ON MULTIPLE CURVES AND SURFACES AS LIMITS

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As particular cases of the general problem of finding the most general limiting tangential form of a degenerate manifold, we discuss here the degeneration of a curve on a surface into one or more multiple components, and the degeneration of a surface on a threefold into a single surface counting multiply, the method used consisting essentially in finding the approximate form of the curve or surface as the limit is approached. The approach system is supposed in all cases to be contained in a linear system of curves or surfaces, on the surface or threefold considered.

1. *Single multiple curve.* We suppose that the limit curve C lies on a surface F and that C^r, or (rC), is approached on F by a variable curve $K(\lambda)$ of the complete system K, whose equation* is

$$K \equiv C^r + \xi_1 K_1 + \xi_2 K_2 + \ldots + \xi_\rho K_\rho = 0, \qquad (1)$$

to which C^r belongs, the parameters ξ_i of $K(\lambda)$ in this system being functions of λ which vanish with λ and are regular at $\lambda = 0$. Expanding these functions in powers of λ, we may write

$$K(\lambda) \equiv C^r + \lambda K^{(1)} + \lambda^2 K^{(2)} + \ldots + \lambda^r K^{(r)} + \lambda^{r+1} R(\lambda) = 0 \qquad (2)$$

as the equation of a general approach system, where $K^{(i)}$ is a fixed curve of the system K and $R(\lambda)$ a variable curve of the same system whose parameters ξ_i, in the system, are finite at $\lambda = 0$.

An approach through a pencil $C^r + \lambda K^{(1)}$, or through a system equivalent in the limit to a pencil, gives a specialized limiting tangential system; and to get the more general limiting tangential form, we must use an *osculating*† approach system of the form

$$K(\lambda) \equiv C^r + \lambda C^{r-1} S^{(1)} + \lambda^2 C^{r-2} S^{(2)} + \ldots + \lambda^r S^{(r)} + \lambda^{r+1} R(\lambda) = 0, \qquad (3)$$

* We shall use freely, throughout this paper, one and the same symbol to denote a curve itself, and a function whose vanishing determines the curve.

† Zeuthen employed systems of this kind in considering the possible limiting forms of plane cubic and quartic curves: cf. *Kjöbenhavn Skrifter* (5), 10 (1873), 287. The general approach system of this kind is characterized geometrically by the fact that, in the representation of the system $|K|$ by the points of a space of ρ dimensions, it is represented by a curve which passes through some point O of the (normal or projected) Veronese manifold M representing r-fold curves of the system $|C|$, while its osculating $[i]$ at O $(i = 1, 2, \ldots, r-1)$ lies in the space $T^{(i)}$ whose points represent all curves of the form $C^{r-i} S^{(i)}$, this space containing, but not necessarily coinciding with, the $(i+1)$-tangential space of M at O. For a triple line in the plane, for example, an approach system of plane cubics, which is to give six distinct limiting tangential points, must be represented in [9] by a curve osculating the Veronesean F^9 representing all threefold lines of the plane.

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where $S^{(i)}$ is a general curve of the system $|iC|$ on F and $R(\lambda)$ is as before. As $K(\lambda)$ approaches C^r in this system, the values of the *function* C on the r sheets of $K(\lambda)$ near any point of the curve C are all of order λ and approximate evidently to the roots of the equation

$$\left(\frac{C}{\lambda}\right)^r + S^{(1)}\left(\frac{C}{\lambda}\right)^{r-1} + \ldots + S^{(r)} = 0. \tag{4}$$

The points of C in whose neighbourhood this equation has two roots tending to coincide correspond to turning points of the approaching r-sheeted curve, along C; and the set of limiting tangential points of C^r, arising from these turning points, is given by the intersection of C with the curve represented by the vanishing of the discriminant of equation (4). The number of them, consequently, is $r(r-1)\gamma$, where γ is the grade of $|C|$ on F; and the limiting tangential system of C^r consists therefore of the tangents to C counted r-fold together with $r(r-1)\gamma$ plane pencils in the tangent planes to F at the limiting tangential points in question.

2. *Curve with several multiple components.* We can find, in a similar way, the general limiting tangential form of a curve $K(\lambda)$ on a surface F which approaches a composite curve consisting of several multiple components, say $C_1, \ldots, C_\sigma$, with multiplicities $r_1, \ldots, r_\sigma$ respectively. The appropriate approach system in this case is of the form

$$K(\lambda) \equiv C_1^{r_1} C_2^{r_2} \ldots C_\sigma^{r_\sigma} + \lambda K_1 + \lambda^2 K_2 + \ldots + \lambda^r K_r + \lambda^{r+1} R(\lambda) = 0, \tag{5}$$

where $r = \Sigma r_i$, and K_i is of the form

$$\Sigma\, C_1^{r_1 - e_1^i} C_2^{r_2 - e_2^i} \ldots C_\sigma^{r_\sigma - e_\sigma^i} S(e_1^i, e_2^i, \ldots, e_\sigma^i),$$

the index e_α^i being, for any α, a positive integer less than r_α^i, the summation extending to every permissible set of e_α^i whose sum is i, and $S(e_1^i, \ldots, e_\sigma^i)$ being a general curve of the system $|\Sigma_\alpha e_\alpha^i C_\alpha|$. In such an approach system the curve $K(\lambda)$ approaches its limiting form *evenly*, in the sense that all the infinitesimal distances of the limit curves C_i from the various sheets of $K(\lambda)$ which approach them are of the order of λ generally*, and a general curve* through any point of intersection, T, of two or more of the curves C_i meets all the branches of $K(\lambda)$ which pass near T in points whose distances from T are of the order of λ.

