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Transient fields produced by a cylindrical electron beam flowing through a plasma

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Abstract

The out-of-equilibrium situation in which an initially sharp-edged cylindrical electron beam, that could, e.g., model electrons flowing within a wire, is injected into a plasma is considered. A detailed computation of the subsequently produced magnetic field is presented. The control parameter of the problem is shown to be the ratio of the beam radius to the electron skin depth. Two alternative ways to address analytically the problem are considered: one uses the usual Laplace transform approach, the other one involves Riemann's method in which causality conditions manifest through some integrals of triple products of Bessel functions.

Keywords: Beam/plasma interaction; Fast ignition studies; Out-of-equilibrium calculations; Plasma-based accelerators; Wave equations

1. INTRODUCTION

1.1. Motivations

Beam-plasma interactions are nowadays receiving some considerable renewed interest especially in the relatively unexplored regimes of high beam intensities and high plasma densities. One particular motivation lies in the fast ignition schemes (FIS) for inertial confinement fusion (Tabak *et al.*, 1994; Deutsch *et al.*, 1996, 1997). These should involve in their final stage the interaction of an ignition beam composed of MeV electrons laser generated at the critical density surface with a dense plasma target. The exploration of the electron beam transport into the overdense plasma is essential to assess the efficiency of the beam energy deposit. In this matter, transverse beam-plasma instabilities could be particularly deleterious in preventing conditions for burn to be met. Related theoretical studies have been mostly devoted to the linear regime of instabilities originating from current and charge neutralized equilibria (See e.g., Bret *et al.,* 2004, 2005*a*, 2005*b*, 2005*c*, 2007, 2008). However, one may argue that the physics of the fast ignition is intrinsically out-of-equilibrium. This was the motivation to consider in (Firpo *et al.*, 2006) the out-of-equilibrium initial value dynamical problem taking place when a radially inhomogeneous electron forward current is launched into a plasma. The aim of this article is to tackle this problem in a rigorous and detailed way in order to obtain a precise picture of the early-time electromagnetic fields produced. Apart from the general inertial fusion and FIS contexts, discussed respectively by Deutsch 2004; Hoffmann *et al.*, 2005; Deutsch *et al.*, 2008; Norreys *et al.*, 2009, this study should be interesting to general physics research studies involving the propagation of a charged particle beam into a plasma such as the very active field of plasma-based accelerators (See Esarey *et al.*, 2009; Kumar *et al.*, 2010). Interestingly enough, the apparently purely academic case of an electron beam with sharp edges considered here happens moreover to be truly relevant to model electron beams constrained to move along a wire. For instance, the recent study presented in Green *et al.* (2007) involves an experimental implementation of a wire plasma that may be relevant to fast-ignition inertial fusion.

1.2. Framework

The physical system considered is that of a radially inhomogeneous electron forward current launched at an initial time into a plasma with no initial current compensation. The direction of the beam is along the z-axis. This study will be devoted to the extreme case where the beam is cylindrical

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with sharp edges so that the current may be written

$$\mathbf{j}_b(r, z, t) = j_0 H(r_b - r) H(t) \mathbf{e}_z, \tag{1}$$

where H denotes the Heaviside function. The focus is put on this early stage where collisions may be neglected. Ions will be assumed to form a fixed neutralizing background. In order to simplify the analysis, the system is taken to be invariant along the beam direction, z (Firpo *et al.*, 2006; Taguchi *et al.*, 2001; Firpo & Lifschitz, 2007). The plasma density n_p will be taken as uniform and constant.

Maxwell equations in cylindrical coordinates read

$$\frac{1}{r}\frac{\partial E_z}{\partial \theta} - \frac{\partial E_\theta}{\partial z} = -\frac{\partial B_r}{\partial t}, \tag{2}$$

$$\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} = -\frac{\partial B_\theta}{\partial t}, \tag{3}$$

$$\frac{1}{r}\left[\frac{\partial}{\partial r}(rE_\theta) - \frac{\partial E_r}{\partial \theta}\right] = -\frac{\partial B_z}{\partial t}, \tag{4}$$

$$\frac{1}{r}\frac{\partial B_z}{\partial \theta} - \frac{\partial B_\theta}{\partial z} = \mu_0 j_{pr} + \mu_0 j_{br} + \frac{1}{c^2}\frac{\partial E_r}{\partial t}, \tag{5}$$

$$\frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} = \mu_0 j_{p\theta} + \mu_0 j_{b\theta} + \frac{1}{c^2}\frac{\partial E_\theta}{\partial t}, \tag{6}$$

$$\frac{1}{r}\left[\frac{\partial}{\partial r}(rB_\theta) - \frac{\partial B_r}{\partial \theta}\right] = \mu_0\left(j_{pz} + j_{bz}\right) + \frac{1}{c^2}\frac{\partial E_z}{\partial t}, \tag{7}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rE_r) + \frac{1}{r}\frac{\partial E_\theta}{\partial \theta} + \frac{\partial E_z}{\partial z} = \frac{e}{\epsilon_0}\left(Zn_{i0} - n_p - n_b\right), \tag{8}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{1}{r}\frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_z}{\partial z} = 0, \tag{9}$$

where the beam current $\mathbf{j}_b$ acts as a *given* source term. Provided that the density of plasma electrons, that are initially at rest, is much larger that the density of beam electrons, the electron plasma current $\mathbf{j}_p = -en_p\mathbf{v}_p$ is given by linear fluid theory in some initial transient stage, so that

$$\frac{\partial \mathbf{j}_p}{\partial t} = \epsilon_0 \omega_p^2 \mathbf{E}, \tag{10}$$

with $\omega_p^2 = n_p e^2/(m_e \epsilon_0)$. The Reader is referred to the Appendix sections A and B for a detailed derivation of Eq. (10) as well as for a discussion on its validity domain.

We Fourier decompose any field $g(r, \theta, z, t)$ through

$$g(r, \theta, z, t) = \sum_m g^{(m)}(r, z, t)\exp(im\theta).$$

After manipulating Eqs. (2)–(7), in order to express the electric field components as functions of the magnetic field components, one is left with a single wave equation for the magnetic field on $m = 0$, namely one has to solve, for $t > 0$,

$$\frac{1}{c^2}\left(\omega_p^2 + \frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial z^2}\right)B_\theta^{(0)} - \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(rB_\theta^{(0)}\right)\right) + \mu_0\frac{\partial j_{bz}^{(0)}}{\partial r} = 0. \tag{11}$$

We focus here on the rotationally invariant part of the magnetic field. The $m \neq 0$ components, that are initially vanishing, would only become non-zero as a result of a possible beam-plasma instability that may appear in a later stage. Putting $\tau = ct$ and introducing the electron skin depth $\lambda_s = c/\omega_p$, (Eq. (11)), that is to be solved for $t > 0$, reads

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial \tau^2} - \frac{1}{r^2} - \lambda_s^{-2}\right]\psi = \mu_0\frac{\partial j_{bz}}{\partial r} \equiv S(r, z, \tau), \tag{12}$$

with $\psi \equiv B_\theta^{(0)}$ for j_{bz} given by (1).

