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A note on random walks in multidimensional time

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1. *Introduction and the result*

Let K_r denote the set of r-tuples $\mathbf{n} = (n_1, n_2, \ldots, n_r)$, $r \geqslant 1$, where the components n_i are positive integers. Let $\{X, X_{\mathbf{n}}, \mathbf{n} \in K_r\}$ be a family of independent and identically distributed random variables with positive mean $EX = \mu < +\infty$ and finite variance $VX = \sigma^2 < +\infty$. In a recent work, M. Maejima and T. Mori [2] have shown that, if X is integer valued, aperiodic and $E|X|^3 < +\infty$, then, for $r = 2$ or 3,

$$\sum_{\mathbf{n} \in K_r} P(S_{\mathbf{n}} = k) \sim \frac{1}{(r-1)!\,\mu} (\log k)^{r-1} \quad \text{as} \quad k \to +\infty, \tag{1}$$

where

$$S_{\mathbf{n}} = \Sigma_{\mathbf{n}} X_{\mathbf{j}} \quad (\mathbf{n} \in K_r), \tag{2}$$

the summation being extended over all members $\mathbf{j} = (j_1, j_2, \ldots, j_r)$ of K_r that satisfy $j_t \leqslant n_t$ for all $1 \leqslant t \leqslant r$.

Note that if, for positive integers n, $d_r(n)$ is the number of solutions of $n = n_1 n_2 \ldots n_r$, $n_i \geqslant 1$ integer, then

$$Q(k) = \sum_{\mathbf{n} \in K_r} P(S_{\mathbf{n}} = k) = \sum_{n=1}^{+\infty} d_r(n) P(S_n = k), \tag{3}$$

where $S_n = X_1 + X_2 + \ldots + X_n$ is the ordinary partial sum of n independent copies X_j of X. The limitation $r = 2$ or 3 in the result of Maejima and Mori is due to the lack of good asymptotic results for the sum of $d_r(n)$ over n. While this same lack of knowledge in number theory imposes limitations on our results, by a new method of proof we establish the following extension of the result of Maejima and Mori.

THEOREM. *Let X be integer valued, aperiodic and $E|X|^3 < +\infty$. Then there is a polynomial*

$$s(u) = \frac{1}{(r-1)!\,\mu} u^{r-1} + b_{r-2} u^{r-2} + \ldots + b_0$$

such that, as $k \to +\infty$,

$$Q(k) = s(\log k\mu^{-1}) + R_k + O(1),$$

where $R_k \to 0$ for $r = 2$ and 3. Furthermore, for arbitrary r,

$$\frac{1}{N} \sum_{k=N+1}^{2N} |R_k| = O(1) \quad \text{as} \quad N \to +\infty,$$

implying, as $N \to +\infty$,

$$\limsup \frac{1}{N} \sum_{k=N+1}^{2N} |Q(k) - s(\log k\mu^{-1})| < +\infty.$$

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In the course of the proof we shall quote a method of explicitly computing the coefficients b_j of $s(u)$. Hence, not only does the first part of the Theorem reproduce the result of Maejima and Mori, but an extension is obtained through explicit error terms in their asymptotic result.

The second part of the Theorem gives an 'average behaviour' of $Q(k)$, valid in all dimensions. It implies that, for a large percentage of large k's, the first part of the Theorem holds. In fact, it follows from the Theorem that if A is the set of those positive integers k for which

$$|Q(k) - s(\log k\mu^{-1})| \geqslant \epsilon \log\log k, \quad \text{say}, \tag{4}$$

where $\epsilon > 0$ is an arbitrary fixed number, then, as $N \to +\infty$,

$$\lim (1/N) \operatorname{card}\{k: k \in A, N < k \leqslant 2N\} = 0. \tag{5}$$

Another consequence of the Theorem is that if its conditions are satisfied then, for $r \geqslant 3$,

$$\sum_{k=T}^{+\infty} \frac{1}{k} \left| \frac{Q(k)}{s(\log k\mu^{-1})} - 1 \right| < +\infty, \tag{6}$$

where T is an integer satisfying $T > \mu$. Indeed, since the major term of $s(\log k\mu^{-1})$ is of the magnitude of $(\log k)^{r-1}$, we get from the Theorem that, for $N \geqslant \mu$,

$$\frac{1}{N} \sum_{k=N+1}^{2N} \left| \frac{Q(k)}{s(\log k\mu^{-1})} - 1 \right| \leqslant \frac{C}{(\log N)^{r-1}}. \tag{7}$$

Here, and throughout the paper, C denotes some positive constant with possibly different values at different places. Now, by choosing $N = 2^t$, $t = 1, 2, \ldots$, in (7), and summing with respect to t, we get (6).

