Math. Struct. in Comp. Science (2008), *vol.* 18, *pp.* 311–324. © 2008 Cambridge University Press
doi:10.1017/S0960129507006391 Printed in the United Kingdom

On asymptotic divergency in equivalential logics

ZOFIA KOSTRZYCKA

University of Technology, Luboszycka 3, 45-036 Opole, Poland
Email: z.kostrzycka@po.opole.pl

Received 29 September 2005; revised 11 June 2007

In this paper we characterise the equivalential reducts of classical and intuitionistic logics over a language with two propositional variables. We then investigate the size of the fraction of the tautologies of these logics against all formulas. Some methods from complex analysis are used to achieve this goal.

1. Introduction

For a given logical calculus, we investigate the proportion of the number of true formulas of a certain length n to the number of all formulas of such length. We are especially interested in the asymptotic behaviour of this fraction when n tends to infinity. If the limit exists, it is represented by a real number between 0 and 1, which we call *the density of truth* for the logic under investigation.

Let L be some logical calculus, $|T_n|$ be the number of tautologies of length n of that calculus and $|F_n|$ be a number of all formulas of that length. We define the density $\mu(L)$ by

$$\mu(L) = \lim_{n\to\infty} \frac{|T_n|}{|F_n|}.$$

This paper is a continuation of quantitative research in different logics. Our main interest is in classical and intuitionistic logic. The *densities of truth* for classical (and intuitionistic) logic of implication of one and two variables have already been determined (see Moczurad *et al.* (2000) and Kostrzycka (2003)), as has the *densities* of implicational-negational fragments of these calculi with one variable (see Zaionc (2004) and Kostrzycka and Zaionc (2004)).

In the case of equivalential reducts of these logics, the problem of the existence of the *density of truth* is more complicated. It has been proved in Matecki (2005) that the equivalential reduct of classical logic is not asymptotically convergent, which means that the *density of truth* (as a limit) does not exist. However, it is possible to determine superior and inferior limits for the fractions of tautologies, which seems to be worth knowing. In this paper we consider the equivalential reduct of intuitionistic logic with two variables and compare the result obtained with that for classical logic.

2. Equivalential reduct of intuitionistic logic INT_E

Equivalential algebras are the algebraic counterpart of the equivalential reducts INT_E of intuitionistic propositional logic, and have been widely investigated (see, for example, Słomczyńska (1996) and Wroński (1993)).

https://doi.org/10.1017/S0960129507006391 Published online by Cambridge University Press

The class E of equivalential algebras is equationally definable by the following identities:

$$\begin{aligned}(x \leftrightarrow x) \leftrightarrow y &= y\\ ((x \leftrightarrow y) \leftrightarrow z) \leftrightarrow z &= (x \leftrightarrow z) \leftrightarrow (y \leftrightarrow z)\\ (x \leftrightarrow y) \leftrightarrow ((x \leftrightarrow z) \leftrightarrow z) \leftrightarrow ((x \leftrightarrow z) \leftrightarrow z) &= x \leftrightarrow y.\end{aligned}$$

The class E forms a variety and is locally finite. The cardinality of the n-generated free algebra $F_E(n)$ is known only for $n = 1, 2$ and 3 and is equal to 2, 9 and 4415434, respectively; see Wroński (1993).

The n-generated free algebra $F_E(n)$ is isomorphic to the Lindenbaum algebra $AL(E(n))$ over a language with n variables. We will consider the algebra $AL(E(n))$ with $n = 2$.

We define the language $\mathscr{F}^{\{\leftrightarrow\}}$ consisting of equivalential formulas with two propositional variables p and q as follows.

Definition 1.

$$\begin{aligned}p, q &\in \mathscr{F}^{\{\leftrightarrow\}},\\ \phi \leftrightarrow \psi &\in \mathscr{F}^{\{\leftrightarrow\}} \text{ iff } \phi \in \mathscr{F}^{\{\leftrightarrow\}} \text{ and } \psi \in \mathscr{F}^{\{\leftrightarrow\}}.\end{aligned}$$

We can introduce an equivalence relation $\equiv$ in the set $\mathscr{F}^{\{\leftrightarrow\}}$ in the conventional way:

$$\alpha \equiv \beta \text{ iff } \alpha \leftrightarrow \beta \in INT_E. \tag{1}$$

The equivalence relation $\equiv$ is also a congruence relation and the quotient algebra $\mathscr{F}^{\{\leftrightarrow\}}/_{\equiv}$ is called the Lindembaum algebra of the logic INT_E. We denote it by

$$AL(INT_E) = \mathscr{F}^{\{\leftrightarrow\}}/_{\equiv}. \tag{2}$$

We have the following theorem from Wroński (1993).