2. CALCULATION OF THE POLOIDAL MAGNETIC FIELD: THE LAPLACE TRANSFORM'S APPROACH

Eq. (12) may be further simplified by introducing the reduced radial variable $\tilde{r} = r/r_b$, the dimensionless variables $\tilde{z} = z/r_b$, $\tilde{\tau} = ct/r_b$, $\tilde{\psi} = \psi/(\mu_0 j_0 r_b)$ and ratio $\eta \equiv r_b/\lambda_s$. It reads then

$$\left[\frac{1}{\tilde{r}}\frac{\partial}{\partial \tilde{r}}\left(\tilde{r}\frac{\partial}{\partial \tilde{r}}\right) + \frac{\partial^2}{\partial \tilde{z}^2} - \frac{\partial^2}{\partial \tilde{\tau}^2} - \frac{1}{\tilde{r}^2} - \eta^2\right]\tilde{\psi} = -\delta(1 - \tilde{r})H(\tilde{\tau}). \tag{13}$$

This dimensionless system will be used in this Section. For the sake of clarity, we shall drop the tildes writing, from now on in this Section, r, z, τ and ψ in place of $\tilde{r}$, $\tilde{z}$, $\tilde{\tau}$ and $\tilde{\psi}$. Let us now proceed to a Laplace transform in time of Eq. (13) and define

$$\hat{g}(r, s) = \int_0^\infty \exp(-s\tau)g(r, \tau)d\tau.$$

Taking into account the invariance of the problem along z, one is left with

$$\mathcal{L}_1\hat{\psi} \equiv \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) - s^2 - \eta^2 - r^{-2}\right]\hat{\psi} = -s^{-1}\delta(1 - r). \tag{14}$$

The Green function $g(r \mid a)$ Duffy (2001) solving $\mathcal{L}_1 g = -\delta(r - a)$ is readily computed as $g(r \mid a) = I_1\left(\sqrt{s^2 + \eta^2}r^<\right)K_1\left(\sqrt{s^2 + \eta^2}r^>\right)$ with $r^< = \min(r, a)$ and $r^> = \max(r, a)$. The solution of Eq. (14) is then

$$\hat{\psi}(r, s) = \begin{cases} -s^{-1}I_1(\sqrt{s^2 + \eta^2}r)K_1(\sqrt{s^2 + \eta^2}) \text{ for } 0 < r < 1 \\ -s^{-1}I_1(\sqrt{s^2 + \eta^2})K_1(\sqrt{s^2 + \eta^2}r) \text{ for } r \geq 1 \end{cases}. \tag{15}$$

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This must be Laplace inverted to obtain the solution in the real time space. From this expression, let us note that it would be possible to obtain the radial behavior of an approximate solution to our problem in separated time-space variables, valid at large enough times, by using that $\lim_{t\to\infty}\psi(r,t)=\lim_{s\to 0}s\hat{\psi}(r,s)$ (See Kuppers *et al.*, 1973; Firpo *et al.*, 2006). That this would be only an approximate solution will become explicit in Section 3. However, this enables to estimate the large time behavior of the $m=0$ component of the poloidal magnetic field for large values of η using, for $x\gg 1$,

$$I_1(x)\sim\frac{e^x}{\sqrt{2\pi x}}\left(1-\frac{3}{8x}+\ldots\right), \tag{16}$$

$$K_1(x)\sim e^{-x}\sqrt{\frac{\pi}{2x}}\left(1-\frac{1}{8x}+\ldots\right). \tag{17}$$

One obtains

$$\lim_{s\to 0}s\hat{\psi}(r,s)=\begin{cases}-I_1(\eta r)K_1(\eta)\sim-\dfrac{e^{-\eta(1-r)}}{2\eta\sqrt{r}} & \text{for } 0<r<1\\ -I_1(\eta)K_1(\eta r)\sim-\dfrac{e^{-\eta(r-1)}}{2\eta\sqrt{r}} & \text{for } r\geq 1\end{cases}. \tag{18}$$

Therefore, except in the vicinity of the border of the beam (for $r\simeq 1$), the magnetic field decreases strongly when the ratio η becomes large, which will be apparent on Figure 1.

The problem of the analytical obtention of $B_\theta^{(0)}(r,t)=\psi(r,t)$ will be addressed in the following section using an alternative approach. Here we shall present results coming from numerical Laplace inversions of Eq. (15). Figure 1 presents the time evolution of the axisymmetric component of the poloidal magnetic field measured at three different radii, namely at $r=0.5$, $r=1$, and $r=2$, for three values of η. Curves have been obtained using a Gaver-Wynn-Rho algorithm for Laplace transform's inversion presented by Valkó and Abate (2004).

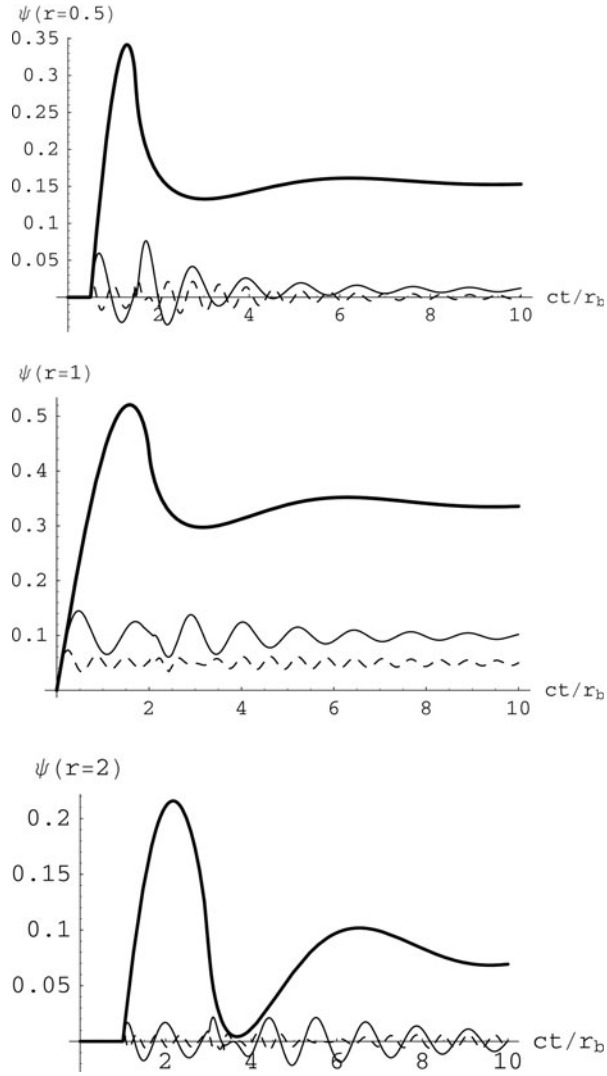


Fig. 1. Initial time behavior of the $m=0$ component of the poloidal magnetic field for three different values of η, namely $\eta=1$ (bold), $\eta=5$ (plain), and $\eta=10$ (dashed line) at different radius: $r=0.5$ (upper plot), $r=1$ (middle plot), and r = 2 (bottom plot).

3. CALCULATION OF THE POLOIDAL MAGNETIC FIELD: THE RIEMANN'S METHOD

3.1. Implementation of Riemann's method

In this Section, the problem will be addressed through a Hankel transform in the radial variable followed by the use of Riemann's method for hyperbolic differential equations in the two independent variables z and τ. In this respect, the present study follows the line of approach pursued in particular in several papers by Borisov and Simonenko (1994) and Borisov (2002).

