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NUMERICAL SEMIGROUPS FROM RATIONAL MATRICES II: MATRICIAL DIMENSION DOES NOT EXCEED MULTIPLICITY

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Abstract

We continue our study of exponent semigroups of rational matrices. Our main result is that the matricial dimension of a numerical semigroup is at most its multiplicity (the least generator), greatly improving upon the previous upper bound (the conductor). For many numerical semigroups, including all symmetric numerical semigroups, our upper bound is tight. Our construction uses combinatorially structured matrices and is parametrised by Kunz coordinates, which are central to enumerative problems in the study of numerical semigroups.

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1. Introduction

Let $\mathrm{M}_d(\cdot)$ denote the set of $d \times d$ matrices with entries in the set $\mathbb{Z}$ of integers or the set $\mathbb{Q}$ of rational numbers, as indicated. Every (additive) subsemigroup of $\mathbb{N} = \{0, 1, 2, \ldots\}$ is the *exponent semigroup*

$$\mathcal{S}(A) = \{n \in \mathbb{N} : A^n \in \mathrm{M}_d(\mathbb{Z})\}$$

of some $A \in \mathrm{M}_d(\mathbb{Q})$ by [6, Corollary 6.5]. In particular, every *numerical semigroup* S, that is, a subsemigroup of $\mathbb{N}$ with finite complement [3, 15], is of the form $S = \mathcal{S}(A)$ with $A \in \mathrm{M}_c(\mathbb{Q})$, in which $c = c(S) = 1 + \max(\mathbb{N}\backslash S)$ is the *conductor* of S [6, Theorem 6.2]. This ensures that the *matricial dimension*

$$\dim_{\mathrm{mat}} S = \min\{d \geq 1 : \text{there is an } A \in \mathrm{M}_d(\mathbb{Q}) \text{ such that } S = \mathcal{S}(A)\}$$

of a numerical semigroup $S \subseteq \mathbb{N}$ is well defined and $\dim_{\mathrm{mat}} S \leq c(S)$.

Each numerical semigroup S has a unique minimal system of *generators*, that is, positive $n_1 < n_2 < \cdots < n_k$ such that $S = \langle n_1, n_2, \ldots, n_k\rangle$ is the smallest additive

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subsemigroup of $\mathbb{N}$ containing $n_1, n_2, \ldots, n_k$. Here, $e(S) = k$ is the *embedding dimension* of S and $m(S) = n_1$ is the *multiplicity* of S.

The main result of this paper is the following dramatic improvement of [6, Theorem 6.2], in which $m(S) = n_1$ replaces $c(S)$.

THEOREM 1.1. *If S is a nontrivial numerical semigroup with multiplicity (minimal generator) $m(S)$, there is an $A \in \mathrm{M}_{m(S)}(\mathbb{Q})$ such that $\mathcal{S}(A) = S$. Thus,* $\dim_{\mathrm{mat}} S \leq m(S)$.

Theorem 1.1 gives an explicit construction in terms of Apéry sets [2], a mainstay in the study of numerical semigroups [12, 17]. At the core of the proof lies a system of inequalities that reflect the fine structure of Apéry sets; these were first introduced by Kunz in [11], and have since been used in enumerative [1, 8, 16] and classification [7, 9] problems in this area.

This paper is organised as follows. Section 2 contains the proof of Theorem 1.1. Section 3 presents several illustrative examples and Corollary 3.6, which extends Theorem 1.1 to arbitrary semigroups in $\mathbb{N}$. We close by computing the matricial dimension for irreducible numerical semigroups in Section 4.

2. Proof of Theorem 1.1

This section contains the proof of Theorem 1.1 and some preliminary remarks. Recall that the *Apéry set* of a numerical semigroup S with multiplicity m is the set

$$\mathrm{Ap}(S) = \{n \in S : n - m \notin S\}.$$

Each element of $\mathrm{Ap}(S)$ is the smallest element of S in its equivalence class modulo m, so one often writes

$$\mathrm{Ap}(S) = \{a_0, a_1, \ldots, a_{m-1}\},$$

in which $a_0 = 0$ and each $a_i \equiv i \bmod m$. It is convenient to interpret the subscripts of the a_i modulo m. For example, it was shown by Kunz in [11] that

$$a_i + a_j \geq a_{i+j} \quad \text{for all } i, j \in \mathbb{Z}, \tag{2.1}$$

and that this system of inequalities (along with the modular requirements and minimality of m) characterise Apéry sets of numerical semigroups.

