamma}_0 k - 1)}\|\psi\|_{H^{1/2}(\Gamma_1)}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\|\phi\|_{H^1(\Omega_{\mathrm{PML}})} &\le C\sup_{\psi_1\in H_0^1(\Omega_{\mathrm{PML}})}\frac{\left|(A\nabla\phi, \nabla\psi_1)_{\Omega_{\mathrm{PML}}} - k^2(J\phi, \psi_1)_{\Omega_{\mathrm{PML}}}\right|}{\|\psi\|_{H^1(B_2)}} \\
&\quad + C(d)ke^{-(\tilde{\gamma}_0 k - 1)}\|\phi\|_{H^1(\Omega_{\mathrm{PML}})}, \\
\frac{\|\psi_1\|_{H^1(\Omega_{\mathrm{PML}})}}{\|\psi\|_{H^1(B_2)}} &\le 1 + C(d)\frac{\|\psi\|_{H^{1/2}(\Gamma_1)}}{\|\psi\|_{H^1(B_2)}} \le 1 + C(d).
\end{aligned}$$

This shows the desired inf-sup condition if $\tilde{\gamma}_0 k$ is sufficiently large. □

The following theorem is the main objective of this section.

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Theorem 4.2. *Let* (H1) *be satisfied and for $q \in H^{1/2}(\Gamma_2)$, let w be the solution of the* PML *equation* (4.1)-(4.2). *Then there exists a constant $C > 0$ depend on σ_0, k but independent of d_j, L_j, $j = 1, 2$ such that*

$$\Big\| \frac{\partial w}{\partial \mathbf{n}_1} \Big\|_{H^{-1/2}(\Gamma_1)} \leq \hat{C}^{-1} C(\sigma_0, k) \|q\|_{H^{1/2}(\Gamma_2)}.$$

Proof. By trace theorem, there exists a function $\zeta \in H^1(\Omega_{\mathrm{PML}})$ such that $\zeta = q$ on Γ_2, $\zeta = 0$ on Γ_1 and satisfies the estimate $\|\zeta\|_{H^1(\Omega_{\mathrm{PML}})} \leq C\|q\|_{H^{1/2}(\Gamma_2)}$, according to the inf-sup condition (4.5) and w satisfies (4.4), $w - \zeta \in H_0^1(\Omega_{\mathrm{PML}})$ satisfies

$$\hat{C}\|w - \zeta\|_{H^1(\Omega_{\mathrm{PML}})} \leq \sup_{\psi \in H_0^1(\Omega_{\mathrm{PML}})} \frac{\big|c(w - \zeta, \psi)\big|}{\|\psi\|_{H^1(\Omega_{\mathrm{PML}})}} = \sup_{\psi \in H_0^1(\Omega_{\mathrm{PML}})} \frac{\big|c(\zeta, \psi)\big|}{\|\psi\|_{H^1(\Omega_{\mathrm{PML}})}}.$$

By Cauchy-Schwarz inequality, we know

$$\begin{aligned} \big|c(\zeta, \psi)\big| &\leq \max_{x \in \Omega_{\mathrm{PML}}} \Big(|A(x)|, |J(x)| \Big) \|\zeta\|_{H^1(\Omega_{\mathrm{PML}})} \|\psi\|_{H^1(\Omega_{\mathrm{PML}})} \\ &\leq C(\sigma_0, k) \|\zeta\|_{H^1(\Omega_{\mathrm{PML}})} \|\psi\|_{H^1(\Omega_{\mathrm{PML}})}. \end{aligned} \tag{4.12}$$

This implies that

$$\|w\|_{H^1(\Omega_{\mathrm{PML}})} \leq \hat{C}^{-1} C(\sigma_0, k) \|\zeta\|_{H^1(\Omega_{\mathrm{PML}})} \leq \hat{C}^{-1} C(\sigma_0, k) \|q\|_{H^{1/2}(\Gamma_2)}. \tag{4.13}$$