The Fourier-Bessel transform, also called Hankel transform, of order m of some function F is defined by

$$\hat{F}_m(s,z,\tau)=\int_0^\infty rJ_m(sr)\,F(r,z,\tau)\,dr,$$

while its inverse transform is obtained through

$$F(r,z,\tau)=\int_0^\infty sJ_m(sr)\,\hat{F}_m(s,z,\tau)\,ds.$$

Applying the Hankel transform of order 1 to Eq. (12) yields

$$\left[\frac{\partial^2}{\partial z^2}-\frac{\partial^2}{\partial\tau^2}-s^2-\lambda_s^{-2}\right]\hat{\psi}_1(s,z,\tau)=\hat{S}_1(s,z,\tau). \tag{19}$$

3.2. First possible expression for the solution

The Riemann function R of the equation

$$\left[\frac{\partial^2}{\partial z^2}-\frac{\partial^2}{\partial\tau^2}-s^2-\lambda_s^{-2}\right]\hat{\psi}_1(s,z,\tau)=0, \tag{20}$$

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may be written, by the virtue of Olevsky's theorem (See Borisov (2002), Olevsky (1952), as the following sum of terms

$$R(s,z,\tau;z',\tau') = R_1(z,\tau;z',\tau') + \int_{\tau-\tau'}^{z-z'} d\xi R_1(z,\xi;z',0) \times \frac{\partial}{\partial\xi} R_2(s,\xi,\tau;0,\tau'), \tag{21}$$

where $R_1(z,\tau;z',\tau') = J_0\left(\lambda_s^{-1}\sqrt{(\tau-\tau')^2-(z-z')^2}\right)$ is the Riemann function of Eq. (20) taking $s=0$ and $R_2(s,z,\tau;z',\tau') = J_0\left(s\sqrt{(\tau-\tau')^2-(z-z')^2}\right)$ is the Riemann function of Eq. (20) taking $\lambda_s^{-1}=0$. Eq. (21) reads then

$$R(s,z,\tau;z',\tau') = J_0\left(\lambda_s^{-1}\sqrt{(\tau-\tau')^2-(z-z')^2}\right) + \int_{\tau-\tau'}^{z-z'} d\xi J_0\left(\lambda_s^{-1}\sqrt{\xi^2-(z-z')^2}\right) \times \frac{\partial}{\partial\xi} J_0\left(s\sqrt{(\tau-\tau')^2-\xi^2}\right).$$

Using Riemann's formula, the solution of Eq. (19) reads

$$\hat{\psi}_1(s,z,\tau) = -\frac{1}{2}\int_0^\tau d\tau' \int_{\tau'+z-\tau}^{-\tau'+z+\tau} dz' R(s,z,\tau;z',\tau')\hat{S}_1(s,z,\tau).$$

It remains to inverse Hankel transform through

$$\psi(r,z,\tau) = \int_0^\infty sJ_1(sr)\hat{\psi}_1(s,z,\tau)\,ds \tag{22}$$

$$= -\frac{1}{2}\int_0^\tau d\tau' \int_{\tau'+z-\tau}^{-\tau'+z+\tau} dz' \int_0^\infty sJ_1(sr)R(s,z,\tau;z',\tau') \times \hat{S}_1(s,z,\tau)\,ds, \tag{23}$$

where one has used

$$\hat{S}_1(s,z,\tau) = \mu_0 \int_0^\infty rJ_1(sr)\frac{\partial j_{bz}}{\partial r}dr.$$

The definition of the beam current given in Eq. (1) gives

$$\frac{\partial j_{b0z}}{\partial r} = -j_0\delta(r_b-r)H(\tau),$$

which yields

$$\hat{S}_1(s,z,\tau) = -\mu_0 j_0 r_b J_1(sr_b)H(\tau).$$

Eq. (23) becomes

$$\psi(r,z,\tau) = -\frac{1}{2}\int_0^\tau d\tau' \int_{\tau'+z-\tau}^{-\tau'+z+\tau} dz' \int_0^\infty sJ_1(sr) \times J_0\left(\lambda_s^{-1}\sqrt{(\tau-\tau')^2-(z-z')^2}\right)\hat{S}_1(s,z,\tau)\,ds - \frac{1}{2}\int_0^\tau d\tau' \int_{\tau'+z-\tau}^{-\tau'+z+\tau} dz' \int_0^\infty sJ_1(sr) \int_{\tau-\tau'}^{z-z'} d\xi\, J_0\left(\lambda_s^{-1}\sqrt{\xi^2-(z-z')^2}\right)\frac{\partial}{\partial\xi} \times J_0\left(s\sqrt{(\tau-\tau')^2-\xi^2}\right)\hat{S}_1(s,z,\tau)\,ds,$$

that is

$$\psi(r,z,\tau) = \psi_1(r,z,\tau) + \psi_2(r,z,\tau), \tag{24}$$

with $\tau>0$. The expressions of the two contributions in Eq. (24) may be simplified since, as expected, they may be put in a form explicitly independent on z when moving from variables τ' and z' to variables $t=\tau-\tau'$ and $u=z-z'$. This yields

$$\psi_1(r,\tau) \equiv \frac{\mu_0 j_0 r_b}{2}\int_0^\tau dt \int_{-t}^t du J_0\left(\lambda_s^{-1}\sqrt{t^2-u^2}\right) \times \int_0^\infty sJ_1(sr)J_1(sr_b)\,ds, \tag{25}$$

$$\psi_2(r,\tau) \equiv \frac{\mu_0 j_0 r_b}{2}\int_0^\tau dt \int_{-t}^t du \int_t^u d\xi J_0\left(\lambda_s^{-1}\sqrt{\xi^2-u^2}\right) \times \int_0^\infty dss J_1(sr)J_1(sr_b)\frac{\partial}{\partial\xi}J_0\left(s\sqrt{t^2-\xi^2}\right). \tag{26}$$

Let us begin with the calculation of ψ_1. Using Gradshteyn and Ryzhik (2007),

$$\int_0^\infty sJ_n(as)J_n(bs)ds = \frac{1}{a}\delta(b-a) \tag{27}$$

gives

$$\psi_1(r,\tau) = \frac{\mu_0 j_0}{2}\delta(r-r_b)\int_0^\tau dt \int_{-t}^t du J_0\left(\lambda_s^{-1}\sqrt{t^2-u^2}\right).$$

The integral

$$\int_{-t}^t du J_0\left(\lambda_s^{-1}\sqrt{t^2-u^2}\right) = t\int_0^\pi \sin\theta\, J_0\left(\frac{t\sin\theta}{\lambda_s}\right)d\theta$$

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is evaluated using the identity

$$\int_0^{\pi} \sin(2\mu x)\, J_{2\nu}(2a\sin\theta)\, dx = \pi \sin(\mu\pi)\, J_{\nu-\mu}(a)\, J_{\nu+\mu}(a).$$

This yields

$$\int_{-t}^{t} du J_0\left(\lambda_s^{-1}\sqrt{t^2-u^2}\right) = \pi t J_{-\frac{1}{2}}\left(\frac{t}{2\lambda_s}\right) J_{\frac{1}{2}}\left(\frac{t}{2\lambda_s}\right).$$

Using then $\int_0^X xJ_{-1/2}(x)\, J_{1/2}(x)\, dx = \sin^2 X/\pi$ finally gives

$$\psi_1(r,\tau) = \frac{\pi\mu_0 j_0}{2}\delta(r-r_b)\int_0^{\tau} tJ_{-\frac{1}{2}}\left(\frac{t}{2\lambda_s}\right) J_{\frac{1}{2}}\left(\frac{t}{2\lambda_s}\right)dt$$
$$= 2\lambda_s^2\mu_0 j_0\delta(r-r_b)\sin^2\left(\frac{\tau}{2\lambda_s}\right).$$

This is just

$$\psi_1(r,t) = \lambda_s^2\mu_0 j_0[1-\cos(\omega_p t)]\,\delta(r-r_b). \tag{28}$$