2. *Further notations and some lemmas*

Put

$$D_r(x) = \sum_{n \leqslant x} d_r(n).$$

It is known (see Titchmarsh [5], p. 263) that

$$D_r(x) = x p_r(\log x) + \Delta_r(x), \tag{8}$$

where

$$p_r(u) = \frac{1}{(r-1)!} u^{r-1} + a_{r-2} u^{r-2} + \ldots + a_0, \tag{9}$$

whose coefficients a_j can explicitly be evaluated from the Laurent series of the Riemann zeta function at $s = 1$ (see Lavrik [1]), and $\Delta_r(x) = O(x^v)$ with some $0 < v < 1$. In order to prove (1) by the method of Maejima and Mori one would need the estimate $v \leqslant \frac{1}{2}$. But, since such an estimate is unavailable, we need a significant departure from their method. However, we utilize the same form of the central limit theorem (with error term) as Maejima and Mori do, which we state below.

LEMMA 1. *Let*

$$\phi_n(k) = \phi_n(k; \mu, \sigma^2) = \frac{1}{(2\pi n)^{\frac{1}{2}} \sigma} \exp\left\{ -\frac{(k - n\mu)^2}{2n\sigma^2} \right\},$$

and

$$\xi_{n,k} = (n\mu - k)/\sigma n^{\frac{1}{2}}.$$

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Then

$$P(S_n = k) = \phi_n(k; \mu, \sigma^2) + \frac{\beta}{6\sigma^2 n^{\frac{1}{2}}}(\xi_{n,k}^3 - 3\xi_{n,k})\,\phi_n(k; \mu, \sigma^2) + \frac{c_{n,k}}{n(1+|\xi_{n,k}^3|)},$$

where $\beta = E(X-\mu)^3$ *and* $c_{n,k}$ *are constants such that* $c_n = \sup_k |c_{n,k}| \to 0$ *as* $n \to +\infty$.

For its proof, see Petrov [3], p. 207.

LEMMA 2. *With the notations of Lemma* 1,

$$|\xi_{n,k}^3 - 3\xi_{n,k}|\,\phi_n(k; \mu, \sigma^2) \leqslant C\phi_n(k; \mu, 2\sigma^2).$$

The proof is evident.

LEMMA 3. *With an arbitrary constant* $\epsilon > 0$,

$$\frac{1}{T}\int_T^{2T} |\Delta_r(x+h) - \Delta_r(x)|^2\,dx \ll h^2 \exp[-(\log T)^{\frac{1}{3}}] + T^{\frac{1}{3}+\epsilon}.$$

For its proof, see Ramachandra [4].

LEMMA 4. *Let* n *be a positive integer satisfying*

$$|n - k/\mu| \leqslant Ck^{\frac{1}{3}}(\log k)^2. \tag{10}$$

Then, for $1 \leqslant j \leqslant k^{\frac{1}{3}}$, *there is an* s (*not necessarily an integer*) *such that* $n \leqslant s \leqslant n+j$, *and*

$$|\phi_{n+j}(k) - \phi_n(k)| \leqslant j(n^{-1} + |\xi_{n,k}|\,n^{-\frac{1}{2}})\,\phi_n(k) + [j^2(\log k)^4/k]\,\phi_s(k).$$

Consequently, for $1 \leqslant M \leqslant k^{\frac{1}{3}}$,

$$\sum_{j=1}^{M} |\phi_{n+j}(k) - \phi_n(k)| \ll M^2(n^{-1} + |\xi_{n,k}|\,n^{-\frac{1}{2}})\,\phi_n(k) + M^3(\log k)^4\,k^{-\frac{3}{2}}.$$

(Here, and throughout the paper, unless otherwise stated, asymptotics are for $k \to +\infty$.)

