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RESEARCH ARTICLE

Analysis for series–parallel systems under Marshall–Olkin run shock model

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This paper develops a reliability framework for series–parallel systems subject to run-dependent Marshall–Olkin (MO) shocks, in which each parallel block may contain an arbitrary number of components sharing a block-common shock source. Within a parallel block, the component lifetimes are dependent, since they share the block-common run-shock time. This dependence prevents a direct application of the phase-type maximum closure. We resolve this obstacle through a structural decomposition lemma that expresses each block lifetime in terms of mutually independent random variables. This lemma allows exact phase-type representations at all system levels through nested Kronecker product constructions. From these representations, we obtain survival functions, mean time to failure, variance, hazard rate, and mean residual life in closed matrix-exponential form. We also show that the conditional mean residual life of any component, given the survival of all other components in its block, depends on their observation times only through their maximum. This is a structural invariance property of the MO parallel-block model. Closed-form dimension formulas describe how the state-space size grows with the system size, the phase dimensions, and the run-length threshold k. The methodology is illustrated on a three-block series–parallel system with Erlang-2 interarrival times.

1. Introduction

Series–parallel systems arise across critical infrastructures such as power networks, manufacturing lines, telecommunication architectures, and supply chains where components within the same subsystem often share environmental stressors that classical independent-failure models cannot represent. The Marshall–Olkin (MO) framework [15] provides a natural mechanism for such dependence: each component is exposed to a private shock source affecting only that component and to a common shock source shared by all components in the same block, so that failure occurs when either source accumulates sufficient damage. When shock damage is governed by a run mechanism requiring k consecutive critical shocks rather than a single harmful event [14], the model captures the inherent resilience of engineered systems to isolated stressors while remaining analytically tractable through the theory of phase-type (PH) distributions [2, 13, 18].

Run shock models have been developed in several directions, including discrete-time formulations [9], generalized δ-shock variants [8], and Polya process arrivals [10]. Closely related δ-shock models include the system reliability analysis of Tuncel and Eryilmaz [28] and the mixed δ-shock model with a change in shock distribution of Chadjiconstantinidis et al. [6]. Eryilmaz and Kan [11] extended the run shock concept by allowing the shock size distribution to change after the first shock exceeding a warning threshold, establishing that the resulting system lifetime has a matrix-exponential

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distribution when interarrival times follow an Erlang law. More recently, Ozkut et al. [23] developed a multi-state system model in which runs of shocks cause partial damage before complete system failure. Fan and Tang [12] studied a mixed run shock model that combines run shocks, extreme shocks, and degradation, and used phase-type distributions to evaluate the hard failure process. A recent review of MO shock models and their applications in reliability is given by Bayramoglu and Ozkut [5]. Bakry and Mokhlis [3] studied the reliability of a k-out-of-n:G system exposed to component-specific and common MO shocks, while also accounting for shock magnitudes. MO type shock models have also been studied by Ozkut and Bayramoglu [20], Bayramoglu and Ozkut [4], Rafiee et al. [24], and Durante et al. [7]. Phase-type distributions, whose closure properties under minimum, maximum, and random summation have been applied to shock models by Montoro-Cazorla et al. [16, 17] and Segovia and Labeau [25], and to two-component systems by Neuts and Meier [19]. More recently, Wang et al. [30] used phase-type distributions to study a heterogeneous system exposed to several shock sources under a mixed shock model. Recent work by Torrado and Ozkut [27] considers series–parallel systems with dependent components. Ozkut and Kan [22] later applied the MO run shock model to coherent systems with phase-type shock interarrival times. Most relevantly, Ozkut and Eryilmaz [21] introduced the MO run shock model for a *two-component system* with PH interarrival times, deriving the joint survival function and mean residual life in closed matrix-exponential form.

The aim of this paper is twofold. First, we extend the MO run shock model from the two-component system of [21] to a general m-block series–parallel system with an arbitrary number of parallel components per block. Second, we derive its reliability measures in closed matrix-exponential form. This is not a routine application of known closure results. The parallel blocks create a dependence that blocks the standard phase-type maximum closure, and removing this obstacle requires the new decomposition lemma below. The paper also proves a new invariance property of the conditional mean residual life. We state both contributions precisely in what follows.

The present paper addresses the natural but technically non-trivial extension of [21] to m-block series–parallel systems in which each block contains an arbitrary number $n_i \geq 1$ of parallel components. This extension introduces a fundamental obstacle absent from [21]. In [21], every block contains a single component, so all random variables involved are independent and only the PH minimum closure is needed. Once a block contains $n_i \geq 2$ parallel components that share a common run-shock time $T_{iC}(k)$, the component lifetimes within the block become *dependent* through this shared term. Direct application of the PH maximum closure, which requires independence of its arguments, is therefore blocked. Lemma 4.1 resolves this obstacle by establishing the representation

$$S_i = \min\left\{T_{iC}(k), \max_{1\leq j\leq n_i} T_{ij}(k)\right\},$$

which expresses the block lifetime entirely in terms of mutually independent run-shock times, restoring full applicability of the PH closure machinery. This lemma is the structural hinge on which the entire hierarchical construction rests, and it has no counterpart in [21]. A second contribution, developed in Section 5, establishes that the conditional mean residual life of any component given the survival of all other components in its block depends on their observation times only through their maximum, a structural invariance property specific to the MO parallel-block architecture with no direct analog in independent-failure models.

The main contributions of this work are as follows.

(i) A structural decomposition that overcomes the dependence obstacle created by shared common shocks in parallel blocks. By reformulating each block lifetime in terms of independent run-shock times, the PH maximum closure, an operation not required in [21], becomes applicable, yielding exact phase-type representations for block lifetimes via nested Kronecker products.
(ii) Exact phase-type representations for component, block, and system lifetimes under the MO run shock mechanism, generalizing [21] from a single-block pair to an arbitrary series–parallel architecture with $n_i \geq 1$ parallel components per block.

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(iii) Survival functions, mean time to failure, variance, hazard rate, and mean residual life are computable without Monte Carlo simulation (equations (17)–(19) and Theorem 4.5).
(iv) Explicit matrix-exponential formulas for the conditional MRL of any component given multi-component survival observations within the same block, together with the maximum-observation-time invariance property (Corollary 5.1).
(v) Closed-form expressions quantifying how state-space complexity scales with the number of blocks, components per block, phase dimensions, and the run-length threshold k.

We close the introduction by stating how these contributions relate to the existing literature. The two-component model of [21] is the closest work. There, every block holds a single component, so the component lifetimes are independent and only the phase-type minimum closure is used. Our model allows any number of parallel components per block. This makes the within-block lifetimes dependent and forces the use of the maximum closure, and Lemma 4.1 is what makes this step possible. This lemma does not appear in [21]. The run shock model with a change in shock size distribution of Eryilmaz and Kan [11] treats a single component and a different mechanism, so the dependence studied here does not arise there. Phase-type methods have been used for shock and wear models by [16, 17, 25] and for two-component systems by Neuts and Meier [19], but not for the MO run shock model on a general series–parallel architecture with block-common shocks. Recent studies have also used phase-type methods for mixed run shocks and multi-source shock environments [12, 30]. However, their failure mechanisms and system structures differ from the block-common MO setting considered here. The maximum closure we use follows the generalized geometric results of Tank and Eryilmaz [26]. To our knowledge, the conditional MRL invariance in Corollary 5.1, which says that the companion observations enter only through their maximum, has not been reported before.

The paper is organized as follows. Section 2 collects the necessary PH distribution theory and Kronecker algebra. Section 3 defines the system model and shock structure. Section 4 derives the hierarchical reliability construction. Section 5 develops the conditional MRL results. Section 6 presents a numerical illustration. Section 7 concludes.

2. Mathematical preliminaries

Throughout, $\mathbf{1}_n$ denotes the $n \times 1$ vector of ones, I_n the $n \times n$ identity matrix, and $\otimes$ the Kronecker product.

2.1. Phase-type distributions

Definition 2.1. *A non-negative continuous random variable X has a* continuous phase-type distribution*, written $X \sim \mathrm{PH}_c(\alpha, A)$, if it represents the time to absorption in a continuous-time Markov chain with m transient states and one absorbing state. The subgenerator matrix $A \in \mathbb{R}^{m\times m}$ satisfies $a_{ii} < 0$, $a_{ij} \geq 0$ for $i \neq j$, and $A\mathbf{1}_m \leq \mathbf{0}_m$; the initial vector $\alpha \in \mathbb{R}^{1\times m}$ satisfies $\alpha\mathbf{1}_m \leq 1$. The survival function is*

$$\bar{F}_X(t) = P(X > t) = \alpha \exp(At)\mathbf{1}_m, \qquad t \geq 0. \tag{1}$$

The integer m is called the order *of the representation.*

The class of continuous phase-type distributions includes exponential, Erlang, hypoexponential, and hyperexponential distributions as special cases and is dense in the set of all distributions on $(0, \infty)$ [1].

Proposition 2.1 ([13]). *Let $X \sim \mathrm{PH}_c(\alpha, A)$ with $\alpha\mathbf{1}_m = 1$. Then*

$$E[X^n] = (-1)^n\, n!\, \alpha A^{-n}\mathbf{1}_m, \qquad n \in \mathbb{N}.$$

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In particular, $E[X] = -\alpha A^{-1}\mathbf{1}_m$ *and* $\mathrm{Var}(X) = 2\alpha A^{-2}\mathbf{1}_m - (\alpha A^{-1}\mathbf{1}_m)^2$.