Since $A(x)$ reduce to the identity matrix on Γ_1. For any $\psi \in H^1(\Omega_{\mathrm{PML}})$ such that $\psi = 0$ on Γ_1, we obtain

$$\begin{aligned} -\int_{\Gamma_1} \frac{\partial w}{\partial \mathbf{n}_1} \bar{\psi} \, \mathrm{d}s(x) &= \int_{\partial \Omega_{\mathrm{PML}}} A \nabla w \cdot \mathbf{n} \bar{\psi} \, \mathrm{d}s(x) \\ &= \int_{\Omega_{\mathrm{PML}}} \big(A \nabla w \cdot \nabla \bar{\psi} + \nabla \cdot (A \nabla w) \bar{\psi} \big) \, \mathrm{d}x \\ &= \int_{\Omega_{\mathrm{PML}}} \big(A \nabla w \cdot \nabla \bar{\psi} - J k^2 w \bar{\psi} \big) \, \mathrm{d}x. \end{aligned}$$

Here we have used the PML equation (4.1) and the integration by parts. Again by Cauchy-Schwarz inequality, we have

$$\Big| \int_{\Gamma_1} \frac{\partial w}{\partial \mathbf{n}_1} \bar{\psi} \, \mathrm{d}s(x) \Big| \leq C(\sigma_0, k) \|w\|_{H^1(\Omega_{\mathrm{PML}})} \|\psi\|_{H^1(\Omega_{\mathrm{PML}})}.$$

By using the argument in (4.12), we obtain

$$\Big\| \frac{\partial w}{\partial \mathbf{n}_1} \Big\|_{H^{-1/2}(\Gamma_1)} \leq \hat{C}^{-1} C(\sigma_0, k) \|w\|_{H^1(\Omega_{\mathrm{PML}})}.$$

This completes the proof by using (4.13). □

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5. Convergence of the PML problem

In this section, we will show the exponential convergence of the solution of problem (2.15)-(2.16) to the solution of the original scattering problem (1.1)-(1.3). We start by introducing the approximate Dirichlet to Neumann operator $\hat{T} : H^{1/2}(\Gamma_1) \to H^{-1/2}(\Gamma_1)$ associated with the PML problem. Given $\chi \in H^{1/2}(\Gamma_1)$, let $\hat{T}\chi = \frac{\partial \phi}{\partial \mathbf{n}_1}\big|_{\Gamma_1}$, where ϕ is the solution of the following Dirichlet problem in the PML layer in weak sense:

$$\nabla \cdot (A\nabla\phi) + Jk^2\phi = 0 \quad \text{in } \Omega_{\text{PML}}, \tag{5.1}$$

$$\phi = \chi \text{ on } \Gamma_1, \qquad \phi = 0 \quad \text{on } \Gamma_2. \tag{5.2}$$

By inf-sup condition (4.5) and Theorem 4.2, there exists a unique solution in problem (5.1)-(5.2) for sufficiently large $\gamma_0 k$, thus $\hat{T}$ is well defined. Then the solution of (2.15)-(2.16) in non-PML domain $\Omega_1 = B_1 \setminus \bar{D}$ is equivalent to solve the following weak formulation: Find $\hat{u} \in H^1(\Omega_1)$ such that $\hat{u} = g$ on Γ_D and

$$\hat{a}(\hat{u}, v) = (f, v)_{\Omega_1}, \qquad \forall v \in H^1_{\Gamma_D}(\Omega_1), \tag{5.3}$$

where $H^1_{\Gamma_D}(\Omega_1) = \{v \in H^1(\Omega_1) : v = 0 \text{ on } \Gamma_D\}$ and the sesquilinear form $\hat{a} : H^1(\Omega_1) \times H^1(\Omega_1) \to \mathbb{C}$ is defined as

$$\hat{a}(\phi, \psi) = \int_{\Omega_1} \left(\nabla\phi \cdot \nabla\bar{\psi} - k^2\phi\psi\right) \mathrm{d}x - \langle \hat{T}\phi, \psi\rangle_{\Gamma_1}, \qquad \forall \phi, \psi \in H^1(\Omega_1).$$