Proof. Define $f(u) = \phi_u(k; \mu, \sigma^2)$ with all real numbers as its domain. Then $f(u)$ is differentiable, and we have

$$\frac{df(u)}{du} = g_k(u)f(u), \tag{11}$$

where $$g_k(u) = (1/2u)[\xi_{u,k}(\mu u^{\frac{1}{2}} - ku^{-\frac{1}{2}}) - 1].$$

Thus, under (10), $$|g_k(n)| \ll (1/n) + |\xi_{n,k}|\,n^{-\frac{1}{2}} \tag{12a}$$

and $$|g_k(n)| \ll (\log k)^2\,k^{-\frac{1}{2}}. \tag{12b}$$

Note that if n satisfies (10) then so does $n+j$ with $j \leqslant k^{\frac{1}{3}}$. Hence, when applying the mean value theorem to

$$f(n+j) - f(n) = \phi_{n+j}(k) - \phi_n(k),$$

we can utilize either of the estimates in (12a, b). Now, writing

$$\begin{aligned} |f(u) - f(v)| &= |u-v|\,f'(w) = |u-v|\,g_k(w)f(w) \\ &\leqslant |u-v|\,|g_k(w)|\,f(v) + |u-v|\,|g_k(w)|\,|f(w) - f(v)| \\ &\leqslant |u-v|\,|g_k(w)|\,f(v) + (u-v)^2\,|g_k(w)|\,|g_k(s)|\,f(s), \end{aligned}$$

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where w and s are some values between u and v, and v and w, respectively, we get the first inequality of the Lemma by choosing $u = n+j$ and $v = n$, and appealing to $(12a, b)$. The second inequality then follows by the rough estimate $\phi_s(k) \ll k^{-\frac{1}{2}}$, and by summation over j. The proof is completed.

3. *Proof of the Theorem*

We estimate $Q(k)$ in several steps.

Let

$$\left.\begin{aligned} Q_1(k) &= \sum_{n=1}^{+\infty} d_r(n)\,\phi_n(k), \\ S_1(k) &= \sum_{n=1}^{+\infty} \frac{d_r(n)}{n^{\frac{1}{2}}}\,\phi_n(k;\mu,2\sigma^2), \end{aligned}\right\} \tag{13}$$

and

$$S_2(k) = \sum_{n=1}^{+\infty} \frac{d_r(n)}{n(1+|\xi_{n,k}^3|)}.$$

On account of Lemmas 1 and 2,

$$|Q(k) - Q_1(k)| \ll S_1(k) + S_2(k). \tag{14}$$

We want to show that the right-hand side of (14) is bounded in k, yielding the relation

$$Q(k) = Q_1(k) + O(1). \tag{15}$$

We split the sums in both $S_1(k)$ and $S_2(k)$ according as $n \in I_k$ or $n \notin I_k$, where

$$I_k = [k/\mu - Ck^{\frac{3}{4}},\ k/\mu + Ck^{\frac{3}{4}}].$$

Note that, if $n \notin I_k$,

$$|\xi_{n,k}| > Ck^{\frac{3}{4}}n^{-\frac{1}{2}}, \quad |\xi_{n,k}| > Cn^{\frac{1}{4}}, \tag{16}$$

where the first inequality is evident from the definition of $\xi_{n,k}$ and of I_k, while the second is a direct consequence of the first one if, in addition to $n \notin I_k$, we also assume $n \leqslant (2/\mu)k$. On the other hand, if $n > (2/\mu)k$, then

$$n\mu - k = \mu[n - (k/\mu)] > \mu(n - \tfrac{1}{2}n) = \tfrac{1}{2}\mu n,$$

and thus the second inequality of (16) follows from the definition of $\xi_{n,k}$ (recall that C is a generic constant). We also utilize $\phi_n^*(k) \ll n^{-\frac{1}{2}}$ (uniformly), where

$$\phi_n^*(k) = \phi_n(k;\mu,2\sigma^2),$$

and $d_r(n) \leqslant n^{\epsilon}$ with arbitrary $\epsilon > 0$. Thus

$$\sum_{n\in I_k} \frac{d_r(n)}{n^{\frac{1}{2}}}\,\phi_n^*(k) \ll \sum_{n\in I_k} \frac{d_r(n)}{n} \ll k^{-1+\epsilon}k^{\frac{3}{4}}. \tag{17}$$

For $n \notin I_k$ we appeal to (16) when estimating $\phi_n^*(k)$. We get $\phi_n^*(k) \ll \exp(-Cn^{\frac{1}{4}}) < n^{-2}$, and thus the contribution of the terms $n \notin I_k$ to $S_1(k)$ is smaller than the sum for all n of $n^{-2\cdot5+\epsilon}$, which is indeed bounded and not dependent on k. A completely similar estimate applies to $S_2(k)$, establishing (15).