Proposition 2.2 ([13]). *Let* $X \sim \mathrm{PH}_c(\alpha, A)$. *For* $t > 0$, *the conditional residual lifetime satisfies*

$$(X - t \mid X > t) \sim \mathrm{PH}_c\left(\frac{\alpha \exp(At)}{\alpha \exp(At)\mathbf{1}_m},\ A\right).$$

In particular, the mean residual life function is

$$E[X - t \mid X > t] = \frac{-\alpha \exp(At)A^{-1}\mathbf{1}_m}{\alpha \exp(At)\mathbf{1}_m}.$$

Definition 2.2. *Let* $A \in \mathbb{R}^{m\times m}$ *and* $B \in \mathbb{R}^{n\times n}$. *The* Kronecker product $A \otimes B \in \mathbb{R}^{mn\times mn}$ *is the block matrix with* (i,j)*-block* $a_{ij}B$. *The* Kronecker sum *is*

$$A \oplus B := A \otimes I_n + I_m \otimes B \ \in \mathbb{R}^{mn\times mn}.$$

Lemma 2.1. *For* $A \in \mathbb{R}^{m\times m}$ *and* $B \in \mathbb{R}^{n\times n}$,

$$\exp\big((A \oplus B)t\big) = \exp(At) \otimes \exp(Bt), \qquad t \geq 0.$$

Proof. The matrices $A \otimes I_n$ and $I_m \otimes B$ commute, since by the mixed-product rule $(A \otimes I_n)(I_m \otimes B) = A \otimes B = (I_m \otimes B)(A \otimes I_n)$. For commuting matrices, the exponential of a sum factors as a product, so $\exp((A \oplus B)t) = \exp((A \otimes I_n)t)\ \exp((I_m \otimes B)t)$. Each factor is $\exp(At) \otimes I_n$ and $I_m \otimes \exp(Bt)$, and the mixed-product rule gives their product as $\exp(At) \otimes \exp(Bt)$. See [13] for details. □

The following three propositions establish the closure results on which the entire hierarchical construction in Section 4 rests.

Proposition 2.3 ([2, 13]). *Let* $X \sim \mathrm{PH}_c(\alpha_X, A_X)$ *and* $Y \sim \mathrm{PH}_c(\alpha_Y, A_Y)$ *be independent, with dimensions* r_X *and* r_Y. *Then*

$$\min(X, Y) \sim \mathrm{PH}_c(\alpha_X \otimes \alpha_Y,\ A_X \oplus A_Y),$$

with dimension $r_X r_Y$.

Proposition 2.4 ([13, 26]). *Under the same conditions as Proposition 2.3, define* $t_0 = -A_X\mathbf{1}_{r_X}$ *and* $s_0 = -A_Y\mathbf{1}_{r_Y}$. *Then*

$$\max(X, Y) \sim \mathrm{PH}_c(\gamma, L),$$

where

$$\gamma = \big(\alpha_X \otimes \alpha_Y,\ \alpha_X(1 - \alpha_Y\mathbf{1}_{r_Y}),\ (1 - \alpha_X\mathbf{1}_{r_X})\alpha_Y\big) \in \mathbb{R}^{1\times(r_Xr_Y + r_X + r_Y)},$$

$$L = \begin{pmatrix} A_X \oplus A_Y & I_{r_X} \otimes s_0 & t_0 \otimes I_{r_Y} \\ 0 & A_X & 0 \\ 0 & 0 & A_Y \end{pmatrix} \in \mathbb{R}^{(r_Xr_Y + r_X + r_Y)\times(r_Xr_Y + r_X + r_Y)}.$$

The dimension of $\max(X, Y)$ *is* $(r_X + 1)(r_Y + 1) - 1$.

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Remark 2.1. The minimum closure gives dimension $r_X r_Y$ (multiplicative), while the maximum closure gives $(r_X+1)(r_Y+1)-1$ (strictly larger). For systems with substantial parallel redundancy, state-space growth is dominated by iterated maximum operations.

Proposition 2.5 ([13]). *Let $\{X_i\}_{i\geq 1}$ be i.i.d. with $X_i \sim \mathrm{PH}_c(\alpha, A)$ of dimension d and $\alpha \mathbf{1}_d = 1$, and let N be an independent counting random variable having a* discrete phase-type distribution $N \sim \mathrm{PH}_d(a, Q)$ *of dimension k with $a\mathbf{1}_k = 1$ (i.e., N is the absorption time in a discrete-time Markov chain with substochastic transition matrix Q). Then*

$$T = \sum_{i=1}^{N} X_i \sim \mathrm{PH}_c\big(\alpha \otimes a,\ A \otimes I_k + (a_0\alpha) \otimes Q\big),$$

where $a_0 = -A\mathbf{1}_d$, with dimension dk.

2.2. Geometric distribution of order k

The run shock mechanism requires counting the number of Bernoulli trials until k consecutive successes first occur; this counting variable follows the *geometric distribution of order k* [13].

Proposition 2.6 ([13]). *Let $\{Y_i\}_{i\geq 1}$ be i.i.d. Bernoulli with success probability $p \in (0,1)$, and let N be the total number of trials until k consecutive successes occur. Then $N \sim \mathrm{PH}_d(a, Q)$ with $a = (1, 0, \ldots, 0) \in \mathbb{R}^{1\times k}$ and*

$$Q = \begin{pmatrix} 1-p & p & 0 & \cdots & 0 & 0 \\ 1-p & 0 & p & \cdots & 0 & 0 \\ 1-p & 0 & 0 & \ddots & & 0 \\ \vdots & & & \ddots & \ddots & \vdots \\ 1-p & 0 & 0 & \cdots & 0 & p \\ 1-p & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \in \mathbb{R}^{k\times k},$$

where state $j \in \{1, \ldots, k\}$ represents a current run of $j-1$ consecutive successes, and absorption corresponds to reaching run length k.

3. System model

3.1. Series–parallel architecture

We consider a system of m blocks connected in series, where block i contains $n_i \geq 1$ components connected in parallel ($i = 1, \ldots, m$; $j = 1, \ldots, n_i$). Let S_{ij}, S_i, and T denote the lifetimes of component (i,j), block i, and the system, respectively. The structural equations are

$$S_i = \max_{1\leq j\leq n_i} S_{ij}, \qquad T = \min_{1\leq i\leq m} S_i = \min_{1\leq i\leq m}\Big(\max_{1\leq j\leq n_i} S_{ij}\Big). \tag{2}$$

The block-size vector $\boldsymbol{n} = (n_1, \ldots, n_m)$ fully characterizes the architecture; Figure 1 illustrates the general case.

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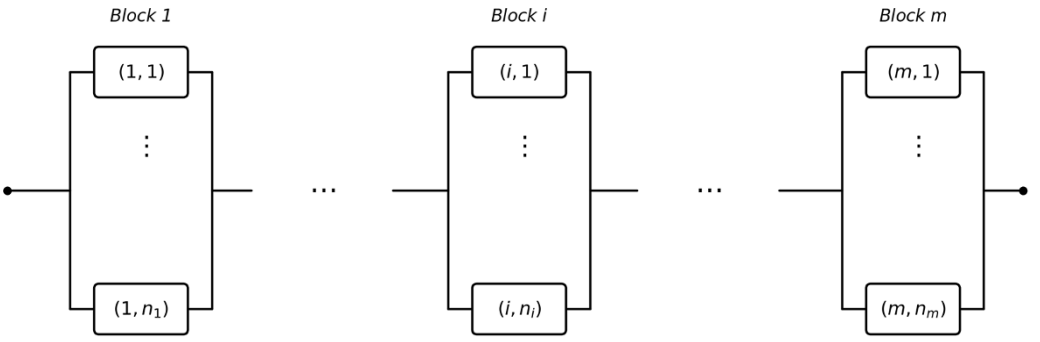


Figure 1. *Series–parallel system with m blocks in series, block i containing n_i parallel components.*

3.2. Marshall–Olkin run shock mechanism

Definition 3.1 (Marshall–Olkin run shock model). *Each component (i,j) is subject to shocks from two mutually independent sources:*

(a) ***Private source*** *(i,j): affects only component (i,j), with i.i.d. interarrival times $X_{ij,n} \sim \mathrm{PH}_c(\alpha_{ij}, A_{ij})$ of phase dimension d_{ij}, and critical shock probability $p_{ij} \in (0,1)$.*
(b) ***Block-common source*** *(i,C): shared by all n_i components in block i, with i.i.d. interarrival times $X_{iC,n} \sim \mathrm{PH}_c(\alpha_{iC}, A_{iC})$ of phase dimension d_{iC}, and critical shock probability $p_{iC} \in (0,1)$.*

Each shock from source r is independently classified as critical *(with probability p_r) or* non-critical *(with probability $1 - p_r$). A non-critical shock resets the consecutive-critical-shock counter to zero. Component (i,j)* fails *when either source (i,j) or source (i,C) first accumulates $k \geq 1$ consecutive critical shocks. All shock processes across different sources are mutually independent.*

The block-common source (i,C) induces positive dependence among the lifetimes of components within block i: any two components (i,j) and (i,j') with $j \neq j'$ share $T_{iC}(k)$, so their lifetimes are positively correlated. Components in different blocks share no common source and are therefore independent.

3.3. Run-shock time distributions

For source r with critical probability p_r, let $N_r(k)$ be the total number of shocks until k consecutive critical shocks occur. By Proposition 2.6, $N_r(k) \sim \mathrm{PH}_d(\boldsymbol{a}, Q_r)$ with $\boldsymbol{a} = (1,0,\ldots,0) \in \mathbb{R}^{1\times k}$ and

$$Q_r = \begin{pmatrix} 1-p_r & p_r & 0 & \cdots & 0 \\ 1-p_r & 0 & p_r & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 1-p_r & 0 & 0 & \cdots & p_r \\ 1-p_r & 0 & 0 & \cdots & 0 \end{pmatrix} \in \mathbb{R}^{k\times k}. \tag{3}$$

The *run-shock time* $T_r(k) = \sum_{n=1}^{N_r(k)} X_{r,n}$ is the time until source r first produces k consecutive critical shocks. By Proposition 2.5, $T_r(k)$ has a phase-type distribution with explicit parameters:

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Proposition 3.1. *The run-shock time $T_r(k)$ satisfies $T_r(k) \sim \mathrm{PH}_c(\boldsymbol{\nu}_r, Z_r)$, where*

$$\boldsymbol{\nu}_r = \alpha_r \otimes \boldsymbol{a}, \qquad Z_r = A_r \otimes I_k + (\boldsymbol{a}_{0,r}\, \alpha_r) \otimes Q_r, \tag{4}$$

with $\boldsymbol{a}_{0,r} = -A_r \mathbf{1}_{d_r}$, and $\bar{d}_r := \dim(Z_r) = d_r \cdot k$.