We are in the position to estimate the error $T\chi - \hat{T}\chi$ for any $\chi \in H^{1/2}(\Gamma_1)$. From the paper [8], we know that

$$T\chi - \hat{T}\chi = \frac{\partial w}{\partial \mathbf{n}_1}\Big|_{\Gamma_1},$$

where $w \in H^1(\Omega_{\text{PML}})$ is the solution of the Dirichlet problem (4.1)-(4.2) with $q = \mathbb{E}(\chi)$ on Γ_2. By theorem 3.1 and theorem 4.2, we obtain

$$\begin{aligned} \left\| \frac{\partial w}{\partial \mathbf{n}_1} \right\|_{H^{-1/2}(\Gamma_1)} &\le \hat{C}^{-1} C(\sigma_0, k) \|\mathbb{E}(\chi)\|_{H^{1/2}(\Gamma_2)} \\ &\le \hat{C}^{-1} C(\sigma_0, k)(1 + kL)^2 e^{-(k\gamma_0 - 1)} \|\chi\|_{H^{1/2}(\Gamma_1)}. \end{aligned} \tag{5.4}$$

Now we are ready to prove the convergence result on this paper.

Theorem 5.1. *Let* (H1) *be satisfied and u be the solution of the problem* (1.1)-(1.3). *The* PML *problem* (2.15)-(2.16) *has a unique solution $\hat{u} \in H^1(\Omega_2)$ for sufficiently large $\gamma_0 k > 0$. Moreover, we have the following error estimate*

$$\|\hat{u} - u\|_{H^1(\Omega_1)} \le \hat{C}^{-1} C(\sigma_0, k)(1 + kL)^2 e^{-(\gamma_0 k - 1)} \|\hat{u}\|_{H^{1/2}(\Gamma_1)}. \tag{5.5}$$

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Proof. We first prove the error estimate (5.5). Let $\hat{u}$ be the solution of the PML problem (2.15)-(2.16), using (2.5), (5.3) and simple integration by parts which inplies

$$a(u-\hat{u},\psi)=\hat{a}(\hat{u},\psi)-a(\hat{u},\psi)=\langle T\hat{u}-\hat{T}\hat{u},\psi\rangle_{\Gamma_1},\qquad \forall\psi\in H_0^1(\Omega_1).$$

By inf-sup condition (2.6) and the estimate (5.4),

$$\|u-\hat{u}\|_{H^1(\Omega_1)}\leq \hat{C}^{-1}C(\sigma_0,k)(1+kL)^2e^{-(\gamma_0 k-1)}\|\hat{u}\|_{H^{1/2}(\Gamma_1)}.$$

We turn to proving the well-posedness of the PML problem. By the Fredholm alternative theorem, we only prove the uniqueness of the PML problem (2.15)-(2.16). For this purpose, we assume (2.15)-(2.16) has a solution $\hat{u}$ for $f=0$ in (2.15) and $g=0$ in (2.16). By the uniqueness of the scattering problem (1.1)-(1.3), the corresponding solution $u=0$ in Ω_1. Thus (5.5) implies

$$\|\hat{u}\|_{H^1(\Omega_1)}\leq \hat{C}^{-1}C(\sigma_0,k)(1+kL)^2e^{-(\gamma_0 k-1)}\|\hat{u}\|_{H^{1/2}(\Gamma_1)}.$$

Then $\hat{u}=0$ in Ω_1 for sufficiently large $\gamma_0 k$, at the same time, by theorem 4.2, $\hat{u}$ also vanishes in Ω_{PML} as $\hat{u}$ satisfies the PML equation (4.1)-(4.2) in the PML layer with the homogenous boundary conditions. This completes the whole proof. □

