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An Upper Bound for Constrained Ramsey Numbers

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Let s and t be integers satisfying $s \geqslant 2$ and $t \geqslant 2$. Let S be a tree of size s, and let P_t be the path of length t. We show in this paper that, for every edge-colouring of the complete graph on n vertices, where $n = 224(s-1)^2 t$, there is either a monochromatic copy of S or a rainbow copy of P_t. So, in particular, the number of vertices needed grows only linearly in t.

1. Introduction

Given any two finite simple graphs G and H, let $f(G,H)$ denote the least integer n (if it exists) such that every edge-colouring of the complete graph on n vertices contains either a monochromatic copy of G (any two edges of the same colour) or a rainbow copy of H (no two edges of the same colour).

It is a straightforward consequence of the Erdős–Rado Canonical Ramsey Theorem [2], that $f(G,H)$ exists if and only if either G is a star or H is a forest.

It is readily seen that if H is the star of size $k+1$, then $f(G,H) = R^k_{\mathrm{loc}}(G)$, the well-known local k-Ramsey number of G, introduced in [4] by Gyárfás, Lehel, Schelp and Tuza. This provides the motivation for considering $f(G,H)$ in general.

In [5] Jamison, Jiang and Ling began a systematic study of $f(G,H)$, including extensions of some local Ramsey theory results from [4]. At the same time, Chen, Schelp and Wei [1] essentially considered $f(T,T)$, where T is a tree on n vertices. Eroh and Oellermann [3] considered a bipartite version of the problem.

One of the main results of Jamison, Jiang and Ling [5] is the following. Let s and t be integers with $s \geqslant 2$ and $t \geqslant 2$, and let S and T be trees with s and t edges, respectively. Then $f(S,T) \leqslant (s-1)(t^2+3t)$.

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Using a standard product colouring (and the fact that every complete graph of even order has a 1-factorization) a lower bound for $f(S,T)$ of order st can easily be obtained. So the question arises as to how fast $f(S,T)$ grows, for fixed S, as a function of $e(T)$.

In certain cases (when at least one of S and T is close to being a star) they establish the exact value of $f(S,T)$ lying near $(s-1)(t-1)$, and it is conjectured in [5] that there exists an absolute constant α such that $f(S,T) \leqslant \alpha(s-1)(t-1)$ for all trees S and T.

It is also suggested [5, Question 6.1] that, as S and T range over all possible trees, the value of $f(S,T)$ is greatest when S and T are both paths. In this paper, therefore, we concentrate on the case when T is the path P_t and show that, for fixed S, $f(S,P_t)$ grows linearly in t.

It is slightly unfortunate that the persistent square factor remains in the upper bound, though in our case it is transferred to S, and it is believed that it is not needed in general.

Theorem 1.1. *Let s and t be integers with $s \geqslant 2$ and $t \geqslant 2$, and let S be a tree and P_t be the path of sizes s and t, respectively. Then $f(S,P_t) \leqslant 224(s-1)^2 t$.*

2. Preliminaries

Let N denote the set of positive integers. For all $a,b \in N$ with $a \leqslant b$, let $[a,b]$ denote the set $\{c \in N : a \leqslant c \leqslant b\}$.

Given a finite simple graph G, let $V(G)$ denote its vertex set and $E(G)$ denote its edge set. Furthermore, let $|G| = |V(G)|$ be its order and $e(G) = |E(G)|$ be its size, as usual. For all $n \in N$, let the complete graph on n vertices be denoted by K_n.

Definition. Let $n \in N$ with $n \geqslant 2$. Given an *edge-colouring* $c : E(K_n) \to N$ of K_n and a colour $i \in c(E(K_n))$, let the *colour subgraph* $G_c(i)$ of colour i be the subgraph of K_n induced by all the edges of colour i. This means that $E(G_c(i))$ consists of all the edges of colour i and $V(G_c(i))$ precisely of all their endpoints.

Note that the order of some colour subgraph of K_n does not necessarily have to be n.

Let $s \in N$ with $s \geqslant 2$, and let S be a tree of size s. A standard simple argument shows that every finite simple graph whose average degree is at least $2(s-1)$ contains a subgraph whose minimum degree is at least s, and that this subgraph contains S. These observations are stated in the following proposition.