As a further approximation to $Q(k)$, we now limit the number of terms in $Q_1(k)$. In fact, defining

$$J_k = [k/\mu - k^{\frac{1}{2}}(\log k)^2,\ k/\mu + k^{\frac{1}{2}}(\log k)^2], \tag{18}$$

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and
$$Q_2(k) = \sum_{n \in J_k} d_r(n)\, \phi_n(k), \tag{19}$$
an easy estimate leads to
$$Q(k) = Q_2(k) + O(1). \tag{20}$$

Namely, if $n \notin J_k$, $|\xi_{n,k}| > C(\log n)^2$, because, for $n \notin J_k$ and $n \leqslant (2/\mu)\, k$,

$$|\xi_{n,k}| > (\mu/\sigma)\, k^{\frac{1}{2}} (\log k)^2\, n^{-\frac{1}{2}} > C(\log n)^2,$$

while for the case $n > (2/\mu)\, k$ we established, following (16), that $|\xi_{n,k}| > Cn^{\frac{1}{4}}$ (with no additional restriction on n), which is of larger magnitude than $(\log n)^2$; therefore, for $n \notin J_k$,
$$\phi_n(k) \ll n^{-\frac{1}{2}} \exp\left(-\tfrac{1}{2}\xi_{n,k}^2\right) \ll n^{-C}$$
with arbitrary $C > 0$. Consequently,

$$\sum_{n \notin J_k} d_r(n)\, \phi_n(k) \ll \sum_{\text{all } n} n^{\varepsilon} n^{-C},$$

which converges for $C \geqslant 2$, and thus (15) implies (20).

Next, we write

$$Q_2(k) = \sum_{n \in J_k} d_r(n) \left\{ \frac{1}{M} \sum_{j=0}^{M-1} \left(\phi_n(k) - \phi_{n+j}(k) + \phi_{n+j}(k)\right) \right\},$$

and put
$$Q_3(k) = \sum_{n \in J_k} d_r(n) \left\{ \frac{1}{M} \sum_{j=0}^{M-1} \phi_{n+j}(k) \right\}, \tag{21}$$

where $M > 0$ is an integer to be specified later. Now, evidently

$$|Q_2(k) - Q_3(k)| \ll \sum_{n \in J_k} d_r(n) \frac{1}{M} \sum_{j=1}^{M-1} |\phi_n(k) - \phi_{n+j}(k)|,$$

the right-hand side of which can be estimated by an appeal to Lemma 4 if we impose $M \leqslant k^{\frac{1}{3}}$. We get

$$|Q_2(k) - Q_3(k)| \ll M \sum_{n \in J_k} d_r(n) \left(\frac{1}{n} + |\xi_{n,k}| n^{-\frac{1}{2}}\right) \phi_n(k) + M^2 (\log k)^4\, k^{-\frac{3}{2}} \sum_{n \in J_k} d_r(n).$$

But since $M \leqslant k^{\frac{1}{3}}$, and, for $n \in J_k$, $d_r(n) \leqslant k^{\varepsilon}$, $|\xi_{n,k}| \ll (\log k)^2$ and $\phi_n(k) \ll k^{-\frac{1}{2}}$, furthermore, the number of terms in J_k is $2k^{\frac{1}{2}}(\log k)^2$, the right-hand side of the last inequality is obtained to be of the magnitude of $k^{-\frac{1}{6}+\varepsilon}$; i.e.

$$|Q_2(k) - Q_3(k)| \ll k^{-\frac{1}{6}+\varepsilon},$$

which, on account of (20), yields

$$Q(k) = Q_3(k) + O(1). \tag{22}$$

Let us interchange the summations in (21). We put $m = n + j$, and, for simplicity of notation, we denote the endpoints of J_k by A_k and B_k (see (18)). We have

$$Q_3(k) = \frac{1}{M} \sum_{m=A_k}^{B_k+M-1} \phi_m(k)\, \Sigma^*\, d_r(n),$$

where in Σ^* summation is over n with $m - M + 1 \leqslant n \leqslant m$ and $n \in J_k$. Note that, if $B_k < m \leqslant B_k + M - 1$, $\phi_m(k) \ll \exp\left[-\frac{1}{2}(\log k)^2\right] \ll k^{-C}$ with arbitrary $C > 0$. Hence, the rough estimate $d_r(n) \ll k^{\varepsilon}$ yields

$$\frac{1}{M} \sum_{m=B_k}^{B_k+M-1} \phi_m(k)\, \Sigma^*\, d_r(n) \ll M k^{\varepsilon - C} \leqslant k^{\frac{1}{3}+\varepsilon-C}, \tag{23}$$