Theorem 2. The Lindembaum algebra $AL(INT_E)$ consists of the following 9 equivalence classes:

$$\begin{aligned}I &= [p]_{\equiv}\\ II &= [q]_{\equiv}\\ III &= [p \leftrightarrow q]_{\equiv}\\ IV &= [(p \leftrightarrow q) \leftrightarrow q]_{\equiv}\\ V &= [(p \leftrightarrow q) \leftrightarrow p]_{\equiv}\\ VI &= [((p \leftrightarrow q) \leftrightarrow q) \leftrightarrow p]_{\equiv}\\ VII &= [((p \leftrightarrow q) \leftrightarrow p) \leftrightarrow q]_{\equiv}\\ VIII &= [(((p \leftrightarrow q) \leftrightarrow q) \leftrightarrow p) \leftrightarrow (((p \leftrightarrow q) \leftrightarrow p) \leftrightarrow q)]_{\equiv}\\ T &= [p \rightarrow p]_{\equiv}.\end{aligned}$$

The behaviour of the operator $\leftrightarrow$ is given in Table 1, and the diagram of the algebra $AL(INT_E)$ is given in Figure 1.

We will now recall the notion of an equivalential filter.

https://doi.org/10.1017/S0960129507006391 Published online by Cambridge University Press

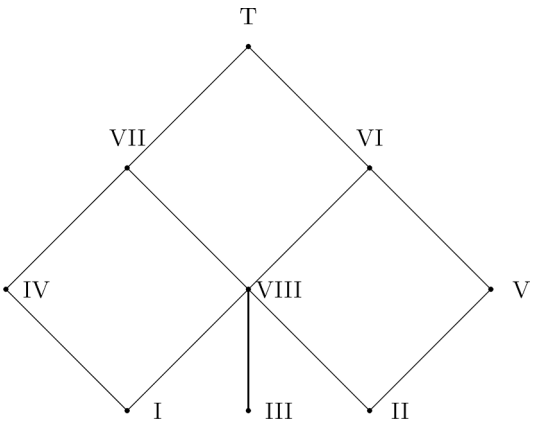


Fig. 1. The diagram of the algebra $AL(INT_E)$

Table 1. *The truth-table of the algebra* $AL(INT_E)$

$\leftrightarrow$	I	II	III	IV	V	VI	VII	VIII	T
I	T	III	V	VI	III	IV	I	IV	I
II	III	T	IV	III	VII	II	V	V	II
III	V	IV	T	V	IV	III	III	III	III
IV	VI	III	V	T	III	I	IV	I	IV
V	III	VII	IV	III	T	V	II	II	V
VI	IV	II	III	I	V	T	VIII	VII	VI
VII	I	V	III	IV	II	VIII	T	VI	VII
VIII	IV	V	III	I	II	VII	VI	T	VIII
T	I	II	III	IV	V	VI	VII	VIII	T

Definition 3. Let A be an equalential algebra. By a filter in A we mean a non-empty subset F of A such that for all $a, x \in A$:

(i) if $a \in F$, then $(a \leftrightarrow x) \leftrightarrow x \in F$,
(ii) if $a \in F$ and $a \leftrightarrow b \in F$, then $b \in F$.

Note that there is only one proper equivalential filter in the algebra $AL(INT_E)$. This is the set $K = \{T, VI, VII, VIII\}$. The new quotient algebra $AL(INT_E)/_K$ is given in Figure 2 and the operation $\leftrightarrow$ is characterised in Table 2.

Lemma 4. The algebra $AL(INT_E)/_K$ is equal to the Lindenbaum algebra $AL(CL_E)$ of the classical propositional logic of equivalence with two variables.

Proof. To prove this lemma, we split the set of all formulas into classes according to the formulas' behaviour on all possible evaluations. Since we have classical formulas built with exactly two propositional variables, we can evaluate them using four valuations: v_1

https://doi.org/10.1017/S0960129507006391 Published online by Cambridge University Press

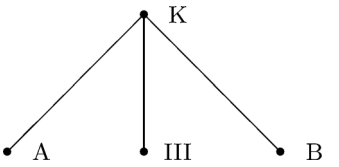


Fig. 2. The diagram of the algebra $AL(INT_E)/_K$, where $A = I \cup IV$, $B = II \cup V$.

Table 2. *The truth-table of the algebra* $AL(INT_E)/_K$

$\leftrightarrow$	A	B	III	K
A	K	III	B	A
B	III	K	A	B
III	B	A	K	III
K	A	B	III	K

associating 0 to p and 0 to q, v_2 associating 0 to p and 1 to q, v_3 associating 1 to p and 0 to q and v_4 associating 1 to p and 1 to q.

For any $i, j, k, l \in \{0, 1\}$, we use $F^{i,j,k,l}$ to mean the set of formulas ϕ from $\mathscr{F}^{\{\leftrightarrow\}}$ such that $v_1(\phi) = i$, $v_2(\phi) = j$, $v_3(\phi) = k$ and $v_4(\phi) = l$.