For those parts of $K(\lambda)$, in fact, for which only λ and C_1 are small, the principal part of equation (5) reduces, except for a factor $C_2^{r_2} \ldots C_\sigma^{r_\sigma}$, to the form

$$C_1^{r_1} + \lambda C_1^{r_1 - 1} S(1, 0, 0, \ldots, 0) + \lambda^2 C_1^{r_1 - 2} S(2, 0, 0, \ldots, 0) + \ldots + \lambda^{r_1} S(r_1, 0, 0, \ldots, 0) = 0, \tag{6}$$

and this is effectively a homogeneous equation of degree r_1 for the ratio $C_1 : \lambda$.

* The distances from C_i of the sheets of $K(\lambda)$ approaching C_i vanish, of course, at the actual intersections of these sheets with C_i and become indeterminate at the intersections of C_i with the other limit curves; and at an intersection T of C_i and C_j a curve in the direction of C_i, for example, meets $K(\lambda)$ in only r_j points near T, instead of in $r_i + r_j$ points.

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Near any point of intersection, T say, of the curves C_1 and C_2 for example, $(r_1 \geqslant r_2)$, equation (5) reduces effectively, except for the factor $C_3^{r_3} \dots C_\sigma^{r_\sigma}$, to a homogeneous equation of order r_1+r_2 in the three relevant infinitesimals C_1, C_2, λ, this being of the form

$$\sum_0^{r_2} C_1^{r_1-i} C_2^{r_2-i} [i] \lambda^i + \sum_1^{r_1-r_2-1} C_1^{r_1-r_2-i} [r_2] \lambda^{r_2+i} + \sum_0^{r_2} [r_2-i] \lambda^{r_1+i} = 0, \qquad (7)$$

each symbol of the form $[\alpha]$ on the left-hand side denoting a homogeneous polynomial, of degree α in C_1 and C_2, whose coefficients can be regarded as constants near T. To a first approximation the surface can be regarded as a plane near T, and the functions C_1, C_2 can be regarded as coordinates in this plane, relative to the tangents to C_1, C_2 at T as axes. Thus the form of $K(\lambda)$ near T can be described as that of $\Phi(\Gamma)$, where Γ is the plane curve, of order r_1+r_2, whose equation is

$$\sum_0^{r_2} X^{r_1-i} Y^{r_2-i} [i] + \sum_1^{r_1-r_2-1} X^{r_1-r_2-i} [r_2] + \sum_0^{r_2} [r_2-i] = 0, \qquad (8)$$

$[\alpha]$ now denoting a homogeneous polynomial of degree α in X and Y, and Φ denotes a transformation of *contraction* towards T in the ratio $\lambda : 1$. Thus a variable curve through T, not touching C_1 or C_2, meets $K(\lambda)$ in r_1+r_2 points near T, whose distances from T are all of the order of λ. And in the case of an intersection of more than two of the curves C_i a similar procedure is applicable.

Restricting ourselves now to the case in which all the intersections of the curves C_i are distinct, and denoting the grade of the system $|C_i|$ by γ_i, we see that the required limiting tangential form consists of (a) the tangents to the various limit curves, counted multiply according to the multiplicities of those curves; (b) plane pencils at a certain set of $r_i(r_i-1)\gamma_i$ limiting tangential points on each curve C_i—the set of intersections of C_i, namely, with a certain discriminantal curve of the system $|r_i(r_i-1)C_i|$; and (c) plane pencils of certain multiplicities at the various intersections of pairs of the curves C_i. To ascertain the multiplicity of any one of the latter, we note that the curve Γ, referred to in the preceding paragraph, has points of multiplicities r_1 and r_2 at the ends of the axes, and is therefore of class $2r_1r_2$; and $\Phi(\Gamma)$ tends therefore tangentially, as λ tends to zero, to a $(2r_1r_2)$-fold tangential point at T. Thus, in the limit, T is also a $(2r_1r_2)$-fold limiting tangential point of $K(\lambda)$; and any intersection, therefore, of any two curves C_i, C_j is the vertex of a plane pencil counting $(2r_ir_j)$-fold in the limiting tangential form of $K(\lambda)$. This completes therefore the description of the required limiting tangential form*.