The second contribution to the field involves an integral of a triple product of Bessel functions of the type

$$\mathcal{I}(r, r_b, \alpha) \equiv \int_0^{\infty} sJ_1(sr)J_1(sr_b)J_0(s\alpha)\, ds, \tag{29}$$

for r, r_b, $\alpha > 0$, through which causality relations will come into play. From Gradshteyn and Ryzhik (2007), this yields

$$\mathcal{I}(r, r_b, \alpha) = \frac{P_{1/2}^{1/2}(\cos v)}{rr_b\sqrt{2\pi\sin v}} = \frac{\cos v}{\pi rr_b \sin v} \text{ for } |r-r_b| < \alpha < r+r_b,$$
$$= 0 \text{ if not,}$$

where $P_{1/2}^{1/2}(.)$ is the Legendre function of the first kind, and the angle v satisfies

$$\cos v = \frac{r^2+r_b^2-\alpha^2}{2rr_b} \in [-1;\, 1],$$

and

$$\sin v = \sqrt{1-\frac{(r^2+r_b^2-\alpha^2)^2}{4r^2r_b^2}} \in [0;\, 1].$$

It happens that the expression of $\mathcal{I}(r,r_b,\alpha)$, as written in Gradshteyn and Ryzhik (2007), should be completed by the singular special case

$$\mathcal{I}(r, r_b, 0) = \frac{1}{r_b}\delta(r-r_b), \tag{30}$$

due to Eq. (27). Let us consider now this second component of the poloidal magnetic field

$$\psi_2(r,\tau) = -\frac{\mu_0 j_0 r_b}{2}\int_0^{\tau} dt \int_{-t}^{t} du \int_t^u d\xi J_0\left(\lambda_s^{-1}\sqrt{t^2-\xi^2}\right)$$
$$\times \int_0^{\infty} dss J_1(sr)J_1(sr_b)\frac{\partial}{\partial\xi}J_0\left(s\sqrt{\xi^2-u^2}\right)$$
$$= \frac{\mu_0 j_0 r_b}{2}\int_0^{\tau} dt \int_{-t}^{t} du \int_t^u d\xi \frac{\xi J_0\left(\lambda_s^{-1}\sqrt{t^2-\xi^2}\right)}{\sqrt{\xi^2-u^2}}$$
$$\times \int_0^{\infty} dss^2 J_1(sr)J_1(sr_b)J_1\left(s\sqrt{\xi^2-u^2}\right).$$

Putting $y = \sqrt{\xi^2-u^2}$, it comes

$$\psi_2(r,\tau) = -\mu_0 j_0 r_b \int_0^{\tau} dt \int_0^{t} du \int_0^{\sqrt{t^2-u^2}} dy J_0\left(\lambda_s^{-1}\sqrt{t^2-u^2-y^2}\right)$$
$$\times \int_0^{\infty} dss^2 J_1(sr)J_1(sr_b)J_1(sy).$$

We have to compute, from Eq. (29),

$$\mathcal{U} \equiv -\int_0^{\sqrt{t^2-u^2}} dy J_0\left(\lambda_s^{-1}\sqrt{t^2-u^2-y^2}\right)\partial_y\mathcal{I}(r, r_b, y). \tag{31}$$

Let us integrate $\mathcal{U}$ by part. Eq. (31) yields

$$\mathcal{U} = -\left[J_0\left(\lambda_s^{-1}\sqrt{t^2-u^2-y^2}\right)\mathcal{I}(r, r_b, y)\right]_0^{\sqrt{t^2-u^2}}$$
$$+ \int_0^{\sqrt{t^2-u^2}} dy\partial_y J_0\left(\lambda_s^{-1}\sqrt{t^2-u^2-y^2}\right)\mathcal{I}(r, r_b, y)$$
$$= -\mathcal{I}\left(r, r_b, \sqrt{t^2-u^2}\right) + J_0\left(\lambda_s^{-1}\sqrt{t^2-u^2}\right)\mathcal{I}(r, r_b, 0)$$
$$+ \int_0^{\sqrt{t^2-u^2}} dy \frac{\lambda_s^{-1} y J_1\left(\lambda_s^{-1}\sqrt{t^2-u^2-y^2}\right)}{\sqrt{t^2-u^2-y^2}}\mathcal{I}(r, r_b, y).$$

The second term in this last expression will yield the opposite of the singular term obtained for ψ_1 in Eq. (28), ensuring thereby the regularity of the field. We are thus left with

$$\frac{\psi(r,\tau)}{\mu_0 j_0 r_b} = \int_0^{\tau} dt \int_0^{t} du\mathcal{I}\left(r, r_b, \sqrt{t^2-u^2}\right) - \int_0^{\tau} dt \int_0^{t} du$$
$$\times \int_0^{\sqrt{t^2-u^2}} dy \frac{yJ_1\left(\lambda_s^{-1}\sqrt{t^2-u^2-y^2}\right)}{\lambda_s\sqrt{t^2-u^2-y^2}}\mathcal{I}(r, r_b, y). \tag{32}$$

Eq. (32) may have been obtained more directly by using Olevsky's theorem interchanging R_1 and R_2 in Eq. (21). The former

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decomposition was amenable to a — relatively — easy estimation of the "purely plasma" component $\psi_1(r, \tau)$ leading to a singular solution with time and space separated. This singular part of the solution was however canceled in $\psi_2(r, \tau)$. Eq. (32), that decomposes into two integrals involving $\mathcal{I}$, expresses in particular the fact that an exact solution in separated variables, time and space, cannot exist. The relevant decomposition (32) shows that the poloidal magnetic field component is the superposition of a "purely vacuum" term

$$\psi_v(r, \tau) \equiv \mu_0 j_0 r_b \int_0^\tau dt \int_0^t du \mathcal{I}\left(r, r_b, \sqrt{t^2 - u^2}\right), \tag{33}$$

and of a term involving the plasma contribution

$$\psi_p(r, \tau) \equiv -\mu_0 j_0 r_b \int_0^\tau dt \int_0^t du \times \int_0^{\sqrt{t^2-u^2}} dy \frac{y J_1\left(\lambda_s^{-1}\sqrt{t^2 - u^2 - y^2}\right)}{\lambda_s \sqrt{t^2 - u^2 - y^2}} \mathcal{I}(r, r_b, y). \tag{34}$$

3.3. Solution decomposing into a vacuum and a plasma parts

Section 3.2 eventually turned into a sort of pedagogically digression. The interchangeability between R_1 and R_2 in Eq. (21) appears as a fake one and one is left with the need to estimate the vacuum contribution (33) and the plasma contribution (34).

3.3.1. Vacuum Contribution

Let us first estimate $\psi_v(r, \tau)$. We have

$$\int_0^\tau dt \int_0^t du\, \mathcal{I}\left(r, r_b, \sqrt{t^2 - u^2}\right) = \int_0^\tau t dt \times \int_0^{\pi/2} \cos\alpha\, d\alpha\, \mathcal{I}(r, r_b, t\cos\alpha).$$