Proof. We obtain the expression from the random-sum closure of Proposition 2.5. The run-shock time is the random sum $T_r(k) = \sum_{n=1}^{N_r(k)} X_{r,n}$, where the interarrival times $X_{r,n} \sim \mathrm{PH}_c(\alpha_r, A_r)$ are i.i.d. of order d_r, and the shock count $N_r(k)$ is independent of the interarrival times. By Proposition 2.6, $N_r(k)$ has the discrete phase-type representation $\mathrm{PH}_d(\boldsymbol{a}, Q_r)$ of order k, with $\boldsymbol{a} = (1, 0, \ldots, 0)$ and Q_r given by (3). We now apply Proposition 2.5 with $\alpha = \alpha_r$, $A = A_r$, $\boldsymbol{a} = (1, 0, \ldots, 0)$, and $Q = Q_r$. With $\boldsymbol{a}_{0,r} = -A_r \mathbf{1}_{d_r}$, this gives the initial vector $\boldsymbol{\nu}_r = \alpha_r \otimes \boldsymbol{a}$ and the subgenerator $Z_r = A_r \otimes I_k + (\boldsymbol{a}_{0,r}\alpha_r) \otimes Q_r$ of order $\bar{d}_r = d_r k$, which is (4). □

Applying Proposition 3.1 to every shock source gives

$$T_{ij}(k) \sim \mathrm{PH}_c(\boldsymbol{\nu}_{ij}, Z_{ij}), \quad \bar{d}_{ij} = d_{ij}k, \qquad T_{iC}(k) \sim \mathrm{PH}_c(\boldsymbol{\nu}_{iC}, Z_{iC}), \quad \bar{d}_{iC} = d_{iC}k, \tag{5}$$

for $i = 1, \ldots, m$ and $j = 1, \ldots, n_i$. Under Definition 3.1, the *lifetime of component* (i, j) is

$$S_{ij} = \min\{T_{ij}(k),\ T_{iC}(k)\}, \qquad i = 1, \ldots, m;\ j = 1, \ldots, n_i. \tag{6}$$

Equation (6) reflects the competing-risks structure: the component fails due to whichever source first accumulates k consecutive critical shocks. Phase-type representations for S_{ij}, S_i, and T are derived in Section 4.

Example 3.1 (Explicit Erlang-2 building blocks). We make the construction explicit for the Erlang$(2, \lambda)$ interarrival times and $k = 2$ used in Section 6. Here $\alpha_r = (0, 1)$, $A_r = \left(\begin{smallmatrix} -\lambda & 0 \\ \lambda & -\lambda \end{smallmatrix}\right)$, so $\boldsymbol{a}_{0,r} = -A_r \mathbf{1}_2 = (\lambda, 0)^\top$, and the run matrix is $Q_r = \left(\begin{smallmatrix} 1-p & p \\ 1-p & 0 \end{smallmatrix}\right)$. By (4), the run-shock time $T_r(2)$ has order $\bar{d}_r = 4$ with

$$\boldsymbol{\nu}_r = (0, 0, 1, 0), \qquad Z_r = \begin{pmatrix} -\lambda & 0 & \lambda(1-p) & \lambda p \\ 0 & -\lambda & \lambda(1-p) & 0 \\ \lambda & 0 & -\lambda & 0 \\ 0 & \lambda & 0 & -\lambda \end{pmatrix}. \tag{7}$$

The exit vector is $-Z_r \mathbf{1}_4 = (0, \lambda p, 0, 0)^\top$, so absorption occurs at rate λp from the single state in which one critical shock has already occurred, and the next interarrival is in its last phase. The component lifetime $S_{ij} = \min(T_{ij}(2), T_{iC}(2))$ then has the explicit order-16 representation (Theorem 4.1) $\alpha_{ij}^{(S)} = \boldsymbol{\nu}_{ij} \otimes \boldsymbol{\nu}_{iC}$ and $A_{ij}^{(S)} = Z_{ij} \oplus Z_{iC}$, where each factor has the form (7) with its own rate and probability. The block and system generators of Theorems 4.2 and 4.3, and the conditional MRL formulas (28)–(29) of Section 5, are obtained from these same 4×4 matrices through the Kronecker operations. The numerical values in Section 6 use exactly this specialization.

4. Reliability analysis

The analysis proceeds hierarchically through three levels. Table 1 summarizes the complete construction pipeline; the subsections below supply the proofs.

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Table 1. *Hierarchical construction of the system lifetime. All operations use PH closure results from Section 2.*

Level	Quantity	Operation	Dimension
Shock source	$T_r(k)$	Random sum (Prop. 2.5)	$d_r k$
Component	$S_{ij} = \min(T_{ij}(k), T_{iC}(k))$	Minimum (Prop. 2.3)	$d_{ij} d_{iC} k^2$
Private max	$W_i = \max_j T_{ij}(k)$	Iterated max (Prop. 2.4)	$\prod_j (\bar{d}_{ij} + 1) - 1$
Block	$S_i = \min(T_{iC}(k), W_i)$	Minimum (Prop. 2.3)	s_i (Lemma 4.2)
System	$T = \min_i S_i$	Iterated min (Prop. 2.3)	$\prod_i s_i$

4.1. Component lifetime

By (6), $S_{ij} = \min\{T_{ij}(k), T_{iC}(k)\}$. Since $T_{ij}(k)$ and $T_{iC}(k)$ are independent PH random variables (Proposition 3.1), Proposition 2.3 applies directly.

Theorem 4.1. *The lifetime of component (i,j) has distribution $S_{ij} \sim \mathrm{PH}_c(\alpha_{ij}^{(S)}, A_{ij}^{(S)})$, where*

$$\alpha_{ij}^{(S)} = \boldsymbol{\nu}_{ij} \otimes \boldsymbol{\nu}_{iC}, \qquad A_{ij}^{(S)} = Z_{ij} \oplus Z_{iC}, \tag{8}$$

with dimension $r_{ij} = \bar{d}_{ij} \cdot \bar{d}_{iC} = d_{ij} d_{iC} k^2$. The survival function and mean time to failure are

$$\bar{F}_{S_{ij}}(t) = \alpha_{ij}^{(S)} \exp(A_{ij}^{(S)} t) \mathbf{1}_{r_{ij}}, \qquad E[S_{ij}] = -\alpha_{ij}^{(S)} \left(A_{ij}^{(S)}\right)^{-1} \mathbf{1}_{r_{ij}}. \tag{9}$$

Proof. Apply Proposition 2.3 to the independent pair $T_{ij}(k) \sim \mathrm{PH}_c(\boldsymbol{\nu}_{ij}, Z_{ij})$ and $T_{iC}(k) \sim \mathrm{PH}_c(\boldsymbol{\nu}_{iC}, Z_{iC})$, giving (8) with dimension $\bar{d}_{ij} \cdot \bar{d}_{iC} = d_{ij} d_{iC} k^2$. Equations (9) follow from Definition 2.1 and Proposition 2.1. □

Although each S_{ij} has an explicit PH representation, the component lifetimes within block i are *not* independent: for $j \neq j'$, both $S_{ij} = \min\{T_{ij}(k), T_{iC}(k)\}$ and $S_{ij'} = \min\{T_{ij'}(k), T_{iC}(k)\}$ share the common factor $T_{iC}(k)$. This dependence prevents direct application of Proposition 2.4 to $S_i = \max_j S_{ij}$. Lemma 4.1 below resolves this obstacle.

4.2. Block lifetime

Lemma 4.1 (Structural decomposition). *The lifetime of block i satisfies*

$$S_i = \min\left(T_{iC}(k), \max_{1 \le j \le n_i} T_{ij}(k)\right). \tag{10}$$

Proof. For any $t \ge 0$, block i survives beyond t iff at least one component does:

$$\{S_i > t\} = \bigcup_{j=1}^{n_i} \{S_{ij} > t\} = \bigcup_{j=1}^{n_i} \left(\{T_{ij}(k) > t\} \cap \{T_{iC}(k) > t\}\right).$$

Factoring out $\{T_{iC}(k) > t\}$, which is common to every term:

$$\{S_i > t\} = \{T_{iC}(k) > t\} \cap \left\{\max_j T_{ij}(k) > t\right\} = \left\{\min\left(T_{iC}(k), \max_j T_{ij}(k)\right) > t\right\}.$$

Since this holds for all $t \ge 0$, equation (10) follows. □

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Lemma 4.1 expresses S_i entirely in terms of the *mutually independent* run-shock times $T_{iC}(k), T_{i1}(k), \ldots, T_{in_i}(k)$, so Propositions 2.3 and 2.4 both become applicable.