There are exactly four classes:

$$F^{0,0,1,1}$$
$$F^{0,1,0,1}$$
$$F^{1,0,0,1}$$
$$F^{1,1,1,1}.$$

The four classes $F^{i,j,k,l}$ are ordered by assuming that $F^{i,j,k,l} \leqslant F^{i',j',k',l'}$ if and only if $i \leqslant i'$, $j \leqslant j'$, $k \leqslant k'$ and $l \leqslant l'$. We can establish an operation of equivalence $\leftrightarrow$ on the classes $F^{i,j,k,l}$ by

$$F^{i,j,k,l} \leftrightarrow F^{i',j',k',l'} = F^{i \Leftrightarrow i', j \Leftrightarrow j', k \Leftrightarrow k', l \Leftrightarrow l'}$$

where $\Leftrightarrow$ stands for the classical equivalence defined on the set $\{0, 1\}$. The above construction gives us a four element lattice, which is exactly the same as the one shown in Figure 2, which gives us the following equalities:

$$\begin{aligned} A &= F^{0,0,1,1} \\ B &= F^{0,1,0,1} \\ III &= F^{1,0,0,1} \\ K &= F^{1,1,1,1}. \end{aligned}$$

Table 2 is identical to the one defined by the operation $\Leftrightarrow$ described above. We can now see that the class K is the class of all classical tautologies in two variables. □

https://doi.org/10.1017/S0960129507006391 Published online by Cambridge University Press

3. Counting formulas

In this section we establish some properties of the values characterising the numbers of formulas in different classes defined in our language. First, we set up the way we will measure the length of formulas.

Definition 5. We use $|\phi|$ to mean the length of the formula ϕ, which is the total number of occurrences of propositional variables in the formula, excluding the equivalence sign and parentheses. Formally,

$$\begin{aligned} |p| &= 1 \\ |q| &= 1 \\ |\phi \leftrightarrow \psi| &= |\phi| + |\psi|. \end{aligned}$$

We will consider the set $F_n \subseteq \mathscr{F}^{\{\leftrightarrow\}}$ of all formulas of length n. Its appropriate subclasses are defined as follows.

Definition 6.

$$Y_n = F_n \cap Y \quad \text{for any} \quad Y \in 2^{AL(INT_E)}.$$

The number of formulas in F_n is finite for any $n \in N$ and will be denoted by $|F_n|$. Consequently, any subset from the class $2^{AL(INT_E)}$ is also finite for any $n \in N$.

Definition 7. We use $|Y_n|$ to mean the number of formulas from the class Y_n.

From Definition 1 we see that any formula from $\mathscr{F}^{\{\leftrightarrow\}}$ may be interpreted as a binary planar tree with the internal nodes labelled by the operator $\leftrightarrow$ and the external nodes labelled by the propositional variables p and q. Then we immediately have the following lemma.

Lemma 8. The numbers $|F_n|$ are given by the following recursion:

$$|F_0| = 0, \quad |F_1| = 2, \tag{3}$$

$$|F_n| = \sum_{i=1}^{n-1} |F_i||F_{n-i}|. \tag{4}$$

Proof. The statement is obvious. □

The numbers $|F_n|$ are closely connected to the Catalan numbers C_n. The well-known non-recursive formula for C_n is

$$C_n = \frac{1}{n}\binom{2n-2}{n-1}. \tag{5}$$

It is easy to see that $|F_n| = 2^n \cdot C_n$.

4. Generating functions

The main tools for dealing with sequences of numbers are *generating functions*: see, for example, Wilf (1994) and Flajolet and Sedgewick (2001). Suppose, we have a system

https://doi.org/10.1017/S0960129507006391 Published online by Cambridge University Press

of non-linear equations $\overrightarrow{y_j} = \Phi_j(z, y_1, ...y_m)$ for $1 \leqslant j \leqslant m$, where any $y_j = \sum_{n=0}^{\infty} a_j z^n$. The following result, which is known as the Drmota–Lalley–Woods theorem, is of great importance in solving the system both explicitly and implicitly; see Flajolet and Sedgewick (2001, Theorem 8.13, page 71).

Theorem 9. Consider a non-linear polynomial system $\overrightarrow{y} = \Phi(\overrightarrow{y})$ that is a-proper, a-positive and a-irreducible. Then all component solutions y_i have the same radius of convergence $\rho < \infty$, so there exist functions h_j analytic at the origin such that

$$y_j = h_j(\sqrt{1 - z/\rho}), \quad (z \to \rho^-). \tag{6}$$

In addition, all other dominant singularities are of the form $\rho\omega$ with ω being a root of unity. If, furthermore, the system is a-aperiodic, all y_j have ρ as their unique dominant singularity. In that case, the coefficients admit a complete asymptotic expansion of the form

$$[z^n]y_j(z) \sim \rho^{-n}\left(\sum_{k\geqslant 1} d_k n^{-1-k/2}\right). \tag{7}$$

The expression on the right-hand side of (7) may be transformed by the so-called Transfer Lemma (Flajolet and Odlyzko 1990) into a formula defining the value of the coefficients $[z^n]y_j(z)$ explicitly. So the a-aperiodicity of a system of equations is a very desirable property. Unfortunately, the system we are considering does not have this property, so we need another formula to approximate the coefficients $[z^n]y_j(z)$. We take advantage of the Szegö Lemma (Szegö 1975, Theorem 8.4); see also Wilf (1994, Theorem 5.3.2).

