Proc. Camb. Phil. Soc. (1970), **68**, 67
PCPS 68–9
Printed in Great Britain

Spaces of cuts

By JOAN GORDON ZABEEH
Queen Elizabeth College, University of London

(*Received* 17 *May* 1968)

Introduction. We will study here for normal T_1 spaces the space of cuts—defined and analysed by Hurewicz (3) for separable metrizable spaces.

A cut of the topological space $(X, \mathscr{T})$ is an unordered pair (A, B) of non-empty subsets A and B of X such that $A = \overline{X - B}$ and $B = \overline{X - A}$, where the bar above denotes closure. Let A/B denote the cut (A, B) of $(X, \mathscr{T})$ and $S(X)$ the set of all cuts of $(X, \mathscr{T})$.

$S(X)$ is topologized as follows. The subsets M and N of $(X, \mathscr{T})$ are said to be separated by the cut A/B if and only if

$$M \cap B = 0 \quad \text{and} \quad N \cap A = 0$$

or

$$M \cap A = 0 \quad \text{and} \quad N \cap B = 0.$$

The collection $\{U_{M,N}\}$ of all sets of the form

$$U_{M,N} = \{A/B \in S(X) : A/B \text{ separates } M \text{ and } N;\ M = \bar{M}, N = \bar{N}\}$$

is a base for a topology for $S(X)$. Since this is the only topology we will consider on $S(X)$ we will not refer to it explicitly again but simply speak of the space $S(X)$.

Hurewicz begins by assuming that $(X, \mathscr{T})$ is separable and metrizable. He proves $S(X)$ is T_2, asserts it is regular, and shows it is Baire. We will generalize these results to the case where $(X, \mathscr{T})$ is normal and T_1 establishing that $S(X)$ is T_2 and regular by embedding it in a quotient space and showing it is subcompact and hence Baire by reason of the generalized Baire theorem of de Groot (1).

1. *Properties of cuts.* All the following results are easily proved (using well known identities, e.g. $\overline{(X - A)} = X - A^\circ$ and $[(F^\circ)^-]^\circ = F^\circ$ when F is closed).

1·1. Theorem. *Let A and B be subsets of the space $(X, \mathscr{T})$. The pair (A, B) is a cut of $(X, \mathscr{T})$ if and only if*

(1) $A = \bar{A}$, $B = \bar{B}$; $A \neq X$, $B \neq X$; $A \cup B = X$.

(2) $F(A) = F(B)$, *where $F(A')$, A' is a subset of the space $(X, \mathscr{T})$, means the boundary of A'*; $A^\circ \cap B = 0$.

1·2. Lemma. *If the pair (A, B) is a cut of the space $(X, \mathscr{T})$, then $A^\circ = X - B$ and $B^\circ = X - A$.*

1·3. *Remark.* Let A be a subset of the space X. Then the pair $(A, \overline{X - A})$ is not a cut unless both $A^\circ \neq 0$ and $X - A \neq 0$. It happens that in the space $(X, \mathscr{T})$, where X is infinite and $\mathscr{T}$ is the finite complement topology, the only closed set with a non-empty interior is the whole space. So there are no cuts in this space.

Note. It follows from 1·2 Lemma that

$$U_{N,M} = \{A/B \in S(X) : M \subset A^\circ \text{ and } N \subset B^\circ \text{ or } M \subset B^\circ \text{ and } N \subset A^\circ;\ M = \bar{M}, N = \bar{N}\}.$$

https://doi.org/10.1017/S0305004100001067 Published online by Cambridge University Press

1·4. THEOREM. *Let A be a subset of a space X. Then the pair $(A, \overline{X-A})$ is a cut of the space X if and only if $A \neq X, A \neq 0$ and $A = \overline{A^\circ}$.*

2. *Embedding $S(X)$ in a quotient space $\pi(X)$.* Let $(X, \mathscr{T})$ be a topological space.

2·1. *Definition.* $2^X = \{A \subset X: 0 \neq A = \bar{A}\}$.