Theorem 4.2. *The lifetime of block i has distribution* $S_i \sim \mathrm{PH}_c(\alpha^{(i)}, A^{(i)})$, *constructed as follows.*

Step 1 (iterated maximum). *Define* $W_i := \max_{1\le j\le n_i} T_{ij}(k)$. *For* $n_i = 1$, $W_i = T_{i1}(k)$. *For* $n_i \ge 2$, *apply Proposition 2.4 iteratively:*

$$V_i^{(1)} := \max(T_{i1}(k), T_{i2}(k)), \quad V_i^{(j)} := \max(V_i^{(j-1)}, T_{i,j+1}(k)), \quad j = 2, \ldots, n_i - 1,$$

each step valid because the arguments are independent. Write $W_i = V_i^{(n_i-1)} \sim \mathrm{PH}_c(\gamma_i^{(W)}, L_i^{(W)})$ *with dimension* $R_i^{(W)}$ *(Lemma 4.2).*

Step 2 (minimum with block-common shock). *Apply Proposition 2.3 to the independent pair* $\big(T_{iC}(k), W_i\big)$*:*

$$S_i = \min(T_{iC}(k), W_i) \sim \mathrm{PH}_c(\alpha^{(i)}, A^{(i)}), \tag{11}$$

where

$$\alpha^{(i)} = \boldsymbol{\nu}_{iC} \otimes \gamma_i^{(W)}, \qquad A^{(i)} = Z_{iC} \oplus L_i^{(W)}. \tag{12}$$

The survival function and mean time to failure of block i are

$$\bar{F}_{S_i}(t) = \alpha^{(i)} \exp(A^{(i)} t)\mathbf{1}_{s_i}, \qquad E[S_i] = -\alpha^{(i)} \big(A^{(i)}\big)^{-1} \mathbf{1}_{s_i}. \tag{13}$$

Proof. Step 1 is valid because $T_{i1}(k), \ldots, T_{in_i}(k)$ arise from distinct private sources and are mutually independent. Step 2 is valid because W_i depends only on private sources $(i,1), \ldots, (i,n_i)$, while $T_{iC}(k)$ arises from the block-common source; mutual independence across distinct sources (Section 3) gives independence of the pair. The expressions in (13) follow from Definition 2.1 and Proposition 2.1. □

Remark 4.1. When $n_i = 1$, Step 1 gives $W_i = T_{i1}(k)$ trivially, and $S_i = \min(T_{i1}(k), T_{iC}(k))$, which recovers Theorem 4.1 as a special case with dimension $s_i = \bar{d}_{iC} \cdot \bar{d}_{i1} = d_{iC} d_{i1} k^2$.

Lemma 4.2. *The dimension of* W_i *is*

$$R_i^{(W)} = \prod_{j=1}^{n_i} (\bar{d}_{ij} + 1) - 1, \tag{14}$$

and the block lifetime dimension is

$$s_i := \bar{d}_{iC} \cdot R_i^{(W)} = \bar{d}_{iC} \cdot \left[\prod_{j=1}^{n_i} (\bar{d}_{ij} + 1) - 1 \right]. \tag{15}$$

Proof. We prove the result by induction on n_i. For $n_i = 2$, Proposition 2.4 gives $R_i^{(1)} = (\bar{d}_{i1} + 1)(\bar{d}_{i2} + 1) - 1 = \prod_{j=1}^{2} (\bar{d}_{ij} + 1) - 1$. If $R_i^{(j-1)} = \prod_{\ell=1}^{j} (\bar{d}_{i\ell} + 1) - 1$, then $R_i^{(j)} = (R_i^{(j-1)} + 1)(\bar{d}_{i,j+1} + 1) - 1 = \prod_{\ell=1}^{j+1} (\bar{d}_{i\ell} + 1) - 1$. Equation (15) follows from the minimum closure dimension $s_i = \bar{d}_{iC} \cdot R_i^{(W)}$. □

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4.3. System lifetime

Since blocks use distinct shock sources (Section 3), the block lifetimes $S_1, \ldots, S_m$ are mutually independent, and $T = \min_i S_i$ is obtained by iterating Proposition 2.3.

Theorem 4.3. *The system lifetime has distribution* $T \sim \mathrm{PH}_c(\alpha_{\mathrm{sys}}, A_{\mathrm{sys}})$, *where*

$$\alpha_{\mathrm{sys}} = \bigotimes_{i=1}^{m} \alpha^{(i)}, \qquad A_{\mathrm{sys}} = \bigoplus_{i=1}^{m} A^{(i)}, \tag{16}$$

with dimension $D_{\mathrm{sys}} = \prod_{i=1}^{m} s_i$. *The survival function, MTTF, and variance are*

$$\bar{F}_T(t) = \alpha_{\mathrm{sys}} \exp(A_{\mathrm{sys}} t) \mathbf{1}_{D_{\mathrm{sys}}}, \tag{17}$$

$$E[T] = -\alpha_{\mathrm{sys}} A_{\mathrm{sys}}^{-1} \mathbf{1}_{D_{\mathrm{sys}}}, \tag{18}$$

$$\mathrm{Var}(T) = 2\alpha_{\mathrm{sys}} A_{\mathrm{sys}}^{-2} \mathbf{1}_{D_{\mathrm{sys}}} - \left(\alpha_{\mathrm{sys}} A_{\mathrm{sys}}^{-1} \mathbf{1}_{D_{\mathrm{sys}}}\right)^2. \tag{19}$$

Proof. We prove the result by induction on m. The base case $m = 2$ follows from a single application of Proposition 2.3. For the inductive step, let $U^{(m-1)} = \min\{S_1, \ldots, S_{m-1}\}$; by hypothesis, $U^{(m-1)} \sim \mathrm{PH}_c\big(\bigotimes_{i=1}^{m-1} \alpha^{(i)}, \bigoplus_{i=1}^{m-1} A^{(i)}\big)$ with dimension $\prod_{i=1}^{m-1} s_i$. Applying Proposition 2.3 to the independent pair $(U^{(m-1)}, S_m)$ gives (16) with dimension $\prod_{i=1}^{m} s_i$. Equations (17)–(19) follow from Definition 2.1 and Proposition 2.1. □

Theorem 4.4. *The system dimension is*

$$D_{\mathrm{sys}} = \prod_{i=1}^{m} s_i = \prod_{i=1}^{m} \left\{ \bar{d}_{iC} \cdot \left[\prod_{j=1}^{n_i} (\bar{d}_{ij} + 1) - 1 \right] \right\}, \tag{20}$$

where $\bar{d}_{ij} = d_{ij}k$ *and* $\bar{d}_{iC} = d_{iC}k$. *For a homogeneous system* ($n_i = n$, $d_{ij} = d_{iC} = d$ *for all* i,j), *this reduces to* $D_{\mathrm{sys}} = \{\bar{d}[(\bar{d}+1)^n - 1]\}^m$ *with* $\bar{d} = dk$.

Proof. Combine Theorem 4.3 and Lemma 4.2. □

Example 4.1. For $m = 3$, $(n_1, n_2, n_3) = (1, 2, 2)$, $k = 2$, and Erlang$(2, \lambda)$ interarrival times ($d = 2$): $\bar{d}_{ij} = \bar{d}_{iC} = 4$ for all sources, giving $s_1 = 16$, $s_2 = s_3 = 96$, and $D_{\mathrm{sys}} = 16 \times 96 \times 96 = 147\,456$. The Kronecker structure allows computing $\exp(A_{\mathrm{sys}}t)$ as $\exp(A^{(1)}t) \otimes \exp(A^{(2)}t) \otimes \exp(A^{(3)}t)$, replacing one large matrix exponential with three small ones.

4.4. Mean residual life and hazard rate

Theorem 4.5. *The mean residual life (MRL) function of the system is*

$$\mu_T(t) := E[T - t \mid T > t] = \frac{-\alpha_{\mathrm{sys}} \exp(A_{\mathrm{sys}} t) A_{\mathrm{sys}}^{-1} \mathbf{1}_{D_{\mathrm{sys}}}}{\alpha_{\mathrm{sys}} \exp(A_{\mathrm{sys}} t) \mathbf{1}_{D_{\mathrm{sys}}}}, \qquad t \geq 0. \tag{21}$$

Proof. By Proposition 2.2, $(T - t \mid T > t) \sim \mathrm{PH}_c\big(\alpha_{\mathrm{sys}} \exp(A_{\mathrm{sys}}t)/[\alpha_{\mathrm{sys}} \exp(A_{\mathrm{sys}}t)\mathbf{1}], A_{\mathrm{sys}}\big)$. Proposition 2.1 then gives (21). □

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Remark 4.2. The hazard rate of T is

$$h_T(t) = \frac{\alpha_{\text{sys}} \exp(A_{\text{sys}} t) \boldsymbol{a}_{0,\text{sys}}}{\alpha_{\text{sys}} \exp(A_{\text{sys}} t) \mathbf{1}_{D_{\text{sys}}}}, \quad \boldsymbol{a}_{0,\text{sys}} = -A_{\text{sys}} \mathbf{1}_{D_{\text{sys}}}.$$

By a standard property of PH distributions [1], both $h_T(t)$ and $\mu_T(t)$ converge as $t \to \infty$ to $\lambda_{\max}$ and $\lambda_{\max}^{-1}$, respectively, where $\lambda_{\max}$ is the largest real eigenvalue of $-A_{\text{sys}}$.

4.5. Computational complexity and feasibility

Theorem 4.4 shows that the nominal state-space dimension D_{sys} grows quickly with the system size and with the run-length threshold k. For the three-block example of Example 4.1 it equals 147 456, and for $k = 4$ it exceeds 2.6×10^7 (Table 5). A dense matrix of order D_{sys} cannot be stored or exponentiated directly. For the example, A_{sys} would need about 2.2×10^{10} double-precision entries, that is, about 170 GB. This is the practical concern raised by such product-form dimensions.

The construction avoids this cost, because the full matrix A_{sys} is never formed. By Lemma 2.1, we have $\exp(A_{\text{sys}} t) = \bigotimes_{i=1}^{m} \exp(A^{(i)} t)$. Since the blocks are independent and $T = \min_i S_i$, the system survival function factors into a product of block survival functions,

$$\bar{F}_T(t) = \prod_{i=1}^{m} \bar{F}_{S_i}(t) = \prod_{i=1}^{m} \alpha^{(i)} \exp(A^{(i)} t) \mathbf{1}_{s_i}. \tag{22}$$

Each factor uses only the block matrix $A^{(i)}$ of order s_i. The mean time to failure and the variance follow from the same factorization by scalar integration,

$$E[T] = \int_0^\infty \prod_{i=1}^{m} \bar{F}_{S_i}(t)\, dt, \qquad E[T^2] = \int_0^\infty 2t \prod_{i=1}^{m} \bar{F}_{S_i}(t)\, dt, \tag{23}$$

so the expressions in (18)–(19) are evaluated without assembling or inverting A_{sys}.