Lemma 10 (Szegö). Szego Let $v(z)$ be analytic in $|z| < 1$ with a finite number of singularities $e^{i\varphi^{(k)}}$, $k = 1, ..., s$ at the circle $|z| = 1$. Suppose that in the neighbourhood of each $e^{i\varphi^{(k)}}$, we have that $v(z)$ has an expansion of the form

$$v(z) = \sum_{p\geqslant 0} v_p^{(k)}(1 - ze^{-i\varphi^{(k)}})^{a^{(k)}+pb^{(k)}},$$

where $a^{(k)} \in C$ and $b^{(k)} > 0$ and the branch chosen above for the expansion equals $v(0)$ for $z = 0$. Then

$$[z^n]\{v(z)\} = \sum_{k=1}^{s}\sum_{p=0}^{\xi(q)} v_p^{(k)}\binom{a^{(k)} + pb^{(k)}}{n}(-e^{i\varphi^{(k)}})^n + O(n^{-q}),$$

with

$$\xi(q) = \max_{k=1,...,s}[(1/b^{(k)})(q - Re(a^{(k)}) - 1)].$$

From the first part of Theorem 9, we see that in the case of generating functions forming a-proper, a-positive and a-irreducible system of equations, we have $a^{(k)} = 0$ and $b^{(k)} = 1/2$. Further simplification depends on the numbers of singularities. In our application we will only have one or two singularities. In both cases we will be satisfied with an error bound of $O(n^{-2})$. Then $\xi(q) = 2$. Moreover, $\binom{0}{n} = \binom{1}{n} = 0$ for $n > 1$. Additionally, in order

https://doi.org/10.1017/S0960129507006391 Published online by Cambridge University Press

to carry out the required calculations, we admit the singularities situated on the circle $|z| = \rho$. If there is only one singularity $z_0 = \rho$, we have the following lemma.

Lemma 11. Let $v(z)$ be analytic in $|z| < \rho$, with $z = \rho$ being the only singularity on the circle $|z| = \rho$. If $v(z)$ in the vicinity of $z = \rho$ has an expansion of the form

$$v(z) = \sum_{p \geqslant 0} v_p (1 - z/\rho)^{\frac{p}{2}}, \tag{8}$$

where the branch chosen above for the expansion equals $v(0)$ for $z = 0$, then

$$[z^n]\{v(z)\} = \left(v_1 \binom{1/2}{n} (-1)^n + O(n^{-2}) \right) \cdot \rho^{-n}. \tag{9}$$

Proof. Consider the function $\widehat{v}(z) = v(z \cdot \rho) = \sum_{p \geqslant 0} v_p (1 - z)^{\frac{p}{2}}$. From the Szegö Lemma, we have

$$[z^n]\{\widehat{v}(z)\} = \left(\widehat{v}_1 \binom{1/2}{n} (-1)^n + O(n^{-2}) \right). \tag{10}$$

Because $v(z) = \sum_{n=0}^{\infty} a_j z^n$, we have $\widehat{v}(z) = \sum_{n=0}^{\infty} a_j (z\rho)^n$. Hence

$$[z^n]\{v(z)\} = [z^n]\{\widehat{v}(z)\} \cdot \rho^{-n}. \tag{11}$$

□

For two singularities $z = \rho$ and $z = -\rho$, we have the following lemma.

Lemma 12. Let $v(z)$ be analytic in $|z| < \rho$ with $z_1 = \rho$ and $z_2 = -\rho$ the only singularities on the circle $|z| = \rho$. If $v(z)$ in the neighbourhood of $z = \rho$ has an expansion of the form

$$v(z) = \sum_{p \geqslant 0} v_p^{(1)} (1 - z/\rho)^{\frac{p}{2}}, \tag{12}$$

where the branch chosen above for the expansion is equal to $v(0)$ for $z = 0$, and again if $v(z)$ in the neighbourhood of $z = -\rho$ has an expansion of the form

$$v(z) = \sum_{p \geqslant 0} v_p^{(2)} (1 + z/\rho)^{\frac{p}{2}}, \tag{13}$$

where the branch chosen above for the expansion is equal to $v(0)$ for $z = 0$, then

$$[z^n]\{v(z)\} = \left(\left(v_1^{(1)} + v_1^{(2)} \cdot (-1)^n \right) (-1)^n \binom{1/2}{n} + O(n^{-2}) \right) \rho^{-n}. \tag{14}$$

Proof. The proof is analogous to the proof of Lemma 11. □

For technical reasons we will need to know the rate of growth of the expression

$$\binom{1/2}{n} (-1)^n$$

appearing in formula (14).