2·2. *Definition.* The Vietoris topology $\mathscr{V}$ for 2^X ((4), p. 160) has as a base the collection

$$\{2^X \cap 2^U \cap 2_{V_1} \cap \ldots \cap 2_{V_n}: U \text{ and } V \text{ are open in } X\},$$

where, for $U, V \subset X$, $2^U = \{A \subset U: 0 \neq A = \bar{A}(\mathscr{T}_U)\}$, thus

$$2^X \cap 2^U = \{A \in 2^X: A \subset U\} \quad \text{and} \quad 2_V = \{A \in 2^X: A \cap V \neq 0\}.$$

2·3. *Definition.* We define an equivalence relation $\sim$ on $2^X \times 2^X$ as follows: $(A, B) \sim (A', B')$ if and only if $A = A'$ and $B = B'$ or $A = B'$ and $B = A'$. (Thus $(A, B) \sim (B, A)$.) Let $R[A, B]$ stand for the equivalence set $\{(A', B'): (A', B') \sim (A, B)\}$.

2·4. *Definition.* Let the quotient set of $2^X \times 2^X$ by the relation $\sim$ be denoted by $\pi(X)$. Let $\alpha: 2^X \times 2^X \to \pi(X)$ be the natural projection. Assign $\pi(X)$ the quotient topology relative to α and the product topology for $2^X \times 2^X$.

Since this is the only topology we will consider on $\pi(X)$ we will simply speak of the space $\pi(X)$ henceforth.

2·5. THEOREM. *The natural projection $\alpha: 2^X \times 2^X \to \pi(X)$, defined by $\alpha((A, B)) = R[A, B]$, is both open and closed.*

Proof. α is open (closed). Let K be open (closed) in $2^X \times 2^X$. $\alpha(K)$ is open (closed) in $\pi(X)$ if $\alpha^{-1}(\alpha(K))$ is open (closed) in $2^X \times 2^X$ by the definition of quotient topology.

$$\alpha^{-1}(\alpha(K)) = \{(A, B): (A, B) \in K \text{ or } (B, A) \in K\} = K \cup \Delta K,$$

where $\Delta: 2^X \times 2^X \to 2^X \times 2^X$ is defined by $\Delta(X, Y) = (Y, X)$. Δ is a homeomorphism, so Δ is open (closed); hence $K \cup \Delta K$ is open (closed).

2·6. THEOREM. *If $(X, \mathscr{T})$ is T_1 and normal, then $\pi(X)$ is Hausdorff and regular.*

Proof. (i) $\pi(X)$ is Hausdorff. $2^X \times 2^X$ is T_1 and regular by the known result that if X is T_1 and normal, then 2^X is T_1 and regular. It then follows that $\pi(X)$ is T_2 by 2·5 Theorem and the known result that if X is a T_1 and regular space and the projection α of X onto X/R the quotient space is both open and closed, then X/R is T_2 ((2), p. 143).

(ii) $\pi(X)$ is regular. Let K be a closed set in $\pi(X)$ and let $R[A, B] \notin K$. $\alpha^{-1}(K)$ is closed in $2^X \times 2^X$. Since $R[A, B] \notin K$, then (A, B) and $(B, A) \notin \alpha^{-1}(K)$. There is an open neighbourhood V of (A, B) such that $\overline{V} \cap \alpha^{-1}(K) = 0$ and so $\alpha(\overline{V}) \cap K = 0$. $\alpha(V)$ is open in $\pi(X)$ and $\alpha((A, B)) = R[A, B] \in \alpha(V)$. $\alpha(\overline{V})$ is closed in $\pi(X)$; thus

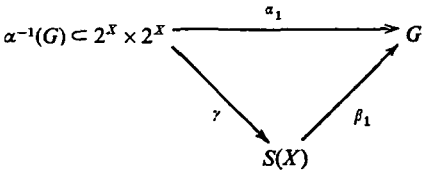
$$\alpha(\overline{V}) = [\alpha(\overline{V})]^-.$$

Thus $\overline{\alpha(V)} \cap K = 0$ and $\alpha(V)$ then is the desired open neighbourhood of $R[A, B]$.