6. Finite element approximation

In this section, we introduce the finite element approximations (2.15)-(2.16). Let $b: H^1(\Omega_2)\times H^1(\Omega_2)\to\mathbb{C}$ be the sesquilinear form given by

$$b(\phi,\psi)=\int_{\Omega_2}\left(A\nabla\phi\cdot\nabla\bar{\psi}-Jk^2\phi\bar{\psi}\right)\mathrm{d}x.$$

The weak formulation of (2.15)-(2.16) is: Given $g\in H^{1/2}(\Gamma_D)$, find $\hat{u}\in H^1(\Omega_2)$ such that $\hat{u}=g$ on Γ_D, $\hat{u}=0$ on Γ_2, and

$$b(\hat{u},\psi)=(f,\psi)_{\Omega_2},\quad \forall\,\psi\in H_0^1(\Omega_2).$$

Let Γ_D^h be a piecewise linear approximation of the boundary Γ_D and Ω_2^h be the domain bounded by Γ_2 and Γ_D^h. Let $\mathcal{M}_h$ be a regualr triangulation of the domain Ω_2^h. Let $V_h\subset H^1(\Omega_2^h)$ be the conforming linear finite element space over Ω_2^h and $\overset{\circ}{V}_h=\{v_h\in V_h: v_h=0 \text{ on } \Gamma_D^h\cup\Gamma_2\}$. The finite element approximation to the PML problem (2.15)-(2.16) is equivalent to: Find $u_h\in V_h$ such that $u_h=g_h$ on Γ_D^h, $u_h=0$ on Γ_2 and

$$b(u_h,\psi_h)=(f,\psi_h)_{\Omega_2},\quad \forall\,\psi_h\in\overset{\circ}{V}_h. \tag{6.1}$$

Here g_h is some piecewise linear approximation of g.

We will use adaptive finite element method based on a posteriori error estimate to solve the discrete problem (6.1). First we introduce a posteriori error estimator used

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in our numerical experiment in the next section. For any $K \in \mathcal{M}_h$, we denote h_K its diameter. Let $\mathcal{B}_h$ denote the set of all sides that do not lie on Γ_D^h and Γ_2. For any $e \in \mathcal{B}_h$, h_e means its length. For any $K \in \mathcal{M}_h$, we define the jump residual:

$$R_h := \nabla \cdot (A\nabla u_h\big|_K) + Jk^2 u_h\big|_K - f\big|_K.$$

For any interior side $e \in \mathcal{B}_h$ which is common side of K_1 and $K_2 \in \mathcal{M}_h$, we define the jump residual across e:

$$J_e := \left(A\nabla u_h\big|_{K_1} - A\nabla u_h\big|_{K_2}\right) \cdot \nu_e.$$

By using the convection that the unit normal ν_e points from K_2 to K_1. For any $K \in \mathcal{M}_h$, the local error estimator η_K is defined as

$$\eta_K = \left(\|h_K R_h\|^2_{L^2(K)} + \frac{1}{2}\sum_{e\subset\partial K} h_e \|J_e\|^2_{L^2(e)}\right)^{1/2}. \tag{6.2}$$

7. Numerical implementation

In this section, we present a numerical example to illustrate the performance of the adaptive new anisotropic PML method. The computations are carried out in MATLAB on Dell Optiplex 780 Inter(R) Core(TM)2 Quad CPU 2.83GHz. The PML parameters are determined as follows. First we choose L_1, L_2 such that $D \subset B_1$. We choose the thickness of the PML layer $d_1, d_2 > 0$. We take $\zeta = 1$ in $\eta = 1 + \zeta\sigma$ and choose the medium property $\sigma(t) = (t\hat{\sigma}(t))'$ with

$$\hat{\sigma}(t) = \sigma_0 \left(\int_0^t s^2(r_0 - s)^2 \,\mathrm{d}s\right)\left(\int_0^{r_0} s^2(r_0 - s)^2 \,\mathrm{d}s\right)^{-1}.$$

The constant σ_0 is such chosen that the exponentially decaying factor satisfies

$$\omega = e^{-\gamma_0 k} = e^{-\frac{\sigma_0 \min(d_1^2, d_2^2)}{\sqrt{(L_1+d_1)^2+(L_2+d_2)^2}}k} \leq 10^{-8}, \tag{7.1}$$

which makes the PML error negligible compared with the finite element discretization error.