Lemma 13.

$$\binom{1/2}{n} (-1)^{n+1} = O(n^{-3/2}).$$

https://doi.org/10.1017/S0960129507006391 Published online by Cambridge University Press

Proof. The statement is obvious. □

From Lemmas 11–13 we get the following lemma.

Lemma 14. Suppose a function $w(z)$ satisfies the assumptions of Lemma 12 and a function $v(z)$ satisfies the assumptions of Lemma 11. Then the limit of

$$\frac{[z^n]\{w(z)\}}{[z^n]\{v(z)\}}$$

is given by the formula

$$\lim_{n\to\infty}\frac{[z^n]\{w(z)\}}{[z^n]\{v(z)\}} = \lim_{n\to\infty}\frac{w_1^{(1)} + w_1^{(2)}\cdot(-1)^n}{v_1}.$$

Proof. By applying the main formulas from the simplified versions of the Szegö Lemma (Lemma 11 and Lemma 12) and the equation from Lemma 13, we get

$$\begin{aligned}\lim_{n\to\infty}\frac{[z^n]\{w(z)\}}{[z^n]\{v(z)\}} &= \lim_{n\to\infty}\frac{\left(\left(w_1^{(1)} + w_1^{(2)}\cdot(-1)^n\right)\cdot(-1)^n\binom{1/2}{n} + O(n^{-2})\right)\rho^{-n}}{\left(v_1\binom{1/2}{n}(-1)^n + O(n^{-2})\right)\rho^{-n}}\\ &= \lim_{n\to\infty}\frac{\left(w_1^{(1)} + w_1^{(2)}\cdot(-1)^n\right)O(n^{-3/2}) + O(n^{-2})}{v_1 O(n^{-3/2}) + O(n^{-2})}\\ &= \lim_{n\to\infty}\frac{w_1^{(1)} + w_1^{(2)}\cdot(-1)^n}{v_1}.\end{aligned}$$

□

5. Calculation of the generating functions

In this section we are going to determine generating functions for the classes of all equivalential formulas, equivalential classical tautologies and equivalential intuitionistic tautologies. A cursory analysis of Table 2 (and Table 1 as well) gives important information about how formulas from each class are built. For example, the formulas p, $(p \leftrightarrow q) \leftrightarrow q$, $q \leftrightarrow (p \leftrightarrow q)$, $(q \leftrightarrow p) \leftrightarrow q$, $q \leftrightarrow (q \leftrightarrow p)$ and so on, belong to the class A. We see that a formula belongs to A if the number of occurrences of the variable q is even while the variable p occurs an odd number of times, or when the variable p is the only variable occurring (an odd number of times) in the formula. This means that the class A consists of formulas of an odd length only. Hence we have the following power series for the generating function f_A:

$$f_A(z) = z + 8z^3 + 224z^5 + \ldots \tag{15}$$

Analogously, the expansion of generating function for tautologies K is as follows:

$$f_K(z) = 2z^2 + 40z^4 + 1344z^6 + \ldots \tag{16}$$

As can be seen, each equivalential classical tautology has an even length. Because $T \subset K$, the same holds for the class T of equivalential intuitionistic tautologies. In this situation, if we want to calculate the densities $\mu(CL_E)$ and $\mu(INT_E)$, we have to limit

https://doi.org/10.1017/S0960129507006391 Published online by Cambridge University Press

attention to the formulas of even length. But even then we have to determine the necessary generating functions f_K, f_T and f.

First, notice that the generating function f for numbers $|F_n|$ is the one for Catalan numbers with variable $2z$.

Lemma 15.

$$f(z) = \frac{1}{2} - \frac{\sqrt{1-8z}}{2}. \tag{17}$$

Proof. The statement is obvious. □

From Table 2 we obtain the following system of equations:

$$(*)\begin{cases} f_A &= 2(f_A f_K + f_B f_{III}) + z \\ f_B &= 2(f_A f_{III} + f_B f_K) + z \\ f_{III} &= 2(f_A f_B + f_{III} f_K) \\ f_K &= f_A^2 + f_B^2 + f_{III}^2 + f_K^2, \end{cases}$$

where the functions f_A, f_B, f_{III}, f_K are the generating functions for the numbers $|A_n|$, $|B_n|$, $|III_n|$, $|K_n|$, respectively.

The system is a-positive, a-proper and a-irreducible but not a-aperiodic. The last fact results, for example, from the expansions (15) and (16).

To determine the function f_K from the above system, we can simplify it significantly. From Figure 2 we can observe the following lemma.