Proposition 2.1. *If a finite simple graph G does not contain S, then $e(G) < (s-1)|G|$.*

Note that it is only this implied property of enforcing a low density that we will make use of; that is, the actual structure of S will be irrelevant for the entire argument.

Corollary 2.2. *Let c be an edge-colouring of K_n without a monochromatic S. Then, for all $i \in c(E(K_n))$, we have $e(G_c(i)) < (s-1)|G_c(i)|$.*

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3. Proof of Theorem 1.1

Let $n = 224(s-1)^2 t$, and let c be an edge-colouring of K_n with neither a monochromatic S nor a rainbow P_t. It is sufficient to contradict the existence of such a c.

3.1. Colour degrees

Given an edge-colouring c of K_n and a vertex $v \in V(K_n)$, let the *colour degree* $d_c(v)$ of v with respect to c be the number of different colours of all edges of the form vw with $w \in V(K_n)$. The largest colour degree is denoted by $\Delta(c)$. Note that the colour degree of a given vertex v is exactly the number of colour subgraphs of which v is a vertex.

For convenience, let the n vertices of K_n be labelled from 1 to n such that, for all $1 \leqslant i < j \leqslant n$, we have $d_c(i) \leqslant d_c(j)$.

Lemma 3.1. *Let $v \in [2, n]$. Then $d_c(v) \geqslant v/2(s-1)$.*

Proof. Consider the complete subgraph K_v of K_n defined by $[1, v]$, and let γ be the restriction of c to $E(K_v)$, which is an edge-colouring of K_v without a monochromatic S. For all $i \in [1, v]$, we trivially have $d_\gamma(i) \leqslant d_c(i)$, and thus $\Delta(\gamma) \leqslant d_c(v)$.

Note that $\sum_{v \in V(K_v)} d_\gamma(v) = \sum_{i \in \gamma(E(K_v))} |G_\gamma(i)|$. It follows with Corollary 2.2 that $(s-1)\sum_{v \in V(K_v)} d_\gamma(v) > \sum_{i \in \gamma(E(K_v))} e(G_\gamma(i)) = e(K_v)$. $\Delta(\gamma) > (v-1)/2(s-1)$ is an immediate consequence, and the result follows. □

3.2. Passing to a structured subgraph

Roughly speaking, it is easy to see how to construct a long rainbow path when the colour degrees are large. However, apart from Lemma 3.1, there is not much to bound the colour degrees from below in our K_n. The crux of the argument will be to pass to a highly structured subgraph, essentially at the cost of losing a factor of s.

Lemma 3.2. *Let $m = 44(s-1)t$. There exists a subset $M \subset V(K_n)$, $|M| = m$, such that the subgraph K_m of K_n with vertex set M has the following properties.*

(i) *There is an injection $\sigma : M \to c(E(K_n))$, such that, for each $v \in M$, every edge of K_m with colour $\sigma(v)$ is incident with v. We call $\sigma(v)$ the* star colour *of v, and let $\Sigma = \{\sigma(v) : v \in M\}$ be the set of star colours. Note that there need not be any edge of K_m coloured $\sigma(v)$; that is, $\Sigma \subset c(E(K_m))$ need not hold.*

(ii) *For all distinct $u, v \in M$, there are at least t vertices $x \in V(K_n) \setminus M$ such that $c(ux) = \sigma(u)$ and $c(vx) = \sigma(v)$.*

(iii) *At most $t-1$ colours of $E(K_m)$ are not star colours; that is, $|c(E(K_m)) \setminus \Sigma| \leqslant t-1$. The colours in $c(E(K_m)) \setminus \Sigma$ are called* rogue colours.

Proof. Let $V_{\text{bottom}} = [1, n/2]$ and $V_{\text{top}} = [n/2+1, n]$. Let P be a rainbow path of maximum length having an endpoint in V_{top}. Clearly, $|P| \leqslant t$, since K_n has no rainbow P_t. Let w be an endpoint of P in V_{top}. Let $F = c(E(P))$ be the set of colours used on P and that are therefore forbidden on any extension of P to a longer rainbow path; then clearly