Let us evaluate

$$\int_0^{\pi/2} \cos\alpha d\alpha\, \mathcal{I}(r, r_b, t\cos\alpha) = \int_{\mathcal{D}_\alpha} d\alpha \frac{\cos\alpha \cos v}{\pi r r_b \sin v},$$

with

$$\frac{2 r r_b \cos v - r^2 - r_b^2}{t^2} = -\cos^2\alpha,$$

and where the definition domain $\mathcal{D}_\alpha$ of the integral is given by the condition $|r - r_b| < t\cos\alpha < r + r_b$. Three cases are to be considered: If $t \le |r - r_b|$, then $\mathcal{D}_\alpha = \varnothing$, from which it will follow that $\psi_v = 0$ for $\tau \le |r - r_b|$. If $|r - r_b| < t$ and $r + r_b > t$, then $\mathcal{D}_\alpha = [0; \arccos(|r - r_b|/t)]$. Finally, if $r + r_b \le t$, then $\mathcal{D}_\alpha = \varnothing$. A change of variables yields

$$\frac{\psi_v(r, \tau)}{\mu_0 j_0 r_b} = -\frac{1}{\pi}\int_0^\tau dt \int_{\mathcal{D}_v} \frac{\cos v}{\sqrt{t^2 - r^2 - r_b^2 + 2 r r_b \cos v}} dv,$$

where $\mathcal{D}_v$ is the definition domain of the integral in the v variable. The last integral is defined provided that $\cos v + a > 0$, where a is given by $a(r, r_b, t) \equiv (t^2 - r^2 - r_b^2)/(2 r r_b)$ and reads then

$$\int_{\mathcal{D}_v} \frac{\cos v}{\sqrt{t^2 - r^2 - r_b^2 + 2 r r_b \cos v}} dv = \frac{1}{\sqrt{2 r r_b}} \int_{\mathcal{D}_v} \frac{\cos v}{\sqrt{\cos v + a}} dv.$$

We shall use the primitive (See Gradshteyn and Ryzhik, 2007),

$$\int \frac{\cos x}{\sqrt{\cos x + a}} dx = \sqrt{2}\left\{2E\left(\gamma, \frac{1}{\varrho}\right) - F\left(\gamma, \frac{1}{\varrho}\right)\right\}, \tag{35}$$

for $|a| < 1$ and $0 \le x < \arccos(-a)$, with $\gamma = \arcsin\left(\sqrt{\frac{1 - \cos x}{1 + a}}\right)$ and $\varrho = \sqrt{2/(1 + a)}$. Here E is the elliptic integral of the second kind $E(\phi, k) = \int_0^\phi \sqrt{1 - k^2 \sin^2\alpha} d\alpha$ and F the elliptic integral of the first kind $F(\phi, k) = \int_0^\phi d\alpha / \sqrt{1 - k^2 \sin^2\alpha}$. Consequently, in the time and space domain satisfying $|r - r_b| < t$ and $r + r_b > t$, one will use Eq. (35) for the primitive. Using reduced (dimensionless) variables, one puts

$$g(\tilde{r}, \tilde{t}) \equiv \int_{\mathcal{D}_v} \frac{\cos v}{\sqrt{\cos v + a}} dv,$$

with, for $|\tilde{r} - 1| < \tilde{t}$ and $\tilde{r} + 1 > \tilde{t}$,

$$g(\tilde{r}, \tilde{t}) = -2\sqrt{2} E\left(\sqrt{\frac{\tilde{t}^2 - (\tilde{r} - 1)^2}{4\tilde{r}}}\right) + \sqrt{2} K\left(\sqrt{\frac{\tilde{t}^2 - (\tilde{r} - 1)^2}{4\tilde{r}}}\right),$$

where E and K denote, respectively, the complete elliptic integral of the second and first kind, and elsewhere, namely for $\tilde{t} \le |\tilde{r} - 1|$ or $\tilde{r} + 1 \le \tilde{t}$,

$$g(\tilde{r}, \tilde{t}) = 0.$$

Finally, the vacuum contribution to the poloidal magnetic field is obtained as a time integral of g, namely through

$$\tilde{\psi}_v(\tilde{r}, \tilde{\tau}) \equiv \frac{\psi_v(\tilde{r}, \tau)}{\mu_0 j_0 r_b} = -\frac{1}{\pi\sqrt{2\tilde{r}}} \int_0^{\tilde{\tau}} d\tilde{t} g(\tilde{r}, \tilde{t}). \tag{36}$$

Figure 2 represents the evaluation of $\tilde{\psi}_v$ in the time-space domain whereas Figure 3 shows the comparison between the

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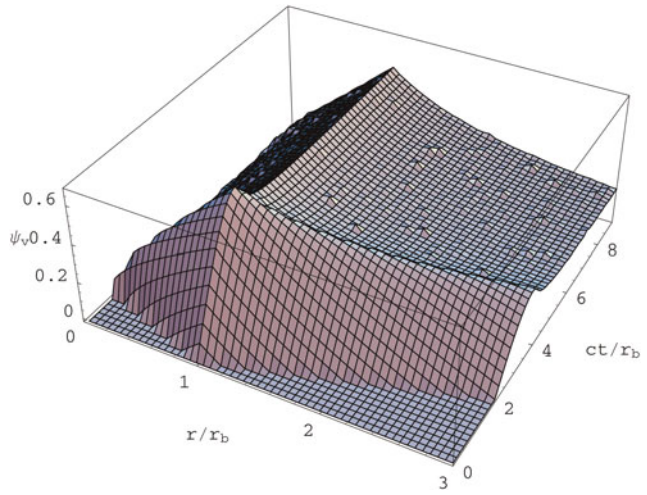


Fig. 2. (Color online) Contribution of the vacuum to the poloidal magnetic field. The representation is given in non-dimensional variables.

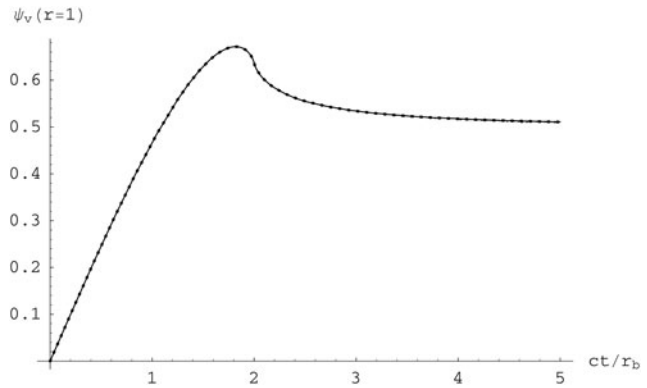


Fig. 3. Contribution of the vacuum to the poloidal magnetic field taken at the radius $\tilde{r} = 1$ using expression (36) (plain line) and inverting Laplace transform (15) putting $\eta = 0$ (bold dashed line). The representation is given in non-dimensional variables.

evaluation of the vacuum contribution of the poloidal magnetic field at the radius $r = r_b$ using Eq. (36) and its evaluation through the numerical Laplace transform inversion of Eqs. (15) for $\eta = 0$.

3.3.2. Plasma Contribution

Let us now consider the plasma contribution to the poloidal magnetic field (34). The plasma contribution involves the double integral

$$\mathcal{K}(r, r_b, t) \equiv \int_0^t du \int_0^{\sqrt{t^2-u^2}} dy \frac{yJ_1\left(\lambda_s^{-1}\sqrt{t^2-u^2-y^2}\right)}{\lambda_s\sqrt{t^2-u^2-y^2}} \mathcal{I}\left(r, r_b, y\right)$$

$$= \frac{t^2}{\pi r r_b \lambda_s} \int_0^{\pi/2} \cos^2\alpha \, d\alpha \int_0^{\pi/2} \sin\beta J_1\left(t\lambda_s^{-1}\cos\alpha\,\cos\beta\right)$$

$$\times \frac{r^2 + r_b^2 - t^2\cos^2\alpha\sin^2\beta}{\sqrt{4r^2r_b^2 - \left(r^2 + r_b^2 - t^2\cos^2\alpha\sin^2\beta\right)^2}} d\beta,$$

in which the product $\cos\alpha\,\sin\beta$ is constrained by the condition $|r - r_b|/t < \cos\alpha\,\sin\beta < (r + r_b)/t$. Let us make the following change of variables: $u = \cos\alpha\,\cos\beta$, $v = \cos\alpha\,\sin\beta$. One gets