Before proceeding to the proof of Theorem 1.1, let us first illustrate the matricial structure employed in the proof. Suppose $z_0, z_1, z_2, z_3 \neq 0$ and

$$A = \begin{bmatrix} 0 & z_1 & 0 & 0 \\ 0 & 0 & z_2 & 0 \\ 0 & 0 & 0 & z_3 \\ z_0 & 0 & 0 & 0 \end{bmatrix} \in \mathrm{M}_4(\mathbb{Q}),$$

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which is a generalised permutation matrix or, equivalently, the adjacency matrix of a weighted directed cycle graph on four vertices. Observe that

$$A^{4q} = \begin{bmatrix} (z_0z_1z_2z_3)^q & 0 & 0 & 0 \\ 0 & (z_0z_1z_2z_3)^q & 0 & 0 \\ 0 & 0 & (z_0z_1z_2z_3)^q & 0 \\ 0 & 0 & 0 & (z_0z_1z_2z_3)^q \end{bmatrix},$$

$$A^{4q+1} = \begin{bmatrix} 0 & z_1(z_0z_1z_2z_3)^q & 0 & 0 \\ 0 & 0 & z_2(z_0z_1z_2z_3)^q & 0 \\ 0 & 0 & 0 & z_3(z_0z_1z_2z_3)^q \\ z_0(z_0z_1z_2z_3)^q & 0 & 0 & 0 \end{bmatrix},$$

$$A^{4q+2} = \begin{bmatrix} 0 & 0 & z_1z_2(z_0z_1z_2z_3)^q & 0 \\ 0 & 0 & 0 & z_2z_3(z_0z_1z_2z_3)^q \\ z_3z_0(z_0z_1z_2z_3)^q & 0 & 0 & 0 \\ 0 & z_0z_1(z_0z_1z_2z_3)^q & 0 & 0 \end{bmatrix} \quad \text{and}$$

$$A^{4q+3} = \begin{bmatrix} 0 & 0 & 0 & z_1z_2z_3(z_0z_1z_2z_3)^q \\ z_3z_0z_2(z_0z_1z_2z_3)^q & 0 & 0 & 0 \\ 0 & z_3z_0z_1(z_0z_1z_2z_3)^q & 0 & 0 \\ 0 & 0 & z_0z_1z_2(z_0z_1z_2z_3)^q & 0 \end{bmatrix}.$$

Writing $p = 4q + r$, with $q, r \in \mathbb{Z}$ and $0 \leq r \leq 3$, the nonzero entries of A^p are

$$(z_0z_1z_2z_3)^q \prod_{\ell=0}^{r-1} z_{i+\ell} \quad \text{for } 0 \leq i \leq 3, \tag{2.2}$$

in which the subscripts are interpreted modulo m.

For the proof of Theorem 1.1, we generalise (2.2) to the $m \times m$ setting, and let each z_i take the form b^{x_i} with $b \in \mathbb{Z}\backslash\{-1, 0, 1\}$ and $x_i \in \mathbb{Z}$, so that the multiplicative structure of (2.2) becomes additive.

Proof of Theorem 1.1. Suppose S is a numerical semigroup with multiplicity $m \geq 2$ and Apéry set $\mathrm{Ap}(S) = \{a_0, a_1, \ldots, a_{m-1}\}$, in which $a_0 = 0$ and $a_i \equiv i \,(\mathrm{mod}\, m)$ for each i. For each $i = 0, 1, \ldots, m-1$, let

$$x_i = m^{-1}(a_{i-1} - a_i + 1),$$

where the subscripts of the x_i are, like those of the a_i, interpreted modulo m. Notice that each $x_i \in \mathbb{Z}$ since $a_{i-1} - a_i + 1 \equiv 0 \,(\mathrm{mod}\, m)$, and telescoping yields

$$\begin{aligned} x_0 + x_1 + \cdots + x_{m-1} &= m^{-1}(a_{m-1} - a_0 + 1) + \cdots + m^{-1}(a_{m-2} - a_{m-1} + 1) \\ &= m^{-1}(\underbrace{1 + \cdots + 1}_{m \text{ times}}) = 1. \end{aligned} \tag{2.3}$$

Fix a base $b \in \mathbb{Z}\backslash\{-1, 0, 1\}$ and let $A \in \mathrm{M}_m(\mathbb{Q})$ denote the matrix

$$A = \begin{bmatrix} 0 & b^{x_1} & 0 & 0 & \cdots & 0 \\ 0 & 0 & b^{x_2} & 0 & \cdots & 0 \\ 0 & 0 & 0 & b^{x_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & b^{x_{m-1}} \\ b^{x_0} & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{2.4}$$