2·7. *Definition.* $\beta: S(X) \to \pi(X)$ by $\beta(A/B) = R[A, B]$. α is the natural projection of $2^X \times 2^X$ onto $\pi(X)$.

https://doi.org/10.1017/S0305004100001067 Published online by Cambridge University Press

2·8. *Definition.* Let $G = \beta(S(X))$; let $\beta_1 : S(X) \to G$ be defined by $\beta_1(A/B) = \beta(A/B)$ and let $\alpha_1 = \alpha|\alpha^{-1}(G)$. Define $\gamma : \alpha^{-1}(G) \to S(X)$ by $\gamma((A,B)) = A/B$ and let $\alpha^{-1}(G)$ have the subspace topology.



Observe that $\beta_1\gamma = \alpha_1$.

2·9. THEOREM. *If X is T_1 and regular, then β is an embedding.*

Proof. (i) β_1 (and hence β) is continuous. Let K be open in G. Then $\alpha_1^{-1}(K)$ is open. We will show that γ is open and since $\beta_1^{-1}(K) = \gamma\alpha_1^{-1}(K)$, it follows that $\beta_1^{-1}(K)$ is open in $S(X)$. Let $\alpha^{-1}(G) \cap (\mathscr{U} \times \mathscr{V})$ be a basic open set in $\alpha^{-1}(G) \subset 2^X \times 2^X$ where

$$\mathscr{U} = 2^X \cap 2^U \cap 2_{V_1} \cap \ldots \cap 2_{V_n}$$

and

$$\mathscr{V} = 2^X \cap 2^{U'} \cap 2_{V'} \cap \ldots \cap 2_{V'_n}.$$

$\mathscr{U}$ and $\mathscr{V}$ are basic open sets in 2^X.

Let $A/B \in \gamma(\alpha^{-1}(G) \cap (\mathscr{U} \times \mathscr{V}))$. Without loss of generality assume $A \in \mathscr{U}$ and $B \in \mathscr{V}$. $A \in \mathscr{U}$ implies that the cut $A/B \in U_{M,N}$ where M is chosen so that M is closed, $M \subset A^\circ$ and $M \cap V_i' \neq 0$ for $i = 1, \ldots, n$, by the regularity of $(X, \mathscr{T})$; and $N = X - U$, so $N = X - U \subset X - A = B^\circ$ (1·2 Lemma). M is constructed as follows:

$$V_i \cap A = V_i \cap A^\circ \neq 0 \quad \text{implies} \quad V_i \cap A^\circ \neq 0 \quad \text{for} \quad i = 1, \ldots, n,$$

where V_i are open, $i = 1, \ldots, n$. For each $i = 1, \ldots, n$, there is a non-empty K_i open in X, such that $\bar{K}_i \subset V_i \cap A^\circ$, by the regularity of X. Let $M = \bigcup_{i=1}^{n} \bar{K}_i$.

$B \in \mathscr{V}$ implies that the cut $A/B \in U_{M',N'}$ where N' is chosen so that N' is closed, $N' \subset B^\circ$ and $N' \cap V_i' \neq 0$ for all $i = 1, \ldots, n$, by the regularity of $(X, \mathscr{T})$; and $M' = X - U'$, so $M' = X - U' \subset X - B = A^\circ$. N' is constructed in a manner similar to M. Thus

$$A/B \in U_{M \cup M', N \cup N'}, \quad \text{i.e.} \quad M \cup M' \subset A^\circ \quad \text{and} \quad N \cup N' \subset B^\circ.$$

$$U_{M \cup M', N \cup N'} \subset \gamma(\alpha^{-1}(G) \cap (\mathscr{U} \times \mathscr{V})).$$

For observe, if $A'/B' \in \mathscr{U}_{M \cup M', N \cup N'}$ then

$$M \subset A'^\circ \quad \text{implies} \quad A' \cap V_i \neq 0 \quad \text{for all} \quad i = 1, \ldots, n$$

and

$$N = X - U \subset B'^\circ = X - A' \quad \text{implies} \quad A' \in \mathscr{U}.$$

By a similar argument we can show $B' \in \mathscr{V}$. Therefore,

$$A'/B' \in \gamma(\alpha^{-1}(G) \cap (\mathscr{U} \times \mathscr{V})).$$

Hence, γ is open.