This lowers the storage cost from order D_{sys}^2 to order $\sum_{i=1}^{m} s_i^2$. For the example, this is $16^2 + 96^2 + 96^2 = 18\,688$ entries, instead of about 2.2×10^{10}. The cost of one evaluation of $\bar{F}_T(t)$ is dominated by the largest block exponential, which is $O(\max_i s_i^3)$ under the scaling-and-squaring method. Here $\max_i s_i = 96$, and for the $k = 4$ case the largest block has order 640. Both are routine sizes.

The building blocks are sparse. The run matrix Q_r in (3) has at most two nonzero entries per row, and the source matrix Z_r in (4) inherits this sparsity through the Kronecker sum. When the blocks are small, as in all cases considered here, a dense exponential of each block is already fast, so sparse storage is not required. For systems with much larger blocks, the same factored form combines with sparse storage and with Krylov methods that compute the action $\exp(A^{(i)} t) \mathbf{1}_{s_i}$ directly, without forming the matrix $\exp(A^{(i)} t)$.

The computation is also numerically stable. Each $A^{(i)}$ is a subgenerator, so its eigenvalues have negative real part and $\exp(A^{(i)} t)$ is substochastic with entries in $[0, 1]$. The block survival values therefore lie in $[0, 1]$, their product in (22) decays to zero, and the integrals in (23) converge, so the quadrature is well conditioned. In our implementation, every reliability measure reported in Section 6 was obtained in well under one second on a standard desktop computer.

5. Conditional mean residual life within a block

For block i with n_i parallel components, suppose all components are observed to be functioning at (possibly different) times $t_1, t_2, \ldots, t_{n_i} \geq 0$. The *conditional mean residual life* (MRL) of component $(i, 1)$,

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$$E\big(S_{i1} - t_1 \mid S_{i1} > t_1,\, S_{i2} > t_2,\, \ldots,\, S_{i,n_i} > t_{n_i}\big)\,,$$

quantifies the expected remaining lifetime of that component given all available survival information within the block—a quantity directly relevant to condition-based maintenance scheduling. Recent work also shows how reliability models under shock environments can support protection and preventive maintenance decisions [29]. A key structural consequence of the MO framework is that all intra-block dependence is mediated exclusively through the shared block-common run-shock time $T_{iC}(k)$; as a result, the observation times of components $(i,2),\ldots,(i,n_i)$ influence the conditional MRL of component $(i,1)$ only through their maximum $t^- := \max(t_2,\ldots,t_{n_i})$, regardless of how many components are in the block or what their individual phase-type parameters are. We first establish the integral representation, then evaluate it in closed matrix-exponential form.

Lemma 5.1 (Conditional MRL: integral representation). *Consider block i with $n_i \geq 2$ and let $t^- := \max(t_2,\ldots,t_{n_i})$. The conditional MRL of component $(i,1)$ satisfies:*

Case 1 *$(t_1 \geq t^-)$:*

$$E\big(S_{i1} - t_1 \mid S_{i1} > t_1, \ldots, S_{i,n_i} > t_{n_i}\big) = \int_0^\infty \frac{P(T_{i1}(k) > x + t_1) \cdot P(T_{iC}(k) > x + t_1)}{P(T_{i1}(k) > t_1) \cdot P(T_{iC}(k) > t_1)}\, dx. \tag{24}$$

Case 2 *$(t_1 < t^-)$:*

$$\begin{aligned} E\big(S_{i1} - t_1 \mid S_{i1} > t_1, \ldots, S_{i,n_i} > t_{n_i}\big) = &\int_0^{t^- - t_1} \frac{P(T_{i1}(k) > x + t_1)}{P(T_{i1}(k) > t_1)}\, dx \\ &+ \int_{t^- - t_1}^\infty \frac{P(T_{i1}(k) > x + t_1) \cdot P(T_{iC}(k) > x + t_1)}{P(T_{i1}(k) > t_1) \cdot P(T_{iC}(k) > t^-)}\, dx. \end{aligned} \tag{25}$$

In both cases, the conditional MRL depends on $t_2,\ldots,t_{n_i}$ only through t^-; the parameters (ν_{ij}, Z_{ij}) of the private shock sources of components $(i,2),\ldots,(i,n_i)$ do not appear.

Proof. By the survival-function representation,

$$E(S_{i1} - t_1 \mid \cdots) = \int_0^\infty P\big(S_{i1} - t_1 > x \mid S_{i1} > t_1, \ldots, S_{i,n_i} > t_{n_i}\big)\, dx.$$

Step 1: Expand the joint event. Since $S_{ij} = \min(T_{ij}(k), T_{iC}(k))$, the identity $\bigcap_j \{X > a_j\} = \{X > \max_j a_j\}$ applied to $T_{iC}(k)$ gives

$$\{S_{i1} > s,\, S_{i2} > t_2, \ldots, S_{i,n_i} > t_{n_i}\} = \{T_{i1}(k) > s\} \cap \bigcap_{j=2}^{n_i} \{T_{ij}(k) > t_j\} \cap \{T_{iC}(k) > \max(s, t^-)\}.$$

Step 2: Apply mutual independence. All run-shock times arise from distinct sources and are mutually independent (Section 3), so

$$P(S_{i1} > s,\, \ldots) = P(T_{i1}(k) > s) \cdot \prod_{j=2}^{n_i} P(T_{ij}(k) > t_j) \cdot P(T_{iC}(k) > \max(s, t^-)). \tag{26}$$

Step 3: Form the ratio and cancel. Substituting (26) with $s = x + t_1$ (numerator) and $s = t_1$ (denominator), the product $\prod_{j=2}^{n_i} P(T_{ij}(k) > t_j)$ is strictly positive and cancels, leaving

$$P(S_{i1} - t_1 > x \mid \cdots) = \frac{P(T_{i1}(k) > x + t_1) \cdot P(T_{iC}(k) > \max(x + t_1, t^-))}{P(T_{i1}(k) > t_1) \cdot P(T_{iC}(k) > \max(t_1, t^-))}, \tag{27}$$

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which depends on $t_2, \ldots, t_{n_i}$ only through t^-.

Step 4: Integrate and separate cases. Case 1 ($t_1 \geq t^-$): $\max(t_1, t^-) = t_1$ and $\max(x+t_1, t^-) = x+t_1$ for all $x \geq 0$; integrating (27) over $x \geq 0$ yields (24). **Case 2** ($t_1 < t^-$): $\max(t_1, t^-) = t^-$; the threshold $x = t^- - t_1$ separates two sub-intervals on which $\max(x + t_1, t^-)$ equals t^- and $x + t_1$, respectively. Integrating each part and summing gives (25). □

Remark 5.1. The cancellation in Step 3 reveals the structural reason for the invariance: all intra-block dependence passes through the single shared random variable $T_{iC}(k)$. The private shock sources of components $(i,2), \ldots, (i,n_i)$ are conditionally independent of S_{i1} given $T_{iC}(k)$, so conditioning on their survival times updates only the conditional distribution of $T_{iC}(k)$, which is fully captured by the single scalar t^-, and leaves $T_{i1}(k)$ unaffected. Neither the number of remaining components $n_i - 1$ nor their phase-type parameters appear anywhere in the result.

Remark 5.2. In Case 1 ($t_1 \geq t^-$), the event $\{S_{i1} > t_1\}$ already implies $\{T_{iC}(k) > t_1\} \supseteq \{T_{iC}(k) > t^-\}$, so the companion survival observations add no new information about $T_{iC}(k)$; the conditional MRL coincides with the marginal MRL $E(S_{i1} - t_1 \mid S_{i1} > t_1)$. In Case 2 ($t_1 < t^-$), the companion observations establish $T_{iC}(k) > t^- > t_1$, which is strictly stronger than what $\{S_{i1} > t_1\}$ alone implies. This deferred-common-shock evidence reduces the immediate threat from the block-common source, so the conditional MRL is no smaller than the marginal MRL:

$$E\big(S_{i1} - t_1 \mid S_{i1} > t_1,\, S_{i2} > t_2, \ldots, S_{i,n_i} > t_{n_i}\big) \;\geq\; E(S_{i1} - t_1 \mid S_{i1} > t_1), \quad t_1 < t^-,$$

with equality as $t^- \to t_1^+$.

Theorem 5.1 (Conditional MRL: matrix-exponential form). *Under the conditions of Lemma 5.1, let* $T_{i1}(k) \sim \mathrm{PH}_c(v_{i1}, Z_{i1})$ *and* $T_{iC}(k) \sim \mathrm{PH}_c(v_{iC}, Z_{iC})$ *be from Proposition 3.1, with dimensions* $\bar{d}_{i1}$ *and* $\bar{d}_{iC}$*, and let* $t^- = \max(t_2, \ldots, t_{n_i})$.