Lemma 16. The following equalities hold between the appropriate generating functions:

$$f_A = f_B \tag{18}$$

$$f_{III} = f_K. \tag{19}$$

Proof. The first equality follows from the symmetry of the diagram of the algebra $AL(CL_E)$ shown in Figure 2. It is easy to see that in any formula $\alpha \in A_i$ we may exchange variables p into q and q into p and obtain the appropriate formula from B_i. To prove the equality (19), observe that applying (18) to the system (*) gives

$$f_{III} = 2(f_A^2 + f_{III} f_K) \tag{20}$$

$$f_K = 2f_A^2 + f_{III}^2 + f_K^2. \tag{21}$$

By reduction of f_A,

$$f_K = f_{III} - 2f_{III} f_K + f_{III}^2 + f_K^2. \tag{22}$$

Solving this with the boundary condition $f_K(0) = 0$, we get $f_K = f_{III}$. □

Hence we have a much simpler system of equations:

$$(**)\begin{cases} f_A = 4f_A f_K + z \\ f_B = f_A \\ f_{III} = f_K \\ f_K = 2(f_A^2 + f_K^2). \end{cases}$$

https://doi.org/10.1017/S0960129507006391 Published online by Cambridge University Press

Lemma 17. The generating function f_A is

$$f_A(z) = \frac{\sqrt{1+8z} - \sqrt{1-8z}}{8}. \tag{23}$$

Proof. From the disjunction of classes A, B, III and K, and from (18) and (19), we get $2f_K = f - 2f_A$. Applying this to the first equation from the system (**), we have

$$f_A(z) = (f(z) - 2f_A(z))f_A(z) + z. \tag{24}$$

After a suitable simplification, we get the following quadratic equation:

$$4f_A^2 + f_A(1-f) - z = 0. \tag{25}$$

Solving this with the boundary condition $f_A(0) = 0$, we have

$$f_A = \frac{f - 1 + \sqrt{(1-f)^2 + 16z}}{8},$$

and replacing f with (17), gives (23). □

We now get the following corollary from the equality $2f_K = f - 2f_A$ and (23).

Corollary 18. The generating function f_K for the numbers $|K_n|$ is

$$f_K(z) = \frac{2 - \sqrt{1+8z} - \sqrt{1-8z}}{8}. \tag{26}$$

We are now ready to analyse Table 1 characterising the algebra $AL(INT_E)$. The appropriate system of equations consists of nine equations written in terms of the functions $f_I - f_{VIII}$ and f_T. We will not write it explicitly; it is enough to mention that the system again is a-positive, a-proper, a-irreducible but not a-aperiodic. We can solve this system of equations by taking advantage of the way we obtain the algebra $AL(INT_E)/_K = AL(CL_E)$.

Observation 19. The following equalities hold between the appropriate generating functions:

$$f_I + f_{IV} = f_{II} + f_V = f_A \tag{27}$$

$$f_{VI} + f_{VII} + f_{VIII} + f_T = f_K\,. \tag{28}$$

From Figure 1 (and the symmetry of that diagram), we have the following observation.

Observation 20. The following equalities hold between the appropriate pairs of generating functions:

$$f_I = f_{II} \tag{29}$$

$$f_{IV} = f_V \tag{30}$$

$$f_{VI} = f_{VII}\,. \tag{31}$$

Further analysis of Table 1 gives us the following lemma.

Lemma 21. The generating function f_{VI} for the numbers $|VI_n|$ is

$$f_{VI} = \frac{1}{32}(S - 2 - A - B) \tag{32}$$

https://doi.org/10.1017/S0960129507006391 Published online by Cambridge University Press

where

$$\begin{aligned} A &= \sqrt{1-8z} \\ B &= \sqrt{1+8z} \\ S &= \sqrt{2-2AB+8\sqrt{1+A+B+AB-12(A-B)z-144z^2}}. \end{aligned}$$

Proof. From Table 1 we obtain the following equalities between appropriate generating functions:

$$f_V = 2(f_I f_{III} + f_{II}(f_{VII} + f_{VIII}) + f_{III} f_{IV} + f_V(f_{VI} + f_T)) \tag{33}$$

$$f_{VI} = 2(f_I f_{IV} + f_{VI} f_T + f_{VII} f_{VIII}) \tag{34}$$

$$f_{VIII} = 2(f_{VI} f_{VII} + f_{VIII} f_T). \tag{35}$$

Applying (31) and (28) to (35) gives

$$f_{VIII} = 2(f_{VI}^2 + f_{VIII}(f_K - f_{VIII} - 2f_{VI})). \tag{36}$$

Applying (27), (30) and (28) to (34) gives

$$f_{VI} = 2((f_A - f_V)f_V + f_{VI}(f_K - 2f_{VI})). \tag{37}$$

Applying (29) – (28) to (33), and using $f_{III} = f_K$, after simplification, we get

$$f_V = 2(f_A f_K + f_A(f_{VI} + f_{VIII}) + f_V(f_K - 2f_{VIII} - 2f_{VI})). \tag{38}$$

Note that the equations (38), (37) and (36) form a system of three equations in the unknown functions f_V, f_{VI} and f_{VIII}.