(ii) β_1 is open. $\beta_1(U_{M,N}) = K$ is open in G if and only if $\alpha_1^{-1}(K)$ is open in $\alpha^{-1}(G)$, i.e., if and only if $\gamma^{-1}(U_{M,N})$ is open in $\alpha^{-1}(G)$. That is, β_1 is, open if and only if γ is continuous. Since $\gamma^{-1}(U_{M,N}) = (2^{X-N} \times 2^{X-M} \cup 2^{X-M} \times 2^{X-N}) \cap \alpha^{-1}(G)$, γ is continuous.

(iii) β_1 is clearly $1-1$.

https://doi.org/10.1017/S0305004100001067 Published online by Cambridge University Press

2·10. COROLLARY. *If $(X, \mathscr{T})$ is T_1 and normal, then $S(X)$ is T_2 and regular.*

Proof. By 2·6 Theorem and 2·9 Theorem.

3. *$S(X)$ is Baire if $(X, \mathscr{T})$ is normal and T_1.* We shall make use of the generalized Baire theorem of de Groot(1) which asserts that a countably subcompact regular T_1-space is Baire. The notion of subcompactness, introduced by de Groot, is defined below.

If $\mathscr{U}$ is a base for the topology of a space X then a non-empty subset $\exists \subset \mathscr{U}$ is a regular filter base relative to $\mathscr{U}$ if the following condition holds: If $U, V \in \exists$ there exists $W \in \exists$ such that $\overline{W} \subset U \cap V$.

A topological space is said to be *countably subcompact* if there exists a base $\mathscr{U}$ with the property that if $\exists$ is any countable regular filter base relative to $\mathscr{U}$ then $\bigcap_{F \in \exists} F \neq 0$.

3·1. LEMMA. *Let $(X, \mathscr{T})$ be a topological space. Then if V and W are non-empty disjoint open subsets of X there is a cut separating them.*

Proof. Since $\overline{V}$ is a regularly closed set, $\overline{V} = [(\overline{V})^\circ]^-$, then $\overline{V}/X - (\overline{V})^\circ$ is a cut separating V and W.

3·2. LEMMA. *Let $(X, \mathscr{T})$ be a normal space. Then if M and N are non-empty disjoint closed subsets of X there is a cut separating them.*

Proof. Since X is normal there are disjoint open subsets V and W of X such that $M \subset V$, $N \subset W$. By 4·1, $\overline{V}/X - (\overline{V})^\circ$ is a cut separating V and W and so separating M and N.

It is easy to prove the following.

3·3. LEMMA. *Let $(X, \mathscr{T})$ be a regular space. If M, N, M', N' are non-empty subsets of X such that $\mathscr{U}_{M,N} \subset \mathscr{U}_{M',N'}$ and $\mathscr{U}_{M,N} \neq 0$, then $M' \subset M$ and $N' \subset N$ (or similarly with M, N interchanged).*

3·4. LEMMA. *Let $(X, \mathscr{T})$ be a normal space. Let V and W be subsets of X such that V and W are non-empty, open and $\overline{V} \cap \overline{W} = 0$. If A/B separates V and W then $A/B \in \overline{U}_{\overline{V},\overline{W}}$.*

Proof. Let $U_{M,N}$ be a neighbourhood of A/B. Without loss of generality assume $M \subset A^\circ$, $N \subset B^\circ$, $\overline{V} \subset A^\circ$ and $W \subset B^\circ$. Thus $(M \cup \overline{V}) \cap (N \cup \overline{W}) = 0$. There is a cut C/D separating $M \cup \overline{V}$ and $N \cup \overline{W}$ (3·2 Lemma). Therefore, C/D separates M and N, hence $C/D \in \mathscr{U}_{M,N}$; and C/D separates $\overline{V}$ and $\overline{W}$, hence $C/D \in U_{\overline{V},\overline{W}}$. Thus $U_{M,N} \cap U_{\overline{V},\overline{W}} \neq 0$; hence $A/B \in \overline{U}_{\overline{V},\overline{W}}$.