Case 1 ($t_1 \geq t^-$)*:*

$$E\big(S_{i1} - t_1 \mid S_{i1} > t_1,\, S_{i2} > t_2,\, \ldots,\, S_{i,n_i} > t_{n_i}\big) = \frac{(v_{i1} \otimes v_{iC})\,\big(-(Z_{i1} \oplus Z_{iC})\big)^{-1}\,\exp\big((Z_{i1} \oplus Z_{iC})\,t_1\big)\,\mathbf{1}}{\big[v_{i1}\exp(Z_{i1}t_1)\mathbf{1}\big]\cdot\big[v_{iC}\exp(Z_{iC}t_1)\mathbf{1}\big]}. \tag{28}$$

Case 2 ($t_1 < t^-$)*:*

$$\begin{aligned} E\big(S_{i1} - t_1 \mid S_{i1} > t_1,\, S_{i2} > t_2,\, \ldots,\, S_{i,n_i} > t_{n_i}\big) = {} & \frac{v_{i1}\,Z_{i1}^{-1}\,\big[\exp(Z_{i1}t^-) - \exp(Z_{i1}t_1)\big]\,\mathbf{1}}{v_{i1}\exp(Z_{i1}t_1)\,\mathbf{1}} \\ & + \frac{(v_{i1} \otimes v_{iC})\,\big(-(Z_{i1} \oplus Z_{iC})\big)^{-1}\,\exp\big((Z_{i1} \oplus Z_{iC})\,t^-\big)\,\mathbf{1}}{\big[v_{i1}\exp(Z_{i1}t_1)\mathbf{1}\big]\cdot\big[v_{iC}\exp(Z_{iC}t^-)\mathbf{1}\big]}. \end{aligned} \tag{29}$$

Proof. Both survival functions are scalars, $P(T_{i1}(k) > t) = v_{i1}\exp(Z_{i1}t)\mathbf{1}$ and $P(T_{iC}(k) > t) = v_{iC}\exp(Z_{iC}t)\mathbf{1}$. We first record two identities.

(i) A scalar equals its own 1×1 Kronecker factor. Using the mixed-product rule $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$ and Lemma 2.1, which gives $\exp(Z_{i1}t) \otimes \exp(Z_{iC}t) = \exp((Z_{i1} \oplus Z_{iC})t)$, we obtain

$$\big[v_{i1}\exp(Z_{i1}t)\mathbf{1}\big]\cdot\big[v_{iC}\exp(Z_{iC}t)\mathbf{1}\big] = (v_{i1} \otimes v_{iC})\exp\big((Z_{i1} \oplus Z_{iC})t\big)\mathbf{1}. \tag{30}$$

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(ii) Let M be a square matrix whose eigenvalues all have strictly negative real part. Then $\exp(Mx) \to 0$ as $x \to \infty$, M is invertible, and

$$\int_0^\tau \exp(Mx)\,dx = M^{-1}\big(\exp(M\tau) - I\big), \qquad \int_0^\infty \exp(Mx)\,dx = -M^{-1}. \tag{31}$$

The eigenvalues of $Z_{i1} \oplus Z_{iC}$ are the pairwise sums of the eigenvalues of Z_{i1} and Z_{iC}. Both Z_{i1} and Z_{iC} are subgenerators, so their eigenvalues have negative real part, and the same holds for $Z_{i1} \oplus Z_{iC}$. Hence (31) applies to both $M = Z_{i1} \oplus Z_{iC}$ and $M = Z_{i1}$.

Case 1 ($t_1 \geq t^-$). By Lemma 5.1, the conditional MRL equals the integral (24). Using (30) at argument $x + t_1$, the numerator integral is

$$\int_0^\infty (v_{i1} \otimes v_{iC}) \exp\big((Z_{i1} \oplus Z_{iC})(x + t_1)\big)\mathbf{1}\,dx = (v_{i1} \otimes v_{iC}) \exp\big((Z_{i1} \oplus Z_{iC})t_1\big) \int_0^\infty \exp\big((Z_{i1} \oplus Z_{iC})x\big)dx\,\mathbf{1}.$$

By (31) the inner integral equals $-(Z_{i1} \oplus Z_{iC})^{-1}$. The factors $\exp((Z_{i1} \oplus Z_{iC})t_1)$ and $(Z_{i1} \oplus Z_{iC})^{-1}$ are functions of the same matrix, so they commute. Dividing by the denominator $P(T_{i1} > t_1)\,P(T_{iC} > t_1)$ gives (28).

Case 2 ($t_1 < t^-$). By Lemma 5.1, the conditional MRL equals the sum of the two integrals in (25).

First term. Since $\int_0^{t^- - t_1} \exp(Z_{i1}(x + t_1))\,dx = \exp(Z_{i1}t_1) \int_0^{t^- - t_1} \exp(Z_{i1}x)\,dx$, identity (31) gives

$$\int_0^{t^- - t_1} v_{i1} \exp(Z_{i1}(x + t_1))\mathbf{1}\,dx = v_{i1}Z_{i1}^{-1}\big[\exp(Z_{i1}t^-) - \exp(Z_{i1}t_1)\big]\mathbf{1}.$$

Dividing by $P(T_{i1} > t_1)$ gives the first term of (29).

Second term. The substitution $u = x - (t^- - t_1)$ moves the lower limit to 0 and replaces $x + t_1$ by $u + t^-$. The integrand becomes $P(T_{i1} > u + t^-)\,P(T_{iC} > u + t^-)$, and by (30) and the same computation as in Case 1,

$$\int_0^\infty (v_{i1} \otimes v_{iC}) \exp\big((Z_{i1} \oplus Z_{iC})(u + t^-)\big)\mathbf{1}\,du = (v_{i1} \otimes v_{iC})\big(-(Z_{i1} \oplus Z_{iC})^{-1}\big) \exp\big((Z_{i1} \oplus Z_{iC})t^-\big)\mathbf{1}.$$

Dividing by $P(T_{i1} > t_1)\,P(T_{iC} > t^-)$ gives the second term of (29). □

Corollary 5.1. *The conditional MRL in Theorem 5.1 depends on $t_2, \ldots, t_{n_i}$ only through $t^- = \max(t_2, \ldots, t_{n_i})$. In particular:*

(a) Adding components to block i (increasing n_i) does not change the conditional MRL of component $(i, 1)$, provided their observation times do not exceed t^-.
(b) Any two observation configurations with the same maximum t^- yield identical conditional MRL values.
(c) As $t^- \to t_1^+$ in Case 2, the conditional MRL converges to the marginal MRL $E(S_{i1} - t_1 \mid S_{i1} > t_1)$.

Proof. Parts (a) and (b) are immediate from the fact that $t_2, \ldots, t_{n_i}$ enter (28)–(29) only through t^-. For part (c): as $t^- \to t_1^+$, the first term of (29) vanishes (upper limit $\to 0$) and the second term converges to (28) at t_1 by continuity of the matrix exponential and right-continuity of $v_{iC}\exp(Z_{iC}t^-)\mathbf{1}$. □

The numerical behavior of Theorem 5.1 is illustrated in Section 6 for the three-block example introduced in Section 3.

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6. Numerical illustration

We illustrate the methodology on a three-block series–parallel system with $m = 3$, $\boldsymbol{n} = (1, 2, 2)$, and Erlang-2 shock interarrival times.

6.1. An engineering application

To fix ideas, we read the three-block example as a small power-supply system, of the kind found in the power networks mentioned in Section 1. The system has three stages in series: an intake stage, a conversion stage, and a distribution stage. The whole chain must work for the system to work, which gives the series structure. Stage 1 has a single unit, and stages 2 and 3 each have two redundant units in parallel, so $\boldsymbol{n} = (1, 2, 2)$.

The shocks are voltage and thermal transients. Each unit faces its own local transients from its own load, which is the private source (i, j). The units in a stage also share a stage-wide stressor, such as transients on the common busbar or heat in the shared enclosure. This is the block-common source (i, C), and it hits every unit in the stage at the same time. A unit withstands an isolated transient, but a run of k consecutive severe transients damages it. This is the run mechanism, with $k = 2$ here. The Erlang-2 interarrival times describe transients that arrive in a fairly regular pattern, less erratic than a Poisson stream.

The results then have a direct reading. The system MTTF is 14.10 time units with the shared transients and 27.19 without them (Table 6), so a model that ignores the stage-wide stressor overestimates the life of the system by about a factor of two. For maintenance, Corollary 5.1 says that, to update the residual-life estimate of one unit, the operator needs only the latest inspection time among the other units in the same stage, not the full inspection history. This keeps condition-based inspection simple.

The model parameters can be estimated from field records. The transient arrival rates λ_r and the critical probabilities p_r follow from event logs and severity thresholds, and phase-type fitting can match the interarrival distribution. A full calibration to a specific data set is left for future work.

6.2. System configuration and parameters

The Erlang$(2, \lambda)$ distribution has phase-type representation $\mathrm{PH}_c(\alpha, A)$ with $d = 2$, $\alpha = (0, 1)$, and $A = \left(\begin{smallmatrix} -\lambda & 0 \\ \lambda & -\lambda \end{smallmatrix}\right)$; its mean is $2/\lambda$. We use the Erlang-2 distribution because it is the simplest phase-type distribution beyond the exponential. It has only two phases, which keeps the matrices small and the example easy to follow, while still exercising the full construction. Unlike the exponential, it is not memoryless and has an increasing hazard rate, so it describes shock interarrivals that are more regular than a Poisson stream. The method itself works for any phase-type interarrival distribution, and Erlang-2 is used here only as a transparent illustration. Table 2 gives the rate and critical probability for each of the eight shock sources; the run-length threshold is $k = 2$ throughout. Figure 2 shows the system architecture.

6.3. Phase-type representations and state-space dimensions

Applying Proposition 3.1 to each source gives $\bar{d}_r = d_r k = 2 \times 2 = 4$ for all eight sources. Theorems 4.1–4.3 then yield the dimensions shown in Table 3.

The system parameters are $\alpha_{\mathrm{sys}} = \alpha^{(1)} \otimes \alpha^{(2)} \otimes \alpha^{(3)} \in \mathbb{R}^{1\times 147456}$ and $A_{\mathrm{sys}} = A^{(1)} \oplus A^{(2)} \oplus A^{(3)} \in \mathbb{R}^{147456\times 147456}$. All subsequent computations exploit the Kronecker factorization $\exp(A_{\mathrm{sys}}t) = \exp(A^{(1)}t) \otimes \exp(A^{(2)}t) \otimes \exp(A^{(3)}t)$, replacing one enormous matrix exponential with three small ones (Remark 4.2). All calculations were performed in MATLAB.