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and thus

$$Q_3(k) = \frac{1}{M} \sum_{m \in J_k} \phi_m(k) \Sigma^* d_r(n) + O(1).$$

Now, if $m \in J_k$ then $n \in J_k$ is superfluous in Σ^* except when $A_k \leqslant m \leqslant A_k + M$. However, whether we ignore $n \in J_k$ or not for these latter values of m makes very little difference in $Q_3(k)$, which follows from (23) upon replacing B_k by A_k. We have thus proved that, with

$$Q_4(k) = \frac{1}{M} \sum_{m \in J_k} \phi_m(k) \sum_{n=m-M+1}^{m} d_r(n), \tag{24}$$

$$Q_3(k) = Q_4(k) + O(1),$$

or, on account of (22),

$$Q(k) = Q_4(k) + O(1). \tag{25}$$

Because

$$\sum_{n=m-M+1}^{m} d_r(n) = D_r(m) - D_r(m-M),$$

in view of (8), $Q_4(k)$ of (24) can also be written as

$$Q_4(k) = Q_5(k) + R_M(k), \tag{26}$$

where

$$Q_5(k) = \frac{1}{M} \sum_{m \in J_k} \phi_m(k) \{m p_r(\log m) - (m-M) p_r(\log (m-M))\} \tag{27}$$

and

$$R_M(k) = \frac{1}{M} \sum_{m \in J_k} \phi_m(k) \{\Delta_r(m) - \Delta_r(m-M)\}. \tag{28}$$

We simplify $Q_5(k)$. Writing

$$\log (m-M) = \log m + \log (1 - M/m) = \log m + \lambda_m, \quad \text{say},$$

where

$$\lambda_m = -\frac{M}{m} + O\left(\frac{M^2}{m^2}\right),$$

and, with some m^* such that $\log m + \lambda_m \leqslant \log m^* \leqslant \log m$,

$$p_r(\log m + \lambda_m) = p_r(\log m) + \lambda_m p_r'(\log m) + c\lambda_m^2 p_r''(\log m^*)$$

$$= p_r(\log m) - \frac{M}{m} p_r'(\log m) + O\left\{\frac{M^2}{m^2} (\log m)^{r-2}\right\},$$

we get

$$m p_r(\log m) - (m-M) p_r(\log (m-M))$$

$$= M\{p_r(\log m) + p_r'(\log m)\} + O\left\{\frac{M^2}{m} (\log m)^{r-2}\right\}.$$

This last expression and (27) thus yield

$$Q_5(k) = Q_6(k) + O\{M \sum_{m \in J_k} [\phi_m(k)/m] (\log m)^{r-2}\}, \tag{29}$$

where

$$Q_6(k) = \sum_{m \in J_k} \phi_m(k) \{p_r(\log m) + p_r'(\log m)\}. \tag{30}$$

But, with the usual estimate $\phi_m(k) \ll k^{-\frac{1}{2}}$ for $m \in J_k$, the error term in (29) is easily seen to be of the magnitude $M k^{-\frac{3}{2}} (\log k)^{r-2}$ times the length of J_k, which is bounded in k. Hence (25), (26), (29) and this last estimate imply

$$Q(k) = Q_6(k) + R_M(k) + O(1). \tag{31}$$

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From the elementary inequality

$$|a^s - b^s| \ll |a-b| \max(|a|^{s-1}, |b|^{s-1}) \quad (1 \leqslant s \leqslant r-1),$$

one can immediately see that

$$\sum_{m \in J_k} \phi_m(k) (\log m)^s - \sum_{m \in J_k} \phi_m(k) [\log (k/\mu)]^s = o(1),$$

and thus

$$Q_6(k) = \left\{ p_r\left(\log \frac{k}{\mu}\right) + p_r'\left(\log \frac{k}{\mu}\right) \right\} \sum_{m \in J_k} \phi_m(k) + O(1). \tag{32}$$

We put

$$s(u) = (1/\mu)\{p_r(u) + p_r'(u)\}. \tag{33}$$

By proving

$$\sum_{m \in J_k} \phi_m(k) = \frac{1}{\mu} + O((\log k)^{-r+1}) \tag{34}$$

and putting $R_k = R_M(k)$, (31) through (34) will complete the proof of the Theorem if we show that

$$R_M(k) \to 0 \quad \text{as} \quad k \to +\infty \quad \text{if} \quad r = 2 \text{ or } 3 \tag{35}$$

and, for arbitrary r,

$$\frac{1}{N} \sum_{k=N+1}^{2N} |R_M(k)| = O(1) \quad \text{as} \quad N \to +\infty. \tag{36}$$