Once the PML region and the medium property are fixed, we use the standard finite element adaptive strategy to modify the mesh according to the posterior error estimate with the local error estimate η_k in (6.2), we define the global a posteriori error estimate

$$\mathcal{E} := \left(\sum_{K\in\mathcal{M}_h} \eta_K^2\right)^{1/2}.$$

Now we introduce the adaptive new anisotropic PML algorithm used in this paper

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Algorithm 7.1. Given tolerance TOL>0 and the initial mesh $\mathcal{M}_0$. Set $\mathcal{M}_h = \mathcal{M}_0$.

(1) Solve the discrete problem (6.1) on $\mathcal{M}_0$.

(2) Compute the local error estimator η_K on each $K \in \mathcal{M}_0$ and the global a posteriori error estimate $\mathcal{E}$.

(4) While $\mathcal{E} >$ TOL do

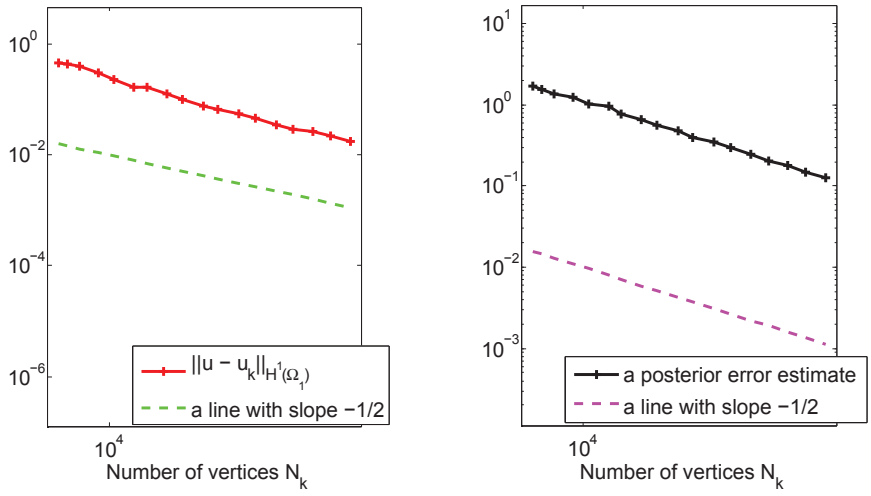
- Refine elements in $\hat{\mathcal{M}}_h \subset \mathcal{M}_h$, where $\hat{\mathcal{M}}_h$ is the minimum subset of $\mathcal{M}_h$ such that

$$\left(\sum_{K \in \hat{\mathcal{M}}_h} \eta_K^2 \right)^{1/2} \geq \frac{1}{2} \mathcal{E}.$$

- Solve the discrete problem (6.1) on $\mathcal{M}_h$.
- Computer the local error estimators η_K on each $K \in \mathcal{M}_h$ and the global a posteriori error estimate $\mathcal{E}$.

end while.