$$|F| = |P| - 1 \leqslant t - 1. \tag{3.1}$$

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For each $x \in V(K_n)$, let $d_f(x)$ denote the number of edges incident with x whose colour is in F. Note that $\sum_{x \in V(K_n)} d_f(x) = 2\sum_{i \in F} e(G_c(i))$, from which it follows by (3.1) and Corollary 2.2 that

$$\sum\nolimits_{x \in V(K_n)} d_f(x) < 2(s-1)(t-1)n. \tag{3.2}$$

Let $B = \{x \in V(K_n) : d_f(x) > n/5\}$. Then it follows from (3.2) that $|B|n/5 < 2(s-1)(t-1)n$; that is,

$$|B| < 10(s-1)(t-1). \tag{3.3}$$

Now choose a set $M \subset V(K_n) \setminus (V(P) \cup B)$ so that the edges from w to M have distinct colours and none of these colours is in F. Since $w \in V_{\text{top}}$, Lemma 3.1 shows $d_c(w) > 56(s-1)t$, and by (3.1) and (3.3) we can choose M so that $|M| \geqslant d_c(w) - |P| - |B| - |F| > 56(s-1)t - t - 10(s-1)(t-1) - (t-1) > 44(s-1)t = m$. So let us choose M such that $|M| = m$. Finally, for each $v \in M$ let $\sigma(v) = c(wv)$.

All that remains is to verify that M and σ have the properties claimed.

Certainly, σ is injective by the construction of M.

Moreover, $M \subset V_{\text{bottom}}$, for if there were a $v \in M \cap V_{\text{top}}$, then we could extend P via w and v to a longer rainbow path with an endpoint in V_{top}, contradicting the definition of P.

Observe also that if $v \in M$ and $x \in V_{\text{top}} \setminus V(P)$ with $c(vx) \notin F$, then $c(vx) = \sigma(v)$, for otherwise we could extend P via w, v and x to a longer rainbow path with an endpoint in V_{top}, contradicting the definition of P.

Now let $uv \in E(K_m)$. Since $u, v \notin B$, the number of vertices $x \in V_{\text{top}} \setminus V(P)$, such that neither $c(ux) \in F$ nor $c(vx) \in F$, is at least $|V_{\text{top}} \setminus V(P)| - 2n/5 \geqslant n/10 - |P| \geqslant t$. Since $M \subset V_{\text{bottom}}$, none of these vertices are in M. Therefore we have $c(ux) = \sigma(u)$ and $c(vx) = \sigma(v)$ for at least t vertices x in $(V_{\text{top}} \setminus V(P)) \setminus M$, which proves property (ii).

Moreover, if $c(uv) \notin F$, then $c(uv) = \sigma(u)$ or $c(uv) = \sigma(v)$, for otherwise P extended by $wuvx$ (for some $x \in V_{\text{top}} \setminus V(P)$ with $c(vx) = \sigma(v)$) is a longer rainbow path ending in V_{top}. This observation proves property (i), since σ is injective, and property (iii) too. □

We can now restrict our attention to the K_m given in Lemma 3.2.

For convenience, we call the edges in $E(K_m)$ that are coloured with a star colour *star edges* and the edges in $E(K_m)$ that are coloured with a rogue colour *rogue edges*.

We now orient each star edge by directing it away from the centre of the monochromatic star it belongs to, and we leave the rogue edges undirected.

It is crucial that we make heavy use of the fact that directed paths are rainbow paths and that two vertex-disjoint directed paths have no edge colour in common.

Remark 1. If there were no rogue edges in $E(K_m)$, we would have a tournament. A well-known fact about tournaments is that they contain a Hamiltonian directed path, which would immediately give a rainbow P_t.

Considering the restriction γ of c to $E(K_m)$ and Corollary 2.2, it is not hard to see that the proportion of rogue edges is, roughly speaking, small. However, even the presence of very few rogue edges can prevent a long directed path.

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Consider, for example, the complete graph on the vertex set $\{v_{i,j} : i \in [1,t], j \in [1,\tau]\}$, where the edges are coloured lexicographically as follows. For all $1 \leqslant i < k \leqslant t$ and $j \in [1,\tau]$, let $v_{i,j}v_{k,j}$ have the rogue colour f_i, and, for all $i,k \in [1,t]$ and $1 \leqslant j < l \leqslant \tau$, let $v_{i,j}v_{k,l}$ have the star colour $\sigma_{i,j}$.