* The tangents from T to the curve $\Phi(\Gamma)$ have all definite directions along which their points of contact tend to T; but of course they need not all be real even when T is real. Of the r_1+r_2 branches of $K(\lambda)$ which pass near T, $2r_2$ turn, after an odd number of bends, from the direction of C_2 to that of C_1, while the remaining r_1-r_2 come from the direction of C_1 and continue in the same direction after an even number of infinitesimal bends near T.

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We note in conclusion moreover that the limiting tangential form obtained has in fact the same rank as that of a general curve of the system $|\Sigma r_i C_i|$ on F. For if π_i, e_i, γ_i are the genus, rank and grade respectively of C_i, and if δ_{ij} is the number of intersections of C_i and C_j, the genus π of a general curve of the system $|\Sigma r_i C_i|$ is given by

$$2\pi - 2 = \Sigma r_i (2\pi_i - 2 - \gamma_i) + \Sigma r_i^2 \gamma_i + 2\Sigma r_i r_j \delta_{ij},$$

the last summation being taken over all pairs i, j, $(i \neq j)$, irrespective of the order in each pair; and this gives for the rank e of the same curve the formula

$$e = \Sigma r_i e_i + \Sigma r_i (r_i - 1) \gamma_i + \Sigma (2 r_i r_j) \delta_{ij},$$

which is seen to be the rank of the limiting tangential form which we have obtained.

3. *Single multiple surface.* By a similar procedure we may obtain the general limiting tangential form of an r-fold surface F approached on a three-fold V by an osculating approach system of surfaces $K(\lambda)$ of the system $|rF|$ on V. The values of the *function* F, on the r sheets of $K(\lambda)$ at any point of F, are given approximately by an equation of the form

$$F^r + \lambda F^{r-1} S^{(1)} + \ldots + \lambda^r S^{(r)} = 0; \tag{9}$$

and a *limiting tangential curve*, L say, of the r-fold surface now arises as the intersection of F with the surface Δ obtained by equating to zero the discriminant of the above equation. This curve L belongs, consequently, to the system $|r(r-1)C|$ on F, where C is a characteristic curve of F. For certain points T of F, whose number is t say, three roots of equation (9) coincide; and for certain other points D of F, whose number is d say, two pairs of roots of the same equation coincide, the points T being cusps and the points D ordinary double points* of L. By considering the conditions for (9) to have two roots in common with its first derivative, we find that T and D, counted twice, are the complete set of intersections of L with a curve L' of the system $|(r-1)(r-2)C|$; so that

$$t + d = \tfrac{1}{2} r (r-1)^2 (r-2) \gamma,$$

where γ is the grade of $|C|$ on F. Furthermore the equation (8) has a triple root if, and only if, it has a root in common with its first and second derived equations. These latter, however, represent surfaces of the systems $|(r-1)F|$ and $|(r-2)F|$ respectively, tending to F^{r-1} and F^{r-2} as λ tends to zero; and the $r(r-1)(r-2)\gamma$ common intersections of these two surfaces and the surface

* Cf., for example, the projection of a surface in ordinary space on to a multiple plane, in which the approach system can be taken as the system of transforms of the given surface by homologies of varying moduli, having the vertex and plane of projection as fundamental elements.

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represented by (9) tend evidently to points of F at which equation (9) has three roots tending to coincidence. Thus

$$t = r(r-1)(r-2)\gamma,$$

and

$$\delta = \tfrac{1}{2}r(r-1)(r-2)(r-3)\gamma.$$

It is easy to see that the double points and cusps of L give rise to no doubly infinite systems of limiting tangent planes of F^r; and the required limiting tangential form of $K(\lambda)$ consists therefore of (a) the tangent planes to F counted r-fold; and (b) those tangent planes to L which lie in the tangent solids to V at their respective points of contact. The curve L has $r(r-1)(r-2)\gamma$ cusps and $\frac{1}{2}r(r-1)(r-2)(r-3)\gamma$ double points, these together constituting its complete intersection with a further curve of the system $|(r-1)(r-2)C|$ on F.

It can easily be verified now that the limiting tangential system thus obtained for F^r has class and type equal to those of a general surface of the system $|rF|$ on V, namely (in the ordinary notation),

$$\bar{\mu}_2 = r\mu_2 + \bar{\epsilon}_1 = r\mu_2 + r(r-1)\epsilon_1 + r(r-1)^2\gamma,$$

$$\bar{\nu}_2 = r\nu_2 + \bar{X} = r\nu_2 + r(r-1)X,$$

where $\bar{\mu}_2$, $\bar{\nu}_2$ refer to the limiting tangential system, μ_2, ν_2 to the surface F, while $\bar{\epsilon}_1$, $\bar{X}$ are the rank of L and its class of immersion in V, and ϵ_1, X are the corresponding characters for a characteristic curve of F. It can also be verified easily that Severi's equations connecting the numerical invariants of two surfaces in algebraic (1, r) correspondence hold for the limiting (1, r) correspondence arising above between F and $K(\lambda)$.

The extension of the above, by the method of § 2, to the case of a surface decomposing into a number of multiple components evidently requires first a solution of the corresponding problem for a decomposition into a number of simple components, a problem of which no complete solution has as yet been given.

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