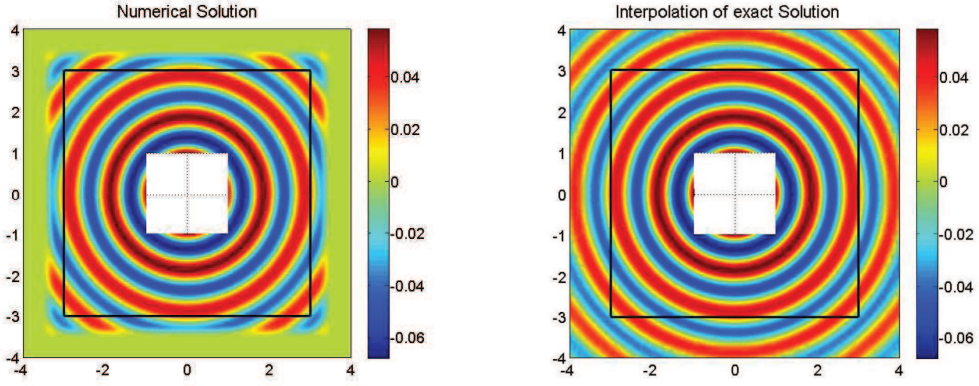
Example 7.1. Let the scatter $D = (2,2)^2$, take $L_1 = 6, L_2 = 6$, the thickness of PML layer $d_1 = 1, d_2 = 1$ and the wavenumber $k = 2\pi$. Fig. 1 shows the setting of domain and the direction of the vector field along which the PML coordinate stretching is performed. We consider the scattering problem whose exact solution is known: $u = H_0^{(1)}(k|x|)$ and $g = H_0^{(1)}(k|x|)\big|_{\Gamma_D}$. We choose the PML parameters $\zeta = 1$, $\sigma_0 = 22$



(a) (b)

Figure 2: (a) The quasi-optimality of the adaptive mesh refinements of the error $||u - u_k||_{H^1(\Omega_1)}$; (b) The quasi-optimality of the adaptive mesh refinements of the a posteriori error estimate.

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(a) (b)

Figure 3: (a) The numerical solution observed from the direction [0,0,1]; (b) The interpolation of exact solution observed from the direction [0,0,1].

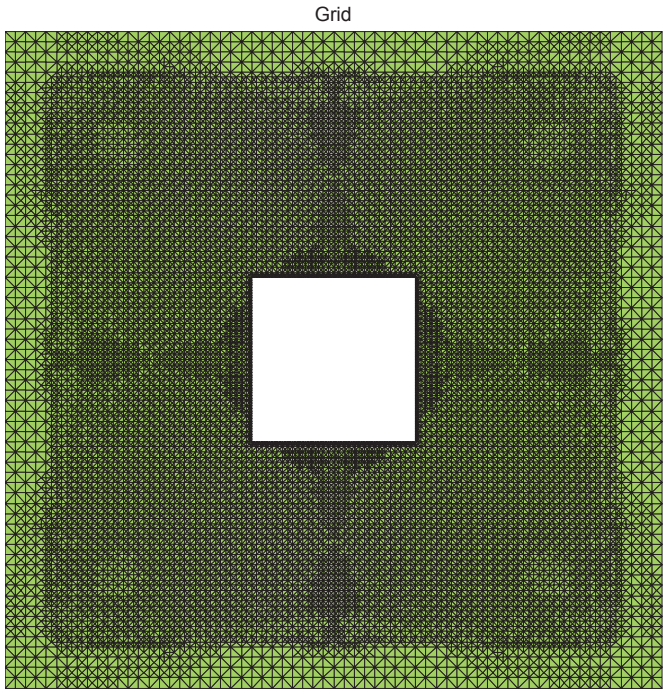


Figure 4: The adaptive mesh with 788,364 nodes.

and $r_0 = 0.8$ to ensure (7.1). We use the complex coordinate stretching transform in Section 2.1 to define the anisotropic PML problem.

Fig. 2 shows log N_k-log $\|u - u_h\|_{H^1(\Omega_1)}$ and log N_k-log $\mathcal{E}$ curves, where N_k is the number of nodes of the mesh $\mathcal{M}_k$ and u_k is the finite element solution over the mesh $\mathcal{M}_k$. It indicates clearly that the mesh and associated numerical complexity are quasi-optimal: $\|u - u_h\|_{H^1(\Omega_1)} \approx CN_k^{1/2}$ and $\mathcal{E} \approx CN_k^{1/2}$ are valid asymptotically.

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Fig. 4 show the adaptive mesh with $788,364$ DOFs. We observe the mesh is very coarse away the inner boundary of the PML layer which is the advantage of adaptive PML method. We also report the real part of numerical and the interpolation of exact solutions in Fig. 3.

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