6.4. Reliability measures

Table 4 reports MTTF values at all levels.

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Table 2. *Erlang-2 rate parameters and critical shock probabilities.*

Block	**Source**	**Rate λ_r**	p_r
1	Private $(1,1)$	0.5	0.30
	Block-common $(1,C)$	0.9	0.25
2	Private $(2,1)$	0.6	0.35
	Private $(2,2)$	0.8	0.30
	Block-common $(2,C)$	1.0	0.20
3	Private $(3,1)$	0.4	0.40
	Private $(3,2)$	0.7	0.25
	Block-common $(3,C)$	1.2	0.15

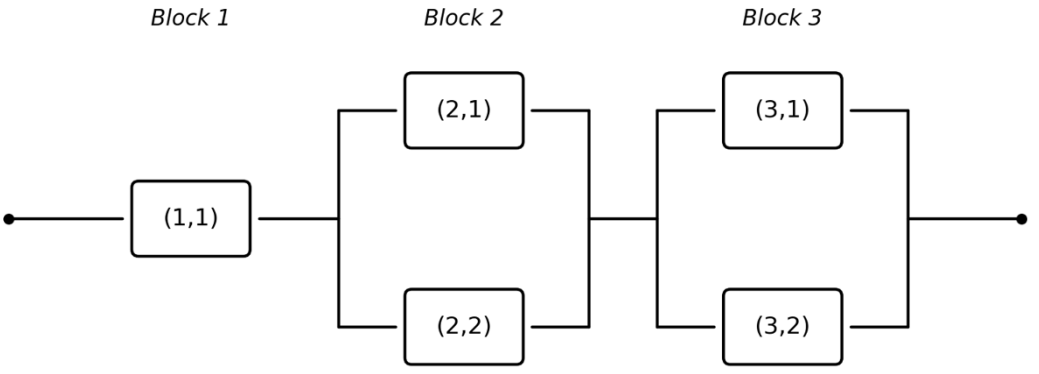


Figure 2. *Three-block series–parallel system, $\boldsymbol{n} = (1,2,2)$.*

Table 3. *State-space dimensions at each hierarchical level.*

Level	**Quantity**	**Dimension**
Run-shock time	$T_r(2)$, each source	4
Component	S_{ij} (all)	$4 \times 4 = 16$
Intermediate max	$W_2 = \max(T_{21}, T_{22})$	$(4+1)(4+1) - 1 = 24$
	$W_3 = \max(T_{31}, T_{32})$	24
Block	S_1 $(n_1 = 1)$	$4 \times 4 = 16$
	S_2 $(n_2 = 2)$	$4 \times 24 = 96$
	S_3 $(n_3 = 2)$	$4 \times 24 = 96$
System	$T = \min(S_1, S_2, S_3)$	$16 \times 96 \times 96 = 147\,456$

Parallel redundancy raises the block MTTF above any individual component MTTF (e.g., $\mathbb{E}[S_2] = 32.00 > 24.10 = \mathbb{E}[S_{21}]$), while the series connection reduces the system MTTF well below any block MTTF. Block 1, with a single component, is the reliability bottleneck. Figure 3 shows MTTF as a function of a common shock rate scale factor λ.

6.5. Sensitivity analysis

Table 5 consolidates the most informative parameter variations: run-length threshold k, common critical probability p (all sources set equal), and block size configuration.

Three conclusions emerge from Table 5. Increasing k yields the largest MTTF gains (a factor of ~ 70 from $k = 1$ to $k = 4$) at the cost of rapidly growing state-space dimension, which scales as $O(k^{2m})$.

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Table 4. *Mean time to failure at component, block, and system level.*

Entity	MTTF (time units)
Component $(1, 1)$	26.7221
Component $(2, 1)$	24.1004
Component $(2, 2)$	23.6682
Component $(3, 1)$	30.4329
Component $(3, 2)$	35.3844
Block 1	26.7221
Block 2	32.0006
Block 3	44.4071
System T	**14.1001**

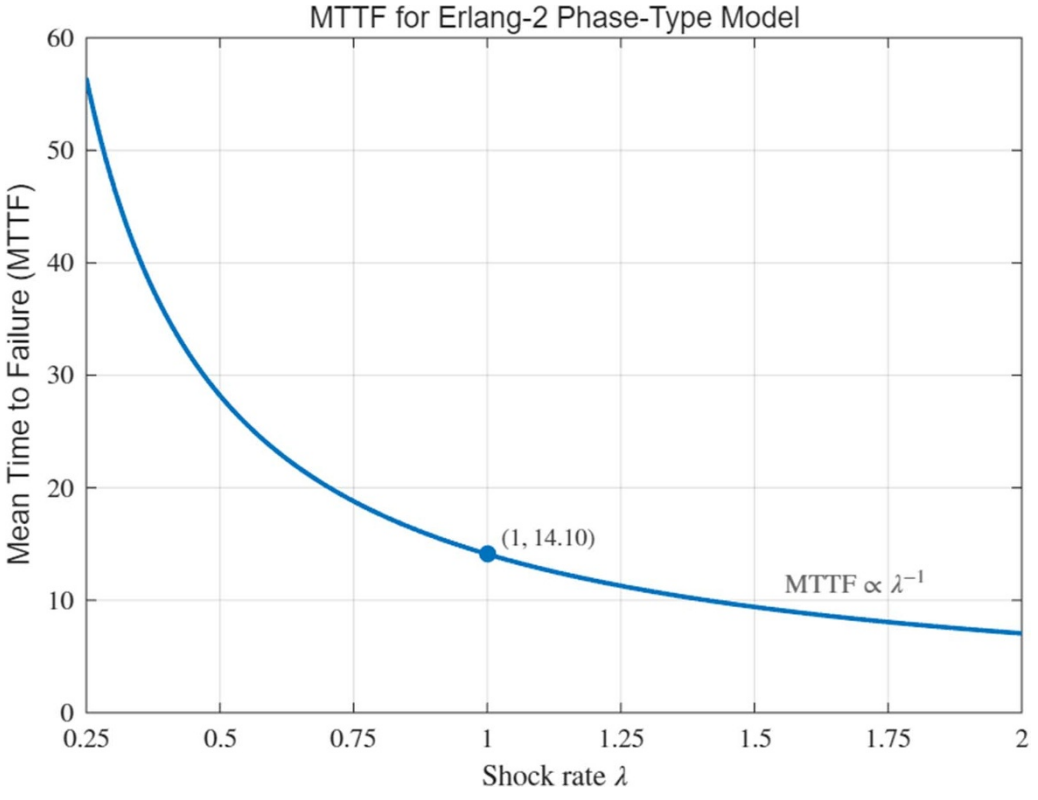


Figure 3. *System MTTF as a function of shock rate scale factor λ. The approximately hyperbolic decay is consistent with the* $\text{MTTF} \propto \lambda^{-1}$ *scaling of PH distributions.*

The critical probability p has a strongly non-linear effect: halving p from 0.4 to 0.2 almost triples the MTTF. Parallel redundancy shows sharply diminishing returns: the first redundant component per block provides the largest improvement, with successive additions yielding progressively smaller gains.

6.6. Comparison with independent failure model

In the independent failure model, block-common shocks are absent and each component fails solely due to its private source: $S_{ij}^{\text{ind}} = T_{ij}(k)$. Table 6 compares key metrics.

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Table 5. *System MTTF and state-space dimension under key parameter variations (baseline = $k = 2$, heterogeneous p_r from Table 2, $\boldsymbol{n} = (1, 2, 2)$).*

Varied parameter	Value	MTTF	D_{sys}
Run-length k	1	2.77	1 024
	2	14.10	147 456
	3	53.56	2 985 984
	4	193.70	26 214 400
Critical prob. p	0.10	58.97	—
(all sources equal)	0.20	17.43	—
	0.30	9.25	—
	0.40	6.20	—
	0.50	4.68	—
Configuration	(1, 1, 1)	11.02	—
$\boldsymbol{n}$	(1, 2, 2)	14.10	147 456
	(2, 2, 2)	15.74	—
	(2, 3, 3)	17.28	—
	(3, 3, 3)	17.84	—

Table 6. *Marshall–Olkin (MO) model vs. independent failure (IND) model.*

Metric	MO model	IND model
$E[S_{11}]$	26.72	57.78
$E[S_{21}]$	24.10	36.73
$E[S_{22}]$	23.67	36.11
$E[S_1]$	26.72	57.78
$E[S_2]$	32.00	53.13
$E[T]$	**14.10**	**27.19**

The MO model yields a system MTTF 48% lower than the independent model. Ignoring block-common shocks produces severely over-optimistic reliability predictions, which could lead to inadequate safety margins in engineering design.

6.7. Conditional MRL within block 2

We apply Theorem 5.1 to Block 2 ($n_2 = 2$), so $t^- = t_2$, using the parameters from Table 2 with $k = 2$. Three reference quantities bracket the conditional MRL family: (i) the marginal MRL $E(S_{21} - t_1 \mid S_{21} > t_1)$ at $t_1 = 0$ equals $E[S_{21}] = 24.10$ time units; (ii) the private-source MRL $E(T_{21}(2) - t_1 \mid T_{21}(2) > t_1)$ at $t_1 = 0$ equals $E[T_{21}(2)] = 36.73$ time units—the upper envelope approached as $t^- \to \infty$; (iii) the conditional MRL for $t_1 \geq t^-$ (Case 1) coincides exactly with the marginal MRL.

Table 7 reports conditional MRL values at $t_1 = 0$ for selected t^-.

Figure 4 displays the full conditional MRL curves over $t_1 \in [0, 3]$ for $t^- \in \{0.5, 1.0, 1.5\}$, together with the marginal MRL (dashed) and the private-source MRL (dotted upper envelope).