By reduction, we get

$$f_V = \frac{f_A(-1 - 4f_K + \sqrt{16f_{VI}^2 + (1 + 4f_{VI} - 2f_K)^2} + 2f_K)}{2\sqrt{16f_{VI}^2 + (1 + 4f_{VI} - 2f_K)^2}} \tag{39}$$

$$f_{VIII} = \frac{1}{4}\left(-1 - 4f_{VI} + \sqrt{16f_{VI}^2 + (1 + 4f_{VI} - 2f_K)^2} + 2f_K\right). \tag{40}$$

After substitution of (39) in the equation (37), we get an algebraic equation of degree 4 in the unknown function f_{VI}. In order to solve this we had to make intensive use of the *Mathematica* package: from four solutions, we chose the one fulfilling the boundary condition $f_{VI}(0) = 0$. After intensive simplification, we finally get (32). □

From Lemma 21 and from (40) we get the following corollary.

Corollary 22. The generating function f_{VIII} for the numbers $|VIII_n|$ is

$$f_{VIII} = \frac{1}{32}\left(\sqrt{2}\sqrt{4 + A^2 + 4B + B^2 + 2A(2 + B) + S^2} - 2 - A - B - S\right) \tag{41}$$

where

$$\begin{aligned} A &= \sqrt{1-8z} \\ B &= \sqrt{1+8z}, \\ S &= \sqrt{2-2AB+8\sqrt{1+A+B+AB-12(A-B)z-144z^2}}. \end{aligned}$$

https://doi.org/10.1017/S0960129507006391 Published online by Cambridge University Press

Finally, we have the following corollary.

Corollary 23. The generating function f_T for the numbers of intuitionistic tautologies $|T_n|$ is

$$f_T = \frac{1}{32}\left(14 - A - B - S - \sqrt{2}\sqrt{4 + A^2 + B(4+B) + 2A(2+B) + S^2}\right) \tag{42}$$

where

$$\begin{aligned} A &= \sqrt{1-8z} \\ B &= \sqrt{1+8z}, \\ S &= \sqrt{2 - 2AB + 8\sqrt{1 + A + B + AB - 12(A-B)z - 144z^2}}\,. \end{aligned}$$

6. Counting asymptotic densities

In this section we expand the derived generating functions around their singularities. We will consider functions f, f_K and f_T.

Lemma 24. $z_1 = \frac{1}{8}$ is the only singularity of the function f located in $|z| \leqslant \frac{1}{8}$.

Proof. The statement is obvious. □

Lemma 25. $z_1 = \frac{1}{8}$ and $z_2 = -\frac{1}{8}$ are the only singularities of f_K and f_T located in $|z| \leqslant \frac{1}{8}$.

Proof. It is easy to see that the function f_K has two singularities $z_1 = \frac{1}{8}$ and $z_2 = -\frac{1}{8}$. Because $f_{III} = f_K$, we get that f_{III} has the same singularities as f_K, only. We know that the function f_{III} appears in the system of equations corresponding to Table 1, so from Theorem 9 we conclude that other functions from the system also only have two singularities. This is also true for the function f_T. □

Theorem 26. The expansion of the function f in a neighbourhood of $z = \frac{1}{8}$ is

$$f(z) = f_0 + f_1\sqrt{1-8z} + \ldots$$

where

$$f_0 = \frac{1}{2}, \quad f_1 = -4.$$

Proof. From Theorem 9, we know that there is a function h_j that is analytic at the origin and such that

$$f(z) = h_j(\sqrt{1-8z}), \quad (z \to \frac{1}{8}^{-}).$$

By the substitution $t := \sqrt{1-8z}$, we obtain a function $h_j(t)$ that is analytic at $t = 0$. Then $f_1 = h_j'(t=0)$. □

Theorem 27. The expansions of the function f_K in the neighbourhoods of $z = \frac{1}{8}$ and $z = -\frac{1}{8}$ are

$$f_K(z) = k_0^{(1)} + k_1^{(1)}\sqrt{1-8z} + \ldots$$

https://doi.org/10.1017/S0960129507006391 Published online by Cambridge University Press

where

$$k_0^{(1)}) = \frac{1}{8}\left(2-\sqrt{2}\right)$$
$$k_1^{(1)} = -1$$

and

$$f_K(z) = k_0^{(2)} + k_1^{(2)}\sqrt{1+8z} + \ldots$$

where

$$k_0^{(2)} = \frac{1}{8}\left(2-\sqrt{2}\right)$$
$$k_1^{(2)} = -1.$$

Proof. The proof is analogous to the proof of Theorem 26. □

Analogously, we can calculate the appropriate coefficients of the generating function f_T of intuitionistic tautologies.