This $K_{t\tau}$ does not contain a monochromatic S (unless S is a star), there are $t-1$ rogue colours, the star edges are directed upwards, there is an amount of $\tau t(t-1)/2$ rogue edges out of $t\tau(t\tau-1)/2$ total edges and hence a small proportion of at most $1/\tau$ rogue edges, and a longest path of directed edges has size $2\tau-2$; that is, we gradually go up from the bottom to the top and then back down to the bottom layer.

Nevertheless, it is easy to see how a long rainbow path can be obtained in this example by using both star edges and rogue edges via the method of repeatedly linking maximal rainbow paths of star edges by rogue edges.

The remaining parts of the argument are largely motivated by the structure of the example in Remark 1.

3.3. Partitioning into directed paths

Choose a directed path P^1 of largest length in K_m. In $M \setminus V(P^1)$ choose a directed path P^2 of largest length. Then in $M \setminus (V(P^1) \cup V(P^2))$ choose a directed path P^3 of largest length and so on, thus partitioning M into a collection $\mathscr{P}$ of directed paths.

Our aim is to construct a rainbow P_t by joining pieces of the P^i together. We can join them at their ends by using property (ii) of Lemma 3.2. What remains is to show how we can join them near their beginnings using rogue edges.

Now label the vertices as follows. For all $p \in [1, |\mathscr{P}|]$, let $P^p = v_1^p v_2^p \cdots v_k^p$ as a directed path, where $k = |P^p|$. By the construction of $\mathscr{P}$, we have $|P^1| \geqslant |P^2| \geqslant \cdots \geqslant |P^{|\mathscr{P}|}|$.

Lemma 3.3. *Let P^p, $P^q \in \mathscr{P}$ with $p < q$. Then, for each $v_i^q \in V(P^q)$, there exists a $v_j^p \in V(P^p)$ with $j \in [i, i + |P^p| - |P^q|]$, such that $v_j^p v_i^q$ is a rogue edge.*

Proof. Suppose not. Then, for some $v_i^q \in V(P^q)$ and for all $v_j^p \in V(P^p)$ with $i \leqslant j \leqslant k = i + |P^p| - |P^q|$, no edge $v_j^p v_i^q$ is rogue; so all these edges are directed. Now $v_i^p v_i^q$ is directed from v_i^p to v_i^q, else the path $v_1^q \cdots v_i^q v_i^p \cdots v_{|P^p|}^p$ would have been chosen in preference to P^p. Also, $v_k^p v_i^q$ is directed from v_i^q to v_k^p, else the path $v_1^p \cdots v_k^p v_i^q \cdots v_{|P^q|}^q$ would have been chosen in preference to P^p. So there must exist an $l \in [i, k-1]$, such that $v_l^p v_i^q$ is directed from v_l^p to v_i^q and $v_{l+1}^p v_i^q$ is directed from v_i^q to v_{l+1}^p. But then $v_1^p \cdots v_l^p v_i^q v_{l+1}^p \cdots v_{|P^p|}^p$ is a longer directed path than P^p, contradicting its definition. □

An immediate consequence of Lemma 3.3 is that between two paths of the same order every horizontal edge, that is, an edge of the form $v_i^p v_i^q$, is rogue. So, for a collection of paths of the same order, every horizontal layer constitutes a complete subgraph of rogue edges exclusively, which reminds us of the example in Remark 1.

3.4. Partitioning into blocks

We now partition the paths in $\mathscr{P}$ into *blocks* as follows. Let $h_1 = |P^1|$. Let B_1 be the set of paths $P \in \mathscr{P}$ with $|P| > 3h_1/4$. Then let $h_2 = \max\{|P| : P \in \mathscr{P} \setminus B_1\}$, and let B_2

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be the set of the remaining paths P with $|P| > 3h_2/4$. Continue in this way until $\mathscr{P}$ is exhausted.

Let $\mathscr{B}$ be the set of blocks. So, if $B_k \in \mathscr{B}$, then B_k consists of all paths $P \in \mathscr{P}$ with $h_k \geqslant |P| > 3h_k/4$. Moreover, $h_k \leqslant (3/4)h_{k-1} \leqslant \cdots \leqslant (3/4)^{k-1}h_1 \leqslant (3/4)^{k-1}t$, since the rainbow path P^1 has at most t vertices. In the example in Remark 1 we have $|\mathscr{B}| = 1$, but, in general, $|\mathscr{B}|$ could be of order $\log t$.