The proof of (34) is quite routine at this stage. The easiest way to proceed is to change the definition of $\phi_m(k)$ to

$$\frac{1}{\{2\pi(k/\mu)\}^{\frac{1}{2}} \sigma} \exp\left\{ -\frac{1}{2} \left[\frac{k - m\mu}{(k/\mu)^{\frac{1}{2}} \sigma} \right]^2 \right\},$$

and first to show that the left hand side of (34) is changed less than the error term of the right-hand side. The advantage of this new expression is that the variable m occurs in it only once. The sum over m can now be approximated by an ordinary normal integral, leading to (34). We omit the details.

Next, we prove (35). By rearranging the terms in (28), we get

$$R_M(k) = \frac{1}{M} \left\{ \sum_{m \in J_k} \Delta_r(m) [\phi_m(k) - \phi_{m+M}(k)] + U_M(k) \right\},$$

where $U_M(k)$ contains M terms of the form $-\phi_m(k)\,\Delta_r(m-M)$ from the lower end, and M terms of the form $\phi_m(k)\,\Delta_r(m)$ from the upper end of J_k. But, for such m,

$$\phi_m(k) \ll k^{-\frac{1}{2}} \exp[-C(\log k)^4],$$

and thus even $|\Delta_r(m)| \leqslant k$ would make $(1/M)\,U_M(k)$ tending to zero as $k \to +\infty$. Turning to the main term of $R_M(k)$, we apply Lemma 4 with $n = m \in J_k$ and $j = M \leqslant k^{\frac{1}{3}}$. Since, for $m \in J_k$, $|\xi_{m,k}| < C(\log k)^2$, and both $\phi_m(k)$ and $\phi_s(k)$ are majorized by $Ck^{-\frac{1}{2}}$, the known estimate $\Delta_r(m) = O(k^{37/75+\epsilon})$ for $r = 2$ and 3 (see [5], pp. 267 and 270) yields

$$R_M(k) \ll k^{\frac{1}{2}} (\log k)^2 k^{37/75+\epsilon} (\log k)^2 / k,$$

where $\epsilon > 0$ is arbitrary. Hence, $R_M(k)$ indeed goes to zero as $k \to +\infty$, establishing (35).

Finally, for proving (36) we note

$$\sum_{k=N+1}^{2N} |R_M(k)| \leqslant \frac{1}{M} \sum_{k=N+1}^{2N} \sum_{m \in J_k} \phi_m(k) |\Delta_r(m) - \Delta_r(m-M)|$$

$$\leqslant \frac{1}{M} \sum_{m \in V_N} |\Delta_r(m) - \Delta_r(m-M)| \sum_{k=N+1}^{2N} \phi_m(k),$$

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where $V_N = [N/2\mu, 4N/\mu]$. In the last inequality we interchanged the summations, and extended the range of m to V_N. Note that the sum of $\phi_m(k)$ over k is bounded uniformly in m; consequently, the inner sum on the extreme right-hand side can be dropped. Thus, by the Cauchy–Schwarz inequality,

$$\sum_{k=N+1}^{2N} |R_M(k)| \ll \frac{1}{M}\left(\sum_{m\in V_N} 1\right)^{\frac{1}{2}}\left(\sum_{m\in V_N} |\Delta_r(m) - \Delta_r(m-M)|^2\right)^{\frac{1}{2}},$$

from which, by splitting V_N into three disjoint intervals of the form $(T, 2T)$, Lemma 3 yields

$$\sum_{k=N+1}^{2N} |R_M(k)| \ll \frac{1}{M} N^{\frac{1}{2}}\{N(M^2 \exp[-(\log cN)^{\frac{1}{3}}] + N^{\frac{1}{3}+\epsilon})\}^{\frac{1}{2}},$$

where $c = 1/2\mu$. Because the only restriction on M so far was $M \leqslant k^{\frac{1}{3}}$, and here $N < k \leqslant 2N$, with the choice $M = N^{\frac{1}{3}}$ (36) now follows. The proof is completed.

Recall that we also gave the explicit form of $s(u)$ at (9) and (33).

In conclusion let us remark that the density version of the result of Maejima and Mori can also be extended to arbitrary r to a form similar to the one in our Theorem.

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