3·5. LEMMA. *Let $(X, \mathscr{T})$ be a T_1 normal space. Let V_0, W_0, V_1, W_1 be non-empty open subsets of X such that*

$$\overline{V}_0 \cap \overline{W}_0 = 0, \quad \overline{V}_1 \cap \overline{W}_1 = 0 \quad \text{and} \quad V_1 = \overline{V}_1^0, \quad W_1 = \overline{W}_1^0.$$

If $\overline{U}_{\overline{V}_1,\overline{W}} \subset U_{\overline{V}_0,\overline{W}_0}$ then $\overline{V}_0 \subset V_1$ and $\overline{W}_0 \subset W_1$.

Proof. There is a cut $\overline{V}_1/X - V_1$ separating V_1 and W_1 (3·1 Lemma). Since

$$U_{\overline{V}_1,\overline{W}_1} \subset U_{\overline{V}_0,\overline{W}_0} \quad \text{then} \quad \overline{V}_0 \subset \overline{V}_1 \quad \text{and} \quad \overline{W}_0 \subset \overline{W}_1.$$

(3·3 Lemma), using the appropriate designation. $\overline{V}_1/X - \overline{V}_1 \in U_{\overline{V}_1,\overline{W}_1}$ (3·4 Lemma). By assumption, therefore, $\overline{V}_1/X - \overline{V}_1 \in U_{\overline{V}_0,\overline{W}_0}$. Thus $\overline{V}_0 \subset \overline{V}_1^0 = V_1$.

There is a cut $\overline{W}_1/X - W_1$ separating V_1 and W_1 (3·1 Lemma). Therefore,

https://doi.org/10.1017/S0305004100001067 Published online by Cambridge University Press

$\overline{W}_1/X - W_1 \in U_{\overline{V}_1,\overline{W}_1}$ (3·4 Lemma). Thus $\overline{W}_1/X - W_1 \in U_{\overline{V}_0,\overline{W}_0}$. Thus $\overline{V}_0 \subset \overline{W}_1^0$ or $\overline{W}_0 \subset \overline{W}_1^0$. But we already know $\overline{W}_0 \subset \overline{W}_1$. Therefore, $\overline{W}_0 \subset \overline{W}_1^0 = W_1$.

3·6 LEMMA. *If $(X, \mathscr{T})$ is a normal T_1 space, then*

$$\mathscr{U} = \{U_{\overline{V},\overline{W}} : (V, W) \text{ is a pair of non-empty open sets such that } \overline{V} \cap \overline{W} = 0,\ \overline{V}^\circ = V,\ \overline{W}^\circ = W\}$$

is a base for $S(X)$.

Proof. Let

$$\mathscr{J} = \{U_{M.N} : M = \overline{M}, N = \overline{N} \text{ and } M \cap N = 0\}$$

where $U_{M.N} = \{A/B \in S(X) : M \subset A^\circ \text{ and } N \subset B^\circ \text{ or } N \subset A^\circ \text{ and } M \subset B^\circ\}$.

$\mathscr{J}$, we already know (see Introduction), generates the topology for $S(X)$.

Let $\mathscr{J} = \mathscr{U}_{M.N} \in \mathscr{J}$. M and N are closed and separated. Let $A/B \in \mathscr{J}$. Without loss of generality assume $M \subset A^\circ$ $N \subset B^\circ$. Because $M \subset A^\circ \neq 0$ and $N \subset B^\circ \neq 0$, and because X is normal and T_1, there exist non-empty open sets V and W in X such that $M \subset V \subset \overline{V} \subset A^\circ$ and $N \subset W \subset \overline{W} \subset B^\circ$ and $\overline{V} \cap \overline{W} = 0$ since $A^\circ \cap B^\circ = 0$. Define $\mathscr{V} = \overline{V}^\circ$ and $\mathscr{W} = \overline{W}^\circ$ so that $\mathscr{V} \neq 0$, $\mathscr{W} \neq 0$ and $\mathscr{V} = \mathscr{V}^\circ$, $\mathscr{W} = \overline{\mathscr{W}}^\circ$. $\mathscr{V} \cap \overline{\mathscr{W}} = 0$ since $\mathscr{V} = (\overline{V}^\circ)^- = \overline{V} \subset A^\circ$ and $\overline{\mathscr{W}} = (\overline{\mathscr{W}}^\circ)^- = \overline{\mathscr{W}} \subset B^\circ$; hence $U_{\overline{\mathscr{V}},\overline{\mathscr{W}}} \in \mathscr{U}$. Clearly A/B separates $\mathscr{V}$ and $\mathscr{W}$ and $U_{\mathscr{V},\overline{\mathscr{W}}} \subset U_{M.N}$.