$$\mathcal{K}(r, r_b, t) = \frac{t^2}{\pi r r_b \lambda_s} \iint_{\mathcal{D}_{uv}} du dv \frac{J_1\left(t\lambda_s^{-1}u\right)}{\sqrt{1-u^2-v^2}}$$

$$\times \frac{v\left[r^2 + r_b^2 - t^2v^2\right]}{\sqrt{4r^2r_b^2 - \left(r^2 + r_b^2 - t^2v^2\right)^2}},$$

where the integration domain $\mathcal{D}_{uv}$ is contained into the upper quarter of the unit circle. We can thus write

$$\mathcal{K}(r, r_b, t) = \frac{t^2}{\pi r r_b \lambda_s} \int_{[0;1]\cap\mathcal{D}_v} dv \frac{v\left[r^2 + r_b^2 - t^2v^2\right]}{\sqrt{4r^2r_b^2 - \left(r^2 + r_b^2 - t^2v^2\right)^2}}$$

$$\times \int_0^{\sqrt{1-v^2}} du \frac{J_1\left(t\lambda_s^{-1}u\right)}{\sqrt{1-u^2-v^2}},$$

with $\mathcal{D}_u \equiv]|r - r_b|/t; (r + r_b)/t\,[$. However, we have, using Eq. (6.552) in Gradshteyn and Ryzhik (2007),

$$\int_0^{\sqrt{1-v^2}} du \frac{J_1\left(t\lambda_s^{-1}u\right)}{\sqrt{1-v^2}\sqrt{1-\frac{u^2}{1-v^2}}} = \int_0^1 dx \frac{J_1\left(\frac{t\sqrt{1-v^2}}{\lambda_s}x\right)}{\sqrt{1-x^2}}$$

$$= \frac{\pi}{2}\left[J_{1/2}\left(\frac{t\sqrt{1-v^2}}{2\lambda_s}\right)\right]^2,$$

so that

$$\mathcal{K}(r, r_b, t) = \frac{t^2}{2r r_b \lambda_s} \int_{[0;1]\cap\mathcal{D}_v} dv \frac{v\left[r^2 + r_b^2 - t^2v^2\right]}{\sqrt{4r^2r_b^2 - \left(r^2 + r_b^2 - t^2v^2\right)^2}}$$

$$\times \left[J_{1/2}\left(\frac{t\sqrt{1-v^2}}{2\lambda_s}\right)\right]^2.$$

In terms of the dimensionless variables introduced in Section 2, we are left with

$$\tilde{\psi}_p(\tilde{r}, \tilde{\tau}) = -\eta \int_0^{\tilde{\tau}} d\tilde{t} \frac{\tilde{t}^2}{2\tilde{r}} \int_{[0;1]\cap\mathcal{D}_v} dv \frac{v\left[\tilde{r}^2 + 1 - \tilde{t}^2v^2\right]}{\sqrt{4\tilde{r}^2 - \left(\tilde{r}^2 + 1 - \tilde{t}^2v^2\right)^2}}$$

$$\times \left[J_{1/2}\left(\eta\frac{\tilde{t}\sqrt{1-v^2}}{2}\right)\right]^2. \quad (37)$$

Figure 4 illustrates the time behavior of the plasma contribution to the poloidal magnetic field at some given radii

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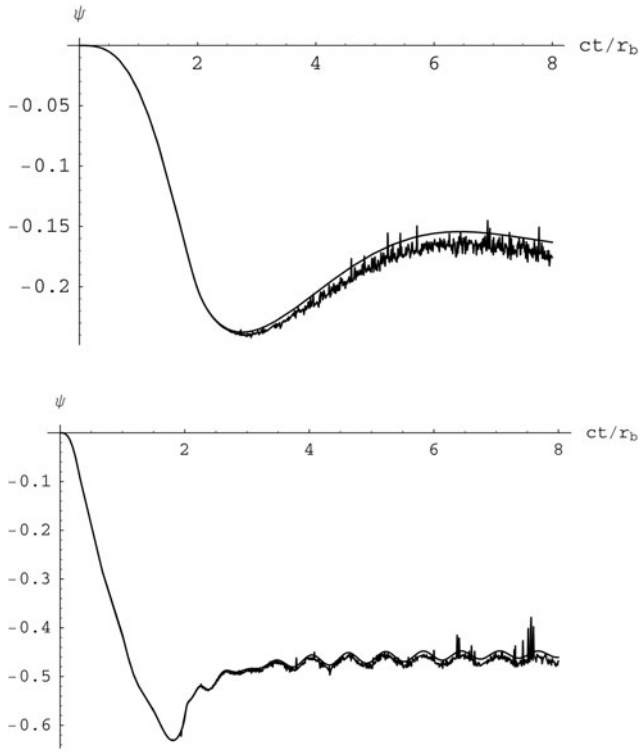


Fig. 4. Time behavior of the "plasma" contribution to the poloidal magnetic field at $r = r_b$ for $\eta = 1$ (upper plot) and $\eta = 10$ (bottom plot). The smooth curves come from the numerical Laplace inverse transform of the full solution given in Eq. (15) subtracted from the vacuum solution (See Fig. 3). The curves showing some numerical noise have been obtained from the numerical evaluation of Eq. (37). The representation is given in non-dimensional variables.

and compares it to the one obtained from the Laplace inversion approach.

3.3.3. The Full Solution

One can now proceed to the computation of the full poloidal magnetic field by summing the so-called vacuum (36) and plasma (37) contributions. The time-space representation of the poloidal magnetic field in non-dimensional coordinates is plotted in Figure 5. In the limit $\eta \to \infty$, one can check that the poloidal magnetic field vanishes, meaning that the plasma contribution compensates exactly the vacuum one. This is in agreement with the asymptotes obtained in the large η limit in Eq. (18). Therefore, the most interesting behavior takes place for intermediary values of η, namely for values of the beam radius of the order of some electron skin depths. The ($m = 0$ component of the) poloidal magnetic field is first created in the initial stage that looks like a choc. In the vacuum limit $\eta \to 0$, the poloidal magnetic field quickly tends to a steady value after this initial choc stage. For low values of η, it displays a transient intermediary stage showing a small number of wave-like variations that rapidly damp. When η gets larger, these plasma-like ripples last longer while their adimensional time period (in terms of ct/r_b) decreases.

4. CONCLUSION

In this article, a detailed computation of the poloidal magnetic field created by the sudden injection of an electron beam with sharp edges into a plasma has been presented.