Theorem 28. The expansions of the function f_T in the neighbourhoods of $z = \frac{1}{8}$ and $z = -\frac{1}{8}$ are

$$f_T(z) = t_0^{(1)} + t_1^{(1)}\sqrt{1-8z} + \ldots$$

where

$$t_0^{(1)} = 0.0517192\ldots$$
$$t_1^{(1)} = -0.4599704\ldots$$

and

$$f_T(z) = t_0^{(2)} + t_1^{(2)}\sqrt{1+8z} + \ldots$$

where

$$t_0^{(2)} = 0.0517192\ldots$$
$$t_1^{(2)} = -0.4599704\ldots$$

We have not quoted the exact values of $t_0^{(1)}$, $t_1^{(1)}$, $t_0^{(2)}$, $t_1^{(2)}$, because they are too long to be written here.

Proof. The above coefficients were found using the *Mathematica* package. □

Theorem 29. The densities of truth of the equivalential reducts of classical and intuitionistic logic do not exist.

Proof. Note that the densities of truth of these logics are given as the following limits (see Lemma 14):

$$\mu(CL_E) = \lim_{n\to\infty}\frac{[z^n]\{f_K(z)\}}{[z^n]\{f(z)\}} = \lim_{n\to\infty}\frac{-1-1\cdot(-1)^n}{-4}$$
$$\mu(INT_E) = \lim_{n\to\infty}\frac{[z^n]\{f_T(z)\}}{[z^n]\{f(z)\}} = \lim_{n\to\infty}\frac{-0.4599704\ldots-0.4599704\ldots\cdot(-1)^n}{-4}.$$

https://doi.org/10.1017/S0960129507006391 Published online by Cambridge University Press

It is easy to see that the above limits do not exist. □

Examining the asymptotic behaviour of the fractions of classical and intuitionistic tautologies (of even length) among all formulas of even length, we get

$$\lim_{n\to\infty} \frac{[z^{2n}]\{f_K(z)\}}{[z^{2n}]\{f(z)\}} = \lim_{n\to\infty} \frac{-\frac{1}{8} - \frac{1}{8}\cdot(-1)^{2n}}{-\frac{1}{2}} = \frac{1}{2} \tag{43}$$

$$\lim_{n\to\infty} \frac{[z^{2n}]\{f_T(z)\}}{[z^{2n}]\{f(z)\}} = \lim_{n\to\infty} \frac{-0.4599704... - 0.4599704...\cdot(-1)^{2n}}{-\frac{1}{2}} \approx 0.23... \tag{44}$$

From (43) and (44) we get the following results.

Corollary 30. The probability of finding an intuitionistic equivalential tautology of even length among all equivalential formulas of such length is asymptotically about 23%.

Corollary 31. The probability of finding a classical equivalential tautology of even length among all equivalential formulas of such length is asymptotically equal to 50%.

Corollary 32. The relative probability of finding an intuitionistic equivalential tautology of even length among the classical ones is asymptotically about 46%.

We find the final result quite surprising. Recall that in the language with two variables and implication, the relative probability is about 97% (Kostrzycka 2003). Analogously, in the language with implication, negation and one variable, the relative probability is about 93% (Kostrzycka and Zaionc 2004). This means that the equivalence connective is much more distinguishable between intuitionistic and classical logics than the other well-known operators.

References

Flajolet, P. and Odlyzko, A.S. (1990) Singularity analysis of generating functions. *SIAM J. on Discrete Math.* **3** (2) 216–240.

Flajolet, P. and Sedgewick, R. (2001) Analytic combinatorics: functional equations, rational and algebraic functions. INRIA, Number 4103.

Kostrzycka, Z. (2003) On the density of implicational parts of intuitionistic and classical logics. *Journal of Applied Non-Classical Logics* **13** (3) 295–325.

Kostrzycka, Z. and Zaionc, M. (2004) Statistics of intuitionistic versus classical logics. *Studia Logica* **76** 307–328.

Matecki, G. (2005) Asymptotic density for equivalence. *Electronic Notes in Theoretical Computer Science* **140** 81–91.

Moczurad, M., Tyszkiewicz, J. and Zaionc, M. (2000) Statistical properties of simple types. *Mathematical Structures in Computer Science* **10** 575–594.

Słomczyńska, K. (1996) Equivalential algebras. Part I: Representation. *Algebra Universalis* **35** 524–547.

Szegö, G. (1975) *Orthogonal polynomials*, Fourth edition, AMS Colloquium Publications **23**.

Wilf, H.S. (1994) *Generating functionology*, Second edition, Academic Press.

Wroński, A. (1993) On the free equivalential algebras with three generators. *Bulletin of the Section of Logic* **22** 37–39.

Zaionc, M. (2004) On the asymptotic density of tautologies in logic of implication and negation. *Reports on Mathematical Logic* **38**.

https://doi.org/10.1017/S0960129507006391 Published online by Cambridge University Press