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By (2.2), for each $p \geq 0$, writing $p = qm + r$ with $q, r \in \mathbb{Z}$ and $0 \leq r \leq m-1$, the exponent of b in each nonzero entry of A^p has the form

$$q + \sum_{\ell=0}^{r-1} m^{-1}(a_{i+\ell} - a_{i+\ell+1} + 1) \quad \text{for some } i = 0, 1, \ldots, m-1.$$

As such, to prove $\mathcal{S}(A) = S$, we must show that the condition

$$q + m^{-1}(a_i - a_{i+r} + r) \geq 0 \quad \text{for every } i = 0, 1, \ldots, m-1, \tag{2.5}$$

holds if and only if $p \in \mathcal{S}(A)$. This clearly holds whenever $m \mid p$ since in this case, $r = 0$ and $q \geq 0$. By the definition of the Apéry set, it suffices to prove that for each $j = 1, \ldots, m-1$, we have

(a) $a_j \in \mathcal{S}(A)$ and
(b) $a_j - m \notin \mathcal{S}(A)$.

Indeed, if $p = a_j$, then for each $i = 0, 1, \ldots, m-1$, (2.5) becomes

$$m^{-1}(a_j - j) + m^{-1}(a_i - a_{i+j} + j) = m^{-1}(a_j + a_i - a_{i+j}) \geq 0,$$

wherein nonnegativity follows from (2.1). Additionally, if $p = a_j - m$, then choosing $i = 0$,

$$m^{-1}(a_j - m - j) + m^{-1}(a_0 - a_j + j) = m^{-1}(a_0 - m) = -1,$$

so condition (2.5) does not hold. This completes the proof. □

3. Examples and remarks

This section contains several remarks and illustrative examples of Theorem 1.1 that demonstrate the effectiveness of our main result, along with an extension of Theorem 1.1 to arbitrary subsemigroups of $\mathbb{N}$. We begin with a careful analysis of how things play out for the so-called McNugget semigroup.

EXAMPLE 3.1. Consider $S = \langle 6, 9, 20 \rangle$. Then, $m(S) = 6$ and $c(S) = 44$, so the construction of [6, Theorem 6.2] produces a $B \in \mathrm{M}_{44}(\mathbb{Q})$ such that $\mathcal{S}(B) = S$. In contrast, Theorem 1.1 produces an $A \in \mathrm{M}_6(\mathbb{Q})$ such that $\mathcal{S}(A) = S$. Indeed, one can check that $\mathrm{Ap}(S) = \{0, 49, 20, 9, 40, 29\}$, so the proof of Theorem 1.1 yields

$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5 \\ -8 \\ 5 \\ 2 \\ -5 \\ 2 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 0 & \frac{1}{256} & 0 & 0 & 0 & 0 \\ 0 & 0 & 32 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{32} & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \\ 32 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

In fact, this establishes $\dim_{\mathrm{mat}} S = 6$ by Corollary 4.2 below.

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EXAMPLE 3.2. If $S = \langle 5, 11\rangle$, then $\mathrm{Ap}(S) = \{0, 11, 22, 33, 44\}$, so Theorem 1.1 yields

$$A = \begin{bmatrix} 0 & b^{-2} & 0 & 0 & 0 \\ 0 & 0 & b^{-2} & 0 & 0 \\ 0 & 0 & 0 & b^{-2} & 0 \\ 0 & 0 & 0 & 0 & b^{-2} \\ b^{9} & 0 & 0 & 0 & 0 \end{bmatrix}.$$

More generally, if $S = \langle m, km + 1\rangle$ with $k \in \mathbb{N}$, then $\mathrm{Ap}(S)$ comprises integer multiples of $km + 1$, so $a_i = i(km + 1)$ for $i = 1, 2, \ldots, m - 1$. The only nonzero integral entry of the resulting matrix is in the lower-left corner; the remaining nonzero entries are identical to each other.