Let $w_k = |B_k|$ be the *width* of the block B_k.

Definition. Given a block $B_k \in \mathscr{B}$, a *rogue set* F_k for B_k is a set of rogue edges with the following four properties.

(i) The colours of the edges of F_k are distinct.
(ii) If $f \in F_k$, then f has endvertices in paths $P^p, P^q \in B_k$, where $p = p(f) < q = q(f)$.
(iii) $|\{p(f), q(f) : f \in F_k\}| = 2|F_k|$; that is, no path in B_k meets more than one edge in F_k (which gives rise to a notion of F_k being strongly independent).
(iv) If $f = v_j^p v_i^q \in F_k$, then $i \leqslant \lceil h_k/4 \rceil$ and $j \leqslant 2\lceil h_k/4 \rceil - 1$.

So, roughly speaking, a rogue set is a set of rogue edges linking the bottom parts of independent pairs of paths in B_k. Obviously, rogue sets of positive size can only occur in blocks of width exceeding 1. Here is a result relating size and width.

Lemma 3.4. *Let $B_k \in \mathscr{B}$ be a block of width $w_k \geqslant 2$. Then there is a rogue set F_k for B_k with $|F_k| \geqslant w_k/6(s-1)$.*

Proof. Note that, for all $P^p, P^q \in B_k$, $p < q$, by construction, $|P^p| - |P^q| \leqslant \lceil h_k/4 \rceil - 1$. Pick $P^p, P^q \in B_k$ with $p < q$ and $v_i^q \in V(P^q)$ with $i \in [1, \lceil h_k/4 \rceil]$. By Lemma 3.3, there is a rogue edge $v_j^p v_i^q$ satisfying properties (ii) and (iv). Every single such edge constitutes a rogue set of size 1, as properties (i) and (iii) trivially hold also. Now choose a rogue set F_k of maximum size $l \geqslant 1$. If $w_k = 2l$ or $w_k = 2l + 1$, then the assertion of the lemma clearly holds. So we may assume that there are at least 2 paths in B_k that do not meet any edge in F_k. Let us restrict our attention to these $w_k - 2l$ paths.

Let E be the set of all rogue edges of the form $v_j^p v_i^q$, where P^p and P^q are two of the remaining paths with $p < q$, $i \leqslant \lceil h_k/4 \rceil$ and $j \leqslant 2\lceil h_k/4 \rceil - 1$. Our observation at the beginning of the proof implies that we have at least $\lceil h_k/4 \rceil (w_k - 2l)(w_k - 2l - 1)/2$ such edges; that is, $|E| \geqslant \lceil h_k/4 \rceil (w_k - 2l)(w_k - 2l - 1)/2$.

Let V be the set of endpoints of the edges in E; so $|V| \leqslant (2\lceil h_k/4 \rceil - 1)(w_k - 2l)$. Thus $|E|/|V| > (w_k - 2l - 1)/4$.

Consider the induced subgraph G of E, with vertex set V, and let τ be the number of rogue colours occurring as edge colours of G. By the choice of F_k, $\tau \leqslant l$. By Proposition 2.1, we can deduce $|E| < \tau(s-1)|V|$.

So $l \geqslant \tau > (w_k - 2l - 1)/4(s-1)$. Thus $w_k - 2l \leqslant 4(s-1)l$. Hence $w_k \leqslant (4s-2)l \leqslant (6s-6)l$, and the result follows. □

3.5. The final construction

We are now ready for the final construction. Let $V_k = \bigcup_{P \in B_k} V(P)$. Note that the number of vertices in paths forming blocks of width 1 is small, namely at most

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$t + (3/4)t + (3/4)^2 t + \cdots \leqslant 4t$; that is,

$$\left| \bigcup_{k \in [1, |\mathscr{B}|], w_k = 1} V_k \right| \leqslant 4t. \tag{3.4}$$

Let $\mathscr{B}_{\geqslant 2}$ be the set of all blocks $B_k \in \mathscr{B}$ of width $w_k \geqslant 2$.