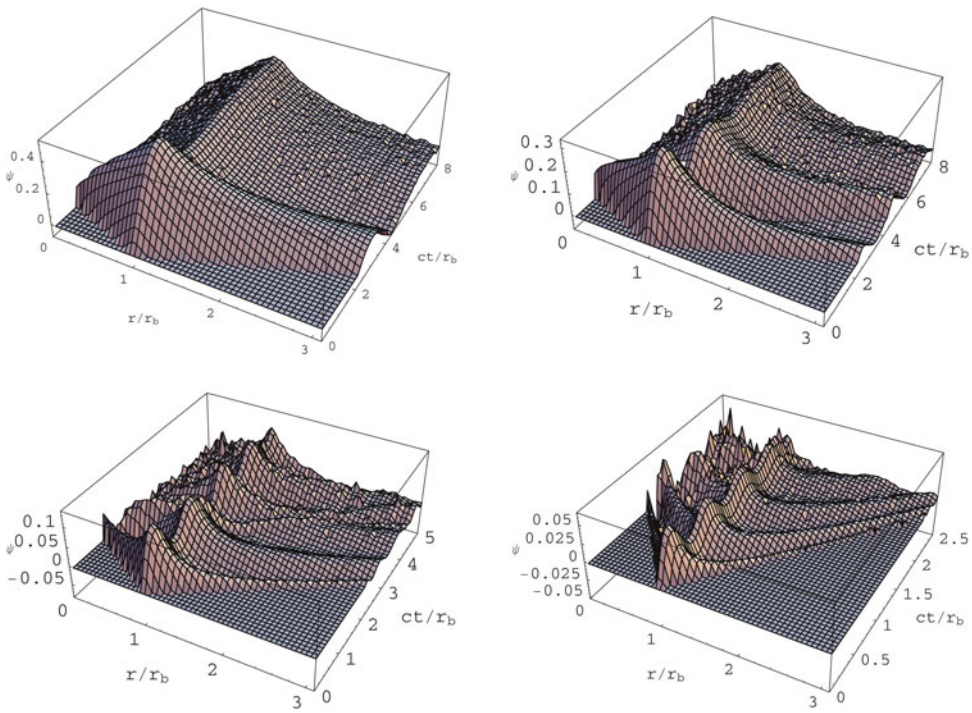


Fig. 5. (Color online) Space and initial time evolution of the total poloidal magnetic field in terms of non-dimensional variables for $\eta = 1$ (left upper plot), $\eta = 2$ (right upper plot), $\eta = 5$ (left bottom plot) and $\eta = 10$ (right bottom plot).

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The control parameter of the problem appears to be η, namely the ratio of the beam radius to the electron skin depth. Two different ways to address analytically this problem have been considered. One way amounts to proceed to an inverse Laplace transform of a solution having a relatively simple expression, given in Eq. (15), in the reciprocal space. Another way involves the Riemann's method. This leads to some rather intricate calculations in which, e.g., causality conditions manifest through some integrals of triple products of Bessel functions. However one advantage of the later approach is to supply a formula that decomposes explicitly, thanks to Olevsky's theorem (Olevsky, 1952), into a vacuum and a plasma contributions, a fact that follows from the linearity of the wave equation but that is not obvious to recover from the Laplace transform approach in Eq. (15). This decomposition provides an easy way to capture the space-time behavior of the poloidal magnetic field.

This study should hopefully provide a calculation useful to benchmark numerical codes or to develop analytical models.

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REFERENCES

Borisov, V.V. (2002). On transient waves in dispersive media produced by moving point sources. *J. Phys. A: Math. Gen.* **35**, 5403–5409.

Borisov, V.V. & Simonenko, I. (1994). Transient waves generated by a source on a circle. *J. Phys. A: Math. Gen.* **27**, 6243.

Bret, A., Firpo, M.C. & Deutsch, C. (2004). Collective electromagnetic modes for beam-plasma interaction in the whole k space. *Phys. Rev. E* **70**, 046401.

Bret, A., Firpo, M.C. & Deutsch, C. (2005*a*). Bridging the gap between two stream and filamentation instabilities. *Laser Part. Beams* **23**, 375–383.

Bret, A., Firpo, M.C. & Deutsch, C. (2005*b*). Characterization of the initial filamentation of a relativistic electron beam passing through a plasma. *Phys. Rev. Lett.* **94**, 115002.

Bret, A., Firpo, M.C. & Deutsch, C. (2005*c*). Transverse beam temperature effects on mixed two-stream/filamentation unstable modes. *Nucl. Instr. Meth. Phys. Res. A* **544**, 427–429.

Bret, A., Firpo, M.C. & Deutsch, C. (2007). About the most unstable modes encountered in beam plasma interaction physics. *Laser Part. Beams* **25**, 117–119.

Bret, A., Gremillet, L., Benisti, D. & Lefebvre, E. (2008). Exact relativistic kinetic theory of an electron-beam–plasma system: Hierarchy of the competing modes in the system-parameter space. *Phys. Rev. Lett.* **100**, 205008.

Deutsch, C. (2004). Penetration of intense charged particle beams in the outer layers of precompressed thermonuclear fuels. *Laser Part. Beams* **22**, 115–120.

Deutsch, C., Bret, A., Firpo, M.C., Gremillet, L., Lefebvre, E. & Lifschitz, A. (2008). Onset of coherent electromagnetic structures in the relativistic electron beam deuterium-tritium fuel interaction of fast ignition concern. *Laser Part. Beams* **26**, 157–165.

Deutsch, C., Furukawa, H., Mima, K., Murakami, M. & Nishihara, K. (1996). Interaction physics of the fast ignitor concept. *Phys. Rev. Lett.* **77**, 2483–2486.

Deutsch, C., Furukawa, H., Mima, K., Murakami, M. & Nishihara, K. (1997). Interaction physics of the fast ignitor concept. *Laser Part. Beams* **15**, 557–564.

Duffy, D. (2001). *Green's Functions with Applications*. New York: Chapman & Hall.

Esarey, E., Schroeder, C. & Leemans, W. (2009). Physics of laser-driven plasma-based electron accelerators. *Phys. Plasmas* **81**, 1229–1285.

Firpo, M.C. & Lifschitz, A.F. (2007). Early out-of beam-plasma evolution: Possible formation of a beam ring structure. *Int. J. of Mod. Phys. B* **21**, 633–636.

Firpo, M.C., Lifschitz, A.F., Lefebvre, E. & Deutsch, C. (2006). Early out-of-equilibrium beam-plasma evolution. *Phys. Rev. Lett.* **96**, 115004.

Gradshteyn, I.S. & Ryzhik, I.M. (2007). *Table of Integrals, Series, and Products*. New York.

Green, J.S., Lancaster, K.L., Akli, K.U., Gregory, C.D., Beg, F.N., Chen, S.N., Clark, D., Freeman, R.R., Hawkes, S., Hernandez-Gomez, C., Habara, H., Heathcote, R., Hey, D.S., Highbarger, K., Key, M.H., Kodama, R., Krushelnick, K., Musgrave, I., Nakamura, H., Nakatsutsumi, M., Patel, N., Stephens, R., Storm, M., Tampo, M., Theobald, W., Van Woerkom, L., Weber, R.L., Wei, M.S., Woolsey, N.C. & Norreys, P.A. (2007). Surface heating of wire plasmas using laser-irradiated cone geometries. *Nat. Phys.* **3**, 853–856.

Hoffmann, D., Blazevic, A., Ni, P., Rosmej., Roth, M., Tahir, N.A., Tauschwitz, A., Udrea, S., Varentsov, D., Weyrich, K. & Maron, Y. (2005). Present and future perspectives for high energy density physics with intense heavy ion and laser beams. *Laser Part. Beams* **23**, 47–53.

Kumar, N., Pukhov, A. & Lotov, K. (2010). Self-modulation instability of a long proton bunch in plasmas. *Phys. Rev. Lett.* **104**, 255003.

Kuppers, G., Salat, A. & Wimmel, H.K. (1973). Current and fields induced in plasmas by relativistic electron beams with arbitrary radial and axial density profiles. *Plasma Phys.* **15**, 429–439.

Norreys, P.A., Beg, F.N., Sentoku, Y., Silva, L.O., Smith, R.A. & Trines, R.M.G.M. (2009). Intense laser-plasma interactions: New frontiers in high energy density physics. *Phys. Plasmas* **16**, 041002.

Olevsky, M.N. (1952). On the Riemann function for the differential equation $u_{xx} - u_{tt} + [p_1(x) + p_2(t)]u = 0$. *Dokl. Akad. Nauk.* **87**, 337.

Tabak, M., Hammer, J., Glinsky, M., Kruer, W., S. C. Wilks, W., Woodworth, J., Campbell, E.M., Perry, M.D. & Mason, R.J., 1994. Ignition and high gain with ultrapowerful lasers. *Phys. Plasmas* **1**, 1626–1634.