REMARK 3.3. The values of $x_0, x_1, \ldots, x_{m-1}$ in the proof of Theorem 1.1 can also be expressed in terms of the so-called *Kunz coordinates* $(k_1, k_2, \ldots, k_{m-1})$ of S, which are defined so that $a_i = k_i m + i$ for each $i = 1, 2, \ldots, m - 1$ [11]. In particular,

$$x_i = \begin{cases} -k_1 & \text{if } i = 1, \\ k_{i-1} - k_i & \text{if } i = 2, 3, \ldots, m - 1, \\ k_{m-1} + 1 & \text{if } i = 0. \end{cases}$$

Translating (2.1) in terms of Kunz coordinates requires the use of cases; this motivates the choice of expression for x_i in the proof of Theorem 1.1.

REMARK 3.4. The parameter $b \in \mathbb{Z}\backslash\{-1, 0, 1\}$ in the proof of Theorem 1.1 is arbitrary. Laplace (cofactor) expansion of (2.4) and using (2.3) ensure that $\det A = (-1)^{m-1} b$, so $\det A$ is arbitrary in $\mathbb{Z}\backslash\{-1, 0, 1\}$ and essentially independent of S. However, $\det A = \pm 1$ implies that S is cyclic [6, Theorem 4.2]. Moreover, $\det A = 0$ whenever A is nilpotent, and [6, Theorem 6.2] ensures that every numerical semigroup is the exponent semigroup of a nilpotent matrix.

EXAMPLE 3.5. A small adjustment to the proof of Theorem 1.1 permits one to find a representing matrix for any given subsemigroup of $\mathbb{N}$, numerical or not. Let us consider $S = \langle 6, 8, 10\rangle$. Since $S = 2T$, in which $T = \langle 3, 7, 11\rangle$ is a numerical semigroup, we have $T = \mathcal{S}(A)$ and $S = \mathcal{S}(B)$ for

$$A = \begin{bmatrix} 0 & 2^{-2} & 0 \\ 0 & 0 & 2^{-1} \\ 2^{4} & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 \cdot 2^{-2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \cdot 2^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{3} \\ 2^{4} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We record this observation here.

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COROLLARY 3.6. *If $S \subseteq \mathbb{N}$ is an additive subsemigroup, then* $\dim_{\text{mat}} S \leq \min(S \backslash \{0\})$.

PROOF. Suppose $S = dT$, where T is a numerical semigroup and $d \in \mathbb{N}$ is positive. Let $m = m(T)$. As in the proof of Theorem 1.1, fix $x_0, x_1, \ldots, x_{m-1} \in \mathbb{Z}$ such that

$$A = \begin{bmatrix} 0 & 2^{x_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 2^{x_{m-1}} \\ 2^{x_0} & 0 & \cdots & 0 \end{bmatrix}$$

has $\mathcal{S}(A) = T$. Let $B \in M_{dm}(\mathbb{Q})$ be the matrix

$$B = \begin{bmatrix} 0 & z_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & z_{dm-1} \\ z_0 & 0 & \cdots & 0 \end{bmatrix}$$

defined by

$$z_i = \begin{cases} 3^{d-1} 2^{x_k} & \text{if } i = kd \text{ with } k \in \mathbb{Z}, \\ \frac{1}{3} & \text{otherwise.} \end{cases}$$

By (2.2), for any $p \in \mathbb{N}$, the nonzero entries in B^{pd} are precisely those that appear in A^p, so $pd \in \mathcal{S}(B)$ if and only if $p \in \mathcal{S}(A)$. However, any power of B not divisible by d has at least one noninteger entry with a power of 3 in the denominator, so $\gcd(\mathcal{S}(B)) = d$. As such, we conclude $\mathcal{S}(B) = S$. □

4. Irreducible numerical semigroups

Fix a numerical semigroup S and let $F = c(S) - 1$. Recall that:

(a) S is *symmetric* if $x \in \mathbb{Z} \backslash S$ implies $F - x \in S$;
(b) S is *pseudosymmetric* if F is even and $x \in \mathbb{Z} \backslash S$ implies $F - x \in S$ or $x = F/2$;
(c) S is *irreducible* if S cannot be written as an intersection of finitely many numerical semigroups properly containing it.

A numerical semigroup is irreducible if and only if it is symmetric or pseudosymmetric [14], and these two families of numerical semigroups are each of interest in commutative algebraic settings (see [4, 10], respectively).