Four key observations follow. First, the conditional MRL is strictly increasing in t^- for fixed $t_1 < t^-$, confirming Corollary 5.1: a later companion observation implies stronger deferral of the block-common shock and uniformly improves the residual life outlook for component $(2, 1)$. Second, each conditional curve merges exactly with the marginal MRL at $t_1 = t^-$ and remains coincident for $t_1 > t^-$, visually confirming Remark 5.2. Third, the conditioning effect at $t_1 = 0$ is modest (increments of 0.002, 0.018, 0.064 time units for $t^- = 0.5, 1.0, 1.5$), because the low block-common probability

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Table 7. *Conditional MRL of component* $(2,1)$ *at* $t_1 = 0$ *for selected observation times* t^- *of component* $(2,2)$. $k = 2$*; parameters from Table 2.*

Quantity	t^-	**MRL (time units)**
Marginal MRL: $E[S_{21}]$	—	24.1004
Conditional: $E(S_{21} \mid S_{22} > t^-)$	0.5	24.1021
Conditional: $E(S_{21} \mid S_{22} > t^-)$	1.0	24.1186
Conditional: $E(S_{21} \mid S_{22} > t^-)$	1.5	24.1643
Private-source MRL: $E[T_{21}(2)]$	∞	36.7347

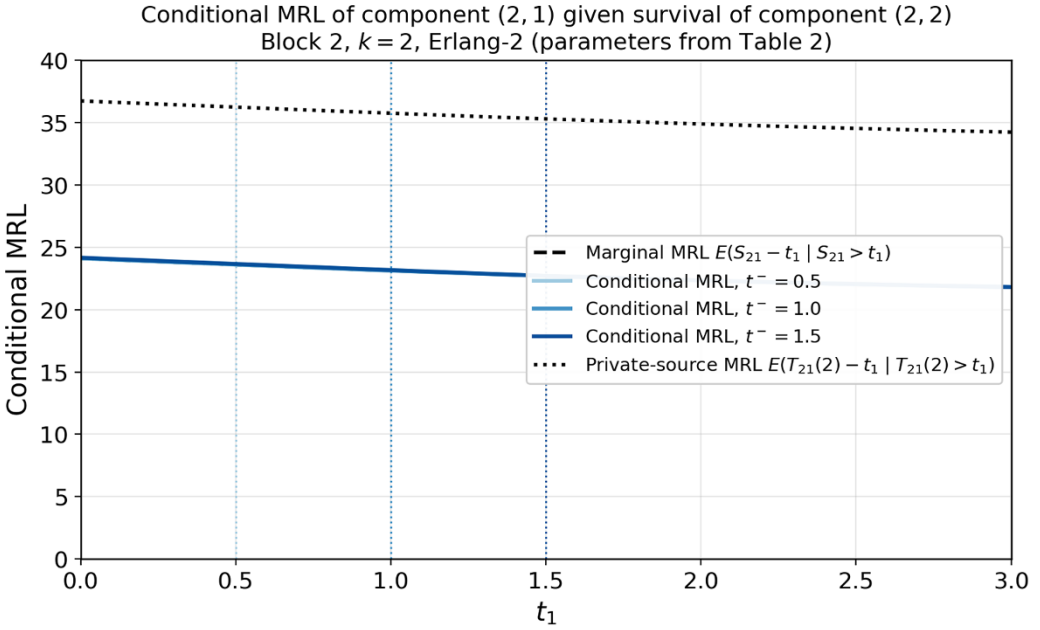


Figure 4. *Conditional MRL of component* $(2,1)$ *given* $S_{22} > t^-$*, for* $t^- \in \{0.5, 1.0, 1.5\}$ *(solid curves, increasing shade). Dashed: marginal MRL. Dotted: private-source MRL (upper envelope). Vertical dotted lines: case boundaries* $t_1 = t^-$*. Block 2,* $k = 2$*.*

$p_{2C} = 0.20$ makes $E[T_{2C}(2)] = 60$ time units far exceed the private mean $E[T_{21}(2)] = 36.73$; in systems where the common shock is the dominant failure mechanism, the conditioning effect would be substantially larger. Fourth, the private-source MRL of 36.73 lies 12.63 time units above the marginal MRL of 24.10, measuring the total contribution of the block-common shock to the failure risk: without it, component $(2,1)$ would be expected to survive 52% longer.

6.8. Monte Carlo validation

We validate the analytical results by direct Monte Carlo simulation. For each shock source we draw the interarrival times as Erlang$(2, \lambda_r)$ and mark each shock as critical with probability p_r. The run-shock time $T_r(k)$ is recorded at the first run of k consecutive critical shocks. The component, block, and system lifetimes are then obtained from $S_{ij} = \min(T_{ij}(k), T_{iC}(k))$, $S_i = \max_j S_{ij}$, and $T = \min_i S_i$. We use $N = 4 \times 10^6$ independent replications with the parameters of Table 2 and $k = 2$. Table 8 compares the analytical MTTF values of Table 4 with the simulated means, the half-width of the 95% confidence interval, and the relative error.

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Table 8. *Analytical MTTF (Theorems 4.1–4.3) versus Monte Carlo estimates ($N = 4\times10^6$ replications).*

Entity	Analytic	MC mean
Component $(1, 1)$	26.7221	26.7116
Component $(2, 1)$	24.1004	24.0891
Component $(2, 2)$	23.6682	23.6620
Component $(3, 1)$	30.4329	30.4169
Component $(3, 2)$	35.3844	35.3857
Block 1	26.7221	26.7116
Block 2	32.0006	31.9963
Block 3	44.4071	44.4111
System T	14.1001	14.0965

The conditional MRL values of Table 7 were reproduced in the same way: at $t_1 = 0$, the simulated $E(S_{21} \mid S_{22} > t^-)$ is 24.09, 24.11, and 24.15 for $t^- = 0.5, 1.0, 1.5$, against the analytical 24.10, 24.12, and 24.16.

The comparison also illustrates the cost of the two approaches. The matrix-exponential method returns the exact values from small block matrices in a fraction of a second. Direct simulation needs several million replications to reach two-decimal accuracy and still leaves a sampling error. The two methods agree, but the analytical method is exact and much cheaper for the same accuracy.

7. Conclusion

This paper has resolved a fundamental obstacle in extending the MO run shock model of [21] from two-component systems to general m-block series–parallel architectures. The obstacle is that component lifetimes within a parallel block are *dependent* through the shared block-common run-shock time $T_{iC}(k)$. This dependence blocks the direct use of the phase-type maximum closure. We removed it with Lemma 4.1, which establishes the representation $S_i = \min(T_{iC}(k), \max_j T_{ij}(k))$ and thereby re-expresses the block lifetime in terms of mutually independent random variables. Building on this lemma, Theorems 4.2–4.3 construct exact phase-type representations for block and system lifetimes via nested Kronecker product operations, and Theorem 4.4 provides closed-form dimension formulas quantifying how state-space complexity scales with system size, phase dimensions, and the run-length threshold k. A second independent contribution, Theorem 5.1, derives closed matrix-exponential formulas for the conditional mean residual life of any component given survival observations of its block companions; Corollary 5.1 establishes the structural invariance property that the result depends on those observations only through their maximum—a property specific to the MO parallel-block architecture with no counterpart in independent-failure models.

We note what is new here relative to the existing literature. Lemma 4.1 does not appear in the two-component model of [21], where the lifetimes are independent, and the maximum closure is not needed. It is the step that makes the hierarchical construction work for blocks with more than one component. The conditional MRL invariance of Corollary 5.1 is, to our knowledge, also new: the survival observations of the other components in a block affect the conditional MRL of a given component only through their maximum, whatever the number of components and their phase-type parameters. As shown in Section 6.1, this gives a simple rule for condition-based maintenance.

The three-block numerical illustration ($m = 3$, $\boldsymbol{n} = (1, 2, 2)$, Erlang-2 interarrival times, $k = 2$) yields a system MTTF of 14.10 time units, compared with 27.19 under the independent failure model—a reduction of 48%, showing that ignoring block-common shocks produces severely over-optimistic predictions. Increasing k from 1 to 4 raises the MTTF by a factor of 70 at the cost of a state-space dimension growing from 1 024 to over 26 million, while halving the critical probability from 0.4 to 0.2

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nearly triples the MTTF. For the conditional MRL within Block 2, the private-source mean of 36.73 time units exceeds the marginal mean of 24.10 by 12.63 time units, quantifying the total contribution of the block-common shock to component failure risk; the conditioning effect grows monotonically with the companion observation time t^-, consistent with Corollary 5.1.

The model has three principal limitations. First, the run-length threshold k is assumed common to all shock sources; allowing source-specific thresholds k_r is conceptually straightforward but requires re-deriving the geometric-order distribution at each source. Second, the phase-type class requires finite-support interarrival distributions; systems with heavy-tailed shock interarrivals would require matrix-exponential approximations with attendant truncation error. Third, the framework is restricted to series–parallel topologies; extension to general coherent systems or k-out-of-n blocks would require replacing the min/max closure pipeline with a more general reliability-function decomposition.

Three concrete extensions emerge naturally from the present work. (i) *Source-specific thresholds*: allowing k_r to vary by source would capture heterogeneous component resilience and require only localized changes to Proposition 3.1 and Lemma 4.1. (ii) *Optimal inspection scheduling*: the conditional MRL invariance in Corollary 5.1 directly motivates a maintenance optimization problem in which inspection times $t_1, \ldots, t_{n_i}$ are chosen to maximize or bound the conditional residual life of the most critical component, exploiting the reduction to the scalar t^- to make the problem analytically tractable. (iii) *Shock–degradation interaction*: combining the run shock mechanism with a cumulative degradation process (Wiener or gamma process) at each component would produce a competing-risks model capturing both sudden-failure and wear-out mechanisms within the same Kronecker algebraic framework.

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Competing interest. The authors have no competing interests to declare.

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