3·7. THEOREM. *If $(X, \mathscr{T})$ is normal and T_1, then $S(X)$ is countably subcompact.*

Proof. Let $\mathscr{U} = \{U_{\overline{V},\overline{W}} : (V, W)$ is a pair of non-empty open sets such that $\overline{V} \cap \overline{W} = 0$, $\overline{V}^\circ = V$, $\overline{W}^\circ = W\}$. $\mathscr{U}$ is a base for $X(X)$ (3·6 Lemma).

Let $\{U_{\overline{V}_i,\overline{W}_i}\}_{i=0}^\infty$ be an arbitrary non-empty countable subfamily of $\mathscr{U}$ such that

$$\{U_{\overline{V}_i.\overline{W}_i}\}_{i=0}^\infty \text{ is a regular filter base relative to } \mathscr{U}.$$

We will show $\bigcap_{i=0}^{\infty} U_{\overline{V}_i,\overline{W}_i} \neq 0$; that is, $S(X)$ is countably subcompact.

From the definition of a regular filter base it easily follows by induction that there exists a sequence of integers $\{K_n;\ n = 0, 1, 2, \ldots\}$ such that, writing $U_r = U_{\overline{V}_r.\overline{W}_r}$, we have

$$(a)\ \ \overline{U}_{K_n} \subset U_{K_{n-1}} \quad \text{for} \quad n \geqslant 1;$$

$$(b)\ \ U_{K_n} \subset \bigcap_{r=1}^{n} U_r, \quad \text{so that} \quad \bigcap_{n=0}^{\infty} U_{K_n} = \bigcap_{r=0}^{\infty} U_r.$$

Then by 3·5 Lemma (supposing the notation is suitably chosen) we have, for $n \geqslant 1$,

$$\overline{V}_{K_{n-1}} \subset V_{K_n} \quad \text{and} \quad \overline{W}_{K_{n-1}} \subset W_{K_n} \subset X - V_{K_n}.$$

Let

$$V_\infty = \bigcup_{n=0}^{\infty} V_{K_n} \text{ and } W_\infty = \bigcup_{n=0}^{\infty} W_{K_n}.$$

V_∞ and W_∞ are open and disjoint. Therefore, there is a cut A/B separating V_∞ and W_∞ (3·1 Lemma) and hence separating each pair V_{K_n}, $\overline{W}_{K_n}$; that is,

$$A/B \in \bigcap_{n=0}^{\infty} U_{K_n} = \bigcap_{r=0}^{\infty} U_r.$$

Therefore, $S(X)$ is countably subcompact.

https://doi.org/10.1017/S0305004100001067 Published online by Cambridge University Press

3·8. COROLLARY. *If* $(X, \mathscr{T})$ *is normal and* T_1, *then* $S(X)$ *is Baire.*

Proof, By 2·10 Corollary, 3·7 Theorem and de Groot's generalized Baire theorem.

The author is indebted to J. M. Day under whose supervision this research was carried out and to A. R. Pears for his advice and suggestions in the preparation of this paper.

REFERENCES

(1) DE GROOT, J. Subcompactness and the Baire category theorem. *Nederl. Akad. Wetensch. Indag. Math.* **25** (1963), 761–767.
(2) DUGUNDJI, J. *Topology* (Allyn and Bacon; Boston, 1966).
(3) HUREWICZ, W. Über Schnitte in topologischen Raumen. *Fund. Math.* **20** (1932), 151–162.
(4) KURATOWSKI, K. *Topology V.I.* (Academic Press: New York and London, 1966).

https://doi.org/10.1017/S0305004100001067 Published online by Cambridge University Press