Pick a rogue set F_k for B_k as given by Lemma 3.4 for each block $B_k \in \mathscr{B}_{\geqslant 2}$ simultaneously. Note that these rogue sets are non-empty.

Let A_k be the set of paths in B_k containing an endvertex of an edge in F_k, and let $U_k = \bigcup_{P \in A_k} V(P)$ be the set of vertices in these paths. Note that, by Lemma 3.4, $|A_k| \geqslant w_k/3(s-1)$. Now $|V_k| \leqslant w_k h_k$, whereas $|U_k| > |A_k|(3/4)h_k \geqslant w_k h_k/4(s-1)$. So $|U_k| > |V_k|/4(s-1)$.

Taking into account the paths lying in blocks of width 1, by (3.4), the number $|\bigcup_{k \in \mathscr{B}_{\geqslant 2}} U_k|$ of vertices left is at least $(m-4t)/4(s-1) \geqslant 10t$.

Let F^* be the set of rogue colours used on at least one rogue edge in $\bigcup_{k \in \mathscr{B}_{\geqslant 2}} F_k$. For each colour $r \in F^*$, let R_r be the set of paths in $\bigcup_{k \in \mathscr{B}_{\geqslant 2}} A_k$ meeting a rogue set edge of colour r; thus $\bigcup_{r \in F^*} R_r = \bigcup_{k \in \mathscr{B}_{\geqslant 2}} A_k$.

Let $P(r)$ and $Q(r)$ be the two longest paths in R_r, and let $f(r)$ be the rogue set edge of colour r that joins them. For convenience, let $P(r)$ be chosen before $Q(r)$ in the construction of $\mathscr{P}$. Let B_k be the block containing $P(r)$ and $Q(r)$. By the definition of the blocks, we have $|P(r)|, |Q(r)| > (3/4)h_k$, whereas $|\bigcup_{P \in R_r} V(P)| \leqslant |P(r)| + |Q(r)| + 2(3/4)h_k + 2(3/4)^2 h_k + \cdots \leqslant |P(r)| + |Q(r)| + 6h_k$. So $|\bigcup_{P \in R_r} V(P)| < 5(|P(r)| + |Q(r)|)$.

Since $|\bigcup_{r \in F^*} \bigcup_{P \in R_r} V(P)| = |\bigcup_{k \in \mathscr{B}_{\geqslant 2}} U_k| \geqslant 10t$, we have $\sum_{r \in F^*} |P(r)| + |Q(r)| > 2t$.

Finally, let $P'(r)$ and $Q'(r)$ be the tail sections of the paths $P(r)$ and $Q(r)$ going out from the endvertices of $f(r)$. To be more precise, if $P(r) = P^p$, $Q(r) = P^q$ and $f(r) = v_j^p v_i^q$, then $P'(r)$ is $v_j^p \cdots v_{|P^p|}^p$ and $Q'(r)$ is $v_i^q \cdots v_{|P^q|}^q$.

As $i \leqslant \lceil h_k/4 \rceil$, $j \leqslant 2\lceil h_k/4 \rceil - 1$ and $|P(r)|, |Q(r)| > (3/4)h_k$, we have $|P'(r)| + |Q'(r)| \geqslant |P(r)| + |Q(r)| - 3(\lceil h_k/4 \rceil - 1) > |P(r)| + |Q(r)| - (3/4)h_k > (|P(r)| + |Q(r)|)/2$.

Thus

$$\sum_{r \in F^*} |P'(r)| + |Q'(r)| > t. \tag{3.5}$$

For each $r \in F^*$, the directed paths $P'(r)$ and $Q'(r)$ joined at their startpoints by $f(r)$ constitute a rainbow path $\Psi(r)$, and, by construction, no two of these paths have an edge colour in common.

By the injectivity of σ, the star colours of the endvertices of the $\Psi(r)$s are pairwise distinct and do not occur on the edges of the $\Psi(r)$s. By property (ii) of Lemma 3.2, we can now (in any order) join these $\Psi(r)$s together at their endpoints via a vertex outside M yielding a longer and longer rainbow path. We can only get stuck by either running out of $\Psi(r)$s, in which case, by (3.5), we must have obtained a rainbow P_t, or having joined t times already, in which case, of course, we must also have obtained a rainbow P_t.

This contradiction to the definition of c finishes the proof of Theorem 1.1.

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