REMARK 4.1. Every numerical semigroup can be written as an intersection of finitely many irreducible numerical semigroups, and such expressions are often far from unique [5, 13]. In some cases, one can use this fact and [6, Theorem 2.3(a)] to obtain a more optimal construction than Theorem 1.1. For example, Theorem 1.1 identifies $A \in \mathrm{M}_{15}(\mathbb{Q})$ with exponent semigroup $\mathcal{S}(A) = \langle 15, 20, 21, 25, 26 \rangle$, but

$$\mathcal{S}(A) = T \cap T' \quad \text{with } T = \langle 3, 11 \rangle \text{ and } T' = \langle 5, 16 \rangle,$$

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so one can obtain a block-diagonal matrix $A' \in \mathrm{M}_8(\mathbb{Q})$ with $\mathcal{S}(A') = \mathcal{S}(A)$ by applying Theorem 1.1 to T and T'. Note that this strategy would be ineffective with the construction in [6, Theorem 6.2] since $c(T \cap T') = \max(c(T), c(T'))$, while $m(T \cap T')$ can be much larger than $m(T) + m(T')$.

Remark 4.1 does not aid in obtaining the matricial dimension of irreducible numerical semigroups, since they cannot be written as an intersection of finitely many other numerical semigroups. Luckily, Theorem 1.1 and [6] together identify the matricial dimension of nearly all such semigroups. We record this here.

COROLLARY 4.2. *If S is a symmetric numerical semigroup, then* $\dim_{\mathrm{mat}} S = m(S)$. *In particular, if $e(S) = 2$, then* $\dim_{\mathrm{mat}} S = m(S)$.

PROOF. This follows from Theorem 1.1 since $\dim_{\mathrm{mat}} S \geq m(S)$ by [6, Theorem 5.3]. Additionally, $e(S) = 2$ implies S is symmetric by [15, Corollary 4.7]. □

COROLLARY 4.3. *Let S be a nontrivial pseudosymmetric numerical semigroup.*

(a) *If $c(S) \leq m(S)$, then* $\dim_{\mathrm{mat}} S = 2$.
(b) *If $m(S) < c(S) \leq 2m(S)$, then* $m(S) - 1 \leq \dim_{\mathrm{mat}} S \leq m(S)$.
(c) *If $c(S) > 2m(S)$, then* $\dim_{\mathrm{mat}} S = m(S)$.

PROOF. Combine Theorem 1.1 with the inequalities in [6, Theorem 5.6]. □

EXAMPLE 4.4. The semigroup $S = \langle 3, 5, 7 \rangle$ is pseudosymmetric with $m(S) = 3$ and $c(S) = 5$. One can readily check that $\mathcal{S}(A) = S$ for

$$A = \begin{bmatrix} 1 & -\frac{3}{16} \\ 16 & 1 \end{bmatrix},$$

so $\dim_{\mathrm{mat}} S = 2 = m(S) - 1$. We conjecture that $\dim_{\mathrm{mat}} S = m(S) - 1$ whenever S is pseudosymmetric and $m(S) < c(S) \leq 2m(S)$.

EXAMPLE 4.5. Consider $S = \langle 7, 54, 66 \rangle$, which has $m(S) = 7$, $c(S) = 192$ and

$$\mathrm{Ap}(S) = \{0, 120, 198, 66, 186, 54, 132\}.$$

Since S is neither symmetric or pseudosymmetric, the previous corollaries do not determine $\dim_{\mathrm{mat}} S$. Theorem 1.1 ensures that $\dim_{\mathrm{mat}} S \leq 7$, whereas [6, Theorem 6.2] provides the much weaker bound $\dim_{\mathrm{mat}} S \leq 192$. However, we can prove that $\dim_{\mathrm{mat}} S = 7$ as follows. Suppose towards a contradiction that $S = \mathcal{S}(A)$, in which $A \in \mathrm{M}_d(\mathbb{Q})$ with $1 \leq d \leq 6$. Then, $185, 186, 187, 188, 189, 190 \in S$ ensures that $\mathcal{S}(A)$ contains all successive natural numbers [6, Theorem 5.1]. This contradicts the fact that $c(S) - 1 = 191 \notin S$. Therefore, $\dim_{\mathrm{mat}} S \geq 7$, so $\dim_{\mathrm{mat}} S = 7$.

EXAMPLE 4.6. These methods are insufficient to compute the matricial dimension of all numerical semigroups. For example, the longest string of consecutive elements in $S = \langle 39, 40, 47 \rangle$ below $c(S) = 390$ is $\{351, 352, \ldots, 381\}$, which has length 31. Thus, $32 \leq \dim_{\mathrm{mat}} S \leq 39$ by [6, Theorem 5.1] and Theorem 1.1. Since S is neither symmetric nor pseudosymmetric, we cannot appeal to the corollaries above.

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