Taguchi, T., Antonsen, T.M., Liu, C.S. & Mima, K. (2001). Structure formation and tearing of an MeV cylindrical electron beam in a laser-produced plasma. *Phys. Rev. Lett.* **86**, 5055–5058.

Valkó, P.P. & Abate, J. (2004). Comparison of sequence accelerators for the gaver method of numerical laplace transform inversion. *Comput. Math. Appl.* **48**, 629–636.

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APPENDIX

A. LINEAR PLASMA RESPONSE

Let us consider the momentum equation for plasma electrons

$$n_p m_e\left[\frac{\partial \mathbf{v}_p}{\partial t}+(\mathbf{v}_p.\nabla)\mathbf{v}_p\right]=-en_p(\mathbf{E}+\mathbf{v}_p\times\mathbf{B})-\nabla.P - \nu m_e n_p \mathbf{v}_p, \tag{38}$$

in which P denotes a stress pressure tensor and ν an effective collision frequency.

At initial time, the plasma is at rest. The linear hypothesis consists in discarding the nonlinear convective term in the left hand side and the Lorentz force that are second order in $\mathbf{v}_p$. In particular, during the validity domain of the model, the plasma electrons are non-relativistic. In the present modeling, one neglects also the pressure tensor term, which amounts to a cold plasma hypothesis, and collisions. Together with the fluid equation of continuity

$$\frac{\partial n_p}{\partial t}+\nabla.(n_p\mathbf{v}_p)=0,$$

and using the definition of the electron plasma frequency

$$\omega_p^2=\frac{n_p e^2}{m_e\varepsilon_0},$$

this yields, keeping only the first order terms in $\mathbf{v}_p$, the desired simple linear plasma response

$$\frac{\partial \mathbf{j}_p}{\partial t}=\varepsilon_0\omega_p^2\mathbf{E}.$$

B. VALIDITY DOMAIN

From Maxwell equations in cylindrical geometry, one obtains

$$\frac{\partial E_z^{(0)}}{\partial r}=\frac{\partial B_\theta^{(0)}}{\partial t}, \tag{39}$$

$$\frac{\partial^2 E_r^{(0)}}{\partial t^2}+\omega_p^2 E_r^{(0)}=-\frac{1}{\varepsilon_0}\frac{\partial j_{br}^{(0)}}{\partial t}. \tag{40}$$

In order to estimate the source term in Eq. (40), one needs to compute $j_{br}^{(0)}$. This requires to solve the fluid equations for the electron beam that read

$$\frac{\partial n_b}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(rn_b v_{br})=0,$$

$$\left(\frac{\partial}{\partial t}+v_{br}\frac{\partial}{\partial r}\right)(\gamma v_{br})=-\frac{e}{m_e}(E_r+v_{bz}B_\theta),$$

$$\left(\frac{\partial}{\partial t}+v_{br}\frac{\partial}{\partial r}\right)(\gamma v_{bz})=-\frac{e}{m_e}(E_z+v_{br}B_\theta).$$

Let us introduce the ratio $\alpha=n_{b0}/n_p$. This will be assumed to be a small parameter. One can then expand the fields in terms of α and get $n_b=n_{b0}+\alpha n_{b1}$, $v_{br}=\alpha v_{br1}$, $j_{br}=\alpha j_{br1}=-\alpha e n_{b0}\, v_{br1}$. One obtains at first order

$$\frac{\partial n_{b1}}{\partial t}-\frac{1}{er}\frac{\partial}{\partial r}(rj_{br1})=0,$$

$$\frac{\gamma_0}{en_{b0}}\frac{\partial j_{b1r}}{\partial t}=\frac{e}{m_e}(E_{1r}+v_{b0z}B_{1\theta}).$$

Therefore one has

$$\frac{1}{\varepsilon_0}\frac{\partial j_{b1r}}{\partial t}=\frac{e^2 n_{b0}}{\gamma_0 m_e\varepsilon_0}(E_{1r}+v_{b0z}B_{1\theta})=\frac{e^2 n_p}{\gamma_0 m_e\varepsilon_0}(E_r+v_{b0z}B_\theta)$$

$$=\frac{\omega_p^2}{\gamma_0}(E_r+v_{b0z}B_\theta).$$

This yields

$$\frac{\partial^2 E_r^{(0)}}{\partial t^2}+\omega_p^2(1+\gamma_0^{-1})E_r^{(0)}=-\frac{\omega_p^2}{\gamma_0}v_{b0z}B_\theta^{(0)},$$

the solution of which reads

$$E_r^{(0)}(r,t)=\frac{\omega_p}{\gamma_0\sqrt{1+\gamma_0^{-1}}}\int_0^t v_{b0z}B_\theta^{(0)}(r,s)\sin\left[\left(\omega_p\sqrt{1+\gamma_0^{-1}}\right)(s-t)\right]ds,$$

We have

$$j_{pz}(r,t)=\int_0^t\varepsilon_0\omega_p^2E_z^{(0)}(r,s)ds=\int_0^t\varepsilon_0\omega_p^2\int^r\frac{\partial B_\theta^{(0)}}{\partial s}dsdr'$$

$$=\varepsilon_0\omega_p^2\int^r B_\theta^{(0)}(r',t)dr'.$$

Therefore, by virtue of the fluid momentum equation for plasma electrons (38), the electrostatic description of plasma electrons is valid as long as the modulus of

$$-en_p\frac{\omega_p}{\gamma_0\sqrt{1+\gamma_0^{-1}}}\int_0^t v_{b0z}B_\theta^{(0)}(r,s)\sin\left[\left(\omega_p\sqrt{1+\gamma_0^{-1}}\right)(s-t)\right]ds \tag{41}$$

$$=\frac{j_0^2}{\alpha}\frac{\eta\mu_0 r_b}{\gamma_0\sqrt{1+\gamma_0^{-1}}}\int_0^{ct/r_b}\tilde{B}_\theta^{(0)}(r,s')\sin\left[\left(\frac{r_b}{c}\omega_p\sqrt{1+\gamma_0^{-1}}\right)\left(s'-\frac{ct}{r_b}\right)\right]ds' \tag{42}$$

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is large in front of the modulus of

$$\varepsilon_0 \omega_p^2 B_\theta^{(0)}(r,t) \int^r B_\theta^{(0)}(r',t)dr' \tag{43}$$

$$= \varepsilon_0 \omega_p^2 \mu_0^2 j_0^2 r_b^2 \tilde{B}_\theta^{(0)}(r,t) \int^r \tilde{B}_\theta^{(0)}(r',t)dr'. \tag{44}$$

In the last Eqs. (42) and (44), one has used the dimensionless notation of Section 2. One has to evaluate the ratio

$$\kappa\left(\frac{r}{r_b}, \frac{ct}{r_b}\right) = \frac{\int_0^{ct/r_b} \tilde{\psi}(r,s') \sin\left[\left(\eta\sqrt{1+\gamma_0^{-1}}\right)\left(s' - \frac{ct}{r_b}\right)\right] ds'}{\alpha\eta\gamma_0\sqrt{1+\gamma_0^{-1}}\tilde{\psi}(r,t)\int_0^{r/r_b} \tilde{\psi}(\tilde{r},t)d\tilde{r}}.$$

Whatever the value of η, one immediately see that κ remains large for a small α even if the numerator tends to cancel on times of the order of some ct/r_b. When η becomes large, one can use the asymptotic behaviours written in Eq. (18). This ensures that κ remains large, since the denominator is of second order in an exponentially small quantity.

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